Logarithmic variance for the height function of square-ice

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Abstract

In this article, we prove that the height function associated with the square-ice model (i.e. the six-vertex model with $a = b = c = 1$ on the square lattice), or, equivalently, of the uniform random homomorphisms from $\mathbb{Z}^2$ to $\mathbb{Z}$, has logarithmic variance. This establishes a strong form of roughness of this height function.

1 Introduction

1.1 Main results

Two-dimensional models for random surfaces are one of the main subjects of interest of modern statistical physics. These models often undergo a phase transition between a *localized* phase where the random surface does not fluctuate (or equivalently, the variance of the height function at a point remains bounded), and a *delocalized* phase where it does, in the sense that the variance goes to infinity as the domain grows. In the latter, the model is usually predicted to have a Gaussian behaviour and to converge in the sense of distributions in the scaling limit to the Gaussian Free Field (GFF).

There are many models of random surfaces but only a few for which it is known whether the model is in its localized or delocalized phase. Even in cases where the random surface was proved to be delocalized, the convergence to GFF is far from understood. The situation is particularly catastrophic in models where the surface is modelled as a function h from the vertices of a graph G to the integers such that $|h_v - h_u| = 1$. We call such functions *homomorphisms* or *height functions*. Indeed, except for the celebrated work of Fröhlich and Spencer [18] that establishes this fact for the high-temperature integer-valued GFF and for Solid-On-Solid models, all known examples belong to the class of height functions of the six-vertex model, that we now briefly define. The six-vertex model was initially proposed by Pauling in 1935 in order to study the thermodynamic properties of ice. Fix an integer n and consider the torus $\mathbb{T}_n := (\mathbb{Z}/n\mathbb{Z})^2$ and its dual graph $\mathbb{T}_n^*$. Let ω be an arrow configuration on the edges of $\mathbb{T}_n^*$ assigning one of two orientations to each edge of the graph.

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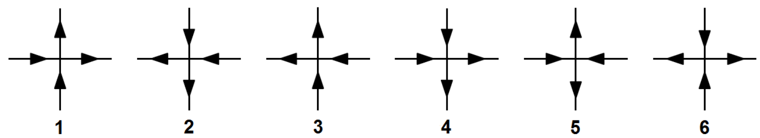

Figure 1: The 6 possibilities for vertices in the six-vertex model. Each possibility comes with a weight a, b or c.

The six-vertex model is given by restricting ω to configurations that have an equal number of arrows entering and exiting each vertex of $\mathbb{T}_n^*$ – a relation we call the *ice rule*. The rule leaves six possible configurations at each such vertex, depicted in Figure 1. Assign the weight a to configurations 1 and 2, b to 3 and 4, and c to 5 and 6. The six-vertex model with weights a, b, c consists in picking such a configuration at random with probability proportional to $a^{n_1+n_2}b^{n_3+n_4}c^{n_5+n_6}$, where n_i is the number of vertices with configuration i in ω, if ω satisfies the ice-rule, and zero otherwise.

Thanks to the ice-rule, six-vertex configurations are naturally associated with a *height function* h on $\mathbb{T}_n$ defined by the property that the increment between the endpoints u and v of the edge e is $+1$ if the associated arrow of the dual edge e^* is crossed from left to right when going from u to v. The height function is technically defined on $\mathbb{Z}^2$, which is the cover of $\mathbb{T}_n$, as a lift of the gradient, which provides a natural definition for $h_u - h_v$ for any $u, v \in \mathbb{T}_n$. On the subset of arrow configurations with as many up arrows as down arrows on each line, and as many left arrows as right arrows on each row, we obtain a well defined height function on $\mathbb{T}_n$. Note that h is defined up to constant, and we will therefore often consider equivalence classes of h for the relation $\sim$, where $h \sim h'$ if and only if $h - h'$ is constant. Also note that the height function partitions the lattice $\mathbb{Z}^2$ into vertices which always take odd values and vertices which always take even values. We call them respectively *odd vertices* and *even vertices*. Throughout the article, we fix the convention that $\{(i,j) \in \mathbb{Z}^2 : (i+j) \mod 2 = 0\}$ is the set of even vertices, and that homomorphisms take even values on even vertices.

When $a = b = 1$ and c is arbitrary, the probability of h is proportional to the number of diagonally connected vertices u and v for which $h_u = h_v$. In particular, when $c = 1$, which corresponds to the famous *square-ice* model, the distribution of h is the uniform measure.

The six-vertex model became the archetypical example of an integrable model after Lieb's solution of the model in 1967 in its anti-ferroelectric and ferroelectric phases [23, 24, 25] using the Bethe ansatz (see [13] and references therein for a review). Since its exact solution, the model has been intensively studied, yet most of the results had fallen short of addressing the question of localization/delocalization of the associated height function. The situation changed in the last two decades. The model at its free fermion point (i.e. when $c = \sqrt{2}$) was directly related to the dimer model, and the height function was proved to converge to GFF (see [9] and reference therein). For $c \geq \sqrt{3}$, the model is related to the critical random-cluster models with $q \geq 1$, where a discontinuous/continuous phase transition was proved in [12] and [11] for $q > 4$ and $1 \leq q \leq 4$, respectively (see also [7]). This immediately implies that the associated height function of the six-vertex model is localized for $c > 2$ and delocalized for $c = 2$. Finally, [30] and later [10] proved that the square-ice height function is delocalized.

In this paper, we wish to study the behaviour of the height function in the height function in the delocalized phase. We start by the following result. Let $\phi_{\mathbb{T}_n}$ be the uniform distribution for height function on the torus.

Theorem 1.1. *There exist $c, C \in (0,\infty)$ such that for every $n \geq 1$ and every $u, v \in \mathbb{T}_n$,*

$$c \log \|u-v\|_1 \leq \phi_{\mathbb{T}_n}[(h_u - h_v)^2] \leq C \log \|u-v\|_1,$$

where $\|\cdot\|_1$ is the L^1 distance in $\mathbb{T}_n$.

In order to state the main result of this paper, we need some more notation. A *path* ($\times$*-path*) is a sequence of vertices $v_0, \dots, v_n$ in $\mathbb{Z}^2$ such that for every $0 \leq i < n$, v_i and v_{i+1} are at a Euclidean distance 1 (resp. $\sqrt{2}$) of each other. When $v_n = v_0$, we call the path a *circuit*. We will often use the notation $[v_i v_j]$ (resp. $(v_i v_j)$) for the subpart of the path made of the vertices $v_i, \dots, v_j$ (resp. $v_{i+1}, \dots, v_{j-1}$). A finite subset D of $\mathbb{Z}^2$ whose boundary ∂D (the boundary is the set of vertices in D with at least one neighbour outside D) is a $\times$-circuit of even (resp. odd) vertices is called an *even* (resp. *odd*) *domain*. A *quad* (D, a, b, c, d) is given by a domain with four marked points in ∂D appearing in counter-clockwise order.

For a quad (D, a, b, c, d), the event $\mathcal{C}_{h\in I}(D, a, b, c, d)$ is the event that there exists a path of vertices in D with height in I that connects $[ab]$ to $[cd]$. We use the shortcut $h = m$, $h \geq m$ and $|h| > 0$ when $I = \{m\}$, $I = [m, \infty)$ and $I = \mathbb{Z} \setminus \{0\}$. We extend this definition to $\times$-paths by introducing the notation $\mathcal{C}^{\times}_{h\in I}(\mathcal{D})$. When D is a rectangle R, we introduce $\mathcal{H}^{\#}_{\#}(R) := \mathcal{C}^{\#}_{\#}(R, a, b, c, d)$ and $\mathcal{V}^{\#}_{\#}(R) = \mathcal{C}^{\#}_{\#}(R, b, c, d, a)$, where a, b, c, d are the four corners of R indexed in counter-clockwise order starting from the top-left one, corresponding to the existence of horizontal and vertical crossings of the rectangle. Also, we write $\Lambda_{n,m} := [-n, n] \times [-m, m]$ for two integers n, m.

Let ϕ^0_D be the uniform distribution on height functions defined on an even domain D which are equal to 0 on ∂D [1]. We consider two possible behaviours:

B1 There exists $c > 0$ such that for every n, k and every even domain $D \supset [-n, n]^2$,

$$\phi^0_D[|h_0| > k] \leq \exp(-n^c).$$

B2 For every $\varepsilon, R, \rho, k > 0$, there exists $c = c(\varepsilon, R, \rho, k) > 0$ such that for every n and every even domain $D \subset \Lambda_{Rn}$ such that the distance between $\Lambda_{\rho n, n}$ and ∂D is at least εn,

$$c \;\leq\; \phi^0_D[\mathcal{H}_{h\geq k}(\Lambda_{\rho n,n})] \;\leq\; 1 - c, \tag{1.1}$$

$$c \;\leq\; \phi^0_D[\mathcal{H}^{\times}_{h=k}(\Lambda_{\rho n,n})] \;\leq\; 1 - c. \tag{1.2}$$

The first case corresponds to a strongly localized behaviour, while the second one corresponds to a delocalized one. For instance, we will see that B2 implies Theorem 1.1 very easily. In fact, it also implies that $\phi^0_D[h_0^2]$ is growing logarithmically in the distance to ∂D. Let us mention that it also easily implies tightness of the family of uniformly chosen height functions when taking the scaling limit of the model, a fact which may be useful to prove convergence to GFF.

We now state what we consider to be the main contribution of this paper.

Theorem 1.2. *For the height function of square-ice, either B1 or B2 occurs.*

[1] This model corresponds to the height function of square-ice when the arrow configurations are defined on the set E^* of edges of $(\mathbb{Z}^2)^*$ bordering the faces of $(\mathbb{Z}^2)^*$ centred on vertices in D, and one applies the *generalized ice-rule* stating that every vertex has the same number of incoming and outgoing arrows.

We insist on stating this result as a dichotomy between two possible behaviours since we believe that this result can be extended to more general random height functions (see Question 1.5 below). Nevertheless, the result of Sheffield [30] (see also [10]) excludes B1, so that we get the following immediate corollary.

Corollary 1.3. *For the height function of square-ice, B2 occurs.*

At a high level, our strategy to prove Theorem 1.2 follows [11], with some inspiration from [16]. It is based on a renormalization argument (which is made more complicated by the height-function structure, see the discussion in Section 4.2) and a Russo-Seymour-Welsh (RSW) theory for height functions. The RSW theory is a study of probabilities of crossing events in planar percolation models. This theory was initially created for the study of Bernoulli percolation [28, 29, 6]. It has blossomed in the past decade and now applies to a wide variety of percolation models [8, 3, 31, 15, 11, 14, 20]. In this paper, we provide the first adaptation of the theory to the study of random height functions.

Theorem 1.4. *For every ρ, there exists $c = c(\rho) > 0$ such that for every even domain D containing $\Lambda_{\rho n,n}$,*

$$\phi^0_{\overline{D}}[\mathcal{H}^\times_{h\geq 2}(\Lambda_{\rho n,n})] \ \geq \ c\phi^0_D[\mathcal{V}^\times_{h\geq 2}(\Lambda_{\rho n,n})]^{\rho/c}, \tag{1.3}$$

where $\overline{D}$ is an even domain containing all the translates of D by $(k,0)$ with $|k| \leq 4\rho n$.

The previous theorem is called a RSW theorem in the sense that it bounds the probability of crossing rectangles in the 'hard' direction in terms of the probability of crossing rectangles in the 'easy' direction. With a little more work, one may replace the right-hand side by a quantity that tends to 1 when the probability of a vertical crossing tends to 1. We will also see that the theorem adapts trivially to other geometry, such as the strip.

1.2 Related results and open questions

The uniform measure on homomorphisms was also introduced independently of the six-vertex model by Benjamini, Häggström and Mossel in [1] (see also [2] for a prior work focusing on the tree) and further investigated in [4, 22, 26, 19, 5, 17, 27] on arbitrary graphs (for which there is no *a priori* connection to square-ice). As mentioned above [30, 10], the model is delocalized on $\mathbb{Z}^2$. In fact, the model undergoes a roughening phase transition; in [27], it was proved that, for every $k \geq 2$ and sufficiently large d, the height function on $\mathbb{T}_n \times (\mathbb{Z}/k\mathbb{Z})^d$ is *localized*.

Related studies consider the behaviour of the class of integer-valued 1-Lipschitz functions. When the base graph is the triangular lattice, delocalization and logarithmic variance has been established through a correpondence with the loop O(2) model with edge weight $x = 1$ [20]. Another delocalization result is obtained in [14] for the height function of the loop O(2) model with weight $1/\sqrt{2}$.

