Parafermionic observables and their applications

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Lattice models have been introduced as discrete models for real life experiments. These models have been used to model a large variety of phenomena, ranging from ferroelectrics to lattice gas. They also provide discretizations of Euclidean and Quantum Field Theories and are as such important from the point of view of theoretical physics. While the original motivation came from physics, they later appeared to be extremely complex and rich mathematical objects, whose study provided an area of cross-fertilization between different fields of mathematics (Algebra, Combinatorics, Probability, Complex Analysis, Spectral Theory to cite a few) and physics (Quantum Field Theory, Condensed Matter Physics, Conformal Field Theory).

The zoo of lattice models is very diverse: it includes models of spin-glasses, quantum chains, random surfaces, spin systems, interacting percolation systems, percolation models, polymers, etc. The special class of models interesting us here is a family of models of interfaces defined on planar lattices. These models undergo a phase transition, at which an extraordinary rich behavior occurs. Through two fundamental examples, we try to illustrate an approach combining probabilistic techniques and ideas coming from analysis on graphs to describe this behavior.

A first example: the Self-Avoiding-Walk (SAW) The SAW model was first introduced by Orr in 1947 as a combinatorial puzzle. In 1953, Nobel prize winner Paul Flory popularized (and rediscovered) SAWs by proposing it as a mathematical model for the spatial position of polymer chains. While very simple to define, the SAW has turned out to be a very interesting model, leading to a rich mathematical theory helping develop techniques that found applications in many other domains of statistical physics. To name but a few examples of tools that emerged from the study of SAWs, the lace expansion technique was developed to understand the SAW in dimension $d > 5$, and the Schramm-Loewner Evolution was

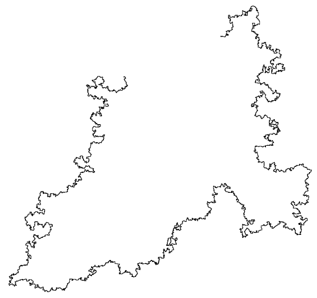

Figure 1: A typical 1000 steps self-avoiding walk.

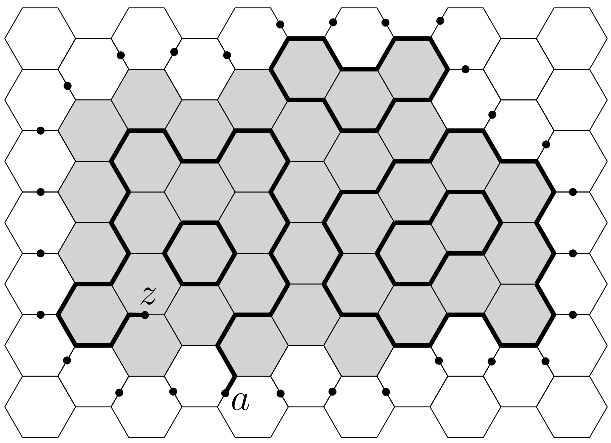

Figure 2: A loop configuration in $\widehat{\mathscr{E}}(\Omega, a, z)$ with an interface from a to z. The dots correspond to the boundary of Ω.

introduced to describe the scaling limit of the 2D loop erased random-walk, a model directly motivated by the SAW.

Let us describe the model more formally (see [MS93] for more details and references). Consider the hexagonal lattice $\mathbb{H}$ (one may also work with the square lattice, but some results presented below use some integrability properties of the model that is specific to the hexagonal lattice). A path is a sequence of neighboring vertices $\gamma_1, \ldots, \gamma_n$. It is self-avoiding if the map $k \mapsto \gamma_k$ is one-to-one. For each n, the model is defined by assigning equal probability to all self-avoiding paths with n vertices starting from 0.

Originally, Flory was interested in the geometric properties of the random path. In particular, he focused on the average distance $\|\gamma_n\|$ between γ_n and 0. Via a clever argument, he predicted that this average distance grows with n roughly like $n^{3/4}$. This prediction was very important since it conjectures a behavior which is very different from the random walk. Interestingly, Flory's prediction was based on two assumptions which are not satisfied by the SAW. Nevertheless, destiny can be sweet and the actual behavior is indeed $n^{3/4}$: the two mistakes made by Flory (one for each assumptions) miraculously cancel each other.

