Introduction to Bernoulli percolation

Hugo Duminil-Copin*

November 2, 2022

Contents

Notation A graph G is given by a set V whose elements are called *vertices*, and a subset E of unordered pairs of elements in V, called *edges*. We usually denote an element of E by e or xy, where x and y are understood as being the two *endpoints* of e. We will mostly work on the infinite graph $(\mathbb{V}, \mathbb{E})$, called the d-dimensional *hypercubic lattice* (when $d = 2$, we speak of the *square lattice*), defined as follows. The vertex set is defined as the set $\mathbb{V} := \mathbb{Z}^d$ of elements of $\mathbb{R}^d$ with integer coordinates, and the edge set $\mathbb{E}$ composed of edges xy with endpoints x and y (in $\mathbb{Z}^d$) satisfying $\|x - y\| = 1$. Below, we use the notation $\mathbb{Z}^d$ to refer both to the lattice and its vertex set. Also introduce $\Lambda_n := [-n, n]^d$ for every integer $n \geq 1$.

For a subgraph $G = (V, E)$ of $\mathbb{Z}^d$, we introduce the boundary of G defined by

$$\partial G := \{x \in V : \exists y \in \mathbb{Z}^d \text{ such that } xy \in \mathbb{E} \setminus E\}.$$

*`duminil@ihes.fr` Institut des Hautes Études Scientifiques and Université de Genève
These lecture notes describe the content of a class given at IHES at a master level. We believe that percolation theory is best learned by exercising on problems. We therefore included a number of them in the notes and recommend that the reader tries them. This teaching was funded by an IDEX Chair from Paris Saclay and by the NCCR SwissMap from the Swiss NSF.

1 Phase transition in Bernoulli percolation

A *percolation configuration* $\omega = (\omega_e : e \in E)$ on $G = (V, E)$ is an element of $\{0,1\}^E$. If $\omega_e = 1$, the edge e is said to be *open*, otherwise e is said to be *closed*. A configuration ω can be seen as a subgraph of G with vertex-set V and edge-set $\{e \in E : \omega_e = 1\}$. A *cluster* will be a connected component of ω.

A *percolation model* is given by a distribution on percolation configurations on G. The simplest example of percolation model is provided by *Bernoulli percolation*: each edge is open with probability p, and closed with probability $1-p$, independently of the states of other edges. This model was introduced by Broadbent and Hammersley in 1957 [9] and has been one of the most studied probabilistic models. We refer to [20, 6, 13] for books on the subject.

We will often define Bernoulli percolation on the infinite lattice $\mathbb{Z}^d$. We therefore consider the probability space $(\{0,1\}^{\mathbb{E}}, \mathfrak{F}, \mathbb{P}_p)$, where $\{0,1\}^{\mathbb{E}}$ is the probability space of percolation configurations on $\mathbb{Z}^d$, $\mathfrak{F}$ is the σ-algebra generated by events depending on finitely many edges, and $\mathbb{P}_p$ is the corresponding product measure, where each coordinate is a Bernoulli random variable of parameter p (its expectation is denoted by $\mathbb{E}_p$).

The first question of interest in Bernoulli percolation is the existence or not of an infinite cluster. Note that this event is measurable with respect to $\mathfrak{F}$ since

$$\begin{aligned}
\{x \text{ is in an infinite cluster}\} &= \bigcap_{n=0}^{\infty} \{\text{the cluster of } x \text{ contains a vertex at distance } n \text{ of } x\} \\
\{\text{there exists an infinite cluster}\} &= \bigcup_{x \in \mathbb{Z}^d} \{x \text{ is in an infinite cluster}\}.
\end{aligned}$$

Define the parameter

$$\theta(p) := \mathbb{P}_p[0 \text{ is in an infinite cluster}]$$

and define

$$p_c := \inf\{p \in [0,1] : \theta(p) > 0\}.$$

Note that for $p < p_c$, the probability that x is in an infinite cluster is zero by invariance under translations. The union bound (on countably many vertices) implies that there is almost surely no infinite cluster.

Theorem 1.1 *For $d \geq 2$, we have that $0 < p_c(d) < 1$.*

A *self-avoiding* path of length n is a sequence of edges $e_1, \dots, e_n$ with $e_i \neq e_j$ for $i \neq j$, such that e_i and e_{i+1} share an endpoint for every $1 \leq i < n$. Let Ω_n be the set of self-avoiding paths of length n starting from the origin.

Proof Fix $n > 0$. If 0 is connected to infinity, there exists a path of open edges of length n starting from 0. Therefore

$$\begin{aligned}
\theta(p) &\leq \mathbb{P}_p[\text{there exists } \gamma \in \Omega_n \text{ such that } \omega_e = 1 \text{ for every } e \in \gamma] \\
&\leq \sum_{\gamma \in \Omega_n} \mathbb{P}_p[\omega_e = 1 \text{ for every } e \in \gamma] \\
&\leq |\Omega_n| p^n \leq (2dp)^n.
\end{aligned}$$

In the second line, we used the union bound and independence for edges, and in the last one, the fact that $|\Omega_n| \leq (2d)^n$.

When $p < \frac{1}{2d}$, the quantity tends to 0 and therefore the origin is connected to infinity with 0 probability. In other words $p \leq p_c(d)$. We therefore proved that $p_c(d) \geq \frac{1}{2d}$.

The argument showing that $p_c < 1$ is more elaborated. We need to prove that when p is close to 1, then $\theta(p) > 0$. Any percolation on $\mathbb{Z}^d$ contains a copy of Bernoulli percolation on $\mathbb{Z}^2$ (simply look at the restriction on $\omega \cap \mathbb{Z}^2$). Hence, if the probability of $0 \longleftrightarrow \infty$ on $\mathbb{Z}^2$ is strictly positive, then $\theta(p) > 0$ on $\mathbb{Z}^d$, and therefore $p_c(d) \le p_c(2)$. It is sufficient to show the result for $d = 2$.

Consider the graph $(\mathbb{Z}^2)^*$ defined as the copy of $\mathbb{Z}^2$ translated by the vector $(1/2, 1/2)$. For every configuration ω on $\mathbb{Z}^2$, define a configuration ω^*, called the *dual configuration* of ω, as follows. Every edge e of $\mathbb{Z}^2$ is naturally associated with an edge e^* of $(\mathbb{Z}^2)^*$ intersecting it in its middle. Set $\omega^*_{e^*} = 1 - \omega_e$. With words, an edge of $(\mathbb{Z}^2)^*$ is open if the corresponding edge of $\mathbb{Z}^2$ is closed. Note that the law of ω^* is a translate by $(1/2, 1/2)$ of $\mathbb{P}_{1-p}$. An example of configuration of ω and ω^* are represented on Fig. 1.

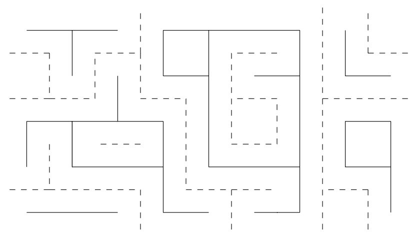

Figure 1: Two configurations ω and ω^*.

For the origin not to be in an infinite cluster (see Fig. 2), there must be a circuit in $(\mathbb{Z}^2)^*$ of open edges in ω^* surrounding the origin. The circuit must intersect $\{(n + \frac{1}{2}, 0) : n \in \mathbb{N}\}$.

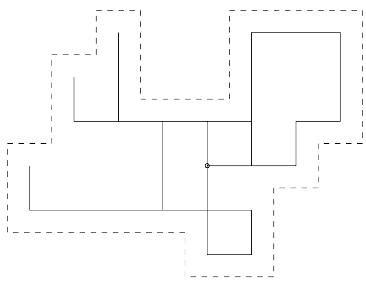

Figure 2: A configuration for which the cluster of the origin is finite, together with a circuit of open edges in ω^* surrounding the origin.

Therefore,

$$\begin{aligned}
1 - \theta(p) &\le \sum_{n \ge 1} \mathbb{P}_p[\omega^* \text{ contains an open circuit surrounding 0 and passing through } (n + \tfrac{1}{2}, 0)] \\
&\le \sum_{n \ge 1} \mathbb{P}_p[\omega^* \text{ contains an open path of length } 2n+4 \text{ passing through } (n + \tfrac{1}{2}, 0)] \\
&\le \sum_{n \ge 1} (4(1-p))^{2n+4}. \qquad (1.1)
\end{aligned}$$

As in the previous argument, we used that the probability that a path of length k is composed of open edges in ω^* has probability $(1-p)^k$. Furthermore, there are less than 4^k such paths. When p is closed enough to 1, this sum is strictly smaller than 1. We deduce that $\theta(p) > 0$. We deduce that $p_c(d) < 1$. □

Remark. The graph $(\mathbb{Z}^2)^*$ is called the *dual graph* of $\mathbb{Z}^2$. This notion will be often used when working with planar percolation (see below).

Exercise 1 *Show that any measurable event can be approximated by events depending on finitely many edges, in the sense that for any A in the product σ-algebra, there exists a sequence (B_n) such that B_n is measurable in terms of the ω_e for e in the box of size n, and $\mathbb{P}_p[A\Delta B_n]$ tends to 0 as n tends to infinity.*

Exercise 2 *Show that $p_c(\mathbb{Z}) = 1$. Show that $p_c(\mathbb{Z}\times\{0,\dots,n\}) = 1$.*

Exercise 3 *Show that $p_c(d) \le 3/4$ for every $d \ge 2$.*

Exercise 4 *Consider the infinite tree of degree $d+1$. Connect the cluster of the origin to a Galton-Watson tree. Use this connexion to compute the critical point.*

Exercise 5 *Show that $|\Omega_{n+m}| \le |\Omega_n|\cdot|\Omega_m|$. Deduce that $|\Omega_n|^{1/n}$ converges as n tends to infinity to $\mu_c \in (1,\infty)$. In dimension d, what is the best bound you can come up with for μ_c.*

Exercise 6 *What site percolation on the square (hexagonal and triangular) lattice could mean? Show that $0 < p_c < 1$. Show that bond percolation on a graph corresponds to site percolation on a modified graph. Show that*

$$p_c(bond) \le p_c(site) \le 1-(1-p_c(bond))^d$$

where d is the degree of the graph.

Exercise 7 *Consider a graph G for which every vertex has degree smaller than or equal to d. A finite connected subset of a graph G is called a* lattice animal*. For $x\in G$ and k,n two integers let $a(n,x)$ be the number of lattice animals with n vertices. Using percolation to show that*

$$\sum_{n=0}^{\infty}[p(1-p)]^{dn}a(n,x) \le 1.$$

Deduce that for every x and n, $a(n,x) \le 4^{dn}$. Show that one can replace $p^d(1-p)^d$ by $p(1-p)^d$ in the previous bound.

Exercise 8 *The goal of this exercise is to implement Peierls' argument in dimension d without using the coupling. A* minimal blocking surface *is a set of edges E such that every path of adjacent edges from the origin to infinity intersects E, and any strict subset of E does not have this property.*

1. Show that any minimal blocking surface is a finite set.

2. Consider such a minimal blocking surface E and assume that it is contained in the box of size n around the origin. Let x be outside of the box. We partition E into two non-empty sets E_1 and E_2. Show that there exists two paths γ_1 and γ_2 of edges from 0 to x such that γ_1 does not intersect E_1, and γ_2 does not intersect E_2. From now on, we call γ the concatenation of γ_1 and the time-reversal of γ_2.

3. Show that the space of loops on $\mathbb{Z}^d$ is a $\mathbb{Z}_2$-vector space. Show that the set of elementary loops, meaning loops of length 4 is generating this vector space. Deduce that there exists a set $\mathcal{L}$ of elementary loops such that

$$\gamma = \sum_{\ell\in\mathcal{L}}\ell.$$

4. By studying the parity of the number of intersections of γ_1, γ_2, γ and each ℓ with E, show that there must exist a loop $\ell\in\mathcal{L}$ intersecting both E_1 and E_2.

5. Deduce that a minimal surface is connected in the graph with vertex-set given by the edges of $\mathbb{Z}^d$, and edge-set given by pairs of edges of $\mathbb{Z}^d$ that are on the same loop of length 4.

6. Use the previous study to implement a version of Peierls' argument in dimension d.

2 Everyone's toolbox

We will need a number of tools in the following sections.

2.1 Increasing coupling

We wish to show that in a certain sense, measures $\mathbb{P}_p$ can be coupled in an increasing way. It will also be interesting to consider Bernoulli percolation of different parameters at the same time. If one considers two independent copies ω and ω' of Bernoulli percolation with parameters $p < p'$, then each edge has probability $p(1-p')$ to be open in ω. Obviously, one would like to argue that there are more and more open edges, and that it should be possible to couple different Bernoulli percolations to guarantee that $\omega \le \omega'$ almost surely.

A configuration ω is *smaller than or equal to* ω' (denoted by $\omega \le \omega'$) if $\omega_e \le \omega'_e$ for every $e \in \mathbb{Z}^d$. The function $f : \Omega = \{0,1\}^E \to \mathbb{R}$ is *increasing* if $f(\omega) \le f(\omega')$ for every $\omega \le \omega'$. The event A on the product σ-algebra is said to be *increasing* if 1_A is non-decreasing. In other words, A is increasing if ω, ω',

$$\omega \in A \text{ and } \omega \le \omega' \Rightarrow \omega' \in A.$$

An event A is *decreasing* if and only if A^c is increasing.

Examples of increasing events:

- The edge e is open.
- The sites x and y are *connected in* ω (i.e. that there exists a path of open edges from x to y), denoted by $x \longleftrightarrow y$.
- There exists a path connecting A to B (where A and B are two sets of vertices), denoted by $A \longleftrightarrow B$.
- The site x is in an infinite connected component (denoted by $x \longleftrightarrow \infty$).
- The number of open edges in F exceeds k.

Proposition 2.1 (increasing coupling) *Fix $p < p'$. There exists a measure $\mathbf{P}$ on $[0,1] \longleftrightarrow \{0,1\}^{\mathbb{Z}^d}$ with marginales $\mathbb{P}_p$, such that*

$$\mathbf{P}[(\omega^p, \omega^{p'}) : \omega^p \le \omega^{p'}] = 1.$$

Proof Consider a family of iid uniform $[0,1]$ random variables $(\mathsf{U}_e)_{e \in \mathbb{Z}^d}$. Construct two configurations $\omega, \omega' \in \{0,1\}^{\mathbb{Z}^d}$ as follows: for every $e \in \mathbb{Z}^d$, set

$$\omega_e^p := \begin{cases} 1 & \text{if } \mathsf{U}_e \le p, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that the law of ω^p is $\mathbb{P}_p$. Indeed, each edge is open with probability p, and closed with probability $1-p$, independently of the state of the other edges. By construction, we have that $\omega^p \le \omega^{p'}$. Define $\mathbf{P}$ to be the law of $p \mapsto \omega^p$. □

Corollary 2.2 *Consider an increasing event $\mathcal{A}$. Then $p \mapsto \mathbb{P}_p[\mathcal{A}]$ is non-decreasing.*

Proof Since $\omega^p \le \omega^{p'}$, if $\omega^p \in \mathcal{A}$, then $\omega^{p'} \in \mathcal{A}$. Hence,

$$\mathbb{P}_p[\mathcal{A}] = \mathbf{P}[\omega^p \in \mathcal{A}] \le \mathbf{P}[\omega^{p'} \in \mathcal{A}] = \mathbb{P}_p[\mathcal{A}].$$

□

In particular, $p \mapsto \theta(p)$ is non-decreasing and

$$p_c = \sup\{p \in [0,1] : \theta(p) = 0\}.$$

2.2 Harris-FKG inequality

To estimate or compute the probability of events, it is often necessary to compute the probability of intersections. Unfortunately, when two events are not independent, it is rarely possible to compute the probability exactly. In the case of percolation, it is nonetheless possible to bound this probability for a large class of events.