It is natural to ask to which extend the techniques developed in this paper help understanding the height function of the six-vertex model for different values of c. We believe that one of the main contributions of the paper lies in the use of the FKG inequality for $|h|$ to implement comparison between boundary conditions and obtain the RSW theory and the renormalization for crossing probabilities. This FKG for h and $|h|$ are valid for every six-vertex model with $a = b = 1$ and $c \geq 1$. We therefore believe that an argument similar to the present paper could lead to an equivalent of Theorem 1.2 in the regime $a = b = 1$ and $c \geq 1$. This would be particularly interesting since some range of $c \geq 1$ corresponds to the random-cluster model with $q \in (0,1)$ which is known not to satisfy the FKG inequality as a percolation model. Unfortunately, the present techniques do not extend in a trivial fashion due to the lack of spatial Markov property when $c > 1$. More precisely, take the example of an even domain. For $c = 1$, the value of h on ∂V is sufficient to decorrelate the outside from the inside, while this is no longer the case for $c > 1$ (one needs to know what are the values in diagonals as well). For this reason, we leave the following interesting problem open.

Question 1.5. *Prove Theorem 1.2 for the height function of the six-vertex model with $a = b = 1$ and $c \geq 1$.*

Of course, we do not address the important open question of proving GFF fluctuations.

Question 1.6. *Prove that the square-ice height function in an even domain $\Omega_\delta \subset \delta\mathbb{Z}^2$ approximating a simply connected open set Ω converges weakly to the GFF on Ω with Dirichlet boundary condition 0 on $\partial\Omega$.*

Organisation of the paper Section 2 contains some preliminaries (FKG and duality properties). Section 3 deals with the proof of Theorem 1.4 while Section 4 presents the proof of the other theorems.

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2 Preliminaries

In this section, we gather some simple facts about homomorphisms. More precisely, the first part proves the FKG inequality while the second discusses certain connectivity issues that will be important in the following sections.

In order to properly state these properties, we introduce a general notion of boundary condition. For $B \subset D$ with D a domain and κ a function from B into the subsets of $\mathbb{Z}$, define $\mathrm{Hom}(D, B, \kappa)$ to be the set of homomorphisms h on D such that $h_v \in \kappa_v$ for every $v \in B$. We call (B, κ) a *boundary condition* and say that the boundary condition is *admissible* if $\mathrm{Hom}(D, B, \kappa) \neq \emptyset$ is finite. For admissible boundary condition (B, κ), we set $\phi_D^{B,\kappa}$ for the uniform measure on $\mathrm{Hom}(D, B, \kappa)$. When $B = \partial D$, we drop it from the notation.

2.1 Monotonicity properties of uniform homomorphisms

We call a function $F : \mathbb{Z}^D \mapsto \mathbb{R}$ *increasing* if for any $h, h' \in \mathbb{Z}^D$ satisfying $h_v \geq h'_v$ for all $v \in D$, $F(h) \geq F(h')$.

Proposition 2.1 (monotonicity for h)**.** *Consider $D \subset \mathbb{Z}^2$ and two admissible boundary conditions (B, κ) and (B, κ') satisfying that for every $v \in B$, $\kappa_v = [a_v, b_v]$ and $\kappa'_v = [a'_v, b'_v]$ with $a_v \leq a'_v$ and $b_v \leq b'_v$ (the previous integers may be equal to $\pm\infty$), then*

(CBC) *For every increasing function F, $\phi_D^{B,\kappa'}[F(h)] \geq \phi_D^{B,\kappa}[F(h)]$;*

(FKG) *For any two increasing functions F, G, $\phi_D^{B,\kappa}[F(h)G(h)] \geq \phi_D^{B,\kappa}[F(h)]\phi_D^{B,\kappa}[G(h)]$.*

The first property is call the *comparison between boundary conditions*, and the second the *Fortuin-Kasteleyn-Ginibre (FKG)* inequality.

Proof. These results follow from Holley's criterion, see [1, Lemma 2.2], since our definition of height function specifies even height on the even sublattice and therefore implies irreducibility (a fact which is required for Holley's criterion). Note that the conditions on the boundary conditions are designed so that Holley's criterion holds for boundary vertices. □

We also crucially use monotonic properties for $|h|$ instead of h. In order to properly state the conditions for such an inequality, we introduce some new notation. We say that the boundary condition κ is $|h|$*-adapted* if there exists a partition $B_{\mathrm{pos}}(\kappa) \sqcup B_{\mathrm{sym}}(\kappa)$ of B such that

- for any $v \in B_{\mathrm{pos}}(\kappa)$, $\kappa_v \subset \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$;
- for any $w \in B_{\mathrm{sym}}(\kappa)$, $\kappa_w = -\kappa_w$.

Proposition 2.2 (monotonicity for $|h|$)**.** *Consider $D \subset \mathbb{Z}^2$ and two admissible $|h|$-adapted boundary conditions (B, κ) and (B, κ') satisfying $B_{\mathrm{pos}}(\kappa) \subset B_{\mathrm{pos}}(\kappa')$ and for every $v \in B$, $[a_v, b_v] := \kappa_v \cap \mathbb{Z}_+$ and $[a'_v, b'_v] := \kappa'_v \cap \mathbb{Z}_+$ satisfy $a_v \leq a'_v$ and $b_v \leq b'_v$,*

(CBC) *For every increasing function F, $\phi_D^{B,\kappa'}[F(|h|)] \geq \phi_D^{B,\kappa}[F(|h|)]$;*

(FKG) *For any two increasing functions F, G, $\phi_D^{B,\kappa}[F(|h|)G(|h|)] \geq \phi_D^{B,\kappa}[F(|h|)]\phi_D^{B,\kappa}[G(|h|)]$.*

Proof. Fix a vertex v. Using [1, Lemmata 2.2 and 2.3] and references therein, it is sufficient to prove that for two homomorphisms $\xi \leq \eta$ in $\mathrm{Hom}(D \setminus \{v\}, B, \kappa)$ and every k,

$$\phi_D^{B,\kappa}\big[|h_v| \geq k \big| |h_{|D\setminus\{v\}}| = \xi\big] \leq \phi_D^{B,\kappa'}\big[|h_v| \geq k \big| |h_{|D\setminus\{v\}}| = \eta\big].$$

To show this, it is enough to consider the case $\xi_u = \eta_u = 1$ for every neighbour u of v. Indeed, otherwise either $|h_v|$ is determined for the measure on the right and the domination is straightforward to check, or $\xi_u = \eta_u \geq 2$ for every neighbour u of v. In the latter case, all neighbours of u must have the same value of h (and not just $|h|$), and thus it reduces to checking the Holley's criterion for h.

We now move to the case $\xi_u = \eta_u = 1$ for every neighbour u of v. Define $k(\xi)$ and $k(\eta)$ to be the number of clusters (in $\mathbb{Z}^2$) of $|h| \geq 1$ containing at least one neighbour of v and that are respectively *not* intersecting $B_{\mathrm{pos}}(\xi)$ and $B_{\mathrm{pos}}(\eta)$. The height function h has constant sign on each cluster of $|h| \geq 1$, and this sign can be any of the two signs with equal probability unless it contains a vertex of $B_{\mathrm{pos}}(\kappa)$, in which case it must be plus. From this fact, it is easy to deduce that

$$\phi_D^{B,\kappa}\big[|h_v| = 0 \big| |h_{|D\setminus\{v\}}| = \xi\big] = \frac{2^{k(\xi)}}{2 + 2^{k(\xi)}}.$$

and the same is true if we replace ξ by η. Since $k(\xi)$ is decreasing in $|h|$ and since $B_{\mathrm{pos}}(\kappa) \subset B_{\mathrm{pos}}(\kappa')$, the proof is complete. □

Remark 2.3. The non-trivial case of the proof above is reminiscent of the proof of FKG for the FK-Ising model and the condition $B_{\text{pos}}(\kappa) \subset B_{\text{pos}}(\kappa')$ is equivalent to "wiring" more subsets of the boundary. In fact, the proof can be generalized to a case in which the boundary condition specifies an arbitrary 'wiring' – i.e. forcing an arbitrary partition of the boundary to take on the same sign without choosing the particular sign.

2.2 Connectivity properties of lattice paths

Our analysis will deal with paths of vertices in the square lattice and will crucially rely on the property that if a certain path does not connect two arcs of a quad, then there must exist a blocking path connecting the two other arcs. The study will be complicated here by the fact that these blocking paths will not necessarily be of the same kind as the original paths. We therefore gather a few technical statements to which we will refer in the next sections.

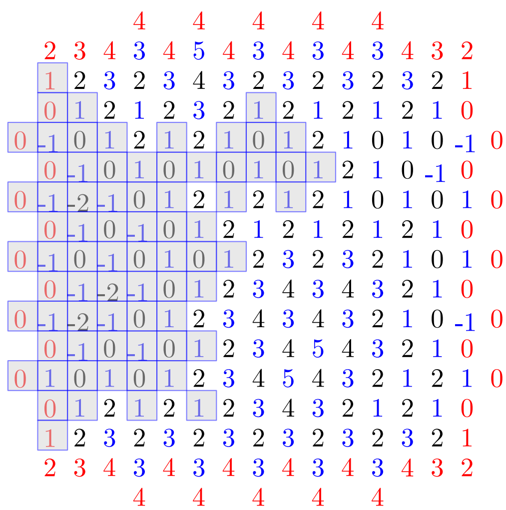

Figure 2: Duality in a square. A top to bottom $h \geq 2$ $\times$-path blocks a left to right $h \leq 1$ path (the cluster of $h \leq 1$ containing the left boundary is shaded). This is a square with symmetric boundary condition (depicted in red) so that the top and bottom boundaries have value 4 and the left and right boundaries have value 0, appropriately modified at the corners. By this duality and symmetry, in a uniform homomorphism, a top to bottom $h \geq 2$ $\times$-path occurs with probability at least $1/2$.

It will be convenient for proofs to introduce a notion of connectivity which is dual to the $\times$-paths. We will say that a path is a $*$-path if successive vertices are at graph distance exactly 2 of each other on $\mathbb{Z}^2$. We introduce the events $\mathcal{C}^*_{h\in I}(D)$ with the notion of $*$-path. The proof of the following lemma is straightforward and left as an exercise (see Figure 2 for an illustration).

Lemma 2.4. *For a quad (D, a, b, c, d) and $m \in \mathbb{Z}$, we have the following properties*

- $\mathcal{C}^\times_{h>m}(D, a, b, c, d)^c = \mathcal{C}_{h\leq m}(D, b, c, d, a) = \mathcal{C}^*_{h<m}(D, b, c, d, a) \supset \mathcal{C}^\times_{h<m}(D, b, c, d, a)$.
- *if $(\partial D, \kappa)$ is an admissible boundary condition which satisfies $\kappa_v \subset [k, \infty]$ for each $v \in [ab] \cup$*

$[cd]$ *and* $\kappa_v \subset [-\infty, k]$ *for each* $v \in [bc] \cup [ad]$*, then for any* $m \geq k$*, on* $\mathrm{Hom}(D, \partial D, \kappa)$*,*

$$\mathcal{C}_{h\geq m}(D,a,b,c,d) = \mathcal{C}_{h\in\{m,m+1\}}(D,a,b,c,d) = \mathcal{C}^*_{h=m+1}(D,a,b,c,d), \tag{2.1}$$

$$\mathcal{C}^\times_{h\geq m}(D,a,b,c,d) = \mathcal{C}^\times_{h=m}(D,a,b,c,d); \tag{2.2}$$

- *If* m *is further assumed to be strictly positive,*

$$\mathcal{C}_{|h|\geq m}(D,a,b,c,d) = \mathcal{C}_{h\geq m}(D,a,b,c,d) \cup \mathcal{C}_{h\leq -m}(D,a,b,c,d).$$

Remark 2.5. The last item tells us that the existence of a $h \geq m$ crossing is *nearly* measurable with respect to the absolute value for any $m \geq 1$. Indeed, $|h|$ determines the connected structure of h, up to the sign of each cluster. Now, if $\mathrm{Hom}(D, B, \kappa)$ is chosen in a manner that determines the sign of a crossing from $[ab]$ to $[cd]$, the event becomes truly measurable with respect to $|h|$. Note that this property does not generalize to every type of connections: while $\times$-crossings of $h \geq 2$ can be decided by h, the same is not true for $\times$-crossings of $h \geq 1$ since $\times$-neighbours may have different signs.

3 Russo-Seymour-Welsh theory

In this section, we prove Theorem 1.4. In Section 3.1, we start by presenting the proof subject to two propositions that we prove in Sections 3.2 and 3.3.