Before discussing the question of the mean-displacement further, let us step back and focus on Orr's original contribution to the problem at hand. In his article, Orr computed the number of SAW (on the square lattice) of length less or equal to 6. With today's computers and clever algorithms, one may be able to enumerate SAWs on the hexagonal lattice up to length 105, but no exact formula giving the number of SAWs of length n in terms of n seems to emerge from such computations. Nevertheless, some non trivial things can still be said about the number of SAWs. For instance, a simple sub-multiplicativity argument (the number of SAWs of length $n+m$ is smaller than the number of SAWs of length n times the number of SAWs of length m) implies that the number of SAWs grows exponentially fast, with a specific rate of growth μ_c depending on the lattice, and called the *connective constant of the lattice*. Much more elaborated physics arguments provided by the Coulomb gas formalism or Conformal Field Theory refine this prediction, and suggest that this number is roughly $n^{11/32} \cdot \mu_c^n$. Interestingly, μ_c will not be the same for the square lattice as for the hexagonal lattice. Nonetheless, the polynomial correction $n^{11/32}$ is present in both cases: the exponent 11/32 is universal.

Despite the precision of the previous predictions, the best results are very far from tight. Hammersley and Welsh proved that the number of SAWs of length n is between μ_c^n and $e^{O(\sqrt{n})}\mu_c^n$ without computing the constant μ_c (their argument dealt originally with the square lattice but it can easily be generalized to the hexagonal lattice). Concerning the mean-displacement, it is not rigorously known whether the average distance to the origin grows faster than $n^{1/2}$. Worse, while the radius of a SAW of length n is obviously larger than $n^{1/2}$, it does not imply much on the end-point, and it is in fact unknown whether the average of $\|\gamma_n\|$ is larger than a constant times $n^{1/2}$, a statement most of us would consider tautological. Concerning upper bounds, it was proved only recently [DH13] that the SAW is sub-ballistic, in the sense that the average of $\|\gamma_n\|$ behaves like $o(n)$ as n tends to infinity. We encourage the reader to try to improve these results on his own (for instance to provide any type of quantitative upper bound). This should illustrate the intrinsic difficulty of the model.

The previous contributions on SAWs are relying on techniques that were developed roughly fifty years ago. Since then, very few new tools have been discovered in two dimensions, with a notable exception that we want to mention now. This idea combines combinatorial techniques that are reminiscent from the original approach with intuition from the theory of discrete holomorphic functions. The main object of interested is a certain *observable* of the model, i.e. the average of a certain random variable. Let us spend some time to define it properly.

From now on, a *discrete domain* will be a collection of half-edges intersecting a family of faces of the hexagonal lattice forming the closure of a simply connected domain of the plane; see Fig. 2 (disregard the definition of $\widehat{\mathscr{E}}(\Omega, a, z)$, it will become relevant only later). Half-edges have two endpoints: one vertex of $\mathbb{H}$ and one *mid-edge*. From now on, a SAW will systematically run between two mid-edges (it boils down to extending the SAW by two half-edges).

Let Ω be a discrete domain and a be a mid-edge on the boundary, i.e. at the end of only one half-edge in Ω (see Fig. 2). Fix $x, \sigma \geq 0$ to be determined later. For a mid-edge $z \in \Omega$, define the *parafermionic observable* via the formula

$$F(z) = F_{\Omega,a,x,\sigma}(z) := \sum_{\omega} \exp(-\mathrm{i}\sigma \mathrm{W}_\omega(a,z))\, x^{\#\,\text{vertices in}\,\omega},$$

where the summation runs over SAWs from a to z staying in Ω. In the definition above, $\mathrm{W}_\omega(a,z)$ is the *winding* or total rotation of the direction in radians when the SAW ω is oriented from a to z. In other words, it is equal to $\pi/3$ times the difference between the number of left and right turns of ω.

The term involving $\mathrm{W}_\omega(a,z)$ may appear as an unnecessary complication. Indeed, for $\sigma = 0$, we obtain the generating function of the SAWs in Ω from a to z which seems like a very natural object to consider. The advantage of this term is that, when σ and x are tuned properly, F satisfies nice local relations as a function of z. Namely, let v be a vertex in the interior of Ω and p, q and r be the three mid-edges next to v. We identify v, p, q, r with their complex affixes. If

$$x = \frac{1}{\sqrt{2+\sqrt{2}}} \qquad \text{and} \qquad \sigma = \frac{5}{8},$$

then F satisfies

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0. \tag{1}$$

The set of equations (1) indexed by vertices v in Ω has a beautiful interpretation in terms of discrete contour integrals. Indeed, fix a sequence $\Gamma := (f_0, \ldots, f_k = f_0)$ of adjacent faces of Ω and define the discrete contour integral of F along Γ by the formula

$$\oint_\Gamma F(z)dz = \sum_{i=0}^{k-1} (f_{i+1} - f_i) F\big(z_i\big) = 0,$$

where f_i denotes the affix of the center of the corresponding face, and z_i the center of the edge between the faces f_i and f_{i+1}.