Proposition 2.3 (Harris inequality) *Let $\mathcal{A}$ and $\mathcal{B}$ be two increasing events, then*

$$\mathbb{P}_p[\mathcal{A} \cap \mathcal{B}] \geq \mathbb{P}_p[\mathcal{A}]\mathbb{P}_p[\mathcal{B}].$$

More generally, if f and g are two bounded increasing functions,

$$\mathbb{E}_p(fg) \geq \mathbb{E}_p(f)\mathbb{E}_p(g).$$

This inequality is often called the FKG (Fortuin, Kasteleyn, Ginibre) inequality. There three authors proved a similar inequality in a more general context. Intuitively, this inequality means that the two increasing events $\mathcal{A}$ and $\mathcal{B}$ are positively correlated. Indeed, the inequality can be restated as

$$\mathbb{P}_p[\mathcal{A}|\mathcal{B}] \geq \mathbb{P}_p[\mathcal{A}],$$

i.e. that the conditional probability of $\mathcal{A}$ on $\mathcal{B}$ is larger or equal to the probability of $\mathcal{A}$. This is not so surprising, the fact that $\mathcal{B}$ occurs suggest that many edges are in fact open, which helps the occurrence of $\mathcal{A}$.

Proof It suffices to show the inequality for increasing functions since the function on events can be obtained by setting $f = 1_{\mathcal{A}}$ and $g = 1_{\mathcal{B}}$. Define $\{e_i, i \geq 1\}$ to be the set of edges E and $\omega_{e_k} = \omega_k$. For any increasing function, $f_n = \mathbb{E}_p(f|\omega_1, \dots, \omega_n)$ and $g_n = \mathbb{E}_p(g|\omega_1, \dots, \omega_n)$ converge to f and g respectively (by the martingale convergence theorem). Since one also has $fg = \lim f_n g_n$, we deduce that the Harris inequality follows from the inequality

$$\mathbb{E}_p(f_n g_n) \geq \mathbb{E}_p(f_n)\mathbb{E}_p(g_n),$$

i.e. from Harris inequality for variables depending on finitely many edges. We prove the later by induction on n.

For $n = 1$, the functions f and g depend on ω_1 only, or if we prefer, are functions of $\{0, 1\}$ to $\mathbb{R}$. It suffices to show the result for $f(0) = g(0) = 0$, since adding a constant to f and/or g does not change the inequality. In such case, $f(1) \geq 0$ and $g(1) \geq 0$ since f and g are increasing. Then,

$$\begin{aligned}\mathbb{E}_p(fg) - \mathbb{E}_p(f)\mathbb{E}_p(g) &= \big[pf(1)g(1) + (1-p)\cdot 0\big] - \big[pf(1) + (1-p)\cdot 0\big]\big[pg(1) + (1-p)\cdot 0\big] \\ &= pf(1)g(1) - p^2 f(1)g(1) \geq 0.\end{aligned}$$

Let us now consider $n \geq 2$ and let us assume that the result is true for $n-1$. Fix first $\omega_1, \dots, \omega_{n-1}$. The definition of conditional expectation implies that

$$\begin{aligned}\mathbb{E}_p(fg|\omega_1, \dots, \omega_{n-1}) &= pf(\omega_1, \dots, \omega_{n-1}, 1)g(\omega_1, \dots, \omega_{n-1}, 1) \\ &\quad + (1-p)f(\omega_1, \dots, \omega_{n-1}, 0)g(\omega_1, \dots, \omega_{n-1}, 0) \\ &= \mathbb{E}_{\omega_n}(f(\omega_1, \dots, \omega_{n-1}, \cdot)g(\omega_1, \dots, \omega_{n-1}, \cdot)),\end{aligned}$$

where $\mathbb{P}_{\omega_n}$ is the law of ω_n (i.e. a Bernoulli percolation of parameter p), and where $f(\omega_1, \dots, \omega_{n-1}, \cdot)$ and $g(\omega_1, \dots, \omega_{n-1}, \cdot)$ depend only on ω_n. Similarly,

$$\begin{aligned}&\mathbb{E}_{\omega_n}(f|\mathcal{F}_{n-1})(\omega_1, \dots, \omega_{n-1}) = \mathbb{E}_{\omega_n}(f(\omega_1, \dots, \omega_{n-1}, \cdot)) \text{ and} \\ &\mathbb{E}_{\omega_n}(g|\mathcal{F}_{n-1})(\omega_1, \dots, \omega_{n-1}) = \mathbb{E}_{\omega_n}(g(\omega_1, \dots, \omega_{n-1}, \cdot)).\end{aligned}$$

For each $(\omega_1, \dots, \omega_{n-1})$, the induction hypothesis applied to $\mathbb{E}_{\omega_n}$ and the increasing functions $f(\omega_1, \dots, \omega_{n-1}, \cdot)$ and $g(\omega_1, \dots, \omega_{n-1}, \cdot)$, which depend on ω_n only, gives that

$$\mathbb{E}_{\omega_n}(f(\omega_1, \dots, \omega_{n-1}, \cdot) g(\omega_1, \dots, \omega_{n-1}, \cdot)) \geq \mathbb{E}_{\omega_n}(f(\omega_1, \dots, \omega_{n-1}, \cdot))\, \mathbb{E}_{\omega_n}(g(\omega_1, \dots, \omega_{n-1}, \cdot))$$

or in other words

$$\mathbb{E}_p(fg|\omega_1, \dots, \omega_{n-1}) \;\geq\; \mathbb{E}_p(f|\omega_1, \dots, \omega_{n-1})\, \mathbb{E}_p(g|\omega_1, \dots, \omega_{n-1}).$$

Now,

$$\begin{aligned}\mathbb{E}_p(fg) &= \mathbb{E}_p(\mathbb{E}_p(fg|\omega_1, \dots, \omega_{n-1}) \\ &\geq \mathbb{E}_p[\mathbb{E}_p(f|\omega_1, \dots, \omega_{n-1})\mathbb{E}_p(g|\omega_1, \dots, \omega_{n-1})] \\ &\geq \mathbb{E}_p[\mathbb{E}_p(f|\omega_1, \dots, \omega_{n-1})]\mathbb{E}_p[\mathbb{E}_p(g|\omega_1, \dots, \omega_{n-1})] \\ &= \mathbb{E}_p(f)\mathbb{E}_p(g),\end{aligned}$$

where we used the induction hypothesis to the variables $\omega_1, \dots, \omega_{n-1}$ and the conditional expectations $\mathbb{E}_p(f|\omega_1, \dots, \omega_{n-1})$ and $\mathbb{E}_p(g|\omega_1, \dots, \omega_{n-1})$, which depend on $n-1$ variables only.

□

Exercise 9 *Show that Harris inequality holds for decreasing events.*

Exercise 10 *Show that increasing events span the product σ-algebra.*

Exercise 11 (Square-root-trick) *Show that for n increasing events $\mathcal{A}_1, \dots, \mathcal{A}_n$. Show that*

$$\max\{\mathbb{P}_p[\mathcal{A}_i], i \leq n\} \geq 1 - \big(1 - \mathbb{P}_p[\mathcal{A}_1 \cup \dots \cup \mathcal{A}_n)\big]^{1/n}. \tag{2.1}$$

The previous inequality, called the (2.1), is very useful. It shows that when the probability of the union of increasing events is close to 1, then so is the maximal probability of events.

2.3 The van den Berg Kesten and the Reimer inequalities

Let $\mathcal{A}$ be an event. Given a configuration ω in $\mathcal{A}$, there is a set $I \subset E$ *is a witness of* $\mathcal{A}$ *for* ω (given $\omega_I \in \mathcal{A}$) if $\omega \in \mathcal{A}$ and any other configuration ω' coinciding with ω on I is also in $\mathcal{A}$.

We say that two events $\mathcal{A}$ and $\mathcal{B}$ are realized *disjointly* if there exist two witnesses $I = I(\omega)$ and $J = J(\omega)$ of $\mathcal{A}$ and $\mathcal{B}$ that are disjoint. This event is denoted by $\mathcal{A} \circ \mathcal{B}$. We insist on the fact that I and J may depend on ω.

If A and B depend on two deterministic disjoint sets of edges I and J, then these sets can be considered as witnesses, and we get that $A \circ B = A \cap B$ (in this case we obtain that $\mathbb{P}_p[A \circ B] = \mathbb{P}_p[A]\mathbb{P}_p[B]$. The BK-Reimer inequality generalizes this fact to events that a priori depend on the same set of edges. In this case, we obtain only an inequality (see below).

Example. Note that $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \circ \mathcal{B}$ are in general different. For $\mathcal{A} = \{x \longleftrightarrow x'\}$ and $\mathcal{B} = \{y \longleftrightarrow y'\}$, the event $\mathcal{A} \circ \mathcal{B}$ means that there are two disjoint paths connecting x to x' and y to y'.

Proposition 2.4 (BK-Reimer inequality) *If $\mathcal{A}$ and $\mathcal{B}$ are events depending on finitely many edges, then*

$$\mathbb{P}_p[\mathcal{A} \circ \mathcal{B}] \leq \mathbb{P}_p[\mathcal{A}]\mathbb{P}_p[\mathcal{B}].$$

Intuitively, the probability that $\mathcal{A} \circ \mathcal{B}$ knowing $\mathcal{B}$ is smaller than or equal to $\mathcal{A}$. This is not surprising, certain edges are used by $\mathcal{A}$ and cannot be used by $\mathcal{B}$, When $\mathcal{A}$ and $\mathcal{B}$ are increasing, the inequality was proved by van den Berg and Kesten. We insist on the fact that this inequality holds only for systems depending on finitely many edges.

The proof of the BK inequality is not the simplest, and we believe that it is best understood when done by oneself. We therefore choose to postpone it to Exercise 12. Also, we will in fact always propose an alternative to using the BK inequality, and therefore this inequality is not really necessary for the next sections. Reimer's inequality is even harder to use and not really necessary, we therefore omit the proof as a whole.

An important application is the following inequality, which is known in the context of other classical models of statistical physics (such as the Ising model) under the name of Simon-Lieb inequality.

We say that $0 \overset{S}{\longleftrightarrow} x$ occurs if 0 is connected to x by edges with both endpoints in $S \subset \mathbb{Z}^d$. Also, let ∂S be the *(vertex-)boundary* of S given by the set of vertices of S that are connected in $\mathbb{Z}^d$ to a vertex outside S.

Corollary 2.5 *Consider a finite set S containing the origin and $x \notin S$, then*

$$\mathbb{P}_p[0 \longleftrightarrow x] \leq \sum_{y \in \partial S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} y]\mathbb{P}_p[y \longleftrightarrow x].$$

Proof Consider $n \geq 1$. If 0 is connected to x in Λ_n, there exists a self-avoiding path γ of open edges from 0 to x. Let y be the first vertex in ∂S on this path. By definition, we have $\{0 \overset{S}{\longleftrightarrow} y\} \circ \{y \overset{\Lambda_n}{\longleftrightarrow} x\}$ since the piece of γ between 0 and y is a witness for the first event, and the remaining of γ a witness for the second one. We deduce that

$$\mathbb{P}_p[0 \overset{\Lambda_n}{\longleftrightarrow} x] \leq \sum_{y \in \partial S} \mathbb{P}_p[\{0 \overset{S}{\longleftrightarrow} y\} \circ \{y \overset{\Lambda_n}{\longleftrightarrow} x\}].$$

The BK inequality (the events depend on finitely many edges) implies that

$$\mathbb{P}_p[0 \overset{\Lambda_n}{\longleftrightarrow} x] \leq \sum_{y \in \partial S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} y]\mathbb{P}_p[y \overset{\Lambda_n}{\longleftrightarrow} x].$$

The claim follows by letting n tend to infinity. □

The previous inequality can also be put in parallel with the following inequality for random walk. Let $G_S(\cdot,\cdot)$ be the Green function of the random walk on $\mathbb{Z}^d$, killed upon exiting S. We have that for every finite set S containing the origin and every $x \notin S$,

$$G_{\mathbb{Z}^d}(0,x) \leq \sum_{y \in \partial S} G_S(0,y) G_{\mathbb{Z}^d}(y,x).$$

As often, the BK inequality is used slightly too quickly and an alternative proof can replace the argument using the BK inequality. Usually, this simpler proof exists when the witness of one of the two events can be explored algorithmically. To illustrate this fact, let us take the example of the previous corollary and provide an elementary proof not relying on the BK inequality.

Proof of Corollary 2.5 (without the BK inequality) Introduce the random variable $\mathsf{C} := \{y \in S : y \overset{S}{\longleftrightarrow} 0\}$ corresponding to the cluster of 0 in S. Since $x \notin S$, one can find an edge $y \in \partial S$ such that $0 \overset{S}{\longleftrightarrow} y$ and $y \overset{\mathsf{C}^c}{\longleftrightarrow} x$. Using the union bound, and then decomposing on the possible realizations of C, we find

$$\begin{aligned}\mathbb{P}_p[0 \longleftrightarrow x] &\le \sum_{y\in\partial S}\sum_{C\subset S}\mathbb{P}_p[\{0 \overset{S}{\longleftrightarrow} y\}\cap\{\mathsf{C}=C\}\cap\{y \overset{C^c}{\longleftrightarrow} x\}]\\ &\le \sum_{y\in\partial S}\sum_{C\subset S}\mathbb{P}_p[\{0 \overset{S}{\longleftrightarrow} y\}\cap\{\mathsf{C}=C\}]\cdot\mathbb{P}_p[y \overset{C^c}{\longleftrightarrow} x]\\ &\le \sum_{y\in\partial S}\sum_{C\subset S}\mathbb{P}_p[\{0 \overset{S}{\longleftrightarrow} y\}\cap\{\mathsf{C}=C\}]\mathbb{P}_p[y \longleftrightarrow x]\\ &\le \sum_{y\in\partial S}\mathbb{P}_p[0 \overset{S}{\longleftrightarrow} y]\mathbb{P}_p[y \longleftrightarrow x].\end{aligned}$$

In the second line, we used that $\{y \overset{C^c}{\longleftrightarrow} x\}$ and $\{0 \overset{S}{\longleftrightarrow} y\}\cap\{\mathsf{C}=C\}$ are independent. Indeed, these events depend on disjoint sets of edges: the first one on edges with both endpoints outside of C, the second one on edges between vertices of S with at least one endpoint in C. In the third line, we used that

$$\mathbb{P}_p[y \overset{C^c}{\longleftrightarrow} x] \le \mathbb{P}_p[y \longleftrightarrow x].$$

In the fourth line, we used that the events $\{0 \overset{S}{\longleftrightarrow} y\}\cap\{\mathsf{C}=C\}$ partition the event $0 \overset{S}{\longleftrightarrow} y$. □

Exercise 12 *Consider $\mathcal{A}$ and $\mathcal{B}$ to be two increasing events depending on edges in $E=\{e_1,\dots,e_n\}$ only. Consider the duplicated set $\{e_1,e_2,\dots,e_n,e'_1,e'_2,\dots,e'_n\}$ and the product measure $\mathbf{P}$ where each coordinate is a Bernoulli random variable. For $j\le n$, set $\omega^j=(\omega_{e_1},\dots,\omega_{e_{j-1}},\omega_{e'_j},\dots,\omega_{e'_n})$ and*

$$\hat{\mathcal{A}}^j=\{\hat\omega\ :\ \omega^j\in\mathcal{A}\}\ \text{and}\ \hat{\mathcal{B}}=\{\hat\omega\ :\omega^{n+1}\in\mathcal{B}\}.$$