3.1 Proof of Theorem 1.4

We prove the result for the rectangle $\Lambda_{\rho n,3n}$. We introduce the rectangles

$$\begin{aligned} R_n^- &= [-3\rho n, 3\rho n] \times [-3n, -n], \\ R_n^0 &= [-3\rho n, 3\rho n] \times [-n, n], \\ R_n^+ &= [-3\rho n, 3\rho n] \times [n, 3n]. \end{aligned}$$

For $\varepsilon < 1/11$ and k, we set the notations (we keep the dependence on n hidden in the notations)

$$\begin{aligned} I_k &:= [\![(2k-1)\varepsilon n, (2k+1)\varepsilon n]\!] \times \{-3n\}, \\ J_k &:= [\![(2k-1)\varepsilon n, (2k+1)\varepsilon n]\!] \times \{-n\}, \\ K_k &:= [\![(2k-1)\varepsilon n, (2k+1)\varepsilon n]\!] \times \{n\}, \\ L_k &:= [\![(2k-1)\varepsilon n, (2k+1)\varepsilon n]\!] \times \{3n\}. \end{aligned}$$

Let us start by a simple observation that will motivate our reasoning below. Set $\mathcal{A}^i$ to be the event that I_i and I_{i+2} are connected by a $\times$-path of $|h| \geq 2$ staying between heights $-3n$ and $3n$. The $\pm$-symmetry and the FKG inequality for $|h|$ implies that

$$2\phi^0_D[\mathcal{H}^\times_{h\geq 2}(\Lambda_{\rho n,3n})] \geq \phi^0_D[\mathcal{H}^\times_{|h|\geq 2}(\Lambda_{\rho n,3n})] \geq \prod_{i=-\lceil\rho/\varepsilon\rceil-1}^{\lceil\rho/\varepsilon\rceil} \phi^0_D[\mathcal{A}^i]. \tag{3.1}$$

Furthermore, for every i_0 and i, the FKG inequality for $|h|$ implies that

$$\phi^0_D[\mathcal{A}^{i_0}] \geq \phi^0_D[\mathcal{A}^{i_0} | h_{|\partial\widetilde{D}^{i_0-i}} = 0] = \phi^0_{\widetilde{D}^{i_0-i}}[\mathcal{A}^{i_0}] = \phi^0_{\widetilde{D}}[\mathcal{A}^i], \tag{3.2}$$

where $\widetilde{D}$ is the union of the translations of D by $(4k\varepsilon n,0)$ with $-1\le k\le 2$ and $\widetilde{D}^i$ is the translate by $(2\varepsilon i,0)$ of $\widetilde{D}$. The reason for introducing $\widetilde{D}$ will become clear after (3.4). Therefore, our goal is to bound $\max_i \phi^0_{\widetilde{D}}[\mathcal{A}^i]$ from below in terms of $\phi^0_D[\mathcal{V}^\times_{|h|\ge 2}(\Lambda_{\rho n,3n})]$.

In order to do that, let $\mathcal{E}_{ijk\ell}$ be the event that there is a vertical $\times$-crossing of $|h|\ge 2$ in $\Lambda_{\rho n,3n}$ that starts from I_i and ends at L_ℓ, and which contains a sub-path crossing going from J_j to K_k in R^0_n. For $\alpha,\beta,\gamma\in\{-,0,+\}$, introduce the event $\mathcal{E}^{\alpha\beta\gamma}_{ijk\ell}$ that $\mathcal{E}_{ijk\ell}$ occurs and in the $\times$-cluster of $|h|\ge 2$ in $\Lambda_{\rho n,3n}$ starting from I_i, one can find

- a vertical $\times$-crossing of R^-_n starting from I_i and staying in $[(2i-11)\varepsilon n,(2i+11)\varepsilon n]\times\mathbb{Z}$ (resp. intersecting $\{(2i-11)\varepsilon n\}\times\mathbb{Z}$ or $\{(2i+11)\varepsilon n\}\times\mathbb{Z}$) if $\alpha=0$ (resp. $\alpha=+$ or $\alpha=-$);
- a vertical $\times$-crossing of R^0_n starting from J_j and staying in $[(2j-11)\varepsilon n,(2j+11)\varepsilon n]\times\mathbb{Z}$ (resp. intersecting $\{(2j-11)\varepsilon n\}\times\mathbb{Z}$ or $\{(2j+11)\varepsilon n\}\times\mathbb{Z}$) if $\beta=0$ (resp. $\beta=+$ or $\beta=-$);
- a vertical $\times$-crossing of R^+_n starting from K_k and staying in $[(2k-11)\varepsilon n,(2k+11)\varepsilon n]\times\mathbb{Z}$ (resp. intersecting $\{(2k-11)\varepsilon n\}\times\mathbb{Z}$ or $\{(2k+11)\varepsilon n\}\times\mathbb{Z}$) if $\gamma=0$ (resp. $\gamma=+$ or $\gamma=-$).

The square-root trick[2] implies that there exist i,j,k,ℓ and α,β,γ such that

$$\phi^0_D[\mathcal{E}^{\alpha\beta\gamma}_{ijk\ell}]\ge 1-\Big(1-\phi^0_D[\mathcal{V}^\times_{|h|\ge 2}(\Lambda_{\rho n,3n})]\Big)^{1/C},\tag{3.3}$$

where $C=C(\varepsilon,\rho)\ge 27\lceil 2\rho/\varepsilon\rceil$. From now on, we fix $i,j,k,\ell,\alpha,\beta,\gamma$ such that (3.3) holds, and set $\mathcal{E}:=\mathcal{E}^{\alpha\beta\gamma}_{ijk\ell}$. We also introduce the translate $\mathcal{E}^k$ of $\mathcal{E}$ by $(2k\varepsilon n,0)$. The FKG inequality for $|h|$ implies that, as in (3.1),

$$\phi^0_{\widetilde{D}}[\mathcal{E}^{-2}\cap\mathcal{E}\cap\mathcal{E}^2\cap\mathcal{E}^4]\ge\phi^0_D[\mathcal{E}]^4.\tag{3.4}$$

We remark that the domain $\widetilde{D}$ was introduced precisely to make this inequality manifest. If either $\phi^0_{\widetilde{D}}[\mathcal{A}^{i-2}]$ or $\phi^0_{\widetilde{D}}[\mathcal{A}^{i+2}]$ is larger than $\frac13\phi^0_D[\mathcal{E}]^4$, we are done thanks to (3.3). Otherwise, we have that

$$\phi^0_{\widetilde{D}}[\mathcal{E}^{-2}\cap\mathcal{E}\cap\mathcal{E}^2\cap\mathcal{E}^4\cap(\mathcal{A}^{i-2})^c\cap(\mathcal{A}^{i+2})^c]\ge\tfrac13\phi^0_D[\mathcal{E}]^4.$$

The rest of the proof will be devoted to the proof of the following inequality:

$$\phi^0_{\widetilde{D}}[\mathcal{A}^i|\mathcal{E}^{-2}\cap\mathcal{E}\cap\mathcal{E}^2\cap\mathcal{E}^4\cap(\mathcal{A}^{i-2})^c\cap(\mathcal{A}^{i+2})^c]\ge\tfrac{1}{32}.$$

Once this inequality is established, we can conclude the argument, since it can be combined with the earlier inequality and (3.3) to provide a lower bound on $\max_i\phi^0_{\widetilde{D}}[\mathcal{A}^i]$

In order to prove this statement, we first state two propositions that will be proved in Sections 3.2 and 3.3 respectively. An *even-quad* is a quad for which D is an even domain.

Proposition 3.1. *For every $n\ge 1$ and every even-quad (D,a,b,c,d) with $[ab]$ and $[dc]$ from top to bottom in $\mathbb{Z}\times[-n,n]$ and $[bc]$ and $[da]$ are the even paths from b to c and d to a in $\mathbb{Z}\times\{-n-1,-n\}$ and $\mathbb{Z}\times\{n,n+1\}$ respectively,*

$$\phi^\kappa_D[\mathcal{C}_{h\ge 1}(D,a,b,c,d)]\ge\tfrac12,$$

[2] We prefer the use of the square-root trick to the use of the union bound since we will refer to this argument later with events having a probability close to 1. We recall that the square-root trick yields that for increasing events $\mathcal{A},\dots,\mathcal{A}_s$ and a measure $\mathbb{P}$ satisfying the FKG inequality,

$$\max_{i\le s}\mathbb{P}[\mathcal{A}_i]\ge 1-(1-\mathbb{P}[\mathcal{A}_1\cup\cdots\cup\mathcal{A}_s])^{1/s}.$$

where κ is equal to 2 on $[ab]\cup[cd]$ and 0 on $(bc)\cup(da)$, if D is in one of the following three configurations:

- *$[ab]\cup[cd]$ is contained in $\Lambda_{n/2,n}$,*
- *$[ab]$ intersects the vertical line containing c,*
- *$[cd]$ intersects the vertical line containing b.*

A quad (D,a,b,c,d) is called *mixed* if $[ab]$ and $[cd]$ are even $\times$-paths, and (bc) and (da) are odd $\times$-paths. Note that in this case D is not quite a domain according to the definition of the introduction, we will therefore refer to it as a *mixed-domain.*

Proposition 3.2. *For every $n\geq 1$ and every mixed-quad (D,a,b,c,d) with $[ab]$ and $[dc]$ from top to bottom in $\mathbb{Z}\times[-n,n]$ and (bc) and (da) remaining outside of the domain bounded between $[ab]$ and $[cd]$ inside $\mathbb{Z}\times[-n,n]$,*

$$\phi_D^{\kappa}[\mathcal{C}^{\times}_{h\geq 2}(D,a,b,c,d)]\geq \tfrac{1}{2},$$

where κ is equal to 2 on $[ab]\cup[cd]$ and 1 on $(bc)\cup(da)$, if D is in one of the following three configurations:

- *$[ab]\cup[cd]$ is contained in $\Lambda_{n/2,n}$,*
- *$[ab]$ intersects the vertical line containing $c-(\frac{1}{2},0)$,*
- *$[cd]$ intersects the vertical line containing $b+(\frac{1}{2},0)$.*

With these two propositions at hand, we can conclude the proof. The argument is divided in three steps: first, we transform our problem into the existence of a $\times$-crossing of $h\geq 2$ in a domain with boundary condition 0/2/0/2. Then, we prove the existence of two crossings of $h\geq 1$ in this domain using the first proposition twice. Finally, we use the second proposition to create a $\times$-crossing of $h\geq 2$.

Condition on $|h|$ on every vertex which is not strictly between the left-most vertical $\times$-crossing $[ab]$ (between $\mathbb{Z}\times\{-3n\}$ and $\mathbb{Z}\times\{3n\}$) of $|h|\geq 2$ intersecting I_i as well as the right-most crossing $[cd]$ between $\mathbb{Z}\times\{-3n\}$ and $\mathbb{Z}\times\{3n\}$ intersecting I_{i+2} . For future reference, we denote by Ω_0 the domain enclosed between these two paths and the two even $\times$-paths $[bc]$ and $[da]$ between b and c and d and a in $\mathbb{Z}\times\{-3n,-3n-1\}$ and $\mathbb{Z}\times\{3n,3n+1\}$ respectively. The $|h|$-adapted boundary condition (B,ξ) induced on Ω_0 by this conditioning fixes $|h|$ on $\partial\Omega_0\cup\Omega_0^c$, with $|h|=2$ on $[ab]\cup[cd]$. Let (B,ξ_0) be the $|h|$-adapted boundary condition equal to $|h|=2$ on $[ab]\cup[cd]$ and 0 on the rest of $\partial\Omega_0\cup\Omega_0^c$. The comparison between boundary conditions for $|h|$ shows that,

$$\phi_{\tilde{D}}^{B,\xi}[\mathcal{A}^i]\geq\phi_{\tilde{D}}^{B,\xi_0}[\mathcal{A}^i]\geq\tfrac{1}{4}\phi_{\Omega_0}^{\kappa_0}[\mathcal{C}^{\times}_{|h|\geq 2}(\Omega_0,a,b,c,d)], \tag{3.5}$$

where κ_0 is the boundary condition on Ω_0 equal to 2 on $[ab]\cup[cd]$ and 0 on $(bc)\cup(da)$. To obtain the last inequality, we used that whatever the $|h|$-adapted boundary condition, the probability that the $|h|\geq 1$-clusters of $[ab]$ and $[cd]$ have a + sign is at least 1/4. Indeed, simply observe that one may discover the rest of the $|h|$ configuration, and find that either the two clusters are distinct in which case the independently assigned sign are + with probability 1/4, or they are connected in which case it is 1/2.

Overall, we see that, in order to conclude the proof, it is sufficient to show that, for any realization of Ω_0,

$$\phi_{\Omega_0}^{\kappa_0}[\mathcal{C}^{\times}_{h\geq 2}(\Omega_0,a,b,c,d)]\geq\tfrac{1}{8}. \tag{3.6}$$

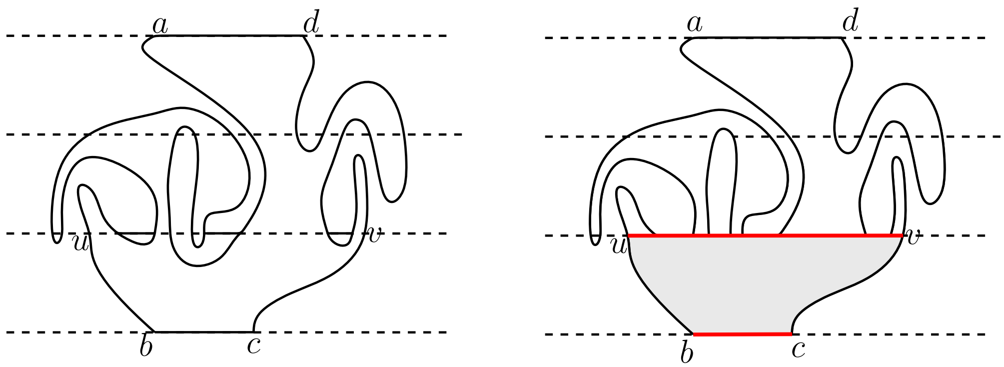

Figure 3: The surgery in the definition of Ω_- (shaded grey). The black paths are even $\times$-paths with value 2 and the red paths are even $\times$-paths with value 0.