Equation (1) corresponds to the fact that the integral of F along the "triangular" contour composed of the three faces around the vertex v is equal to 0. Since any contour integral can be written as the sum of the triangular contours inside it, the relations (1) imply that the integral of F along *any* discrete contour vanishes. This property is reminiscent of a classical property of holomorphic functions. For this reason, one may think of F as a discrete version of a holomorphic function.

A word of caution: imagine for a moment that we wish to determine F using only its boundary values and the relations (1). We have one unknown variable $F(z)$ by mid-edge, and one relation per vertex. For generic domains, this is vastly insufficient, and we are therefore apparently facing a dead end: the fact that the discrete contour integrals vanish is providing little information on the observable F. In conclusion, a function satisfying the relations (1) can be seen as some kind of weakly discrete holomorphic function, but the relations do not allow us to do as much as the standard notion of holomorphicity does.

Fortunately, the property above is not meaningless. A careful analysis of contour integrals going along the boundary of well chosen domains Ω implies that the value $\sqrt{2+\sqrt{2}}$ mentioned above has to be the connective constant of $\mathbb{H}$. We refer to [DS12b] for the proof of this result. Let us mention that the value of the connective constant was predicted by Nienhuis in [Nie82, Nie84] using completely different techniques. The fact that μ_c has such a simple form can almost be considered as an anomaly. Except for trees and one dimensional lattices, the connective constant is not predicted to have any special form (except for the 3.12^2 lattice, which is obtained from the hexagonal lattice by a simple transformation). As an example, the connective constant of the square lattice can be approximated but no prediction currently exists concerning its exact value. In fact, it is even unknown whether it should be rational or algebraic for instance.

Computing the connective constant should be considered as a stepping stone towards a bigger goal since physicists and mathematicians are ultimately interested in the critical behavior of the model. Let us depart from our combinatorial question (counting SAWs) to enter the realm of phase transitions in statistical physics.

Consider a simply connected domain Ω together with two points a and b on its boundary. Also consider the graph $\Omega_\delta = \Omega \cap \delta\mathbb{H}$ for $\delta > 0$. Let a_δ and b_δ be two mid-edges on the boundary of Ω_δ close to a and b. We think of the family of triplets $(\Omega_\delta, a_\delta, b_\delta)$ as more and more refine (as $\delta \searrow 0$) discrete approximations of (Ω, a, b). Let us assume that the graphs Ω_δ are discrete domains[1]. We define a model of random interface $\gamma_{(\Omega_\delta, a_\delta, b_\delta)}$ as follows:

[1]Even though they obviously have no reason to be, one may easily alter the definition of Ω_δ so that the next discussion is still valid. We therefore prefer to ignore this difficulty.

SAWs from a_δ to b_δ in Ω_δ have probability proportional to $x^{\#\text{ vertices}}$ while other paths have probability zero.

If x is too small, the SAW is too penalized by its length, and $\gamma_{(\Omega_\delta,a_\delta,b_\delta)}$ converges in law to the geodesic between a and b in Ω. On the other hand if x is too large, then the SAW is not penalized enough and $\gamma_{(\Omega_\delta,a_\delta,b_\delta)}$ converges to a space-filling curve. The phase transition between these two possible behaviors occurs exactly at the value $x_c = 1/\mu_c$. While the previous statements about $x \neq x_c$ are now mathematical theorems, the behavior at the "critical value" x_c is still conjectural. Let us describe briefly what is expected to happen at this special value.

At $x = x_c$, Conformal Field Theory predicts that $\gamma_{(\Omega_\delta,a_\delta,b_\delta)}$ converges *in the scaling limit* (i.e. as δ tends to 0) to a random, continuous, fractal, simple curve $\gamma_{(\Omega,a,b)}$ from a to b staying in Ω. Furthermore, the family of random curves $\gamma_{(\Omega,a,b)}$ indexed by the triplets (Ω,a,b) is expected to be conformally invariant in the following sense: for any (Ω,a,b) and any conformal (i.e. holomorphic and one-to-one) map $\psi:\Omega\to\mathbb{C}$,

$$\psi(\gamma_{(\Omega,a,b)}) \text{ has the same law as } \gamma_{(\psi(\Omega),\psi(a),\psi(b))}.$$

This prediction can be rephrased as follows: the random curve obtained by taking the scaling limit in $(\psi(\Omega),\psi(a),\psi(b))$ has the same law as the image by ψ of the random curve obtained by taking the scaling limit in (Ω,a,b). This is clear for a transformation corresponding to a symmetry of the lattice (for instance the rotation by $k\frac{2\pi}{3}$ for some $k\in\mathbb{Z}$), but this claim implies that the result is true for any conformal transformation (therefore in particular for a rotation by any angle).