1. *Show that $\mathbf{P}[\hat{\mathcal{A}}^1\circ\hat{\mathcal{B}}]=\mathbb{P}_p[\mathcal{A}]\mathbb{P}_p[\mathcal{B}]$ and $\mathbf{P}[\hat{\mathcal{A}}^{n+1}\circ\hat{\mathcal{B}}]=\mathbb{P}_p[\mathcal{A}\circ\mathcal{B}]$.*
2. *We wish to show that $j\mapsto\mathbf{P}[\hat{\mathcal{A}}^j\circ\hat{\mathcal{B}}]$ is decreasing in j by constructing a measure-preserving injection $\hat\omega\mapsto s(\hat\omega)$ from $\hat{\mathcal{A}}^{j+1}\circ\hat{\mathcal{B}}$ to $\hat{\mathcal{A}}^j\circ\hat{\mathcal{B}}$. Let $\hat\omega$ be a configuration in $\hat{\mathcal{A}}^{j+1}\circ\hat{\mathcal{B}}$. Let I and J be witnesses of $\hat{\mathcal{A}}^{j+1}$ and $\hat{\mathcal{B}}$ (respectively) for $\hat\omega$. Assume that there exists I not containing e_j. In this case, simply set $s(\omega)=\omega$. Assume now that any I contains e_j. Define $s(\hat\omega)$ obtained by exchanging e_j and e'_j. Check that s is one-to-one and that $\mathbb{P}[\hat\omega]=\mathbb{P}_p[s(\hat\omega)]$.*
3. *Deduce the BK inequality.*

Exercise 13 *Find two events A and B depending on finitely many edges for which*

$$\mathbb{P}_p[A\circ B]<\mathbb{P}_p[A]\mathbb{P}_p[B].$$

Exercise 14 *Consider bond percolation on $p=1/2$ and set $R:=[-n,n]\times[0,2n-1]$.*

1. *Let $\mathcal{H}_n$ be the event that there is an open path from left to right in R. Show that $\mathbb{P}_{1/2}(\mathcal{H}_n)=1/2$. Deduce that*

$$\mathbb{P}_{1/2}(0\longleftrightarrow\partial\Lambda_{2n})\ge 1/(2n).$$

2. *Let $\mathcal{H}'_n$ be the event that there exists a self-avoiding path from left to right in R. What can be said on $\mathbb{P}_{1/2}(\mathcal{H}'_n)$?*
3. *Let $\mathcal{C}_n(x)$ be the event that there exists an open path from x to $x+\partial\Lambda_n$. Show that*

$$\mathbb{P}_{1/2}[\mathcal{C}_n(0)\circ\mathcal{C}_n(0)]\ge 1/(2n).$$

What can we say about $\mathbb{P}_{1/2}[\mathcal{C}_n(0)]$?

Exercise 15 *Using Corollary 2.5, show that if there exists a set S with $\psi_p(S):=\sum_{y\in\partial S}\mathbb{P}_p[0\overset{S}{\longleftrightarrow}y]<1$, then there exists $c>0$ such that for every $n\ge1$,*

$$\mathbb{P}_p[0\longleftrightarrow\partial\Lambda_n]\le\exp(-cn).$$

2.4 Margulis-Russo formula

In this section, G is a finite graph. For a boolean function $\mathbf{f}:\{0,1\}^E\to\{0,1\}$ and $p\in[0,1]$, set $f(p):=\mathbb{E}_p[\mathbf{f}]$.

Proposition 2.6 *For any $p \in [0,1]$ and E finite, we have that for every $\mathbf{f} : \{0,1\}^E \to \{0,1\}$,*

$$f'(p) = \tfrac{1}{p(1-p)} \sum_{e \in E} \mathrm{Cov}[\mathbf{f}, \omega_e].$$

Proof Set $|\omega| = \sum_{e\in E} \omega_e$. Differentiating $f(p) = \sum_\omega f(\omega) p^{|\omega|}(1-p)^{|E|-|\omega|}$ with respect to p immediately gives

$$f'(p) = \tfrac{1}{p}\mathbb{E}_p[\mathbf{f}(\omega)|\omega|] - \tfrac{1}{1-p}\mathbb{E}_p[\mathbf{f}(\omega)(|E| - |\omega|)] = \tfrac{1}{p(1-p)} \sum_{e\in E} \mathbb{E}_p[\mathbf{f}(\omega)(\omega_e - p)]. \tag{2.2}$$

□

When $\mathbf{f}$ is the indicator function of an increasing event, the previous formula has a more geometric interpretation in terms of so-called pivotal points. For a configuration ω and an edge e, we set $\omega^{(e)}$ (resp. $\omega_{(e)}$) for the configuration coinciding with ω except at e, where it is equal to 1 (resp. 0). We say that $\omega \in \mathrm{Piv}_e(\mathcal{A})$ if $\omega^{(e)} \in \mathcal{A}$ and $\omega_{(e)} \notin \mathcal{A}$. The edge e is said to be *pivotal* for $\mathcal{A}$ if $\omega \in \mathrm{Piv}_e(\mathcal{A})$.

Example. Consider the event $\mathcal{A} := \{\omega_e = 1\}$. For this event, $\mathrm{Piv}_f(\mathcal{A})$ is either empty if $f \neq e$, and to the full space if $f = e$. Note that the event that e is pivotal for $\mathcal{A}$ is independent of ω_e. In fact, one can easily check that this is always the case: $\mathrm{Piv}_e(\mathcal{A})$ is independent of ω_e.

Example. Consider the event $\{0 \longleftrightarrow \partial\Lambda_n\}$ on $\mathbb{Z}^2$. In this case, $\mathrm{Piv}_e(\mathcal{A})$ is the event that both endpoints of e are connected respectively to 0 and $\partial\Lambda_n$ in ω; furthermore, the endpoints of e^* are connected by a circuit (minus e^*) which is open in ω^* that surrounds the origin.

Proposition 2.7 (Margulis [27], Russo [30]) *For any $p \in [0,1]$ and any increasing event $\mathcal{A}$ depending on finitely many edges, we have*

$$f'(p) = \sum_{e\in E} \mathbb{P}_p[\mathrm{Piv}_e(\mathcal{A})].$$

In words, this formula means that the derivative of the probability of an increasing event is equal to the expected number of pivotal points for the event.

Proof Set $\mathbf{f} := \mathbb{1}_{\mathcal{A}}$. Note that

$$\mathbb{E}_p[\mathbf{f}(\omega)(\omega_e - p)\mathbb{1}_{\mathrm{Piv}_e(\mathcal{A})^c}] = \mathbb{E}_p[\omega_e - p] \cdot \mathbb{E}_p[\mathbf{f}(\omega)\mathbb{1}_{\mathrm{Piv}_e(\mathcal{A})^c}] = 0$$

(in the first equality, we used that $\mathcal{A} \cap \mathrm{Piv}_e(\mathcal{A})^c$ depends on edges different from e only). Therefore,

$$\mathbb{E}_p[\mathbf{f}(\omega)(\omega_e - p)] = \mathbb{E}_p[\mathbf{f}(\omega)(\omega_e - p)\mathbb{1}_{\mathrm{Piv}_e(\mathcal{A})}].$$

For $\omega \in \mathrm{Piv}_e(\mathcal{A})$, the fact that $\mathbf{f}$ is increasing implies that $\mathbf{f}(\omega) = 0$ is $\omega_e = 0$, and $\mathbf{f}(\omega) = 1$ if $\omega_e = 1$. We deduce that

$$\mathbb{E}_p[\mathbf{f}(\omega)(\omega_e - p)] = (1-p)\mathbb{P}_p[\mathrm{Piv}_e(\mathcal{A}) \cap \{\omega_e = 1\}].$$

Since ω_e and $\mathrm{Piv}_e(\mathcal{A})$ are independent, we conclude that

$$\mathbb{E}_p[\mathbf{f}(\omega)(\omega_e - p)] = p(1-p)\mathbb{P}_p[\mathrm{Piv}_e(\mathcal{A})].$$

Inserting this expression in (2.2) implies the claim. □

Exercise 16 *Prove Russo's formula using the increasing coupling between the percolations of parameters p and $p+\varepsilon$ (where ε converges to 0).*

2.5 Ergodicity

Let τ_x be a translation of the lattice by $x \in \mathbb{Z}^d$. This translation induces a shift on the space of configurations $\{0,1\}^{\mathbb{E}}$. Define $\tau_x\mathcal{A} := \{\omega \in \{0,1\}^{\mathbb{E}} : \tau_x^{-1}\omega \in \mathcal{A}\}$. An event $\mathcal{A}$ is *invariant under translations* if for any $x \in \mathbb{Z}^d$, $\tau_x\mathcal{A} = \mathcal{A}$. A measure μ is *invariant under translations* if $\mu[\tau_x\mathcal{A}] = \mu[\mathcal{A}]$ for any event $\mathcal{A}$ and any $x \in \mathbb{Z}^d$. The measure is said to be *ergodic* if any event invariant under translation has probability 0 or 1.

Examples. The existence of infinitely many edges, of an infinite connected component, of k infinite connected components are three examples of events that are invariant under translations.

Lemma 2.8 *The measure $\mathbb{P}_p$ is invariant under translations and ergodic.*

Proof Let $\mathcal{A}$ be an increasing event depending on finitely many edges, and $x \in \mathbb{Z}^d$. We obviously have $\mathbb{P}_p[\mathcal{A}] = \mathbb{P}_p[\tau_x\mathcal{A}]$ for events depending on finitely many edges. Since events depending on finitely many edges span the σ-algebra of measurable events, we obtain that $\mathbb{P}_p$ is invariant under translations.

Let us now show that the probability of an event $\mathcal{A}$ which is invariant under translations is either 0 or 1. For that, we prove that $\mathbb{P}_p[\mathcal{A}] \le \mathbb{P}_p[\mathcal{A}]^2$.

Fix $\varepsilon > 0$. By definition of $\mathfrak{F}$, $\mathcal{A}$ can be approximated by events depending on finitely many edges (see Exercise 10). Therefore, choose $\mathcal{B}$ depending on finitely many edges such that $\mathbb{P}_p[\mathcal{A}\Delta\mathcal{B}] \le \varepsilon$, where $\mathcal{A}\Delta\mathcal{B} := (\mathcal{B} \setminus \mathcal{A}) \cup (\mathcal{A} \setminus \mathcal{B})$ is the *symmetric difference* between $\mathcal{A}$ and $\mathcal{B}$. Since $\mathcal{B}$ depends on a finite set E of edges, there exists x large enough so that E does not intersect the translate of E by x, so that

$$\mathbb{P}_p[\mathcal{B} \cap \tau_x\mathcal{B}] = \mathbb{P}_p[\mathcal{B}]\mathbb{P}_p[\tau_x\mathcal{B}] = \mathbb{P}_p[\mathcal{B}]^2.$$

We deduce that

$$\begin{aligned}\mathbb{P}_p[\mathcal{A}] &= \mathbb{P}_p[\mathcal{A} \cap \mathcal{A}] = \mathbb{P}_p[\mathcal{A} \cap \tau_x\mathcal{A}] \\ &\le \mathbb{P}_p[\mathcal{B} \cap \tau_x\mathcal{B}] + 2\varepsilon = \mathbb{P}_p[\mathcal{B}]^2 + 2\varepsilon \le \mathbb{P}_p[\mathcal{A}]^2 + 4\varepsilon.\end{aligned}$$

By letting ε tend to 0, we deduce that $\mathbb{P}_p[\mathcal{A}] \le \mathbb{P}_p[\mathcal{A}]^2$ which implies that $\mathbb{P}_p[\mathcal{A}] \in \{0,1\}$. □

Corollary 2.9 *For $p > p_c$, there exists an infinite cluster in ω almost surely.*

Note that we may have used Kolmogorov's law to deduce that the infinite cluster in ω exists almost surely when $p > p_c$ since this event is in the asymptotic σ-algebra. Ergodicity will be in fact crucial in the next section where Kolmogorov's law will not be sufficient anymore.

Exercise 17 *Show that the existence of a vertex in the lattice from which two edge-disjoint self-avoiding paths of open edges start has a probability which is either 0 or 1.*

3 The non-critical phases

3.1 Uniqueness of the infinite connected component

Theorem 3.1 *If $p \in [0,1]$ is such that $\theta(p) > 0$, then $\mathbb{P}_p[\exists$ a unique infinite cluster$] = 1$.*

This result was first proved by Aizenman, Kesten and Newman in [2]. It was later obtained via different types of arguments. The beautiful argument presented here is due to Burton and Keane [10].

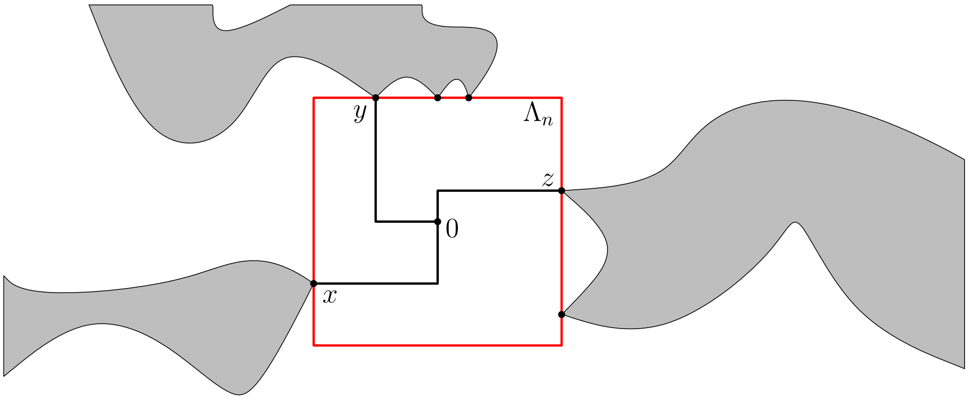

Figure 3: Construction of a trifurcation at the origin starting from three disjoint infinite clusters (in gray) intersecting Λ_n. The three paths inside Λ_n are vertex-disjoint, except at the origin.

Proof of Theorem 3.1 We present the proof in the case of wired boundary conditions and for $p \in (0,1)$ (the result is obvious for p equal to 0 or 1). Let $\mathcal{E}_{\leq 1}$, $\mathcal{E}_{<\infty}$ and $\mathcal{E}_{\infty}$ be the events that there are no more than one, finitely many and infinitely many infinite clusters respectively. Since having no infinite cluster is an event which is invariant under translations, it has probability 0 or 1 by ergodicity, and it is therefore sufficient to prove that $\mathbb{P}_p[\mathcal{E}_{\leq 1}] = 1$.

Let us start by showing that $\mathbb{P}_p[\mathcal{E}_{<\infty} \setminus \mathcal{E}_{\leq 1}] = 0$. By ergodicity, $\mathcal{E}_{<\infty}$ and $\mathcal{E}_{\leq 1}$ both have probability equal to 0 or 1. Since $\mathcal{E}_{\leq 1} \subset \mathcal{E}_{<\infty}$, we only need to prove that $\mathbb{P}_p[\mathcal{E}_{<\infty}] > 0$ implies $\mathbb{P}_p[\mathcal{E}_{\leq 1}] > 0$. Let $\mathcal{F}$ be the event that all (there may be none) the infinite clusters intersect Λ_n. Since $\mathcal{F}$ is independent of E_n, we get that

$$\mathbb{P}_p[\mathcal{F} \cap \{\omega_e = 1, \forall e \in E_n\}] \geq \mathbb{P}_p[\mathcal{F}]\, p^{\,|E_n|}.$$

Now, assume that $\mathbb{P}_p[\mathcal{E}_{<\infty}] > 0$. Since any configuration in the event on the left contains zero or one infinite cluster (all the vertices in Λ_n are connected), choosing n large enough that $\mathbb{P}_p[\mathcal{F}] \geq \frac{1}{2}\mathbb{P}_p[\mathcal{E}_{<\infty}] > 0$ implies that $\mathbb{P}_p[\mathcal{E}_{\leq 1}] > 0$.