Let u and v be the vertices of $[ab]$ and $[cd]$ such that $[ub]$ and $[cv]$ are vertical crossings of the strip between height $-3n$ and $-n$ and let Ω_- be the part of $\mathbb{Z}^2$ enclosed by $[ub] \cup [bc] \cup [cv]$ and the even $\times$-path $[vu]$ between u and v in $\mathbb{Z} \times \{-n, -n+1\}$ with boundary condition κ_- equal to 0 on the bottom and top arcs, and 2 on the rest of the boundary. We find that

$$\begin{aligned}\phi^{\kappa_0}_{\Omega_0}[\mathcal{C}_{h\geq 1}(\Omega_-, u, b, c, v)] &\geq \phi^{\kappa_0}_{\Omega_0}[\mathcal{C}_{h\geq 1}(\Omega_-, u, b, c, v) | h_{|\partial\Omega_- \setminus \partial\Omega_0} = 0] \\ &= \phi^{\kappa_0}_{\Omega_0}[\mathcal{C}^*_{h=2}(\Omega_-, u, b, c, v) | h_{|\partial\Omega_- \setminus \partial\Omega_0} = 0] \\ &\geq \phi^{\kappa_-}_{\Omega_-}[\mathcal{C}^*_{h=2}(\Omega_-, u, b, c, v)] \\ &= \phi^{\kappa_-}_{\Omega_-}[\mathcal{C}_{h\geq 1}(\Omega_-, u, b, c, v)]. \end{aligned} \tag{3.7}$$

In the first line, we use FKG for $|h|$ (the conditioning is fixing some values to be 0, while the event is increasing in $|h|$). The next equality follows from Lemma 2.4, the third line follows from FKG for $|h-2|$ and the spatial Markov property, as we are taking vertices where $h = 2$ and setting them to take the value 0. The fourth line is Lemma 2.4, again.

The crucial observation is that the occurrence of events $\mathcal{E}^{-2} \setminus \mathcal{A}^{i-2}$ and $\mathcal{E}^4 \setminus \mathcal{A}^{i+2}$ restricts the geometry of the quad (Ω_-, u, b, c, v). More specifically, when the events occur, the quad with boundary condition κ_- is necessarily in the first configuration of Proposition 3.1 if $\alpha = 0$, in the second if $\alpha = +$, and in the third if $\alpha = -$. In any case, we deduce that with probability $1/2$, there is a crossing of $h \geq 1$ from $[ub]$ to $[cv]$ in Ω_-. One can do the same in a domain Ω_+ defined in a similar fashion in the strip $\mathbb{Z} \times [n, 3n]$, and FKG implies that both crossings occur with probability at least $1/4$.

We now assume that the event $\mathcal{C}(\Omega_-, u, b, c, v)$ and the analogous event for the top domain occur in Ω_0. By Lemma 2.4, this implies that Ω_- and Ω_+ both contain a $\times$-crossings of $h = 1$ from $[ub]$ to $[cv]$. Condition on the bottom-most and top-most such $\times$-crossings of $h = 1$ and let Ω_1 be the subdomain of Ω_0 enclosed between these paths. We denote by κ_1 the boundary condition induced by the conditioning, which is 2 on the even vertices of the boundary and 1 on the odd ones. Let $[a'b']$ and $[c'd']$ be subpaths of $[ab]$ and $[cd]$ that

- are contained in $\mathbb{Z} \times [-n, n]$,
- are crossing $\mathbb{Z} \times [-n, n]$ from $J_j \cup J_{j+1} \cup J_{j+2}$ to $K_k \cup K_{k+1} \cup K_{k+2}$,
- are such that $[a'b']$ is on the left of $[c'd']$ and there is no additional crossing of $\mathbb{Z} \times [-n, n]$ in $[ab] \cup [cd]$ in between.

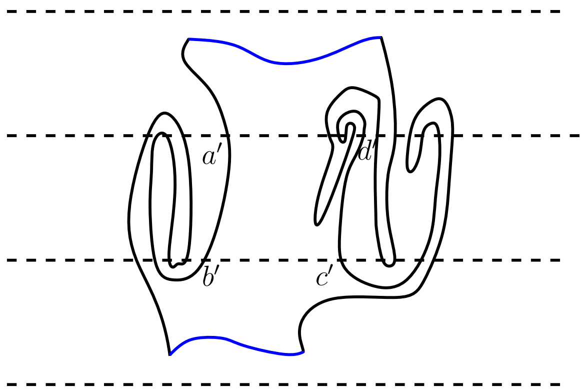

Figure 4: The black paths are even $\times$-paths with value 2 and the blue paths are odd $\times$-paths with value 1. Left: The domain Ω_1. Right: The domain Ω_2 in shaded grey obtained after the surgery.

The existence of such subpaths is easy to obtain. Indeed, take the leftmost and rightmost crossings satisfying the second item above and consider a curve starting at a point immediately to the left of the leftmost crossing and ending at a point to the right of the rightmost crossing, staying inside $\mathbb{Z} \times [-n, n]$ and avoiding $[ab] \cup [cd]$ except for the crossings of $\mathbb{Z} \times [-n, n]$. This curve must intersect a crossing coming from $[cd]$ immediately after intersecting a crossing coming from $[ab]$ at some point and these two crossings can be taken to be $[a'b']$ and $[c'd']$.

Let Ω_2 be the domain obtained as the union of Ω_1, the vertices Ω_1' of $\mathbb{Z} \times [-n, n]$ between $[a'b']$ and $[c'd']$, as well as all the vertices that are surrounded by $\Omega_1 \cup \Omega_1'$. In words, Ω_2 corresponds to cutting all the "tongues" of $[ab]$ and $[cd]$ entering Ω_1' by "pushing them away". Note that, since we took $[a'b']$ and $[c'd']$ to be successive crossings, none of these tongues crosses Ω_1'. Further, in the bottom strip, no part of the boundary with $h = 1$ is affected by this procedure since we considered crossings from $[ub]$ to $[cv]$[3]. Similarly, no part of the boundary with $h = 1$ is affected in the top strip. Let κ_2 be the boundary condition equal to 2 on even vertices of $\partial\Omega_2$, and 1 on odd ones.

Exactly as for (3.7), FKG for $|h - 2|$ enables us to push away the $h = 2$ boundary condition to get that

$$
\begin{aligned}
\phi_{\Omega_1}^{\kappa_1}[\mathcal{H}_{h\geq 2}^{\times}(\Omega_1, a, b, c, d)] &= \phi_{\Omega_1}^{\kappa_1}[\mathcal{H}_{h=2}^{\times}(\Omega_1, a, b, c, d)] \\
&= \phi_{\Omega_2}^{\kappa_2}[\mathcal{H}_{h=2}^{\times}(\Omega_1, a, b, c, d) | h_{|\partial\Omega_1 \setminus \partial\Omega_2} = 2] \\
&\geq \phi_{\Omega_2}^{\kappa_2}[\mathcal{H}_{h=2}^{\times}(\Omega_2, a, b, c, d)] \\
&= \phi_{\Omega_2}^{\kappa_2}[\mathcal{H}_{h\geq 2}^{\times}(\Omega_2, a, b, c, d)] \\
&\geq \phi_{\Omega_2}^{\kappa_2}[\mathcal{H}_{h\geq 2}^{\times}(\Omega_2, a', b', c', d')]. \qquad (3.8)
\end{aligned}
$$

Let us elaborate a bit. In the first equality, we used the second item of Lemma 2.4, the second equality is simply using the spatial Markov property, the first inequality is FKG for $|h - 2|$ and inclusion (a horizontal crossing of Ω_2 guarantees a horizontal crossing of Ω_1), the last equality is again the second item of Lemma 2.4 and the final inequality is simply inclusion.

[3]More precisely, we have that $h \geq 2$ on $\partial\Omega_1 \setminus \partial\Omega_2$. Indeed, on the one hand any boundary vertex with $h = 1$ is connected to $\mathbb{Z} \times \{-3n\}$ by a path staying in $\Omega_0 \cap (\mathbb{Z} \times [-3n, -n])$, while on the other hand, by definition $\partial\Omega_1 \setminus \partial\Omega_2 \subset \mathbb{Z} \times [-n, n]$ and any vertex in $\Omega_2 \setminus \partial\Omega_1'$ is disconnected from $\mathbb{Z} \times \{-3n\}$ by $\partial\Omega_1'$.

Now, let Ω_3 be the mix-domain composed of Ω_2 together with the odd vertices outside $\mathbb{Z}\times[-n,n]$ that are on the exterior boundary of Ω_2 (meaning that they do not belong to the set but are neighbours of a vertex belonging to the set). If κ_3 is the boundary condition on Ω_3 equal to 2 on $[a'b']\cup[c'd']$ and 1 on $(b'c')\cup(d'a')$, we may use the FKG inequality for h to show that

$$\begin{aligned}\phi_{\Omega_2}^{\kappa_2}[\mathcal{H}_{h\geq 2}^{\times}(\Omega_2,a',b',c',d')] &= \phi_{\Omega_3}^{\kappa_3}[\mathcal{H}_{h\geq 2}^{\times}(\Omega_2,a',b',c',d')|h_{|\partial\Omega_2\setminus\partial\Omega_3}=2]\\ &= \phi_{\Omega_3}^{\kappa_3}[\mathcal{H}_{h\geq 2}^{\times}(\Omega_2,a',b',c',d')|h_{|\partial\Omega_2\setminus\partial\Omega_3}\geq 2]\\ &\geq \phi_{\Omega_3}^{\kappa_3}[\mathcal{H}_{h\geq 2}^{\times}(\Omega_3,a',b',c',d')].\end{aligned}\tag{3.9}$$

Since $\mathcal{E}^{-2}\setminus\mathcal{A}^{i-2}$ and $\mathcal{E}^{4}\setminus\mathcal{A}^{i+2}$ occur, we are now facing a quad which is in the first configuration of Proposition 3.2 if $\beta=0$ (resp. second if $\beta=+$, third if $\beta=-$). We deduce that

$$\phi_{\Omega_3}^{\kappa_3}[\mathcal{H}_{h\geq 2}^{\times}(\Omega_3,a',b',c',d')]\geq \tfrac{1}{2},$$

which together with the last displayed equations, concludes the proof.

3.2 Proof of Proposition 3.1

We prove the first two cases of the proposition; the third can be proven analogously to the second.

First case Let Λ_n^{even} be the set of vertices inside (or on) the even circuit in $\Lambda_{n+1}\setminus\Lambda_{n-1}$ surrounding the origin. Consider the boundary condition ξ on Λ_n^{even} equal to 2 on left and right (including vertices on $y=x$), and 0 on the rest. By the second item of Lemma 2.4, and FKG for $|h-2|$ (the reasoning is the same as in (3.7)), we find that

$$\phi_D^{\kappa}[\mathcal{C}_{h\geq 1}(D,a,b,c,d)]=\phi_D^{\kappa}[\mathcal{C}_{h=2}^{*}(D,a,b,c,d)]\geq \phi_{\Lambda_n^{\text{even}}}^{\xi}[\mathcal{H}_{h=2}^{*}(\Lambda_n^{\text{even}})]=\phi_{\Lambda_n^{\text{even}}}^{\xi}[\mathcal{H}_{h\geq 1}(\Lambda_n^{\text{even}})].\tag{3.10}$$

Now, using the first item of Lemma 2.4 again, we find that

$$\begin{aligned}\phi_{\Lambda_n^{\text{even}}}^{\xi}[\mathcal{H}_{h\geq 1}(\Lambda_n^{\text{even}})] &= 1-\phi_{\Lambda_n^{\text{even}}}^{\xi}[\mathcal{V}_{h\leq 0}^{\times}(\Lambda_n^{\text{even}})]\\ &\geq 1-\phi_{\Lambda_n^{\text{even}}}^{\xi}[\mathcal{V}_{h\leq 1}(\Lambda_n^{\text{even}})],\\ &\geq 1-\phi_{\Lambda_n^{\text{even}}}^{\xi}[\mathcal{H}_{h\geq 1}(\Lambda_n^{\text{even}})].\end{aligned}\tag{3.11}$$

In order to deduce the last inequality, we used the FKG inequality for h and the symmetry of Λ_n^{even} by $\pi/2$ rotation and the fact that the boundary condition ξ' obtained by rotating and applying the transformation $2-h$ is smaller than the boundary condition ξ. Overall, (3.11) implies that $\phi_{\Lambda_n^{\text{even}}}^{\xi}[\mathcal{H}_{h\geq 1}(\Lambda_n^{\text{even}})]\geq\frac{1}{2}$, which concludes the proof.