The emergence of these additional symmetries in the scaling limit has tremendous implications. In particular, Schramm [Sch00] managed to identify a natural candidate for the possible conformally invariant family of continuous non self-crossing curves. Together with Lawler and Werner [LSW04], he was thus able to predict that $\gamma_{(\Omega,a,b)}$ should be the Schramm-Loewner Evolution (SLE) of parameter 8/3. This object, which is directly related to many other lattice models in dimension 2 (in particular simple random walks), is very well understood. Proving the convergence of $\gamma_{(\Omega_\delta,a_\delta,b_\delta)}$ to SLE(8/3) would therefore provide deep insight into the behavior of the model at x_c, and as a byproduct into the behavior of the uniformly sampled SAW for large n (the two models are closely related). In particular, it would probably enable one to determine the critical exponents 11/32 and 3/4.

The previous discussion on conformal invariance seems to have carried us away from our original discussion concerning parafermionic observables, but in fact the two discussions are deeply related. Indeed, the parafermionic observable is expected to have a conformally covariant scaling limit. Namely, set F_δ for the observable in the domain Ω_δ with $a = a_\delta$, and $f_\delta = F_\delta(\cdot)/F_\delta(b_\delta)$ (which depends on Ω_δ, a_δ and b_δ). Smirnov conjectured that if $\sigma = 5/8$ and $x = x_c$, then

$$\lim_{\delta\to 0} f_\delta \;=\; (\psi')^{5/8}, \tag{2}$$

where ψ is the conformal map from Ω to the upper half-plane sending a to infinity, b to 0, and with $\psi'(b) = 1$ (conformal covariance follows readily). Above, the convergence is uniform on any compact of the domain Ω. To come back to the discussion about the fact that the observable shared the property of vanishing contour integrals with holomorphic maps, we see that it is in fact expected to converge (when properly renormalized) in the scaling limit to such a holomorphic map.

In fact the previous conjecture represents the main step in a program dedicated to the proof of convergence of $\gamma_{(\Omega_\delta,a_\delta,b_\delta)}$ to SLE(8/3). From this point of view, [DS12b] is indeed a first step towards a bigger goal. Unfortunately, proving convergence of the observable seems out of reach at the moment. Nevertheless, a similar program has been carried out for a different model, and we propose to switch now to this model to discuss parafermionic observables further. While the connection to the story above will not be immediately apparent, it will become clearer as the discussion progresses.

A second example: the Ising model The Ising model was introduced by Lenz in 1920 to model Curie's temperature. It has been used to model a wide variety of phenomena in physics, ranging from ferromagnetism to spin glasses. In fact, the Ising model finds new applications in other fields of science (such as biology, neuroscience, etc) every single day. We will focus on the nearest-neighbor ferromagnetic Ising model on the hexagonal lattice. Let $G = (V, E)$ be a finite subgraph of $\mathbb{H}$. Define the Hamiltonian $H_G(\sigma)$ of a configuration $\sigma = (\sigma_u : u \in V)$ of spins $\sigma_u \in \{\pm 1\}$ by the formula

$$H_G(\sigma) := -\sum_{\{u,v\}\in E} \sigma_u\sigma_v.$$

For $\beta > 0$ and $f : \{\pm 1\}^V \longrightarrow \mathbb{R}$, let

$$\langle f\rangle_{G,\beta} := \frac{\displaystyle\sum_{\sigma\in\{\pm1\}^V} f(\sigma)e^{-\beta H_G(\sigma)}}{\displaystyle\sum_{\sigma\in\{\pm1\}^V} e^{-\beta H_G(\sigma)}}.$$

The measure $\langle\cdot\rangle_{G,\beta}$ is called the Ising measure on the graph G at inverse-temperature $\beta > 0$.