We now exclude the possibility of an infinite number of infinite clusters. Consider $n > 0$ large enough that

$$\mathbb{P}_p[K \text{ infinite clusters intersect the box } \Lambda_n] \geq \tfrac{1}{2}\mathbb{P}_p[\mathcal{E}_{\infty}], \tag{3.1}$$

where $K = K(d)$ is large enough that three vertices x, y, z of $\partial\Lambda_n$ at a distance at least three of each other that are connected to infinity in $\omega_{|\mathbb{E}\setminus E_n}$. Using these three vertices, one may modify[1] the configuration in E_n as follows:

1. Choose three paths in Λ_n intersecting each other only at the origin, and intersecting $\partial\Lambda_n$ only at one point, which is respectively x, y and z.
2. Open all the edges of these paths, and close all the other edges in E_n.

We deduce from this construction that leaves

$$\mathbb{P}_p[\mathcal{T}_0] \geq [p(1-p)]^{\,|E_n|} \cdot \tfrac{1}{2}\mathbb{P}_p[\mathcal{E}_{\infty}], \tag{3.2}$$

where $\mathcal{T}_0$ is the following event: $\mathbb{Z}^d \setminus \{0\}$ contains three distinct infinite clusters which are connected to 0 by an open edge. A vertex $x \in \mathbb{Z}^d$ is called a *trifurcation* if $\tau_x\mathcal{T}_0 =: \mathcal{T}_x$ occurs.

[1] Note that one may wish to pick $K = 3$ in (3.1) instead of a (a priori) larger K, but that this choice would make the construction of the trifurcations described below more difficult due to the fact that the three clusters may arrive very close to each other on the corner of Λ_n, and therefore prevent us from "rewiring them" to construct a trifurcation at the origin.

Fix $n \geq 1$ and denote the number of trifurcations in Λ_n by T. By invariance under translation, $\mathbb{P}_p[\mathcal{T}_x] = \mathbb{P}_p[\mathcal{T}_0]$ and therefore

$$\mathbb{E}_p[\mathsf{T}] = \mathbb{P}_p[\mathcal{T}_0] \times |\Lambda_n|. \tag{3.3}$$

Let us now bound deterministically T. In order to do this, first perform the following two "peelings" of the set $F_0 := \{e_1, \dots, e_r\}$ of edges in E_n that are open in ω.

- For each $1 \leq i \leq r$, if e_i is on a cycle formed by edges in F_{i-1}, set $F_i = F_{i-1} \setminus \{e_i\}$, otherwise, set $F_i = F_{i-1}$. In the end, the set $\tilde{F}_0 := F_r = \{f_1, \dots, f_s\}$ is a forest.
- For each $1 \leq j \leq s$, if $\tilde{F}_{j-1} \setminus \{f_j\}$ contains a cluster not intersecting $\partial\Lambda_n$, then set $\tilde{F}_j$ to be $\tilde{F}_{j-1} \setminus \{f_j\}$ and the cluster in question. Otherwise, set $\tilde{F}_j = \tilde{F}_{j-1}$. At the end, $\tilde{F}_s$ is a forest whose leaves belong to $\partial\Lambda_n$.

Since the trifurcations are vertices of degree at least three in this forest, we deduce that T is smaller than the number of leaves in the forest, i.e. $\mathsf{T} \leq |\partial\Lambda_n|$. This gives

$$\mathbb{P}_p[\mathcal{T}_0] \overset{(3.3)}{=} \frac{\mathbb{E}_p[\mathsf{T}]}{|\Lambda_n|} \leq \frac{|\partial\Lambda_n|}{|\Lambda_n|} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Combined with (3.2), this implies that $\mathbb{P}_p[\mathcal{E}_\infty] = 0$. The claim follows. □

Exercise 18 *We say that an (countable) infinite locally finite transitive graph $\mathbb{G}$ is amenable if*

$$\inf_{G \subset \mathbb{G}} \frac{|\partial G|}{|G|} = 0.$$

Show that Theorem 3.1 still holds in this context. What about graphs which are not amenable (give some examples of such graphs), do we always have uniqueness of the infinite cluster?

3.2 Continuity properties of $\theta(p)$

Proposition 3.2 *The map $p \mapsto \theta(p)$ is continuous on $[0,1] \setminus \{p_c\}$ and right continuous at p_c.*

Proof Set $\theta_n(p) = \mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n)$. Since $0 \longleftrightarrow \partial\Lambda_n$ depends only on the edges in Λ_n, the function $\theta_n(p)$ is polynomial and therefore continuous. Moreover, $\theta_n(p)$ decreases to $\theta(p)$ as n tends to infinity. The function θ is therefore the decreasing limit of continuous increasing functions. This implies that θ is right continuous (see Exercise 19).

To prove the left continuity, we use the uniqueness of the infinite cluster. For $p < p_c$, the result is obvious. Fix $p_0 > p_c$ and consider the increasing coupling $\mathbf{P}$. We wish to show that

$$\lim_{p \to p_0-} \theta(p) = \theta(p_0).$$

In this case,

$$\begin{aligned}\theta(p_0) - \lim_{p \to p_0-} \theta(p) &= \mathbf{P}(0 \in \mathsf{C}_{p_0}) - \mathbf{P}(0 \in \mathsf{C}_p \text{ for a certain } p < p_0)\\ &= \mathbf{P}(0 \in \mathsf{C}_{p_0} \text{ and } 0 \notin \mathsf{C}_p \text{ for all } p < p_0).\end{aligned}$$

Choose $p_1 \in (p_c, p_0)$. Almost surely, ω_{p_1} has a (unique) infinite connected component C_{p_1}. Let us assume that $0 \in \mathsf{C}_{p_0}$ but that $0 \notin \mathsf{C}_p$ for every $p < p_0$. By uniqueness of the infinite connected component in ω_{p_0}, there exists a finite open set from 0 to C_{p_1}, but that this is not the case for $p < p_0$. This implies that $\mathsf{U}_e = p_0$ for one of the edges in this path (where U_e is the uniform random variable used to define e). Then,

$$\mathbf{P}(\exists x \in \mathbb{Z}^d : x \in \mathsf{C}_{p_0} \text{ and } x \notin \mathsf{C}_p \text{ for all } p < p_0) = \mathbf{P}(\exists e \in \mathbb{Z}^d : \mathsf{U}_e = p_0) = 0.$$

Thus, $\lim_{p \to p_0 -} \theta(p) = \theta(p_0)$. □

There is one question that the previous theorem does not answer: is θ left continuous at p_c? This is equivalent to asking that $\theta(p_c) = 0$. We will show this fact when $d = 2$. For $\mathbb{Z}^d$ with $d \geq 3$, the absence of an infinite cluster at criticality was proved using lace expansion for $d \geq 19$ by Hara and Slade[22] (it was recently improved to $d \geq 11$ [29]). The technique involved in the proof is expected to work until $d \geq 6$. For $d \in \{3, 4, 5\}$, the strategy will not work and the following conjecture remains one of the major open questions in our field.

Conjecture 1 *For any $d \geq 2$, $\theta(p_c) = 0$.*

Some partial results going in the direction of this conjecture were obtained in the past decades. For instance, it is known that the probability at p_c of an infinite cluster in $\mathbb{N} \times \mathbb{Z}^2$ is zero [3]. Let us also mention that $\mathbb{P}_{p_c(\mathbb{Z}^2 \times G)}[0 \longleftrightarrow \infty]$ was proved to be equal to 0 on graphs of the form $\mathbb{Z}^2 \times G$, where G is finite; see [15], and on graphs with exponential growth in [4] and [23] (see also Exercise 27).

Exercise 19 *Show that a decreasing limit of continuous increasing functions is right continuous.*

Exercise 20 *Show that $p \mapsto \theta(p)$ is strictly increasing when $p > p_c$.*

3.3 Exponential decay in the subcritical regime

Theorem 3.3 (Exponential decay in diameter) *Fix $d \geq 2$. For every $p < p_c$, there exists $c_p > 0$ such that for all $n \geq 1$, $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq \exp(-c_p n)$. Furthermore, there exists $c > 0$ such that for $p > p_c$, $\theta(p) \geq c(p - p_c)$.*

Theorem 3.3 was first proved by Aizenman and Barsky [1] and Menshikov [28] (these two proofs are presented in [19]). Here, we choose to present a new argument from [17, 18]. We also refer to [14] for an alternative proof. The second inequality is called the *mean-field lower bound.*

Of course, the probability of $\{0 \longleftrightarrow \partial\Lambda_n\}$ is at least p^n (since p^n is the probability that the path goes straight in one direction is open). We deduce that the decay cannot be faster.

Proof Define $\theta_n(p) := \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$. Recall that $0 \stackrel{S}{\longleftrightarrow} x$ means that 0 is connected to x using only edges between vertices of S. Denote the edge-boundary of S by $\Delta S = \{xy \subset \mathbb{E} : x \in S, y \notin S\}$. For $p \in [0,1]$ and $S \subset \mathbb{Z}^d$, define

$$\varphi_p(S) := p \sum_{xy \in \Delta S} \mathbb{P}_p[0 \stackrel{S}{\longleftrightarrow} x] \tag{3.4}$$

(note that $\varphi_p(S) = 0$ if $0 \notin S$). The proof will be based on the following two claims.

Claim 1. *If there exists a finite set S containing 0 such that $\varphi_p(S) < 1$, then there exists $c_p > 0$ such that for every $n \geq 1$, $\theta_n(p) \leq \exp(-c_p n)$.*

Claim 2. *For $n \geq 1$, introduce $\mathsf{S} := \{z \in \Lambda_n : z \not\longleftrightarrow \partial\Lambda_n\}$ to be the set of points* not *connected to the boundary of the box of size n. For every $p \in (0,1)$, we have that*

$$\theta_n'(p) = \tfrac{1}{p(1-p)} \mathbb{E}_p[\varphi_p(\mathsf{S})]. \tag{3.5}$$

Remark 3.4 *On the one hand, Claim 2 motivates the introduction of the quantity $\varphi_p(S)$. As we will see in the proof, $\varphi_p(S)$ can be interpreted (up to constant) as the expected number of closed pivotal points for the event $0 \longleftrightarrow \partial\Lambda_n$ conditionally on the fact that the connected component of the boundary in Λ_n is equal to S. On the other hand, $\varphi_p(S) < 1$ for a single finite set $S \ni 0$ is a sufficient condition to imply exponential decay (this observation was actually made in the case of $S = \Lambda_n$ in one of the first papers on Bernoulli percolation, namely Hammersley's paper from 1957 [21]), which provides a second motivation for the introduction of $\varphi_p(S)$.*

Before proving these two claims, let us show how they imply the proof. Set

$$\tilde{p}_c := \sup\big\{p \in [0,1] : \exists \text{ finite set } S \text{ containing } 0 \text{ with } \varphi_p(S) < 1\big\}. \tag{3.6}$$

By Claim 1, for any $p < \tilde{p}_c$, there exists $c_p > 0$ such that $\theta_n(p) \le \exp(-c_p n)$ for every $n \ge 1$. In particular, $\theta(p) = 0$ and therefore $p \le p_c$. As a consequence $\tilde{p}_c \le p_c$. We therefore only need to prove that $\tilde{p}_c \ge p_c$ to conclude the proof of the theorem, or equivalently that $\theta(p) > 0$ for every $p > \tilde{p}_c$. In order to do this, fix $p > \tilde{p}_c$, for which $\varphi_p(S) \ge 1$ for every S containing 0, and $\varphi_p(S) = 0$ for every S not containing 0. The identity (3.5) becomes

$$\theta_n'(p) \ge \tfrac{1}{p(1-p)} \mathbb{P}_p[0 \in \mathsf{S}] = \tfrac{1}{p(1-p)}(1 - \theta_n(p)) \tag{3.7}$$

which can be rewritten as

$$\big[\log\big(\tfrac{1}{1-\theta_n}\big)\big]' \ge \big[\log\big(\tfrac{p}{1-p}\big)\big]'.$$

Integrating between $\tilde{p}_c$ and p implies that for every $n \ge 1$, $\theta_n(p) \ge \frac{p-\tilde{p}_c}{p(1-\tilde{p}_c)}$. By letting n tend to infinity, we obtain the desired lower bound on $\theta(p)$. We now prove the two claims.

Proof of Claim 1. Choose S such that $\varphi_p(S) < 1$ and $L > 0$ such that $S \subset \Lambda_{L-1}$. A proof similar to the proof of Corollary 2.5 (see Exercise 23) implies that

$$\theta_{kL}(p) \le p \sum_{xy \in \Delta S} \mathbb{P}_p\big[0 \overset{S}{\longleftrightarrow} x\big] \mathbb{P}_p[y \longleftrightarrow \partial\Lambda_{kL}] \le \varphi_p(S)\theta_{(k-1)L}(p). \tag{3.8}$$

(In the second inequality, we used that y is at a distance at least $(k-1)L$ of $\partial\Lambda_{kL}$. Induction on k gives $\theta_{kL}(p) \le \varphi_p(S)^k$, thus proving exponential decay at p.

Proof of Claim 2. Let E_n be the set of edges between vertices of Λ_n. Russo's formula (Proposition 2.7) implies that

$$\theta_n'(p) = \sum_{e \in E_n} \mathbb{P}_p[e \text{ is pivotal for } 0 \leftrightarrow \partial\Lambda_n] \tag{3.9}$$

$$= \tfrac{1}{(1-p)} \sum_{e \in E_n} \mathbb{P}_p[\omega_e = 0 \text{ and } e \text{ is pivotal for } 0 \leftrightarrow \partial\Lambda_n], \tag{3.10}$$

where in the second equality we used that e is pivotal is independent of the state of e. Now, the edge e is pivotal and closed if

P1 one of the endpoints of e is connected to 0,
P2 the other one is connected to $\partial\Lambda_n$,
P3 0 is not connected to $\partial\Lambda_n$.

We deduce that

$$\theta_n'(p) = \tfrac{1}{(1-p)} \sum_{\substack{x,y \in \Lambda_n \\ xy \in E_n}} \mathbb{P}_p[0 \longleftrightarrow x, y \longleftrightarrow \partial\Lambda_n, 0 \not\longleftrightarrow \partial\Lambda_n]. \tag{3.11}$$

Now, fix a set S. Let S be the (random) set of $x \in \Lambda_n$ which are *not* connected to $\partial\Lambda_n$. The intersection of $\{\mathsf{S} = S\}$ with the event on the right-hand side of (3.11) can be interpreted nicely. Indeed, the conditions get rephrased as 0 and x belong to S and y does not belong to S. This can be rewritten as $xy \in \Delta S$ and 0 is connected to x in S. Thus, partitioning the event on the right of (3.11) into the possible values of S gives

$$\begin{aligned}
\theta_n'(p) &= \tfrac{1}{(1-p)} \sum_{S \subset \Lambda_n} \sum_{xy \in \Delta S} \mathbb{P}_p\big[0 \overset{S}{\longleftrightarrow} x, \mathsf{S} = S\big] \\
&= \tfrac{1}{(1-p)} \sum_{S \subset \Lambda_n} \Big(\sum_{xy \in \Delta S} \mathbb{P}_p\big[0 \overset{S}{\longleftrightarrow} x\big]\Big) \cdot \mathbb{P}_p[\mathsf{S} = S] \\
&= \tfrac{1}{(1-p)} \mathbb{E}_p\big[\tfrac{1}{p}\varphi_p(\mathsf{S})\big],
\end{aligned}$$

where in the second line we used that $0 \xleftrightarrow{S} x$ is measurable in terms of edges with both endpoints in S, and $\mathsf{S} = S$ is measurable in terms of the other edges. This concludes the proof of Claim 2. $\square$

Remark 3.5 *Since $\varphi_p(\{0\}) = 2dp$, we find $p_c(d) \geq 1/2d$. Also, $p_c(d) \leq p_c(2) = \frac{1}{2}$.*

Remark 3.6 *The set of parameters p such that there exists a finite set $0 \in S \subset \mathbb{Z}^d$ with $\varphi_p(S) < 1$ is an open subset of $[0,1]$. Since this set is coinciding with $[0,p_c)$, we deduce that $\varphi_{p_c}(\Lambda_n) \geq 1$ for any $n \geq 1$. As a consequence, the expected size of the cluster of the origin satisfies at p_c,*

$$\sum_{x \in \mathbb{Z}^d} \mathbb{P}_{p_c}[0 \longleftrightarrow x] \geq \frac{1}{dp_c} \sum_{n \geq 0} \varphi_{p_c}(\Lambda_n) = +\infty.$$

In particular, $\mathbb{P}_{p_c}[0 \leftrightarrow x]$ cannot decay faster than algebraically (see Exercise 26 for more detail).