Second case Let ℓ be the vertical line passing by c. Assume that $[ab]$ and $[cd]$ do not intersect — otherwise, there is nothing to prove. Denote the reflection with respect to the line ℓ by σ, and note that σ maps the even lattice to itself. Let a' be the first intersection of $[ab]$ when going from bottom to top (i.e. when going from b to a) with ℓ and let Ω be the quad enclosed by $[a'b]$, $\sigma([a'b])$ and the $\times$-path of even vertices in $\mathbb{Z}\times\{-n-1,-n\}$ between b and $\sigma(b)$. By definition, $(\Omega,b,c,\sigma(b),a')$ is symmetric under σ.

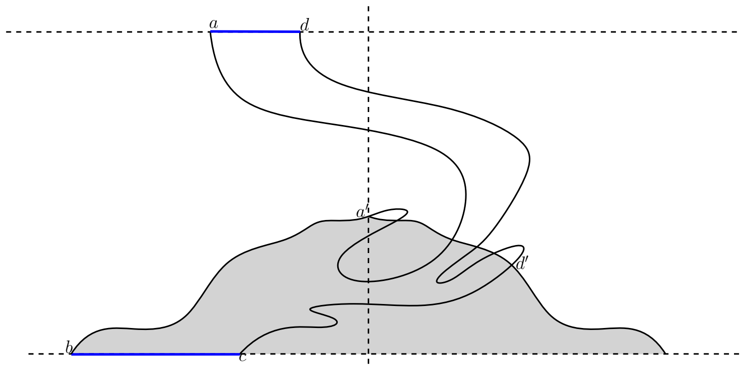

Figure 5: The black paths are even paths with value 2 and the blue paths are odd paths with value 1. The symmetric domain Ω is shaded grey.

Finally, let D' be the domain bounded by $[a'b]$, $[bc]$, $[cd']$ where d' the first intersection of $[cd]$ with $\sigma([a'b])$, and $\sigma([a'b])$.

Using the inclusion of events and the FKG inequality applied to $|h|$ (note that the event $\mathcal{H}_{h\geq 1}(D') = \mathcal{H}_{|h|\geq 1}(D')$ is increasing in terms of $|h|$), we find that, like in (3.7),

$$\begin{aligned}\phi_D^\kappa[\mathcal{H}_{h\geq 1}(D,a,b,c,d)] &\geq \phi_D^\kappa[\mathcal{H}_{h\geq 1}(D',a',b,c,d')] \\ &\geq \phi_D^\kappa[\mathcal{H}_{h\geq 1}(D',a',b,c,d')|h_{|\partial D'\setminus\partial D}=0] \\ &= \phi_{D'}^{\kappa'}[\mathcal{H}_{h\geq 1}(D',a',b,c,d')],\end{aligned}$$

where κ' is the boundary condition equal to 2 on $[a'b]\cup[cd']$, and 0 on $(bc)\cup(d'a')$. Now, following a reasoning similar to (3.10) and then (3.11), we find that

$$\phi_{D'}^{\kappa'}[\mathcal{H}_{h\geq 1}(D',a',b,c,d')] \geq \phi_\Omega^\xi[\mathcal{H}^*_{h=2}(\Omega,a',b,c,\sigma(b))] \geq \tfrac{1}{2},$$

where ξ is the boundary condition equal to 2 on $[a'b]\cup[c\sigma(b)]$ and 0 on $(bc)\cup(\sigma(b)a')$.

3.3 Proof of Proposition 3.2

Again, we prove the first two cases of the proposition, as the third is analogous to the second.

First case Let Λ_n^{mix} be the domain enclosed between the two even $\times$-paths in $\{n,n+1\}\times\mathbb{Z}$ and $\{-n-1,-n\}\times\mathbb{Z}$ connecting the vertices $(\pm n,\pm n)$ and the two odd $\times$-paths obtained as the translations by $(0,1)$ of the reflections with respect to $y=x$ of the two previous paths. We denote the corners $(\pm n,\pm n)$ of this box by r,s,u,v, starting from the top-left corner and going counterclockwise.

Let U_{top} be the domain enclosed by the (odd) $\times$-path $[da]$ in the quad D and the (odd) $\times$-path from a to d in $\mathbb{Z}\times\{n,n+1\}$. We define U_{right} as the translation by $(0,1)$ of the reflection of U_{top} with respect to the line $y=x$. Similarly, we define U_{bottom} and U_{left} in a straightforward fashion.

We now introduce

$$\Omega = \Lambda_n^{\mathrm{mix}} \uplus U_{\mathrm{top}} \uplus U_{\mathrm{left}} \uplus U_{\mathrm{bottom}} \uplus U_{\mathrm{right}},$$

where $\uplus$ denotes the disjoint gluing of the graphs along the top and bottom parts of Λ_n^{mix}. Note that Ω is planar but may not be embeddable in an isometric fashion in $\mathbb{R}^2$, and that it contains

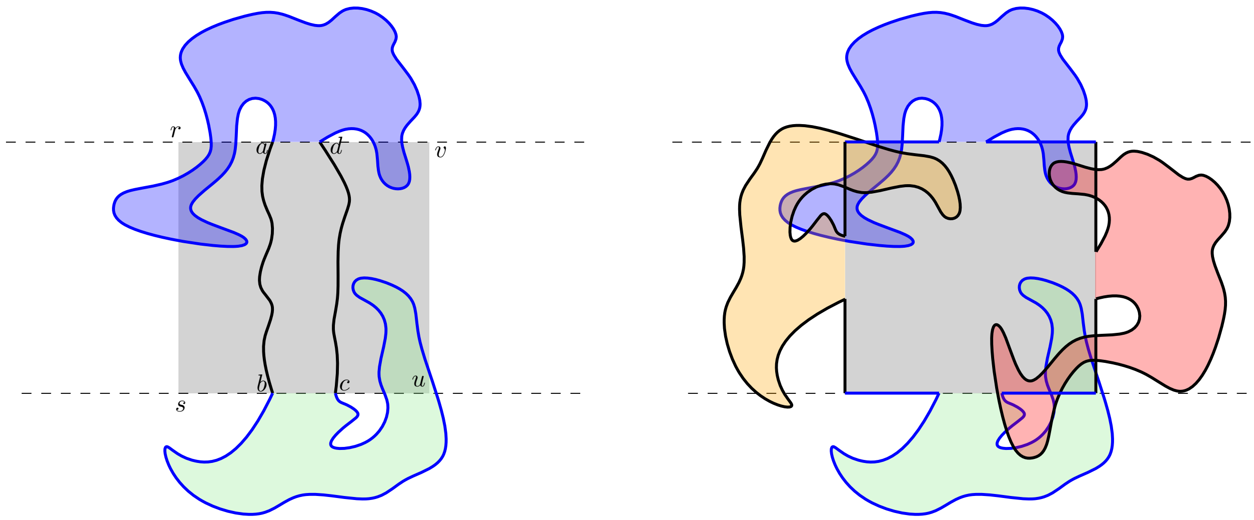

Figure 6: Blue paths are odd paths with $h = 1$ and black paths are even paths with $h = 2$. Left: The domain D with the square Λ_n in shaded grey. Right: The symmetric domain Ω.

a natural copy of the graph D. The graph Ω also satisfies some symmetry for the reflection with respect to the line $y = x$ shifted by $(0, 1)$. Let ξ be the boundary condition on Ω equal to 2 on even vertices of $\partial\Omega$ (here $\partial\Omega$ denotes the set of vertices with at most three neighbours in Ω) and 1 on odd ones.

Using the same reasoning as in (3.7), we find that

$$\phi_D^\kappa[\mathcal{C}_{h\geq 2}^\times(D, a, b, c, d)] \geq \phi_\Omega^\xi[\mathcal{C}_{h\geq 2}^\times(\Omega, r, s, u, v)]. \tag{3.12}$$

It remains to observe that by the first item of Lemma 2.4,

$$\begin{aligned}\phi_\Omega^\xi[\mathcal{C}_{h\geq 2}^\times(\Omega, r, s, u, v)] &= 1 - \phi_\Omega^\xi[\mathcal{C}_{h\leq 1}(\Omega, v, r, s, u)] \\ &\geq 1 - \phi_\Omega^\xi[\mathcal{C}_{h\leq 1}^\times(\Omega, v, r, s, u)]. \end{aligned} \tag{3.13}$$

By symmetry, this implies that the probability of the former is larger than $\frac{1}{2}$. This concludes the proof.

Remark 3.3. In Section 3.2, we used symmetries that are mapping the even vertices to even vertices, and symmetric even-domains with boundary condition that are made of 0s and 2s. In this section, we used symmetries that are mapping even vertices to odd vertices, and symmetric mix-quads with boundary condition that are made of 1s and 2s.

Remark 3.4. Unlike the proofs in Section 3.2, we are not allowed to 'push in' boundary condition larger than or equal to 1 on top and bottom, because $\times$-crossings of $h \geq 1$ cannot be transformed into increasing events in $|h|$. We therefore need to symmetrize the domain D by pushing boundary condition 2 away only.

Second case Let ℓ' be the vertical line passing by $c - (\frac{1}{2}, 0)$. We start by defining an equivalent of the symmetric domain introduced in the second case of Proposition 3.1. Assume that $[ab]$ and

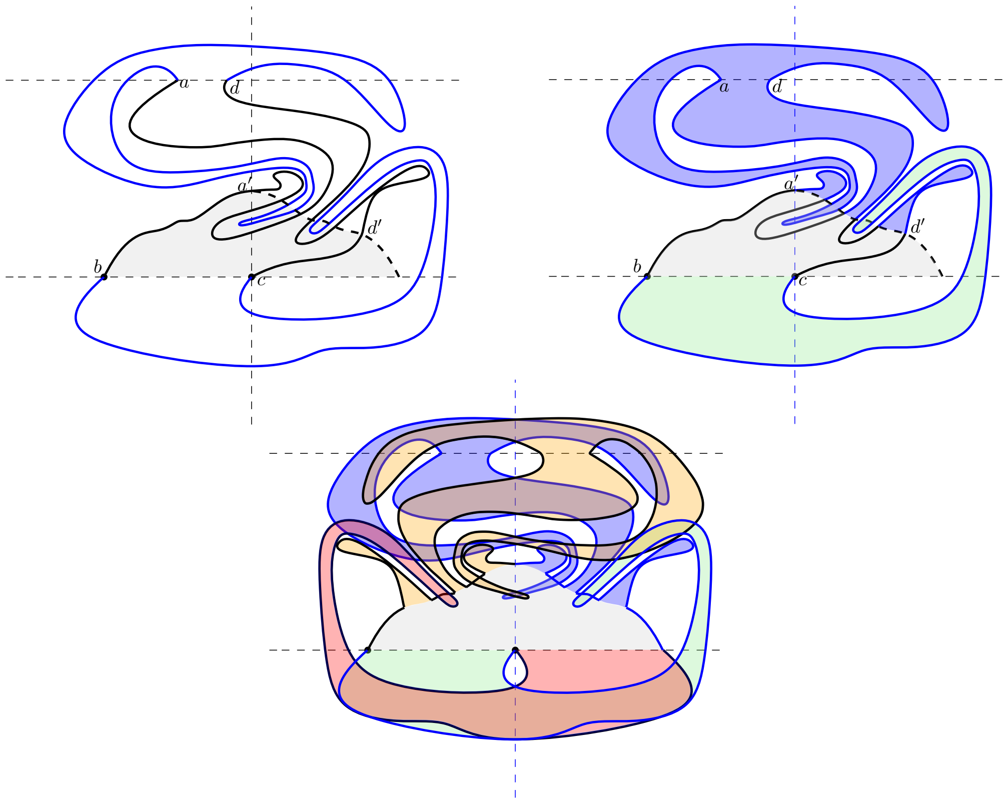

Figure 7: Blue curves denote odd $\times$ paths with boundary condition 1 while the black curves are even $\times$ paths boundary condition 2. Top left: The symmetric domain S shaded in grey. Top right: The portion $U_{\text{top-right}}$ is shaded blue and $U_{\text{bottom-left}}$ is shaded green. Bottom: The final symmetric domain Ω. $U_{\text{top-left}}$ is shaded orange and $U_{\text{bottom-right}}$ is shaded red.