When working with the Ising model, one is usually interested in quantities of the form $\langle \prod_{u\in A}\sigma_u\rangle$, where $A \subset V$. The operator σ_u associated to a vertex u characterizes the phase transition and is as such an *order* operator. From the point of view of Field Theory, it is convenient to consider a different type of operators associated to faces, which is corresponding to *disorder* operators. Let f, g be two faces and introduce a cut $\mathscr{C}$ from f to g, i.e. a sequence of adjacent faces starting from f and ending at g. Consider $\mu_f\mu_g$ to be the operator reversing the value of the coupling constants of the edges between successive faces of the cut (we identify the cut with this set of edges). In other words,

$$\mu_f(\sigma)\mu_g(\sigma) := \exp\Big(-2\beta \sum_{\{u,v\}\in\mathscr{C}} \sigma_u\sigma_v\Big).$$

Observe that the operator depends on the cut $\mathscr{C}$ and on β. The use of such disorder operators goes back as far as the original exact solutions to the 2D Ising model and is fundamental in the study of the critical behavior of the model (since it pops up everywhere, we do not give a specific reference).

We would like to manipulate order and disorder operators. To do this, we consider the high-temperature expansion of the Ising model, which we present briefly now. As observed by van der Waerden, the identity

$$\exp(\beta\sigma_u\sigma_v) = \cosh(\beta)(1 + \tanh(\beta)\sigma_u\sigma_v)$$

allows to express the partition function of the Ising model as follows

$$\begin{aligned}\sum_{\sigma\in\{\pm1\}^V} e^{-\beta H_G(\sigma)} &= \cosh(\beta)^{|E|} \sum_{\sigma\in\{\pm1\}^V} \prod_{e=\{u,v\}\in E} (1 + \tanh(\beta)\sigma_u\sigma_v)\\ &= \cosh(\beta)^{|E|} \sum_{\omega\subset G} \Big(\prod_{e\in\omega} \tanh(\beta)\Big)\Big(\sum_{u\in\omega} \sigma_u^{|\{v:\{u,v\}\in\omega\}|}\Big).\end{aligned}$$

For any $u \in V$, associating the configuration σ with the same configuration except that the spin at u is flipped implies that the last sum is equal to

$$\begin{cases} 2^{|V(G)|} & \text{if } \omega \in \mathscr{E}(G),\\ 0 & \text{otherwise,}\end{cases}$$

where $\mathscr{E}(G)$ denotes the set of even subgraphs of G, that is, the set of subgraphs ω of G such that every vertex of G is incident to an even number of edges of ω. Note that on a subgraph of the hexagonal lattice, $\omega \in \mathscr{E}(G)$ is the disjoint union of self-avoiding loops. We deduce that

$$\sum_{\sigma\in\{\pm1\}^V} e^{-\beta H_G(\sigma)} = \cosh(\beta)^{|E|}\, 2^{|V|} \sum_{\omega\in\mathscr{E}(G)} x^{|\omega|}, \tag{3}$$

where $x := \tanh(\beta)$. A similar computation shows that for $A \subset V$,

$$\sum_{\sigma \in \{\pm 1\}^V} \Big(\prod_{u \in A} \sigma_u \Big) e^{-\beta H_G(\sigma)} = \cosh(\beta)^{|E|} \, 2^{|V|} \sum_{\omega \in \mathscr{E}(G,A)} x^{|\omega|},$$

where $\mathscr{E}(G,A)$ denotes the set of subgraphs ω of G such that every vertex not in A (resp. in A) is incident to an even (resp. odd) number of edges in ω. Altogether, we get

$$\langle \prod_{u \in A} \sigma_u \rangle_{G,\beta} = \frac{\sum_{\omega \in \mathscr{E}(G,A)} x^{|\omega|}}{\sum_{\omega \in \mathscr{E}(G)} x^{|\omega|}}.$$

In other words, correlations between order operators can be expressed in terms of ratios of weighted sums over subgraphs of G. But what happens when one mixes order and disorder operators? Let us take a specific example. Consider a discrete domain Ω and a vertex u on its boundary. Also consider a vertex $v \in \Omega$ and a cut $\mathscr{C}$ between a face f outside Ω and bordered by u and a face g bordered by v. When doing the same expansion as above, one obtains that

$$\langle \sigma_u \sigma_v \, \mu_f \mu_g \rangle_{G,\beta} = \frac{\sum_{\omega \in \mathscr{E}(G,\{u,v\})} (-1)^{|\omega \cap \mathscr{C}|} x^{|\omega|}}{\sum_{\omega \in \mathscr{E}(G)} x^{|\omega|}}. \tag{4}$$

Since $\omega \in \mathscr{E}(G,\{u,v\})$ is the disjoint union of self-avoiding loops and a self-avoiding path from u to v, the loops do not surround v and therefore contribute an even number to $|\omega \cap \mathscr{C}|$. As a consequence, only the self-avoiding path from u to v can contribute an odd number, which corresponds modulo 2 to the number of turns that the path does around the face f.