This theorem implies that when n is very large and $p < p_c$, the largest clusters are of order $\log n$, i.e. they are much smaller than n (see Exercise 22). The previous result implies the following exponential decay in volume.

Theorem 3.7 (Exponential decay volume) *Let C be the connected component of the origin. For every $p < p_c$, there exists $c_p' > 0$ such that for every n,*

$$\mathbb{P}_p[|\mathsf{C}| \geq n] \leq \exp(-c_p' n).$$

The proof of this result is a perfect example of the use of coarse graining. We also present an alternative proof based on the BK inequality in Exercice 25. In the next proof, $\Lambda_n(x)$ denotes the box of size n around k.

Proof Let k be an integer to be chosen in (3.12). Define the graph $\mathcal{G}_k$ as being the graph with vertex set $k\mathbb{Z}^d$, and edges between two vertices $x, y \in \mathcal{G}_k$ if the boxes $\Lambda_{2k}(x)$ and $\Lambda_{2k}(y)$ intersect. The graph is of degree D, where D is independent of k. We will be considering the set $\mathcal{A}(m)$ of animals of cardinality m in $\mathcal{G}_k$. For each animal in $\mathcal{A}(m)$, let $T(A)$ be a maximal set of sites containing at most one endpoint of each edge of $\mathcal{G}_k$.

A site of $\mathcal{G}_k$ is *good* (in ω) if there exists a path from $\Lambda_k(x)$ to $\Lambda_{2k}(x)$ in ω. Note that every site x of $\mathcal{G}_k$ such that $\mathsf{C} \cap \Lambda_k(x) \neq \emptyset$ and $0 \notin \Lambda_{2k}(x)$ is necessarily good in ω. In particular, if $|\mathsf{C}| \geq n$, then there exists a connected animal in $\mathcal{G}_k$ of good sites of cardinality $m \geq \frac{n - |\Lambda_{2k}|}{|\Lambda_k|}$. We deduce that

$$\begin{aligned}
\mathbb{P}_p[|\mathsf{C}| \geq n] &\leq \mathbb{P}_p[\exists A \in \mathcal{A}(m) \text{ such that } \forall x \in A, x \text{ is good}] \\
&\leq \sum_{A \in \mathcal{A}(m)} \mathbb{P}_p[\forall x \in A, x \text{ is good}] \\
&\leq \sum_{A \in \mathcal{A}(m)} \mathbb{P}_p[\forall x \in T(A), x \text{ is good}] \\
&\leq \sum_{A \in \mathcal{A}(m)} \mathbb{P}_p[\Lambda_k \longleftrightarrow \partial\Lambda_{2k}]^{|T(A)|},
\end{aligned}$$

where in the last inequality we have used that the states (good or not good) of sites in $T(A)$ are independent since the boxes of size $2k$ centered on these sites are all disjoint. By Exercise 7, $|\mathcal{A}(m)| \leq 4^{Dm}$. Also, one can clearly find $T(A)$ of cardinality larger than or equal to m/D (this fact can be proved by induction). Now, by picking k large enough, one can choose k in such a way that

$$\mathbb{P}_p[\Lambda_k \longleftrightarrow \partial\Lambda_{2k}] \leq |\partial\Lambda_k| \exp(-ck) < \tfrac{1}{e} \cdot 4^{-D^2}. \tag{3.12}$$

Altogether, we find that

$$\mathbb{P}_p[|\mathsf{C}| \geq n] \leq |\mathcal{A}(m)|\, \mathbb{P}_p[\Lambda_k \longleftrightarrow \partial\Lambda_{2k}]^{m/D} \leq \exp\Big(- \frac{n - |\Lambda_{2k}|}{D|\Lambda_k|}\Big).$$

□

Exercise 21 *Show* (3.5) *using Russo formula.*

Exercise 22 *Show that for every $p < p_c$, there exists $C_p > 0$ such that the probability that there exists a cluster of radius larger than $C_p \log n$ in the box of size Λ_n tends to 0 as n tends to infinity.*

Exercise 23 *Provide two proofs of* (3.8)*, one with and one without the BK inequality.*

Exercise 24 (Percolation with long-range interactions) *Consider a family $(J_{xy})_{x,y\in\mathbb{Z}^d}$ of non-negative coupling constants which is invariant under translations, meaning that $J_{xy} = J(x-y)$ for some function J. Let $\mathbf{P}_\beta$ be the bond percolation measure on $\mathbb{Z}^d$ defined as follows: for $x, y \in \mathbb{Z}^d$, $\{x,y\}$ is* open *with probability $1 - \exp(-\beta J_{xy})$, and* closed *with probability $\exp(-\beta J_{xy})$.*

1. Define the analogues β_c, $\tilde\beta_c$ and $\varphi_\beta(S)$ of p_c, $\tilde p_c$ and $\varphi_p(S)$ in this context.

2. Show that there exists $c > 0$ such that for any $\beta \geq \tilde\beta_c$, $\mathbf{P}_\beta[0 \longleftrightarrow \infty] \geq c(\beta - \tilde\beta_c)$.

3. Show that if the interaction is finite range (i.e. that there exists $R > 0$ such that $J(x) = 0$ for $\|x\| \geq R$), then for any $\beta < \tilde\beta_c$, there exists $c_\beta > 0$ such that $\mathbf{P}_\beta[0 \longleftrightarrow \partial\Lambda_n] \leq \exp(-c_\beta n)$ for all n.

4. In the general case, show that for any $\beta < \tilde\beta_c$, $\sum_{x\in\mathbb{Z}^d} \mathbf{P}_\beta[0 \longleftrightarrow x] < \infty$.

Hint. *Consider S such that $\varphi_\beta(S) < 1$ and show that for $n \geq 1$ and $x \in \Lambda_n$, $\sum_{y\in\Lambda_n} \mathbf{P}_\beta[x \overset{\Lambda_n}{\longleftrightarrow} y] \leq \dfrac{|S|}{1 - \varphi_\beta(S)}$.*

Exercise 25 *The goal of this exercise is to show Theorem 3.7. In this exercise, $p < p_c$ is fixed.*

1) What is the best bound for $\mathbb{P}_p(|\mathsf{C}| \geq n)$ given by Theorem 3.3?

2) Show that it is sufficient to show that $\mathbb{E}_p(e^{\varepsilon|\mathsf{C}|}) < \infty$ for some $\varepsilon > 0$ sufficiently small.

3) Show that $\mathbb{E}_p(|\mathsf{C}|) = \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(0 \longleftrightarrow x) < \infty$ and more generally

$$\mathbb{E}_p[|\mathsf{C}|^n] = \sum_{x_1,\dots,x_n\in\mathbb{Z}^d} \mathbb{P}_p(0 \longleftrightarrow x_1 \longleftrightarrow \cdots \longleftrightarrow x_n) < \infty.$$

4) Show that if 0, x and y are in the same connected component, then there exists $u \in \mathbb{Z}^d$ such that

$$\{u \longleftrightarrow 0\} \circ \{u \longleftrightarrow x\} \circ \{u \longleftrightarrow y\}.$$

Deduce that $\mathbb{E}_p[|C|^2] \leq \mathbb{E}_p[|C|]^3$.

In other words, the preceding reasoning shows that there exists a skeleton, in this case a tree, in $\mathbb{Z}^d$ such that all the edges are open, which contains the points 0, x and y, and u is the unique vertex of degree three.

5) Try to generalize this notion of a skeleton to show that $\mathbb{E}_p[|C|^3] \leq 3\mathbb{E}_p[|C|]^5$.

6) More generally, show that $\mathbb{E}_p(|C|^n) \leq A_n \mathbb{E}_p(|C|)^{2n-1}$ where A_n denotes the number of skeletons with n leaves $x_1, \dots, x_n$.

7) Show that $A_n \leq 2^n n!$ and that $\mathbb{E}_p(|C|^n) \leq 2^n \mathbb{E}_p(|C|)^{2n-1} n!$.

8) Conclude.

The next two exercises illustrate the use of a very simple, yet very powerful technique: sub-additivity (and its corresponding notion sub-multiplicativity).

Exercise 26 (Definition of the correlation length) *Fix $d \geq 2$ and set $e_1 = (1, 0, \dots, 0)$.*

1. Prove that, for any $p \in [0,1]$ and $n, m \geq 0$, $\mathbb{P}_p[x_0 \longleftrightarrow (m+n)e_1] \geq \mathbb{P}_p[x_0 \longleftrightarrow me_1] \cdot \mathbb{P}_p[x_0 \longleftrightarrow ne_1]$

2. Deduce that $\xi(p) = \Big(\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}_p[0 \longleftrightarrow ne_1]\Big)^{-1}$ and that $\mathbb{P}_p[0 \longleftrightarrow ne_1] \leq \exp(-n/\xi(p))$.

3. Assume that $0 \longleftrightarrow \partial\Lambda_{n+m}$, show that there exists $x \in \partial\Lambda_n$ such that $\{0 \longleftrightarrow x\} \circ \{x \longleftrightarrow x + \partial\Lambda_m\}$. Deduce that for all n, m,

$$\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_{m+n}) \le |\partial\Lambda_n| \mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n)\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_m).$$

Show that

$$\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n) \ge e^{-n/\xi_p}/(2^d d(2n+1)^{d-1}).$$

Show that for every $n \in \mathbb{N}$ and $x \in \partial\Lambda_n$,

$$\xi_p \ge \frac{n}{-\log \mathbb{P}_p[0 \stackrel{\Lambda_n}{\longleftrightarrow} x]}.$$

Deduce that ξ_p tends to infinity as $p \nearrow p_c$. Show that $p \mapsto \xi_p$ is continuous on $[0, p_c)$.

4. Prove that for any $x \in \partial\Lambda_n$,

$$\mathbb{P}_p[0 \longleftrightarrow 2ne_1] \ge \mathbb{P}_p[0 \longleftrightarrow x]^2.$$

Deduce that

$$\mathbb{P}_p[0 \leftrightarrow x] \ge \frac{c}{\|x\|_\infty^{2d(d-1)}} \exp(-\|x\|_\infty/\xi_p).$$

5. Deduce that for every $x \in \mathbb{Z}^d$, $\mathbb{P}_{p_c}[0 \leftrightarrow x] \ge \frac{c}{\|x\|_\infty^{2d(d-1)}}$.

Much more precise (and more difficult) estimates are known for $\mathbb{P}_p(0 \longleftrightarrow ne_1)$. These estimates, known as Orstein-Zernike estimates, state that there exists $c = c(p) > 0$ such that

$$\mathbb{P}_p[0 \longleftrightarrow ne_1] = \frac{c}{n^{(d-1)/2}} \exp(-n/\xi_p) \cdot (1 + o(1)).$$

This exercise presents the beautiful proof due to Tom Hutchcroft of absence of percolation at criticality for amenable locally finite transitive graphs with exponential growth. We say that $\mathbb{G}$ has exponential growth if there exists $c_{\rm vg} > 0$ such that $|\Lambda_n| \ge \exp(c_{\rm vg} n)$.

Exercise 27 ($\mathbb{P}_{p_c}[0 \leftrightarrow \infty] = 0$ for amenable Cayley graphs with exponential growth) *Let $\mathbb{G}$ be an amenable infinite locally finite transitive graphs with exponential growth.*

1. Use amenability to prove that $\theta(p_c) > 0 \Longrightarrow \inf\{\mathbb{P}_{p_c}[x \leftrightarrow y], x, y \in \mathbb{G}\} > 0$. Hint: *use Exercise 18.*

2. Use the FKG inequality to prove that $u_n(p) = \inf\{\mathbb{P}_{p_c}[x \leftrightarrow 0], x \in \partial\Lambda_n\}$ satisfies that for every n and m,

$$u_{n+m}(p) \ge u_n(p) u_m(p).$$

3. Adapt Step 1 of the proof of Theorem 3.3 (see also Question 4 of Exercise 24) to get that for any $p < p_c$, $\sum_{x \in \mathbb{G}} \mathbb{P}_p[0 \longleftrightarrow x] < \infty$.

4. Use the two previous questions to deduce that for any $p < p_c$, $u_n(p) \le \exp(-c_{\rm vg} n)$ for every $n \ge 1$.

5. Conclude.

4 Critical percolation on $\mathbb{Z}^2$

4.1 Kesten's theorem

In this section, we focus on the case $d = 2$. Recall the definition of the dual lattice $(\mathbb{Z}^2)^* := (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ of the lattice $\mathbb{Z}^2$ defined by putting a vertex in the middle of each face, and edges between nearest neighbors. Each edge $e \in \mathbb{E}$ is in direct correspondence with an edge e^* of the dual lattice crossing it in its middle. For a finite graph $G = (V, E)$, let G^* be the graph with edge-set $E^* = \{e^*, e \in E\}$ and vertex-set given by the endpoints of the edges in E^*. The configuration ω is naturally associated with a dual configuration ω^*: every edge e which is closed (resp. open) in ω corresponds to an open (resp. closed) edge e^* in ω^*. More formally,

$$\omega^*_{e^*} := 1 - \omega_e \qquad \forall e \in E.$$

Note that if ω is sampled according to $\mathbb{P}_p$, then ω^* is sampled according to (a translate of) $\mathbb{P}_{1-p}$. This duality relation suggests that the critical point of Bernoulli percolation on $\mathbb{Z}^2$ is equal to $1/2$. We discuss different levels of heuristic leading to this prediction.

Heuristic level 0 The simplest non-rigorous justification of the fact that $p_c = 1/2$ invokes the uniqueness of the phase transition, i.e. the observation that the model should undergo a single

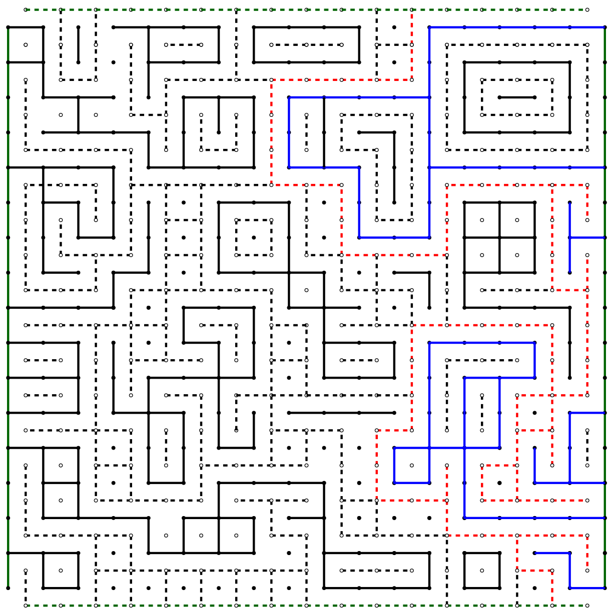

Figure 4: The rectangle R_n together with its dual R_n^* (the green edges on the boundary are irrelevant for the crossing, so that we may consider only the black edges, for which the dual graph is isomorphic to the graph itself (by rotating it). The dual edges (in red) of the edge boundary of the cluster of the right boundary in ω (in blue) is a cluster in ω^* crossing from top to bottom in R_n^*.

change of macroscopic behavior as p varies. This implies that p_c must be equal to $1 - p_c$, since otherwise the model will change at p_c (with the appearance of an infinite cluster in ω), and at $1 - p_c$ (with the disappearance of an infinite cluster in ω^*). Of course, it seems difficult to justify why there should be a unique phase transition. This encourages us to try to improve our heuristic argument.