$[cd]$ do not intersect otherwise there is nothing to prove. Denote the reflection with respect to the vertical line ℓ' passing by $c - (\frac{1}{2}, 0)$ by σ' (σ' maps the even lattice to the odd lattice). Let a' be the vertex just before the first intersection of $[ab]$ (when going from bottom to top and when seen as a continuous path) with ℓ' and let S be the quad enclosed by $[a'b]$, $\sigma'([a'b])$, the odd $\times$-path of $\mathbb{Z} \times \{-n-1, -n\}$ between the right neighbour of b and the left neighbour of c, and the even $\times$-path between c and $\sigma'(b)$. Finally, let d' be the last vertex before $[cd]$ exits S for the first time.

Let $U_{\text{top-right}}$ be the union of odd domains whose boundaries are defined as follows. Let s be the odd path from d' to a' using the odd vertices neighbouring from the exterior the even vertices of $[dd']$, then the odd path (da), then the odd vertices neighbouring from the exterior the vertices of $[aa']$. Let t be the odd $\times$-path going along $\sigma([a'b])$ from d' to the neighbour of a' and observe that $t \cap D$ is divided into segments. Now, let $[u_1 v_1], \ldots, [u_i v_i]$ be the segments of t that can be reached from a' (or equivalently d') while staying in $D \cap S$. For $1 \leq j \leq i$, consider the domain U_j enclosed by $[u_j v_j]$ and the part of s going from u_j to v_j. We then define $U_{\text{top-right}}$ to be the union of the U_j for $1 \leq j \leq i$ and $U_{\text{top-left}} = \sigma'(U_{\text{top-right}})$. Similarly, we define $U_{\text{bottom-left}}$ and $U_{\text{bottom-right}} = \sigma'(U_{\text{bottom-left}})$ in a straightforward fashion (in this case the definition is even simpler: there is only one domain since (bc) does not cross the odd $\times$-path of $\mathbb{Z} \times \{-n-1, -n\}$

between the right neighbour of b and the left neighbour of c).

Introduce

$$\Omega = S \biguplus U_{\text{top}-\text{left}} \biguplus U_{\text{bottom}-\text{left}} \biguplus U_{\text{bottom}-\text{right}} \biguplus U_{\text{top}-\text{right}}$$

(the gluing of the different pieces is made along the segments $[u_j v_j]$ defined above for the top pieces, and the odd $\times$-path of $\mathbb{Z} \times \{-n-1, -n\}$ between the right neighbour of b and the left neighbour of c for the bottom parts). Let ξ be the boundary condition on Ω equal to 2 on even vertices of the boundary and 1 on odd ones.

To complete the proof, we must compare the crossing probabilities in D' and Ω by pushing away the boundary condition $h = 2$, and then applying a symmetry argument. The proof follows the same procedure as the previous case, and we therefore omit the details for the sake of brevity.

4 Proofs of the theorems

4.1 Two useful crossing probabilities

Proposition 4.1. *For every $\rho > 0$, there exists $c = c(\rho) > 0$ such that for every n and every even domain D containing $\Lambda_{\rho n,n}$,*

$$\phi_D^0[\mathcal{H}_{h\geq 0}(\Lambda_{\rho n,n})] \geq c. \tag{4.1}$$

Remark 4.2. It is worth mentioning that we do not require ∂D to be far away from $\Lambda_{\rho n,n}$. In fact, the boundary of ∂D may partially coincide with the boundary of $\Lambda_{\rho n,n}$ without raising any issues.

Proof. The first item of Lemma 2.4 implies that

$$p := \phi_D^0[\mathcal{H}_{h\geq 0}(\Lambda_{\rho n,n})] = 1 - \phi_D^0[\mathcal{V}^{\times}_{h<0}(\Lambda_{\rho n,n})].$$

As in (3.3), the square-root trick implies that there exist i, j, α such that

$$\phi_D^0[\mathcal{E}] \geq 1 - p^{1/C},$$

where $\mathcal{E} = \mathcal{E}_{ij}^{\alpha}$ is defined as $\mathcal{E}_{ijk\ell}^{\alpha\beta\gamma}$, but with $h < 0$ instead of $|h| \geq 2$, and with only I_i, J_j and α involved (we trust that the reader will easily figure out the precise definition). For the rest of the proof, call a $\times$-path γ *achieving* if it guarantees the occurrence of $\mathcal{E}$. For a subset I of I_i, let $\mathcal{E}(I)$ be the event that there exists an achieving $\times$-path starting from I.

We now claim that I_i can be splitted in eight intervals $I^1, \dots, I^8$ ordered from left to right and intersecting at their extremities (they may have different sizes) such that $\phi_D^0[\mathcal{E}(I^k)] \geq 1 - p^{1/(8C)}$ for each k. Indeed, first split the interval $I_i = [ab]$ in two by choosing the left-most x such that the probability that $\phi_D^0[\mathcal{E}([ax])] \geq \phi_D^0[\mathcal{E}((xb])]$. Then, the square-root trick implies that $\phi_D^0[\mathcal{E}([ax])]$ and $\phi_D^0[\mathcal{E}([xb])]$ are larger than $1 - p^{1/(2C)}$. One can iterate this reasoning with each one of the intervals twice to get the claim.

If $\widetilde{\mathcal{E}}(I)$ denotes the image of the event $\mathcal{E}(I)$ after flipping all the signs, the flip symmetry and the union bound give that

$$\phi_D^0[\mathcal{E}(I^1) \cap \widetilde{\mathcal{E}}(I^2) \cap \mathcal{E}(I^3) \cap \widetilde{\mathcal{E}}(I^4) \cap \mathcal{E}(I^5) \cap \widetilde{\mathcal{E}}(I^6) \cap \mathcal{E}(I^7)] \geq 1 - 7p^{1/(8C)}.$$

Nevertheless, on this event, none of the intervals I^2, I^4 or I^6 are bridged by a $\times$-path of $h < 0$. Yet, proceeding[4] as in the proof of Theorem 1.4 starting after (3.4), we can show that the probability that one of these intervals is bridged by $|h+1| > 0$ is larger than $\frac{1}{96}(1 - p^{1/(8C)})^4$ [5]. Since the boundary condition on D is 0, FKG for h implies that the probability that this is achieved by a $h > 0$ $\times$-path is larger than by a $h < -2$ path, so that $\phi^0_D[\mathcal{F}] \le 1 - \frac{1}{192}(1-(1-p)^{1/(8C)})^4$ where $\mathcal{F}$ is the event in the display above. Together with the last displayed equation, this provides a lower bound on p, which is what we were looking for. □

The second estimate we wish to obtain is the following result. When n is even, consider the approximation $R^g_{n,m}$ of $[-n,n] \times [0,m]$ obtained by taking what is inside

- the even $\times$-path going from $(n,0)$ to $(-n,0)$ following $\{n, n+1\} \times \mathbb{Z}$, then $\mathbb{Z} \times \{m, m+1\}$ and finally $\{-n-1, n\} \times \mathbb{Z}$,
- if g is even (resp. odd), the even (resp. odd) $\times$-path from $(n,0)$ to $(-n,0)$ in $\mathbb{Z} \times \{-1, 0\}$.

A similar domain can easily be defined for n odd by replacing n above by $2\lfloor n/2 \rfloor$. Also, let $0/g$ be the boundary condition equal to g on the bottom of $R^g_{n,m}$, 0 on the left, right and top boundary, except at the bottom-left and bottom-right corners where the boundary condition is interpolating between 0 and g in the shortest way. Since the superscript will always be obvious from context (for instance because it is the only one compatible with the boundary conditions), we will write $R_{n,m}$ instead of $R^g_{n,m}$.

Proposition 4.3. *For any $g \in \mathbb{N}$, $H > \delta > 0$, there exits $c = c(H, \delta, g) > 0$ such that for all $n \ge 1$,*

$$\phi^{0/g}_{R_{n,Hn}}[\mathcal{H}^\times_{h=0}(R_{n,\delta n})] \ge \phi^{0/g}_{R_{n,Hn}}[\mathcal{H}_{h \le 0}(R_{n,\delta n})] \ge c.$$

Proof. The first inequality clearly follows by inclusion. For the second, the result for general g follows from the result for $g = 1$ by an easy induction. Indeed, assume that we already proved the existence of $c(H, \delta, 1) > 0$. The FKG for h implies the existence of an horizontal crossing of $h \le g-1$ within $R_{n,\delta n}$ with positive probability $c(H, \delta, 1) > 0$. Condition on the bottom-most crossing of $h \le g-1$. We wish to find a crossing of $h \le g-2$ above this crossing. Noting that the event "there exists a crossing of $R_{n,2\delta n} \setminus R_{n,\delta n}$ of $h \le g-2$" is increasing in terms of $|h-g+1|$, we can use FKG for $|h-g+1|$ to push the boundary conditions to get the translate of the $0/(g-1)$ boundary condition on the translate by $(0, \delta n)$ of $R_{n,(H-\delta)n}$, so that the conditional probability of finding a crossing of $h \le g-2$ inside $R_{n,2\delta n} \setminus R_{n,\delta n}$ (and therefore inside $R_{n,2\delta n}$) is bounded from below by $c(H-\delta, \delta, 1)$. Iterating, we see that we can find a crossing of $h \le 0$ inside $R_{n,g\delta n}$ with probability at least $c(H, g\delta, g) := \prod_k c(H-k\delta, \delta, 1) > 0$.

We now focus on proving the existence of $c(H, \delta, 1) > 0$. Assume without loss of generality that Hn is divisible by 4. Let S_n be the infinite strip bounded by the even vertices of $\mathbb{Z} \times \{Hn+1, Hn+2\}$ and the odd vertices of $\mathbb{Z} \times \{0, -1\}$ and let $0/1$ be the boundary condition equal to 0 on the top and 1 on the bottom. The existence of the measure $\phi^{0/1}_{S_n}$ is a straightforward exercise as the domain

[4] To see this more easily, use the flip symmetry and the shift by 1 to convert events $h < 0$ and boundary condition 0 to $h \ge 2$ and boundary condition 1. Then, the only changes are that the $\pm$ -symmetry used before (3.1) should be replaced by the comparison between boundary conditions (to say that a $h \ge 2$ $\times$-crossing is more likely than a $h \le -2$), and that all boundary conditions induced by conditions on $|h|$ are $|h|$-adapted since B_{pos} involves only positive values, so that we can use FKG for $|h|$. Note that the hard part of the proof of Theorem 1.4, which consists in the two propositions, remains unchanged.

[5] Let us mention that this argument does not use Proposition 3.2.

is essentially one dimensional and the homomorphism model enjoys a version of the finite energy property. Let $R'_n := [-n,n]\times[\frac{1}{4}Hn,\frac{3}{4}Hn]$.

We claim that there exists a constant $c = c(H) > 0$ such that for all $n \geq 1$,

$$\phi^{0/1}_{S_n}[\mathcal{H}^{\times}_{h=0}(R_{n,3Hn/4})] \geq \phi^{0/1}_{S_n}[\mathcal{H}^{\times}_{h\leq 0}(R'_n)] \geq c. \tag{4.2}$$

Indeed, the first inequality follows from the inclusion of events (induced by boundary conditions). For the second, assume that it does not hold with $c = 1/2$. Then, the first item of Lemma 2.4 and the symmetry of the measure imply that

$$\phi^{0/1}_{S_n}[\mathcal{V}^{\times}_{h\leq 0}(R'_n)] \geq \phi^{0/1}_{S_n}[\mathcal{V}_{h\leq 0}(R'_n)] = \phi^{0/1}_{S_n}[\mathcal{V}_{h\geq 1}(R'_n)] = 1-\phi^{0/1}_{S_n}[\mathcal{H}^{\times}_{h\leq 0}(R'_n)] \geq \tfrac{1}{2}.$$

The proof of Theorem 1.4 applies *mutatis mutandis* in this context[6] and we obtain (4.2) with a constant $c = c(\rho) > 0$.

The FKG inequality for $|h|$ enables us to bring boundary conditions in to find that

$$\phi^{0/1}_{R_{n,Hn}}[\mathcal{H}^{\times}_{h=0}(R_{n,3Hn/4})] \geq \phi^{0/1}_{S_n}[\mathcal{H}^{\times}_{h=0}(R_{n,3Hn/4})] \geq c. \tag{4.3}$$

Now, condition on the top-most horizontal $\times$-crossing of $h = 0$ in $R_{n,3Hn/4}$. Applying spatial Markov property and FKG of $|h|$, the probability of seeing a $\times$-crossing of $h = 0$ in $R_{n,(3/4)^2Hn}$ is larger than $c(\frac{3}{4}H)$. We iterate this step to find that for all $n \geq 1$,

$$\phi^{0/1}_{R_{n,Hn}}[\mathcal{H}^{\times}_{h=0}(R_{n,\delta n/2})] \geq \prod_{k\leq \log_{4/3}(2H/\delta)} c((\tfrac{3}{4})^k H) > 0.$$

To conclude, it remains to create a $h \leq 0$ crossing in $R_{n,\delta n}$. Explore from the bottom to find the lowest such $\times$-crossing of $h = 0$ and call the domain above it D. Conditioned on this lowest crossing, the boundary condition on D is 0 everywhere. We therefore can apply Proposition 4.1 to show that the probability of a crossing of $h \leq 0$ in $R_{n,\delta n}\setminus R_{n,\delta n/2}$ is bounded from below by $c' > 0$, thus proving that

$$\phi^{0/1}_{R_{n,Hn}}[\mathcal{H}_{h\leq 0}(R_{n,\delta n})] \geq c'\phi^{0/1}_{R_{n,Hn}}[\mathcal{H}^{\times}_{h=0}(R_{n,\delta n/2})].$$

Combined with the previous displayed equation, this concludes the proof. □

4.2 The renormalization proposition

As described in the introduction, we wish to follow the renormalisation argument from [11] to complete the argument. Unfortunately, a new difficulty appears in our setting: one could imagine that the existence of a long $\times$-crossing of $h \geq 2$ inside a box forces the height function to be much larger than 2 everywhere inside. In practice, it manifests in the fact that to apply Proposition 4.3, we need a bound on the boundary values which is not *a priori* clear.