Now, Smirnov introduced an observables at mid-edges by considering the following quantity: let Ω be a discrete domain, a a mid-edge on its boundary and z a mid-edge inside. Consider the set $\widehat{\mathscr{E}}(\Omega,a,z)$ of "subgraphs of Ω" obtained as the union of disjoint self-avoiding loops plus a SAW from a to z avoiding the loops. Let $|\omega|$ be the number of vertices in ω (note that it is also the number of vertices, if the two half-edges arriving at a and z contribute $1/2$). Also set $\mathrm{W}_\omega(a,z)$ for the winding of the SAW from a to z. Then define

$$F(z) \;=\; F_{\Omega,a,x}(z) \;:=\; \sum_{\omega \in \widehat{\mathscr{E}}(\Omega,a,z)} \exp(-\tfrac{\mathrm{i}}{2} \mathrm{W}_\omega(a,z)) \; x^{|\omega|}.$$

The observable has a structure similar to the one of the SAW, with $\sigma = 1/2$ instead of $\sigma = 5/8$ and the sum on SAWs replaced by a sum on subgraphs $\omega \in \widehat{\mathscr{E}}(\Omega,a,z)$. Consider the specific case of configurations for which the SAW arrives from one endpoint (say v) of the edge corresponding to z. In such case, the term corresponding to the winding contributes $-\lambda$ or λ depending on the parity of the number of turns around the mid-edge z. A small leap of faith (or a small computation using the previous observation, which we leave to the reader) shows that F is in fact a complex linear combination of quantities of the form (4), where v is one of the two endpoints of the edge of z, and g one of the two faces bordered by z. To summarize, an observable similar to the parafermionic observable for SAWs can be defined in the Ising model as a linear combination of order-disorder operators.

The similarity between the observables for SAW and Ising is uncanny. It does not come as a surprise that for a certain value x_c of x, the Ising observable also satisfies the relations (1). This value is in fact equal to $1/\sqrt{3} = \tanh(\beta_c)$, where β_c is the critical inverse-temperature of the Ising model on $\mathbb{H}$. Exactly as in the case of the SAW, one may ask whether, when considering a sequence $(\Omega_\delta, a_\delta, b_\delta)$ approximating (Ω, a, b), $f_\delta = F_\delta(\cdot)/F_\delta(b_\delta)$ converges.

The Ising model has a tactical advantage compared to SAWs. The value of σ is $1/2$ instead of $5/8$. This apparently small difference was harvested by Chelkak and Smirnov to prove that the observable f_δ satisfies additional relations, and that it is now discrete holomorphic in the standard sense, not only weakly. In particular, f_δ is determined uniquely by its boundary conditions and these relations. Let us mention that discrete holomorphicity goes far back. Discrete holomorphic functions have also found several applications in geometry, analysis, combinatorics, and probability. We refer the interested reader to [DS12a] for more references on this beautiful theory.

Anyway, Chelkak and Smirnov [CS12] were able to describe f_δ as the solution of a discrete "Riemann-Hilbert" Boundary Value Problem. With some additional work, they also showed that such a solution must converge to the holomorphic solution of the corresponding continuous Riemann-Hilbert Boundary Value Problem. As a consequence, they were able to rigorously prove that

$$\lim_{\delta \to 0} f_\delta = \sqrt{\psi'},$$

where ψ was defined in (2).

Using a program similar to the one that could potentially be used for SAW, interfaces of the Ising model with Dobrushin boundary conditions were proved to converge to SLE(3) in [CDH+14]. In other words, conformal invariance of interfaces can be proved rigorously in the case of the Ising model.

Let us conclude this part by mentioning that since the breakthrough of [CS12], conformal invariance of many observables of the Ising model has been derived: crossing probabilities [BDH14], energy and spin fields [HS13, Hon10, CI13, CHI15]), etc.

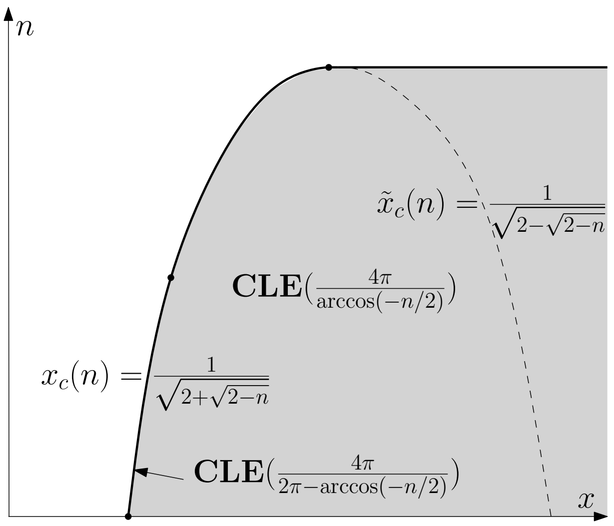

Figure 3: Phase diagram of the loop $O(n)$ model on the hexagonal lattice.