Heuristic level 1 One may invoke a slightly more subtle argument. On the one hand, assume for a moment that $p_c < 1/2$. In such case, for any $p \in (p_c, 1 - p_c)$, there (almost surely) exist infinite clusters in both ω and ω^*. Since the infinite cluster is unique almost surely, this seems to be difficult to have coexistence of an infinite cluster in ω and an infinite cluster in ω^*, and it therefore leads us to believe that $p_c \geq 1/2$. On the other hand, assume that $p_c > 1/2$. In such case, for any $p \in (1 - p_c, p_c)$, there (almost surely) exist no infinite cluster in both ω and ω^*. This seems to contradict the intuition that if clusters are all finite in ω, then ω^* should contain an infinite cluster. This reasoning is wrong in general (there may be no infinite cluster in both ω and ω^*), but it seems still believable that this should not occur for a whole range of values of p. Again, the argument is fairly weak here and we should improve it.

Heuristic level 2 Consider the event, called $\mathcal{H}_n$, corresponding to the existence of a path of open edges of ω in $R_n := [0, n] \times [0, n-1]$ going from the left to the right side of R_n. Observe that the complement of the event $\mathcal{H}_n$ is the event that there exists a path of open edges in ω^* going from top to bottom in the graph R_n^*; see Fig. 5. Using the rotation by $\pi/2$, one sees that at $p = 1/2$, these two events have the same probability, so that

$$\mathbb{P}_{1/2}[\mathcal{H}_n] = \tfrac{1}{2} \quad \forall n \geq 1. \tag{4.1}$$

Now, one may believe that for $p < p_c$, the clusters are so small that the probability that one of them contains a path crossing R_n from left to right tends to 0, which would imply that the

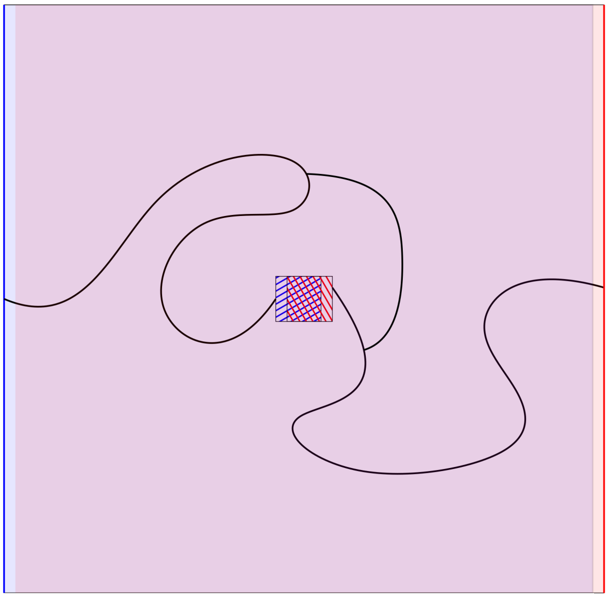

Figure 5: Construction in the proof of Proposition 4.1. One path connects the left side of R_n (in blue) to the blue hatched area. The other one on the right side of R_n (in red) to the red hatched area. The two paths must be in the same cluster (of R_n) by uniqueness, which therefore must contain a path from left to right.

probability of $\mathcal{H}_n$ would tend to 0, and therefore that $p_c \le 1/2$. On the other hand, one may believe that for $p > p_c$, the infinite cluster is so omnipresent that it contains with very high probability a path crossing R_n from left to right, thus implying that the probability of $\mathcal{H}_n$ would tend to 1. This would give $p_c \ge 1/2$. Unfortunately, the first of these two claims is difficult to justify. Nevertheless, the second one can be proved as follows.

Proposition 4.1 *Assume that* $\theta(p) > 0$, *then* $\lim_{n\to\infty} \mathbb{P}_p[\mathcal{H}_n] = 1$.

Proof Fix $n \ge k \ge 1$. Since a path from Λ_k to Λ_n ends up either on the top, bottom, left or right side of Λ_n, the square root trick implies that

$$\mathbb{P}_p[\Lambda_k \text{ is connected in } \Lambda_n \text{ to the left of } \Lambda_n] \ge 1 - \mathbb{P}_p[\Lambda_k \not\leftrightarrow \infty]^{1/4}.$$

Set $n' = \lfloor (n-1)/2 \rfloor$. Consider the event $\mathcal{A}_n$ that $(n', n') + \Lambda_k$ is connected in R_n to the left of R_n, and $(n'+2, n') + \Lambda_k$ is connected in R_n to the right of R_n. We deduce that

$$\mathbb{P}_p[\mathcal{A}_n] \ge 1 - 2\mathbb{P}_p[\Lambda_k \not\leftrightarrow \infty]^{1/4}.$$

The uniqueness of the infinite cluster implies[2] that

$$\liminf_{n\to\infty} \mathbb{P}_p[\mathcal{H}_n] = \liminf_{n\to\infty} \mathbb{P}_p[\mathcal{A}_n] \ge 1 - 2\mathbb{P}_p[\Lambda_k \not\leftrightarrow \infty]^{1/4}.$$

Letting k tend to infinity and using that the infinite cluster exists almost surely, we deduce that $\mathbb{P}_p[\mathcal{H}_n]$ tends to 1. □

[2]The event $\mathcal{A}_n \setminus \mathcal{H}_n$ is included in the event that there are two distinct clusters in R_n going from Λ_k to ∂R_n. The intersection of the latter events for $n \ge 1$ is included in the event that there are two distinct infinite clusters, which has zero probability. Thus, the probability of $\mathcal{A}_n \setminus \mathcal{H}_n$ goes to 0 as n tends to infinity.

Exercise 28 (Zhang argument) *1. Show that*

$$\mathbb{P}_{1/2}[\textit{top of } \Lambda_n \textit{ is connected to infinity outside } \Lambda_n] \geq 1 - \mathbb{P}_{1/2}[\Lambda_n \not\longleftrightarrow \infty]^{1/4}.$$

2. Deduce that the probability of the event $\mathcal{B}_n$ that there exist infinite paths in ω from the top and bottom of Λ_n to infinity in $\mathbb{Z}^2 \setminus \Lambda_n$, and infinite paths in ω^* from the left and right sides to infinity satisfies

$$\mathbb{P}_{1/2}[\mathcal{B}_n] \geq 1 - 4\mathbb{P}_{1/2}[\Lambda_n \not\longleftrightarrow \infty]^{1/4}.$$

3. Using the uniqueness of the infinite cluster, prove that $\mathbb{P}_{1/2}[\Lambda_n \not\leftrightarrow \infty]$ cannot tend to 0.

Exercise 29 *Consider Bernoulli percolation on $\mathbb{Z}^2$. Use the Borel-Cantelli lemma to show that for every $p < p_c$, there exists an infinite connected component in ω^* almost surely. Deduce that $p_c \leq 1/2$.*

This proposition together with (4.1) implies the following corollary

Corollary 4.2 *There is no infinite cluster at $p = 1/2$. In particular, $p_c \geq 1/2$.*

We are in a position to state Kesten's theorem.

Theorem 4.3 (Kesten [24]) *For Bernoulli percolation on $\mathbb{Z}^2$, p_c is equal to 1/2. Furthermore, there is no infinite cluster at p_c.*

As mentioned above, the last thing to justify rigorously is the fact that for $p < p_c$, $\mathbb{P}_p[\mathcal{H}_n]$ tends to 0.

Proof By Theorem 3.3, for any $p < p_c$, there exists $c_p > 0$ such that for all $n \geq 1$,

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \leq \exp(-c_p n).$$

Then, $\mathbb{P}_p[\mathcal{H}_n]$ tends to 0 as n tends to infinity since

$$\begin{aligned}\mathbb{P}_p[\mathcal{H}_n] &\leq \sum_{k=0}^{n-1} \mathbb{P}_p[(0,k) \text{ is connected to the right of } R_n] \\ &\leq n\,\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \leq n\exp(-c_p n).\end{aligned}$$

By (4.1), this implies that $p < 1/2$. □

Remark 4.4 *Note that we just proved that $\theta(p_c) = 0$ on $\mathbb{Z}^2$.*

Exercise 30 ($p_c(\mathbb{G}) + p_c(\mathbb{G}^*) = 1$) *In this exercise, we use the notation $A \overset{B}{\longleftrightarrow} C$ the event that A and C are connected by a path using vertices in B only. Consider Bernoulli percolation on a planar lattice $\mathbb{G}$ embedded in such a way that $\mathbb{Z}^2$ acts transitively on $\mathbb{G}$. We do not assume any symmetry of the lattice. We call the left, right, top and bottom parts of a rectangle* Left, Right, Top *and* Bottom. *Also, $\mathcal{H}(n,k)$ and $\mathcal{V}(n,k)$ are the events that $[0,n]\times[0,k]$ is crossed horizontally and vertically by paths of open edges.*

1. Use the Borel-Cantelli lemma and Theorem 3.3 (one may admit the fact that the theorem extends to this context) to prove that for $p < p_c(\mathbb{G})$, there exists finitely many open circuits surrounding a given vertex of $\mathbb{G}^*$. Deduce that $p_c(\mathbb{G}) + p_c(\mathbb{G}^*) \leq 1$.

We want to prove the converse inequality by contradiction. From now on, we assume that both $p > p_c(\mathbb{G})$ and $p^* > p_c(\mathbb{G}^*)$.

2. For $s > 0$ and $x \in \mathbb{Z}^2$, define $S_x = x + [0,s]^2$. Prove that for any rectangle R, there exists $x = x(R) \in R \cap \mathbb{Z}^2$ such that there exists x' and x'' neighbors of x in $\mathbb{Z}^2$ satisfying

$$\mathbb{P}_p[S_x \overset{R}{\longleftrightarrow} \mathsf{Bottom}] \geq \mathbb{P}_p[S_x \overset{R}{\longleftrightarrow} \mathsf{Top}] \qquad \mathbb{P}_p[S_x \overset{R}{\longleftrightarrow} \mathsf{Left}] \geq \mathbb{P}_p[S_x \overset{R}{\longleftrightarrow} \mathsf{Right}], \tag{4.2}$$

$$\mathbb{P}_p[S_{x'} \overset{R}{\longleftrightarrow} \mathsf{Top}] \geq \mathbb{P}_p[S_{x'} \overset{R}{\longleftrightarrow} \mathsf{Bottom}] \qquad \mathbb{P}_p[S_{x''} \overset{R}{\longleftrightarrow} \mathsf{Right}] \geq \mathbb{P}_p[S_{x''} \overset{R}{\longleftrightarrow} \mathsf{Left}]. \tag{4.3}$$

3. Set $\mathbb{H} := \mathbb{R}_+ \times \mathbb{R}$, $\ell_+ := \{0\}\times\mathbb{R}_+$, $\ell_- := \{0\}\times\mathbb{R}_-$ and $\ell = \ell_- \cup \ell_+$. Prove that there exists $x = x(m)$ with first coordinate equal to m satisfying

$$\mathbb{P}_p[S_x \overset{\mathbb{H}}{\longleftrightarrow} \ell_-] \geq \mathbb{P}_p[S_x \overset{\mathbb{H}}{\longleftrightarrow} \ell_+] \quad \textit{and} \quad \mathbb{P}_p[S_{x+(0,1)} \overset{\mathbb{H}}{\longleftrightarrow} \ell_-] \leq \mathbb{P}_p[S_{x+(0,1)} \overset{\mathbb{H}}{\longleftrightarrow} \ell_+].$$

4. Using the square root trick, deduce that

$$\mathbb{P}_p[S_x \overset{\mathbb{H}}{\longleftrightarrow} \ell_-] \geq 1 - \sqrt{\mathbb{P}_p[S_x \not\leftrightarrow \ell]} \quad \textit{and} \quad \mathbb{P}_p[S_{x+(0,1)} \overset{\mathbb{H}}{\longleftrightarrow} \ell_+] \geq 1 - \sqrt{\mathbb{P}_p[S_{x+(0,1)} \not\leftrightarrow \ell]}.$$

5. Using the fact that there exists a unique infinite cluster in ω almost surely, prove that the probability that $\{0\}\times[0,1]$ is connected in $\omega^* \cap \mathbb{H}$ to infinity is tending to 0.

6. Prove that the distance between $x(R)$ and the boundary of R is necessarily tending to infinity as $\min\{n,k\}$ tends to infinity.

7. Using $x(R)$, prove that $\max\{\mathbb{P}_p[\mathcal{V}(n,k)],\mathbb{P}_p[\mathcal{H}(n,k+1)]\}$ tends to 1 and $\min\{\mathbb{P}_p[\mathcal{V}(n,k)],\mathbb{P}_p[\mathcal{H}(n,k)]\}$ tends to 0 as $\min\{k,n\}$ tends to infinity. Hint. *Use the square root trick and the uniqueness criterion like in the previous questions.*

8. By considering the largest integer k such that $\mathbb{P}_p[\mathcal{V}(n,k)] \geq \mathbb{P}_p[\mathcal{H}(n,k)]$, reach a contradiction. Deduce that $p_c(\mathbb{G}) + p_c(\mathbb{G}^*) \geq 1$.

Exercise 31 (Critical points of the triangular and hexagonal lattices) *Define p such that $p^3 + 1 = 3p$ and set p_c for the critical parameter of the triangular lattice.*

1. Consider a graph G and add a vertex x inside the triangle u,v,w. Modify the graph F by removing edges uv, vw and wu, and adding xu, xv and xw. The new graph is denoted G'. Show that the Bernoulli percolation of parameter p on G can be coupled to the Bernoulli percolation of parameter p on G' in such a way that connections between different vertices of G are the same.

2. Using exponential decay in subcritical for the triangular lattice, show that if $p < p_c$, the percolation of parameter $1-p$ on the hexagonal lattice contains an infinite cluster almost surely. Using the transformation above, reach a contradiction.

3. Prove similarly that $p \leq p_c(\mathbb{T})$.

4. Find a degree three polynomial equation for the critical parameter of the hexagonal lattice.

4.2 The Russo-Seymour-Welsh theory

We saw that the probability of crossing (almost) squares was equal to 1/2. This raises the question of probabilities of crossing more complicated shapes, such as a rectangle with an aspect ratio $\rho \neq 1$. While this could look like a technical question, studying crossing probabilities is instrumental in the study of critical random cluster models.

We begin with some general notation. For a rectangle $R := [a,b]\times[c,d]$, introduce the event $\mathcal{H}(R)$ that R is crossed horizontally, i.e. that the left side $\{a\}\times[c,d]$ is connected by a path in $\omega\cap R$ to the right side $\{b\}\times[c,d]$. Similarly, define $\mathcal{V}(R)$ be the event that R is crossed vertically, i.e. that the bottom side $[a,b]\times\{c\}$ is connected by a path in $\omega\cap R$ to the top side $[a,b]\times\{d\}$. When $R = [0,n]\times[0,k]$, we rather write $\mathcal{V}(n,k)$ and $\mathcal{H}(n,k)$.