[6]To see this more easily, use the symmetry with respect to 1 to convert events $h \leq 0$ and 0/1 boundary condition to $h \geq 2$ and 2/1 boundary condition. Then, the only changes are that the $\pm$ -symmetry used before (3.1) should be replaced by the comparison between boundary conditions (to say that a $h \geq 2$ $\times$-crossing is more likely than a $h \leq -2$), that (4.3) is no longer useful since we have invariance under translations, and that all boundary conditions induced by conditions on $|h|$ are $|h|$-adapted since 2/1 involves only positive values, so that we can use FKG for $|h|$. Note that the hard part of the proof of Theorem 1.4, which consists in the two propositions, remains unchanged.

To deal with this issue, we distinguish two cases. If the probability of a crossing of $h \geq 2$ is similar to the one of a $\times$-crossing of $2 \leq h \leq g$ for some g, which is the expected behaviour, we can apply the original argument of [11] with appropriate modifications. If not, then the cost of a $\times$-crossing of $h \geq 2$ is similar to the cost of a crossing of $h \geq g$. In this case we obtain $(g-2)/2$ crossings "for free", an event whose probability can be easily bounded.

Define $\mathcal{A}_n$ to be the event that there exists a $\times$-loop of $h \geq 2$ in the annulus $A_n := \Lambda_{2n} \setminus \Lambda_n$ and let $a_n := \phi^0_{\Lambda_{5n}}[\mathcal{A}_n]$. We also let $\mathcal{A}_n(x)$ denote the event $\mathcal{A}_n$ shifted by $(x,0)$. Finally, we introduce $\Lambda_n(x)$ and $A_n(x)$ for the box Λ_n and the annulus A_n shifted by $(x,0)$.

Remark 4.4. By Proposition 4.1, we have that $a_n \leq 1-c$ for some constant $c > 0$ independent of n.

Proposition 4.5. *There exists a constant $C > 0$ such that for all $n \geq 1$,*

$$a_{10n} \leq C a_n^2. \tag{4.4}$$

We start with a lemma. Let $\mathcal{E}^{\times}_{h\geq k}(n)$ be the event that there exists a $\times$-cluster of $h \geq k$ of diameter at least n.

Lemma 4.6. *There exists $\rho > 0$ such that the following holds. For any $r > 10$, there exists $C = C(r) > 0$ such that for any k and n,*

$$\phi^0_{\Lambda_{rn}}[\mathcal{E}^{\times}_{h\geq k}(n)] \leq (Ca_n)^{k/\rho}. \tag{4.5}$$

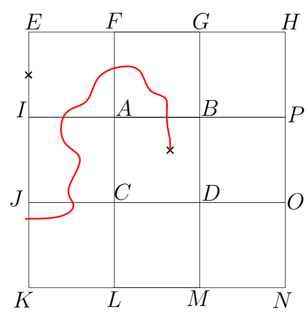

Figure 8: The red curve has diameter at least n and hence has to exit the square EKNH. Without loss of generality let the curve exit the square ABCD for the last time through AB before it exits EKNH. After this assume it exits the square AFGB through AF or GB, since otherwise we are done and without loss of generality assume it is AF. Then we can assume it exits IBGE through IA since otherwise we are also done. After this, the curve must necessarily include an easy crossing of either IJDB or EJCF before exiting EKNH.

Proof. In any configuration of $\mathcal{E}^{\times}_{h\geq k}(n)$, there must exist $k/2$ nested $\times$-loops with increasing heights $2i \leq k$. Let D_{2i} be the interior of the outer-most $\times$-loops of $h = 2i$ in Λ_{rn} and let κ_i denote the boundary condition equal to $2i$ on D_{2i}. On the event described above, the domains D_{2i} exist for every $2i \leq k$ and if the diameter of the largest connected component of these domains is denoted by d_{2i}, we find that

$$\phi^0_{\Lambda_{rn}}[\mathcal{E}^{\times}_{h\geq k}(n)] \leq \phi^0_{\Lambda_{rn}}\big[\mathbb{1}_{d_2\geq n}\phi^{\kappa_2}_{D_2}[\mathbb{1}_{d_4\geq n}\phi^{\kappa_4}_{D_4}[\cdots]]\big].$$

Using the symmetry in D_{2i} with respect to $2i$ and the FKG for $|h-2i|$, we may bound each expectation by $2\phi_{\Lambda_{rn}}[d_2 \geq n]$ (the factor 2 arises from switching between $|h|$ and h), giving overall

$$\phi^0_{\Lambda_{rn}}[\mathcal{E}^{\times}_{h\geq k}(n)] \leq (2\phi^0_{\Lambda_{rn}}[d_2 \geq n])^{k/2}. \tag{4.6}$$

Consider the set T of translates of rectangles included in Λ_{rn} of sizes $n \times n/2$ and $n/2 \times n$ by vertices in $\frac{n}{2}\mathbb{Z}^2$. A topological argument (see Figure 8) easily implies that if $d_2 \geq n$, there exists a rectangle in T that is crossed in the 'easy' direction, meaning vertically if it has size $n \times n/2$ or horizontally if it has size $n/2 \times n$. For this reason, in order to bound $\phi^0_{\Lambda_{rn}}[d_2 \geq n]$, it suffices to consider a rectangle R in T, which we assume without loss of generality has size $n \times n/2$, and to prove that there exists $C_0 > 0$ such that

$$\phi^0_{\Lambda_{rn}}[\mathcal{V}^{\times}_{h=2}(R)] \leq C_0 a_n^{1/C_0}. \tag{4.7}$$

Let $\mathcal{A}$ be the event that there exists a $\times$-circuit of $h \leq 0$ surrounding R in the n neighbourhood of R (if R intersects the boundary, the $\times$-circuit can use the boundary of Λ_{rn}, which has value 0). We have that

$$\phi^0_{\Lambda_{rn}}[\mathcal{V}^{\times}_{h=2}(R)] \leq \frac{\phi^0_{\Lambda_{rn}}[\mathcal{V}^{\times}_{h\geq 2}(R)|\mathcal{A}]}{\phi^0_{\Lambda_{rn}}[\mathcal{A}|\mathcal{V}^{\times}_{h=2}(R)]} \leq \frac{\phi^0_{\Lambda_{2n}}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{n,n/2})]}{\phi^0_{\Lambda_{rn}}[\mathcal{A}|\mathcal{V}^{\times}_{h=2}(R)]} \leq C_1\phi^0_{\Lambda_{2n}}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{n,n/2})]. \tag{4.8}$$

Indeed, the first inequality follows from inclusion. The second holds because the $\times$-loop of $h \leq 0$ induced by $\mathcal{A}$ can be replaced by a $\times$-loop of $h = 0$ by FKG for h and then pushed away using FKG for $|h|$ (we already presented several arguments like that and omit the details). In the third inequality, we bounded the probability of the denominator as follows. Condition on $|h-2|$ in the even vertices R^{even} in R. Any realization of this conditioning is a measure of the form $\phi^{\kappa}_{\Lambda_{rn}\setminus R^{\text{even}}}$ with κ being $|h|$-adapted (the intervals are containing one value on $\partial\Lambda_{rn}$ and two symmetric values in R^{even}). Since the single-valued vertices all receive value $h = 0$, we may use the comparison between boundary conditions with $h = 2$ on ∂R^{even} to bound the conditional probability from below by the $\phi^0_{\Lambda_{rn}\setminus R^{\text{even}}}$-probability in $\Lambda_{rn} \setminus R^{\text{even}}$ with boundary condition equal to 2 on ∂R^{even} and 0 in $\partial\Lambda_{rn}$. Using Proposition 4.3, we have a positive probability that $\mathcal{A}$ occurs, hence the third inequality.

Now, Theorem 1.4 implies that

$$\phi^0_{\Lambda_{5n,2n}}[\mathcal{H}^{\times}_{h\geq 2}(\Lambda_{2n,n/2})] \geq c_0\phi^0_{\Lambda_{2n}}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{n,n/2})]^{\rho}. \tag{4.9}$$

Finally, FKG for h implies that

$$a_n \geq \phi^0_{\Lambda_{5n,2n}}[\mathcal{H}^{\times}_{h\geq 2}(\Lambda_{2n,n/2})]^4. \tag{4.10}$$

Together with (4.8) and (4.9), (4.10) implies (4.10) and therefore of the claim. □

Proof of Proposition 4.5. We start by claiming that there exists $c_1 > 0$ such that for all $n \geq 1$,

$$a_{10n} \leq c_1\phi^0_{\Lambda_{50n}}[\mathcal{A}_n(-7n) \cap \mathcal{A}_n(7n)] \tag{4.11}$$

since $\mathcal{A}_{10n}$ implies that there exists a $\times$-loop of $h = 2$ in $\Lambda_{50n} \setminus \Lambda_{10n}$ and hence we can apply Proposition 4.1.

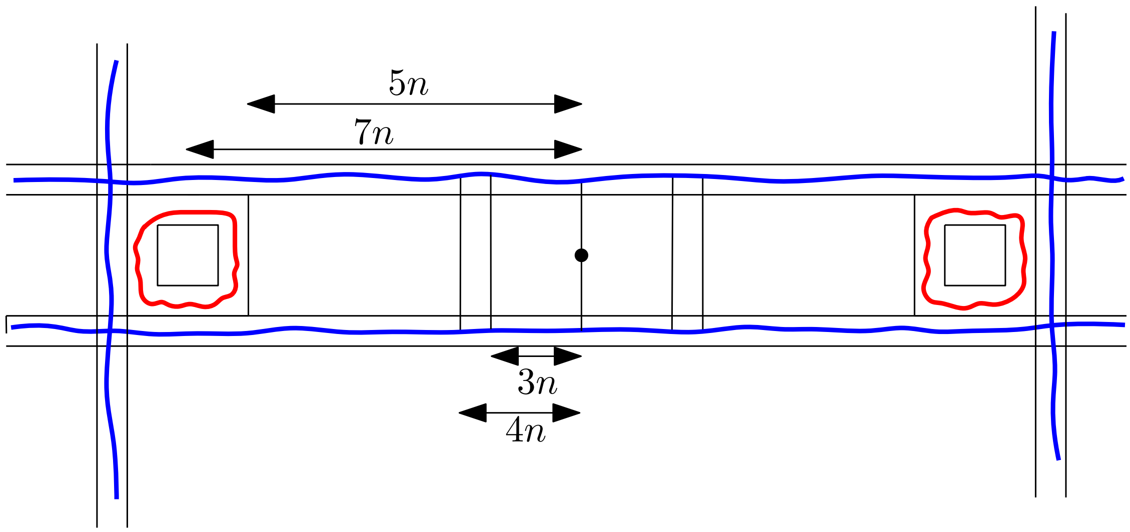

Figure 9: The red paths are even $\times$-loops of $h \geq 2$ coming from the event $\mathcal{A}_n(-7n) \cap \mathcal{A}_n(7n)$. The blue paths are $\times$-paths of $h \leq 0$ coming from the events (4.13)–(4.16) (i.e. the event $\mathcal{C}$).

Define events $\mathcal{B}_n(x)$ similarly to $\mathcal{A}_n(x)$, where we add the restriction that the $\times$-loops must satisfy $2 \leq h \leq k_0 := 2^{\ell_0}$, where $k_0 > 2\rho$ with ρ provided by Lemma 4.6. With this choice of k_0, Lemma 4.6 gives that

$$\phi^0_{\Lambda_{50n}}[\mathcal{A}_n(\pm 7n) \setminus \mathcal{B}_n(\pm 7n)] \leq \phi^0_{\Lambda_{50n}}[\mathcal{E}^{\times}_{h \geq k_0}(n)] \leq C_1 a_n^2.$$

In order to conclude the proof, we now need to prove that

$$\phi^0_{\Lambda_{50n}}[\mathcal{B}_n(-7n) \cap \mathcal{B}_n(7n)] \leq C_2 a_n^2. \tag{4.12}$$

Suppose we are on the event $\mathcal{B}_n(-7n) \cap \mathcal{B}_n(7n)$ and let $\ell_\pm$ be the two innermost $\times$-loops with values in $2 \leq h \leq k_0$ in $\Lambda_n(\pm 7n)$.