The parallel between the stories of the SAW and the high-temperature expansion of the Ising model leaves little doubt about a connection between the two models. A model indeed interpolates between the two examples above, and we propose to discuss it briefly below.

The loop $O(n)$-model The high-temperature expansion of the Ising model and the SAW are both part of a wider family of statistical models, called the loop $O(n)$-model. In this model, a configuration ω is an element of $\mathscr{E}(G)$ and the probability of ω is proportional to $x^{\#\text{ edges}}n^{\#\text{ loops}}$. For $n = 0$ and $n = 1$, we recover the SAW and the high-temperature expansion of the Ising model respectively. The phase diagram (Fig. 3) of the loop $O(n)$ model on the hexagonal lattice was predicted by Nienhuis in [Nie82, Nie84]:

1. For $n \le 2$ and $x = x_c(n) := 1/\sqrt{2+\sqrt{2-n}}$, the probability of having a loop of length ℓ passing through the origin decays as an inverse power of ℓ. Furthermore, the scaling limit of the loops is described by a conformally invariant family of *simple* loops called CLE(κ) (where κ depends on n and ranges from 8/3 to 4).

2. For $n \le 2$ and $x > x_c(n)$, the probability of having a loop of length ℓ passing through the origin decays as an inverse power of ℓ. Furthermore, the scaling limit of the loops is described by a conformally invariant family of *self-touching* loops called CLE(κ) (where κ depends on n but not on $x > x_c(n)$ and ranges from 4 to 8). Except for

$n = 2$, the exponent in the inverse power is not the same as the one at $x_c(n)$.

3. Otherwise, the probability decays exponentially fast. In particular, for $n > 2$ the probability of having large loops is always decaying exponentially fast.

Most of the previous diagram is still conjectural. Nevertheless, a generalization of the previous observables provides some understanding on what is going on. Exactly like in the examples of the SAW and Ising, one may introduce an observable

$$F(z) = F_{\Omega,a,x,n,\sigma}(z) := \sum_{\omega\in\widehat{\mathscr{E}}(\Omega,a,z)} \exp(-\mathrm{i}\sigma \mathrm{W}_\omega(a,z))\, x^{\#\,\mathrm{edges}}\, n^{\#\,\mathrm{loops}}.$$

For $n \le 2$, two values of (x,σ) play a special role in the sense that the corresponding observable has vanishing contour integrals. The first one is for $x = x_c(n)$ and $\sigma = \sigma(n)$ (the value is irrelevant here). The other value is at $x = \tilde{x}_c(n) = 1/\sqrt{2-\sqrt{2-n}}$ and $\tilde{\sigma} = \tilde{\sigma}(n)$. One expects that the observable f_δ defined as above would converge to $(\psi')^\sigma$ for $(x_c(n),\sigma)$ and $(\psi')^{\tilde{\sigma}}$ for $(\tilde{x}_c(n),\tilde{\sigma})$. The values of σ and $\tilde{\sigma}$ allow to predict the dependency of the value κ of the CLE(κ) on n (see Fig. 3 for the precise values). Furthermore, proving convergence of the observable represents the main step towards a proof of conformal invariance for the whole family of loops.

Interestingly, no good observable seems to be available for $n > 2$. It is therefore unclear how to prove that there is exponential decay at every x for $n > 2$. Nevertheless, we should mention a recent result proving this for $n \gg 1$ [DPSS14]. This result should be compared to a conjecture of Polyakov concerning the spin $O(n)$ models, that yields that spin-spin correlations decay exponentially fast at every inverse temperature in the 2D spin $O(n)$ model as soon as $n > 2$. While the previous result does not answer this conjecture, it is worth noting that the loop $O(n)$ model can be seen as an approximative high-temperature expansion of the spin $O(n)$ model for integer values of n.