Theorem 4.5 (Box-crossing property) *Let $\rho > 0$, there exists $c = c(\rho) > 0$ such that for every $n \geq 1$,*

$$c \leq \mathbb{P}_{1/2}\big[\mathcal{H}(\rho n, n)\big] \leq 1 - c.$$

Note that as soon as we have to our disposal a uniform lower bound (in n) for some $\rho > 1$ on crossing horizontally rectangles of the form $[0,n]\times[0,\rho n]$, then one can easily combine crossings in different rectangles to obtain a uniform lower bound for any $\rho' > 1$. Indeed, set $\varepsilon = \rho - 1$ and define (for integers $i \geq 0$) the rectangles $R_i := [i\varepsilon n, (i\varepsilon+\rho)n]\times[0,n]$ and the squares $S_i := R_i \cap R_{i+1}$. Then,

$$\mathbb{P}_{1/2}[\mathcal{H}(\rho' n, n)] \geq \mathbb{P}_{1/2}\Big[\bigcap_{i\leq\rho'/\varepsilon}(\mathcal{H}(R_i)\cap\mathcal{V}(S_i))\Big] \overset{\text{(Harris)}}{\geq} c(\rho)^{\lfloor\rho'/\varepsilon\rfloor}.$$

One may even prove lower and upper bounds for crossing probabilities in arbitrary topological rectangles (see Exercise 32 below).

Note that combining crossings in squares is much harder. This will in fact be the major obstacle: the main difficulty of this theorem lies in passing from crossing squares with probabilities bounded uniformly from below to crossing rectangles in the hard direction with probabilities bounded uniformly from below. A statement claiming that crossing a rectangle in the hard direction can be expressed in terms of the probability of crossing squares is called a Russo-Seymour-Welsh type theorem.

Theorem 4.6 (Russo-Seymour-Welsh) *For every* $n \geq 1$,

$$\mathbb{P}_{1/2}[\mathcal{H}(3n,2n)] \geq \tfrac{1}{128}.$$

For Bernoulli percolation on the square lattice, such a result was first proved in [30, 31] (maybe with a different lower bound on the right-hand side). Since then, many proofs have been produced, among which [7, 5, 8, 33, 32]. We present a recent proof [7], which is the shortest one (for the square lattice) we are aware of.

Proof Let us introduce the three rectangles

$$R := [-n,2n] \times [-n,n] \qquad S := [0,n]^2 \qquad S' := [-n,n]^2.$$

Define $\mathcal{A} = \mathcal{H}(S)$ and $\mathcal{B}$ the event that there exists a horizontal crossing of S which is connected to the left L of S'. For a path γ from top to bottom in S, and $\sigma(\gamma)$ the reflection of this path with respect to $\mathbb{Z} \times \{0\}$, define the set $V(\gamma)$ of vertices $x \in S'$ on the left of $\gamma \cup \sigma(\gamma)$. Now, on $\mathcal{A}$, condition on the right-most crossing Γ of S. We find that

$$\begin{aligned}
\mathbb{P}_{1/2}[\mathcal{B}] &\geq \sum_{\gamma} \mathbb{P}_{1/2}[\mathcal{B} \,|\, \mathcal{A} \cap \{\Gamma = \gamma\}]\, \mathbb{P}_{1/2}[\mathcal{A} \cap \{\Gamma = \gamma\}] \\
&\geq \sum_{\gamma} \mathbb{P}_{1/2}[\gamma \overset{V(\gamma)}{\longleftrightarrow} L]\, \mathbb{P}_{1/2}[\{\Gamma = \gamma\} \cap \mathcal{A}] \\
&\geq \tfrac{1}{4} \sum_{\gamma} \mathbb{P}_{1/2}[\{\Gamma = \gamma\} \cap \mathcal{A}] = \tfrac{1}{4}\mathbb{P}_{1/2}[\mathcal{A}] \geq \tfrac{1}{8}.
\end{aligned}$$

In the third line, to deduce the lower bound 1/4 we used the facts that conditioned on $\mathcal{A} \cap \{\Gamma = \gamma\}$, the configuration in $V(\gamma)$ is a Bernoulli percolation of parameter 1/2 (since $\mathcal{A} \cap \{\Gamma = \gamma\}$ is measurable with respect to edges on γ or above γ), the symmetry and the fact that the probability of an open path from left to right in $V(\gamma)$ is larger than 1/2 (by (4.1) applied to S'). In the last inequality, we used (4.1) applied to S'.

Now, the event $\mathcal{H}(R)$ occurs if the three events $\mathcal{V}(S)$, $\mathcal{B}$ and $\mathcal{B}'$ occur, where $\mathcal{B}'$ is the event that there exists a horizontal crossing of S which is connected to the right in $[0,2n]^2$. By symmetry,

$$\mathbb{P}_{1/2}[\mathcal{B}'] = \mathbb{P}_{1/2}[\mathcal{B}] \geq \tfrac{1}{8}.$$

Harris inequality (used in the second inequality) implies that

$$\mathbb{P}_{1/2}[\mathcal{H}(R)] \geq \mathbb{P}_{1/2}[\mathcal{V}(S) \cap \mathcal{B} \cap \mathcal{B}'] \geq \mathbb{P}_{1/2}[\mathcal{V}(S)]\mathbb{P}_{1/2}[\mathcal{B}]\mathbb{P}_{1/2}[\mathcal{B}'] \geq \frac{1}{128}.$$

□

Corollary 4.7 *There exists* $\alpha \in (0,\infty)$ *such that for every* $n \geq 1$,

$$\frac{1}{2n} \leq \mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_n] \leq \frac{1}{n^{\alpha}}.$$

Proof For the lower bound, simply observe that

$$n\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_n] \geq \mathbb{P}_{1/2}[\mathcal{H}([0,n]\times[0,n-1])] = 1/2.$$

For the upper bound, introduce the event $\mathcal{A} := \mathcal{V}([-3n,3n]\times[2n,3n])$. If $\partial\Lambda_n$ is connected to $\partial\Lambda_{4n}$, then one of the four rotated versions of the event $\mathcal{A}$ must also occur (where the angles of the rotation are $\frac{\pi}{2}k$ with $0 \leq k \leq 3$). Therefore, the FKG inequality implies that

$$\mathbb{P}_{1/2}\left[\partial\Lambda_n \longleftrightarrow \partial\Lambda_{4n}\right] \leq 1 - \mathbb{P}_{1/2}[\mathcal{A}^c]^4 \leq 1-c,$$

where $c > 0$ exists thanks to Russo-Seymour-Welsh. Using the independence, we obtain that

$$\mathbb{P}_{1/2}\left[0 \longleftrightarrow \partial\Lambda_n\right] \leq \prod_{4^k \leq n} \mathbb{P}_{1/2}\left[\partial\Lambda_{4^{k-1}} \longleftrightarrow \partial\Lambda_{4^k}\right] \leq (1-c)^{\lfloor \log_4 n \rfloor} \leq n^{-\alpha}. \tag{4.4}$$

□

Exercise 32 *Consider a simply connected domain with a smooth boundary Ω with four distinct points a, b, c and d on the boundary. Let $(\Omega^\delta, a^\delta, b^\delta, c^\delta, d^\delta)$ be the finite graph with four marked points on the boundary defined as follows: Ω^δ is equal to $\Omega \cap \delta\mathbb{Z}^2$ (we assume here that it is connected and of connected complement, so that the boundary is a simple path) and a^δ, b^δ, c^δ, d^δ be the four points of $\partial\Omega^\delta$ closest to a, b, c and d.*

Prove that there exists $c = c(q,\Omega,a,b,c,d) > 0$ such that for any $\delta > 0$,

$$\phi^1_{p_c,q}[(a^\delta b^\delta) \overset{\Omega^\delta}{\longleftrightarrow} (c^\delta d^\delta)] \geq c,$$

where $(a^\delta b^\delta)$ and $(c^\delta d^\delta)$ are the portions of $\partial\Omega^\delta$ from a^δ to b^δ, and from c^δ to d^δ, when going counterclockwise around $\partial\Omega^\delta$.

Exercise 33 *Consider Bernoulli percolation (of parameter p) on a planar transitive locally finite infinite graph with $\pi/2$ symmetry.*

1. *Using the rectangles $R_1 = [0,n]\times[0,2n]$, $R_2 = [0,n]\times[n,3n]$, $R_3 = [0,n]\times[2n,4n]$, $R_4 = [0,2n]\times[n,2n]$ and $R_5 = [0,2n]\times[2n,3n]$, show that*
$$\mathbb{P}_p[\mathcal{H}(n,4n)] \leq 5\mathbb{P}[\mathcal{H}(n,2n)].$$
2. *Deduce that $u_{2n} \leq 25u_n^2$ where $u_n = \mathbb{P}_p[\mathcal{H}(n,2n)]$. Show that (u_n) decays exponentially fast as soon as there exists n such that $u_n < \frac{1}{25}$.*
3. *Deduce that $u_n \geq \frac{1}{25}$ for every n or* (EXP$_p$). *What did we prove at p_c?*

Exercise 34 1. *Prove that there exists $c > 0$ such that $\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_n] \leq c\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_{2n}]$.*

2. *Prove that there exist $c_1, c_2 > 0$ such that for any $x \in \partial\Lambda_n$,*
$$c_1\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_n]^2 \leq \mathbb{P}_{1/2}[0 \longleftrightarrow x] \leq c_2\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_n]^2.$$
3. *(quasi-multiplicativity) Prove that there exists $c_1 > 0$ such that for any $1 \leq n \leq N/2$,*
$$\frac{\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_N]}{\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_n]} \leq \mathbb{P}_{1/2}[\Lambda_n \longleftrightarrow \partial\Lambda_N] \leq c_1 \frac{\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_N]}{\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_n]}.$$

Exercise 35 1. *show that the probability that there exists a horizontal crossing of $[0,4n]\times[-n,0]$ in ω and a horizontal crossing of $[-\frac{1}{2}, 4n+\frac{1}{2}]\times[\frac{1}{2}, n-\frac{1}{2}]$ in ω^*.*

2. *Use reasoning similar to Exercise 13 applied to the upper boundary of a cluster crossing $[0,4n]\times[-n,0]$ to get that $\mathbb{P}_{1/2}[0 \longleftrightarrow \partial\Lambda_n] \geq cn^{-1/3}$ for all $n \geq 1$.*

Exercise 36 *Consider a measure μ on $\{0,1\}^{\mathbb{E}}$ which is invariant under the graph isomorphisms of $\mathbb{Z}^2$ onto itself. We further assume that μ satisfies the FKG inequality. We assume that $\inf_n \mu[\mathcal{H}(n,n)] > 0$. The goal of this exercise is to prove that*

$$\limsup_n \mu[\mathcal{H}(3n,n)] > 0. \tag{4.5}$$

1. *Let $\mathcal{E}_n$ be the event that the left side of $[-n,n]^2$ is connected to the top-right corner (n,n). Use the FKG inequality to prove that $\limsup_n \mu[\mathcal{E}_n] > 0$ implies (4.5).*

2. Assume the limit superior above is zero. Now, for any $-n \leq \alpha < \beta \leq n$, define the event $\mathcal{F}_n(\alpha,\beta)$ to be the existence of a crossing from the left side of $[-n,n]^2$ to the segment $\{n\} \times [\alpha,\beta]$. We consider the function

$$h_n(\alpha) = \mu[\mathcal{F}_n(0,\alpha)] - \mu[\mathcal{F}_n(\alpha,n)].$$

Show that h_n is an increasing function, and that there exists $c_0 > 0$ such that $h_n(n) > c_0$ for all n.

3. Assume that $h_n(n/2) < c_0/2$. Use (FKG) *to prove that* (4.5)*.*

4. Assume that $h_n(n/2) > c_0/2$, and let $\alpha_n = \inf\{\alpha : h(\alpha) > c_0/2\}$. Define the event $\mathcal{X}_n(\alpha)$ by the existence of a cluster in $[-n,n]^2$ connecting the four segments $\{-n\} \times [-n,-\alpha]$, $\{-n\} \times [\alpha,n]$, $\{n\} \times [-n,-\alpha]$, and $\{n\} \times [\alpha,n]$. Prove that there exists a constant $c_1 > 0$ independent of n such that $\mu[\mathcal{X}_n(\alpha)] \geq c_1$.

5. Prove that, for infinitely many n's, $\alpha_n < 2\alpha_{2n/3}$.

6. Prove that, whenever $\alpha_n < 2\alpha_{2n/3}$, there exists a constant c_2 such that $\mu[\mathcal{H}(8/3n,2n)] > c_2$. Conclude.

4.3 Conformal invariance of two-dimensional percolation

In 1992, the observation that properties of interfaces should also be conformally invariant led Langlands, Pouliot and Saint-Aubin ([26]) to publish numerical values in agreement with the conformal invariance in the scaling limit of crossing probabilities in the percolation model. More precisely, consider a Jordan domain Ω with four points A, B, C and D on the boundary. The 5-tuple (Ω, A, B, C, D) is called a *topological rectangle*. The authors checked numerically that the probability $\mathcal{C}^\delta(\Omega, A, B, C, D)$ of having a path of adjacent open sites between the boundary arcs AB and CD of a finite piece of a lattice of mesh size δ converges as δ goes to 0 towards a limit which is the same for (Ω, A, B, C, D) and $(\Omega', A', B', C', D')$ if they are images of each other by a conformal map. Notice that the existence of such a crossing property can be expressed in terms of properties of a well-chosen interface, thus keeping this discussion in the frame proposed earlier.

The paper [26], while only numerical, attracted many mathematicians to the domain. The authors attribute the conjecture on conformal invariance of the limit of crossing probabilities to Aizenman. The same year (1992), Cardy [11] proposed an explicit formula for the limit. In 2001, Smirnov proved Cardy's formula rigorously for critical site percolation on the triangular lattice, hence rigorously providing a concrete example of a conformally invariant property of the model. In this section, we switch our attention to site percolation on the triangular lattice. Note that the Russo-Seymour-Welsh is also true in this context.

Below, a *domain* always denotes a finite piece Ω of $\mathbb{H}$ cut from $\mathbb{H}$ by taking the finite connected component of $\mathbb{H} \setminus \Gamma$, where Γ is a self-avoiding polygon on $\mathbb{T} := \mathbb{H}^*$. The set $\partial\Omega$ will denote the set of mid-edges on Γ. A *topological rectangle* is a domain with four points a, b, c, d on $\partial\Omega$. The arc (ab) denotes the counterclockwise arc of $\partial\Omega$ from a to b. We extend this definition to subdomains of the triangular lattice $\delta\mathbb{T}$ with mesh size $\delta > 0$. In this case, we set $(\Omega^\delta, a^\delta, b^\delta, c^\delta, d^\delta)$.

We see site percolation on the triangular lattice as a coloring in black and white of the hexagons of $\mathbb{H}$ or $\delta\mathbb{H}$. In this context, we extend the measure $\mathbb{P}_p$ to $\delta\mathbb{T}$.

Set $j = e^{2\pi i/3}$. Let $\mathbf{T}$ be the equilateral triangle with vertices 1, j and j^2.

Theorem 4.8 (Smirnov) *Consider a family of topological rectangles $(\Omega^\delta, a^\delta, b^\delta, c^\delta, d^\delta)_{\delta>0}$ converging to $(\mathbf{\Omega}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ in the Caratheodory sense. Then,*

$$\lim_{\delta\to 0} \mathbb{P}_{1/2}[(a^\delta b^\delta) \stackrel{\Omega^\delta}{\longleftrightarrow} (c^\delta d^\delta)] = x,$$

where d is mapped to $x + j^2(1-x)$ by the conformal map from Ω to $\mathbf{T}$ mapping a to 1, b to j and c to j^2.

Below, we present a version of Smirnov's proof of this theorem due to Mikhail Khristoforov.

Let us highlight a connection between site percolation on the triangular lattice and the so-called loop $O(1)$ model on the hexagonal lattice. A *loop configuration* on a domain Ω is a collection of non-intersecting loops. Equivalently, it is an even subgraph of Ω, i.e. a graph with even degree everywhere. We denote the set of loop configurations by $\mathcal{E}_\Omega^\varnothing$. Each percolation configuration on $\mathbb{H}$ can be seen as a loop configuration by considering all the edges of $\mathbb{H}$ separating two faces with different colors. A percolation configuration on a finite domain Ω can be seen as a percolation configuration on $\mathbb{H}$ by coloring all the hexagons outside of Ω in white. We therefore can associate a percolation configuration in Ω with a loop configuration in $\mathcal{E}_\Omega^\varnothing$. We deduce that $|\mathcal{E}_\Omega^\varnothing| = 2^N$ where N is the number of faces in Ω.

We extend the definition of loop configurations to include paths from mid-edges to mid-edges. More precisely, let $\mathcal{E}_\Omega^{uv}$ (resp. $\mathcal{E}_\Omega^{uv,rs}$) be the set of configurations with loops together with one self-avoiding path from u to v (resp. and one self-avoiding path from r to s) avoiding the loops.

Before diving into the proof, let us introduce an object which will be important. Fix z a mid-edge of Ω and three mid-edges a, b, c on $\partial\Omega$. Define $F_a(z) := 2^{-N}\mathrm{card}(\mathcal{E}_\Omega^{az,bc})$ and similarly F_b and F_c. Finally, define

$$F(z) := F_a(z) + jF_b(z) + j^2F_c(z).$$

The function F is called the *parafermionic observable with three marked points* in the domain Ω. Among other things, these parafermionic observables have been used to prove conformal invariance of the Ising model [12] and to compute the connective constant of the hexagonal lattice [16].

Remark 4.9 *We have that $F_a(z) + F_b(z) + F_c(z) = 1$ since*

$$\mathrm{card}(\mathcal{E}_\Omega^{az,bc}) + \mathrm{card}(\mathcal{E}_\Omega^{bz,ca}) + \mathrm{card}(\mathcal{E}_\Omega^{cz,ab}) = \mathrm{card}(\mathcal{E}_\Omega^\varnothing) = 2^N.$$

The first identity is given by the fact that for any $\eta_0 \in \mathcal{E}_\Omega^{az,bc} \cup \mathcal{E}_\Omega^{bz,ca} \cup \mathcal{E}_\Omega^{cz,ab}$, the map $\eta \mapsto \eta\Delta\eta_0$ is a bijection from $\mathcal{E}_\Omega^{az,bc} \cup \mathcal{E}_\Omega^{bz,ca} \cup \mathcal{E}_\Omega^{cz,ab}$ to $\mathcal{E}_\Omega^\varnothing$.

Lemma 4.10 *For any vertex v in Ω, if p, q and r denote the three mid-edges around v, indexed in counterclockwise order,*

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0. \tag{4.6}$$

We propose two proofs of this statement.

Proof (number 1) Fix a path γ from b to c and let $\mathcal{E}_\Omega^{az,bc}[\gamma]$ and $\mathcal{E}_\Omega^{bc}[\gamma]$ be the sets of loop configurations in $\mathcal{E}_\Omega^{az,bc}$ and $\mathcal{E}_\Omega^{bc}$ with path γ from b to c. We obtain that

$$\mathrm{card}(\mathcal{E}_\Omega^{az,bc}[\gamma]) = \begin{cases} 0 & \text{if there is no lattice path from } a \text{ to } z \text{ disjoint from } \gamma \\ \mathrm{card}(\mathcal{E}_\Omega^{bc}[\gamma]) & \text{otherwise.} \end{cases} \tag{4.7}$$

Indeed, the first line is trivial and the second simply follows from the fact that for any path ψ from a to z in the complement of γ, the map $\eta \mapsto \eta\Delta\psi$ is an involution from $\mathcal{E}_\Omega^{az,bc}[\gamma]$ to $\mathcal{E}_\Omega^{bc}[\gamma]$. Together with the observation that if γ does not go through v, it either separates simultaneously p, q and r from v, or it does not, we deduce that

$$(p - v)F_a(p) + (q - v)F_a(q) + (r - v)F_a(r) = (p - v)T_a(p) + (q - v)T_a(q) + (r - v)T_a(r), \tag{4.8}$$

where $T_a(p)$ if equal to 2^{-N} times the number of configurations in $\mathcal{E}^{az,bc}$ for which the path from b to c visits v.

But any configuration in $\mathcal{E}_\Omega^{ap,bc}$ is naturally in bijection with configurations in $\mathcal{E}_\Omega^{bq,ca}$ and $\mathcal{E}_\Omega^{cr,ab}$ by simply rotating the two mid-edges incident to v. We deduce that

$$(p - v)T_a(p) + (q - v)jT_b(q) + (r - v)j^2T_c(r) = 0.$$

The proof follows by doing the same for b and c instead of a, and by inserting this in (4.8). □

Proof (number 2) Let $c(\omega) := \tau(\omega)(z-v)2^{-N}$ be the contribution of the configuration ω to the left-hand side of (4.6), where $\tau(\omega)$ is equal to 1, j or j^2 depending on which mid-edge of the boundary is paired with z.

We group configurations in triplets $(\omega_1, \omega_2, \omega_3)$ in such a way that $c(\omega_1) + c(\omega_2) + c(\omega_3) = 0$. In the next cases, we assume that ω_1 contains a path ending at p. There are three cases:

- If no edge in ω_1 is incident to v, then add pv and vq to obtain ω_2, and pv and vr to obtain ω_3. One easily obtains that $c(\omega_1) = j^2 c(\omega_2) = jc(\omega_3)$.
- If a loop of ω_1 passes through v, then add pv and remove vq to obtain ω_2, and add pv and remove vr to obtain ω_3. Again, one obtains that $c(\omega_1) = j^2 c(\omega_2) = jc(\omega_3)$.
- If the path from boundary to boundary passes through v, then add pv and remove vq to obtain ω_2, and add pv and remove vr to obtain ω_3. We find $\tau(\omega_2) = j\tau(\omega_1)$ and $\tau(\omega_3) = j^2\tau(\omega_1)$ so that $c(\omega_1) = jc(\omega_2) = j^2 c(\omega_3)$.

If ω_1 contains a path ending at q or r, one can adapt the previous construction by permuting the indices. Note that any configuration is in one of these triplets, so that the claim follows. □

Remark 4.11 *The result of Lemma 4.10 can be understood in the following way. Coefficients in* (4.6) *are three cubic roots of unity multiplied by $p-v$, so that the left-hand side can be seen as a discrete integral along an elementary contour on the dual lattice in the following sense. For a closed path $\mathbf{z} = (z_i)_{i\le n}$ of vertices in the triangular lattice $\mathbb{T}$ dual to $\mathbb{H}$, define the discrete integral of a function F on mid-edges by*

$$\oint_{\mathbf{z}} F(z)dz := \sum_{i=0}^{n-1} F\left(\tfrac{z_i+z_{i+1}}{2}\right)(z_{i+1}-z_i). \tag{4.9}$$

Equation (4.6) *at v implies that the discrete contour integral going around the face of $\mathbb{T}$ corresponding to v is zero. Decomposing a closed contour into a sum of elementary triangles (this can always be done by the definition of Ω) gives that the discrete integral along any closed path vanishes.*

Remark 4.12 *Note that* (4.7) *implies that $F_a(z) = G_a(z)$, where $G_a(z)$ is equal to 2^{-N} times the number of configurations in $\mathcal{E}_\Omega^{bc}$ for which the path from b to c does not disconnect a from z.*

Proof of Theorem 4.8 Consider the domain Ω^δ and three mid-edges $a^\delta, b^\delta, c^\delta$ on $\partial\Omega^\delta$. Define F_a^δ, F_b^δ, F_c^δ and F^δ as before. Extend F^δ to the interior of the polygon $\partial\Omega^\delta$ by convexly interpolating between mid-edges. Remark 4.9 implies that F^δ maps Ω^δ to $\mathbf{T}$. Furthermore $F_b^\delta(z)$ (resp. $F_c^\delta(z)$ and $F_a^\delta(z)$) is obviously equal to 0 on $(c^\delta a^\delta)$ (resp. $(a^\delta b^\delta)$ and $(b^\delta c^\delta)$). As a consequence $F^\delta(z) \in [j^2, 1]$ (resp. $[1, j]$ and $[j, j^2]$) for $z \in (c^\delta a^\delta)$ (resp. $(a^\delta b^\delta)$ and $(b^\delta c^\delta)$). In other words, $\partial\Omega^\delta$ is mapped to $\partial\mathbf{T}$.

We also have that for $z \in (c^\delta a^\delta)$,

$$F_a^\delta(z) = \mathbb{P}_{1/2}[(a^\delta b^\delta) \stackrel{\Omega^\delta}{\longleftrightarrow} (c^\delta d^\delta)] = 1 - F_c^\delta(z).$$

(The first identity comes from the fact that $\mathcal{E}_\Omega^{az,bc}$ is in bijection with percolation configurations having a crossing from (ab) to (cd), and the second from Remark 4.9 again.) Using Lemma 4.13 below, we may consider a sub-sequential limit F of $(F^\delta)_{\delta>0}$. By Lemma 4.10 and Morera's theorem, F is holomorphic. The previous observation gives us that F is the holomorphic map from $\mathbf{\Omega}$ to $\mathbf{T}$ mapping $\partial\mathbf{\Omega}$ in an injective way (this follows from the fact that $F^\delta(z)$ maps $\partial\mathbf{\Omega}$ to $\partial\mathbf{T}$ in an injective way). Furthermore, a is mapped to 1, b to j, c to j^2. From standard results of complex analysis (more precisely the principle of corresponding boundaries, see for instance

[25, Theorem 4.3]), we deduce that F is the conformal map from $\mathbf{\Omega}$ to $\mathbf{T}$ mapping a, b and c to 1, j and j^2 respectively. We deduce that

$$F(d) = x + j^2(1 - x) = \lim_{\delta\to 0}\left[F_a^\delta(d^\delta) + j^2(1 - F_a^\delta(d^\delta))\right],$$

and the result follows readily. □

Lemma 4.13 *The family $(F^\delta)_{\delta>0}$ is precompact for the uniform convergence on any compact subset of $\overline{\Omega}$.*

Proof We prove that $(F_a^\delta)_{\delta>0}$ is precompact. By Remark 4.12, $F_a^\delta = G_a^\delta$. We deduce that

$$F_a^\delta(z) = G_a^\delta(z) = \mathbb{P}_{1/2}[\gamma \text{ does not disconnect } z \text{ from } a],$$

where γ is obtained as the interface between the black clusters touching $(b^\delta c^\delta)$ and the white clusters connecting $(c^\delta b^\delta)$, and disconnecting means that there exists no self-avoiding path from z to a not intersecting γ.

By the Russo-Seymour-Welsh theorem, one can then prove that

$$|F_a^\delta(z) - F_a^\delta(z')| \le \mathbb{P}_{1/2}[\gamma \text{ disconnects exactly one of } z \text{ or } z' \text{ from } a] \le C|z - z'|^\alpha.$$

Indeed, if there exists a circuit of black hexagons surrounding both z and z', or a path of black hexagons from boundary to boundary and disconnecting both z and z' from a, then the interface must either disconnect simultaneously z and z', or disconnect none of them.

This implies that (F_a^δ) is equicontinuous and the result follows by Arzela-Ascoli. □

References

[1] M. Aizenman and D. J. Barsky. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, 108(3):489–526, 1987.

[2] M. Aizenman, H. Kesten, and C. M. Newman. Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Comm. Math. Phys.*, 111(4):505–531, 1987.

[3] D. J. Barsky, G. R. Grimmett, and Charles M. Newman. Percolation in half-spaces: equality of critical densities and continuity of the percolation probability. *Probab. Theory Related Fields*, 90(1):111–148, 1991.

[4] I. Benjamini, Russell Lyons, Y. Peres, and Oded Schramm. Critical percolation on any nonamenable group has no infinite clusters. *Ann. Probab.*, 27(3):1347–1356, 1999.

[5] Béla Bollobás and Oliver Riordan. The critical probability for random Voronoi percolation in the plane is 1/2. *Probab. Theory Related Fields*, 136(3):417–468, 2006.

[6] Béla Bollobás and Oliver Riordan. *Percolation*. Cambridge University Press, New York, 2006.

[7] Béla Bollobás and Oliver Riordan. A short proof of the Harris-Kesten theorem. *Bull. London Math. Soc.*, 38(3):470–484, 2006.

[8] Béla Bollobás and Oliver Riordan. Percolation on self-dual polygon configurations. In *An irregular mind*, volume 21 of *Bolyai Soc. Math. Stud.*, pages 131–217. János Bolyai Math. Soc., Budapest, 2010.

[9] S. R. Broadbent and J. M. Hammersley. Percolation processes. I. Crystals and mazes. *Proc. Cambridge Philos. Soc.*, 53:629–641, 1957.

[10] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.*, 121(3):501–505, 1989.

[11] John L. Cardy. Critical percolation in finite geometries. *J. Phys. A*, 25(4):L201–L206, 1992.

[12] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.*, 189(3):515–580, 2012.

[13] H. Duminil-Copin. Lectures on the Ising and Potts models on the hypercubic lattice. arXiv:1707.00520.

[14] H. Duminil-Copin, A. Raoufi, and V. Tassion. Sharp phase transition for the random-cluster and Potts models via decision trees. arXiv:1705.03104, 2017.

[15] H. Duminil-Copin, V. Sidoravicius, and V. Tassion. Absence of infinite cluster for critical Bernoulli percolation on slabs. *Communications in Pure and Applied Mathematics*, 69(7):1397–1411, 2016.

[16] H. Duminil-Copin and S. Smirnov. The connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$. *Ann. of Math. (2)*, 175(3):1653–1665, 2012.

[17] H. Duminil-Copin and V. Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Communications in Mathematical Physics*, 343(2):725–745, 2016.

[18] H. Duminil-Copin and V. Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$. *Enseignement Mathématique*, 62(1-2):199–206, 2016.

[19] G. Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.

[20] G. R. Grimmett. Inequalities and entanglements for percolation and random-cluster models. In *Perplexing problems in probability*, volume 44 of *Progr. Probab.*, pages 91–105. Birkhäuser Boston, Boston, MA, 1999.

[21] J. M. Hammersley. Percolation processes: Lower bounds for the critical probability. *Ann. Math. Statist.*, 28:790–795, 1957.

[22] Takashi Hara and Gordon Slade. Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.*, 128(2):333–391, 1990.

[23] Tom Hutchcroft. Critical percolation on any quasi-transitive graph of exponential growth has no infinite clusters. *Comptes Rendus Mathematique*, 354(9):944–947, 2016.

[24] H. Kesten. The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.*, 74(1):41–59, 1980.

[25] Serge Lang. *Complex analysis*, volume 103 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, fourth edition, 1999.

[26] R. Langlands, P. Pouliot, and Y. Saint-Aubin. Conformal invariance in two-dimensional percolation. *Bull. Amer. Math. Soc. (N.S.)*, 30(1):1–61, 1994.

[27] Grigorii A. Margulis. Probabilistic characteristics of graphs with large connectivity. *Problemy Peredaci Informacii*, 10(2):101–108, 1974.

[28] M. V. Menshikov. Coincidence of critical points in percolation problems. *Dokl. Akad. Nauk SSSR*, 288(6):1308–1311, 1986.

[29] Fitzner R. and van der Hofstad R. Mean-field behavior for nearest-neighbor percolation in d>10. arXiv:1506.07977.

[30] L. Russo. A note on percolation. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 43(1):39–48, 1978.

[31] P. D. Seymour and D. J. A. Welsh. Percolation probabilities on the square lattice. *Ann. Discrete Math.*, 3:227–245, 1978. Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).

[32] Vincent Tassion. *Planarité et localité en percolation.* PhD thesis, ENS Lyon, 2014.

[33] Vincent Tassion. Crossing probabilities for voronoi percolation. *Annals of Probability*, 44(5):3385–3398, 2016. arXiv:1410.6773.