Conclusion The go home message is the following: the order-disorder operators of the Ising model give rise, when written in terms of the high-temperature expansion, to discrete holomorphic observables. As a consequence, one may prove that they converge in the scaling limit to conformally invariant objects, a fact which leads to conformal invariance of interfaces. Certain generalizations of these quantities to loop models are still discretizations of conformal maps. Proving their convergence in the scaling limit would imply conformal invariance of loops in the corresponding model, but unfortunately, in basically any case except the Ising model, the properties of the observables are insufficient to derive rigorously the convergence. Still, weaker properties of the observables can be used to derive interesting features such as critical points and bounds for critical exponents.

Let us conclude by mentioning that the name parafermionic observable was coined in [FK80], were these observables were introduced initially.

Let us mention that parafermionic observables are not restricted to the loop $O(n)$ model and can be used in many other models. Maybe the most notable example is provided by the Fortuin-Kasteleyn percolation and Potts models, where they were used to determine the order of the phase transition, see [DST15].

References

[BDH14] S. Benoist, H. Duminil-Copin, and C. Hongler, *Conformal invariance of crossing probabilities for the Ising model with free boundary conditions*, arXiv:1410.3715, 2014.

[CDH+14] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen, and S. Smirnov, *Convergence of Ising interfaces to Schramm's SLE curves*, C. R. Acad. Sci. Paris Math. **352** (2014), no. 2, 157–161.

[CHI15] Dmitry Chelkak, Clément Hongler, and Konstantin Izyurov, *Conformal invariance of spin correlations in the planar Ising model*, Ann. of Math. (2) **181** (2015), no. 3, 1087–1138. MR 3296821

[CI13] Dmitry Chelkak and Konstantin Izyurov, *Holomorphic spinor observables in the critical Ising model*, Comm. Math. Phys. **322** (2013), no. 2, 303–332. MR 3077917

[CS12] Dmitry Chelkak and Stanislav Smirnov, *Universality in the 2D Ising model and conformal invariance of fermionic observables*, Invent. Math. **189** (2012), no. 3, 515–580. MR 2957303

[DH13] H. Duminil-Copin and A. Hammond, *Self-avoiding walk is sub-ballistic*, Communications in Mathematical Physics **324** (2013), no. 2, 401–423.

[DPSS14] H. Duminil-Copin, R. Peled, W. Samotij, and Y. Spinka, *Exponential decay of loop lengths in the loop $o(n)$ model with large n*, arXiv:1412.8326, 12 2014.

[DS12a] H. Duminil-Copin and S. Smirnov, *Conformal invariance of lattice models*, Probability and statistical physics in two and more dimensions, Clay Math. Proc., vol. 15, Amer. Math. Soc., Providence, RI, 2012, pp. 213–276. MR 3025392

[DS12b] H. Duminil-Copin and S. Smirnov, *The connective constant of the honeycomb lattice equals* $\sqrt{2+\sqrt{2}}$, Ann. of Math. (2) **175** (2012), no. 3, 1653–1665. MR 2912714

[DST15] H. Duminil-Copin, V. Sidoravicius, and V. Tassion, *Continuity of the phase transition for planar random-cluster and Potts models with* $1 \leq q \leq 4$, arXiv:1505.04159, 2015.

[FK80] Eduardo Fradkin and Leo P Kadanoff, *Disorder variables and para-fermions in two-dimensional statistical mechanics*, Nuclear Physics B **170** (1980), no. 1, 1–15.

[Hon10] C. Hongler, *Conformal invariance of Ising model correlations*, Ph.D. thesis, université de Genève, 2010, p. 118.

[HS13] Clément Hongler and Stanislav Smirnov, *The energy density in the planar Ising model*, Acta Math. **211** (2013), no. 2, 191–225. MR 3143889

[LSW04] Gregory F. Lawler, Oded Schramm, and Wendelin Werner, *On the scaling limit of planar self-avoiding walk*, Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2, Proc. Sympos. Pure Math., vol. 72, Amer. Math. Soc., Providence, RI, 2004, pp. 339–364. MR MR2112127 (2006d:82033)

[MS93] Neal Madras and Gordon Slade, *The self-avoiding walk*, Probability and its Applications, Birkhäuser Boston Inc., Boston, MA, 1993. MR 1197356 (94f:82002)

[Nie82] Bernard Nienhuis, *Exact Critical Point and Critical Exponents of* $\mathrm{O}(n)$ *Models in Two Dimensions*, Physical Review Letters **49** (1982), no. 15, 1062–1065.

[Nie84] B. Nienhuis, *Coulomb gas description of 2D critical behaviour*, J. Statist. Phys. **34** (1984), 731–761.

[Sch00] Oded Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288. MR 1776084 (2001m:60227)