Consider the event $\mathcal{C}$ (see Figure 9) which is the intersection of the following four events

$$\mathcal{H}^{\times}_{h \leq 0}([-50n, 50n] \times [2n, 3n]), \tag{4.13}$$
$$\mathcal{H}^{\times}_{h \leq 0}([-50n, 50n] \times [-3n, -2n]), \tag{4.14}$$
$$\mathcal{V}^{\times}_{h \leq 0}([-10n, -9n] \times [-50n, 50n]), \tag{4.15}$$
$$\mathcal{V}^{\times}_{h \leq 0}([9n, 10n] \times [-50n, 50n]). \tag{4.16}$$

Conditionally on ℓ_+ and ℓ_- (which involves information on h inside the two loops only), we claim that $\phi^0_{\Lambda_{50n}}[\mathcal{C}] \geq c_3$. Let us provide a lower bound on the conditional probability of the first event, since the bound for the others is similar and that one can use FKG to deduce a bound on $\phi^0_{\Lambda_{50n}}[\mathcal{C}]$. Since the values of h on the loops ℓ_+ and ℓ_- are between 2 and k_0, the FKG inequality for $|h - k_0|$ enables us to bound the probability of the first event in $\mathcal{C}$ from below if we assign (the translate of) boundary condition $0/k_0$ on the rectangle $[-50n, 50n] \times [2n, 50n]$ as in Proposition 4.3. In other words, it is enough to prove the lower bound for the same event but in the domain $[-50n, 50n] \times [2n, 50n]$ with $0/k_0$ boundary condition, which is exactly what is given by Proposition 4.3.

Remark 4.7. Note that this step crucially relies on the fact that the values on ℓ_+ and ℓ_- are bounded between 2 and k_0 since applying FKG for $|h-k_0|$ requires all the boundary values to have the same sign in Proposition 2.2.

Overall, the argument in the previous paragraph gives us the existence of $c_4 > 0$ such that for all n,

$$\phi^0_{\Lambda_{50n}}[\mathcal{C}|\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)] \geq c_4. \tag{4.17}$$

On $\mathcal{C}\cap\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)$, let Ω be the connected component of the origin inside the outermost realisations of the crossings in (4.13), (4.14), (4.15), (4.16) minus the loops ℓ_+ and ℓ_- (see Figure 9). As in [11], we want to separate ℓ_+ and ℓ_- with an $h \leq 0$ $\times$-path. However, since the values on ℓ_- and ℓ_+ can be as high as k_0, we need several steps to find this path. We do so iteratively, each time dividing the value of the separating path by a factor of 2. We now provide the details. Let R_-, R_0 and R_+ be the subsets of Ω made of vertices with first coordinates in $[-4n,-3n]$, $[-3n,3n]$, and $[3n,4n]$ respectively. We write $\mathcal{V}^\times_{h\leq 2^k}(R_\#)$ for the existence of a vertical $\times$-crossing of this quad between the bottom and top boundaries of $\partial\Omega$. Let

$$\mathcal{D}(k) := \mathcal{C}\cap\mathcal{V}^\times_{h\leq 2^k}(R_-)\cap\mathcal{V}^\times_{h\leq 2^k}(R_+),$$

and on this event, set Ω_k to be the part of Ω between the left-most vertical $\times$-crossing of $h\leq 2^k$ of R_- and the right-most vertical $\times$-crossing of R_+. We also use the conventions $\mathcal{D}(\ell_0) = \mathcal{C}$ and $\Omega_{\ell_0} := \Omega$.

We wish to show iteratively that there exist $c_{\ell_0},\dots,c_1 > 0$ such that for every $1\leq k<\ell_0$,

$$\phi^0_{\Lambda_{50n}}[\mathcal{D}(k)|\mathcal{C}\cap\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)] \geq c_k. \tag{4.18}$$

Fix $k\geq 1$ and assume that the previous result was obtained for every $k' > k$. The boundary condition induced on Ω_k is such that

$$\mathcal{V}^\times_{h\leq 2^k}(R_0) = \mathcal{V}^\times_{|h-2^{k+1}|\geq 2^k}(R_0)$$

since the sign of the path must be the same as the boundary by the third item of Lemma 2.4. Therefore, we can put boundary conditions 2^{k+1} on the left and right sides of R_0 using FKG for $|h|$. Rewriting this event as $|h|\leq 2^k$ using Lemma 2.4 again, we can push out the zeros on $\partial\Omega$ to the top and bottom boundaries of R_0 [7]. By duality (like in the argument for the first case of Proposition 3.1), we deduce that

$$\phi^0_{\Lambda_{50n}}[\mathcal{V}^\times_{h\leq 2^k}(R_0)|\mathcal{D}(k+1)\cap\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)] \geq \tfrac{1}{2},$$

which together with the induction hypothesis gives

$$\phi^0_{\Lambda_{50n}}[\mathcal{V}^\times_{h\leq 2^k}(R_0)|\mathcal{C}\cap\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)] \geq \tfrac{1}{2}c(k+1).$$

On this event, let Ω_+ be the subregion of Ω on the right of the left-most vertical $\times$-crossings of R_0 of $h\leq 2^k$. Conditionally on Ω_+, we can make the event $\mathcal{V}^\times_{h\leq 2^k}(R_+)$ less probable by putting boundary conditions 2^k to the left, top and bottom sides of $[-3n,4n]\times[-3n,3n]$, and k_0 to the right (which is again made compatible near the corners and the parity chosen appropriately) by using

[7] It is easy to check that if $n\geq 2^{k+1}$, it is possible to design boundary conditions that interpolate between 2^{k+1} and 0 in a symmetric fashion in the corners but we voluntarily suppress this issue for the sake of clarity).

- FKG for $|h-k_0|$ ($\mathcal{V}^\times_{h\le 2^k}(R_+)=\mathcal{V}^\times_{|h-k_0|\ge k_0-2^k}(R_+)$ is increasing in $|h-k_0|$ for the boundary conditions on Ω_+) to put $h=k_0$ boundary conditions on the right of $\Omega'_+:=\Omega_+\cap(R_0\cup R_+)$,
- Comparison between boundary conditions for h ($\mathcal{V}^\times_{h\le 2^k}(R_+)$ is decreasing in h) to put $h=2^k$ on the rest of $\partial\Omega'_+$,
- FKG for $|h-2^k|$ ($\mathcal{V}^\times_{h\le 2^k}(R_+) = \mathcal{V}^\times_{h=2^k}(R_+)$ is decreasing for $|h-2^k|$ for the boundary conditions) to push the boundary conditions to the top, left, and bottom of the rectangle $[-3n,4n]\times[-3n,3n]$.

We can apply Proposition 4.3 to get that

$$\phi^0_{\Lambda_{50n}}[\mathcal{V}^\times_{h\le 2^k}(R_+)|\mathcal{V}^\times_{h\le 2^k}(R_0)\cap\mathcal{C}\cap\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)]\ge c_1. \tag{4.19}$$

Similarly, one can condition on the right of the right-most crossing of $\mathcal{V}^\times_{h\le 2^k}(R_0)$ to get

$$\phi^0_{\Lambda_{50n}}[\mathcal{V}^\times_{h\le 2^k}(R_-)|\mathcal{V}^\times_{h\le 2^k}(R_+)\cap\mathcal{V}^\times_{h\le 2^k}(R_0)\cap\mathcal{C}\cap\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)]\ge c_1. \tag{4.20}$$

Forgetting the occurrence of $\mathcal{V}^\times_{h\le 2^k}(R_0)$, we deduce (4.18) for $c(k):=\frac{1}{2}c_1^2c(k+1)>0$.

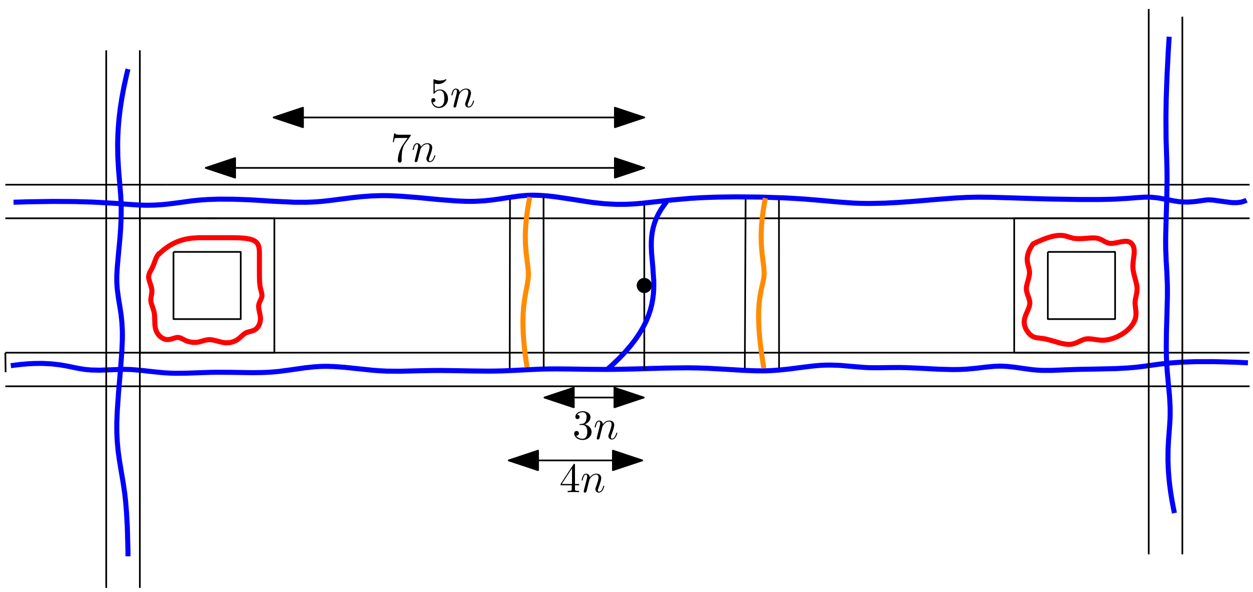

Figure 10: The event $\mathcal{D}(k)$ involves finding the orange $\times$-paths depicted above taking value at most 2^k. Given $\mathcal{D}(1)$, we wish to find a blue $\times$-path between them taking value at most 0 using the bridging Proposition 3.1.

Now that we have the existence of $c(1)>0$, suppose we are on $\mathcal{D}(1)\cap\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)$. The first case of Proposition 3.1 in Ω_1 implies that with probability 1/2, one can construct vertical $\times$-crossings of $h\le 0$ in $R:=R_-\cup R_0\cup R_+$. Forgetting about the occurrence of $\mathcal{D}(1)$ and conditioning on the left-most $\times$-crossing of $h\le 0$ in R, we can again construct a domain Ω^+ and this time deduce the existence of a $\times$-crossing of $h\le 0$ in $[4n,5n]\times[-3n,3n]$ via a reasoning similar to the one leading to (4.19). Then, one conditions on the right-most $\times$-crossing and deduces a similar claim for $[-5n,-4n]\times[-3n,3n]$. Overall, if $\mathcal{E}$ is the event that $A_{2n}(-7n)$ and $A_{2n}(7n)$ contain $\times$-circuits of $h\le 0$, the previous reasoning together with (4.18) gives that

$$\phi^0_{\Lambda_{50n}}[\mathcal{C}\cap\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)]\le C_2\phi^0_{\Lambda_{50n}}[\mathcal{E}\cap\mathcal{A}_n(-7n)\cap\mathcal{A}_n(7n)]. \tag{4.21}$$

As in [11], conditioned on the outer-most circuits of $h\le 0$ in $\Lambda_{5n}(-7n)$ and $\Lambda_{5n}(7n)$, the FKG for $|h|$ implies that $\mathcal{A}_n(-7n)$ and $\mathcal{A}_n(7n)$ are decoupled events and the probability of $\mathcal{A}_n(-7n)$ and $\mathcal{A}_n(7n)$ are each bounded by $\phi^0_{\Lambda_{5n}}[\mathcal{A}_n]$, so that

$$\phi^0_{\Lambda_{50n}}[\mathcal{E}\cap\mathcal{B}_n(-7n)\cap\mathcal{B}_n(7n)]\le Ca_n^2.$$

Combining this inequality with (4.17) and (4.21), we obtain (4.12), a fact which conclude the proof. □

4.3 Proofs of the theorems

Proof of Theorem 1.2. Assume that there exists m_0 such that $a_{m_0}<(2C)^{-1}$ with C being the constant from the renormalisation equation (Proposition 4.5). Iterating (4.4), there exist $c_1,C_1>0$ such that for every $r\in\mathbb{Z}_+$,

$$a_{10^r m_0}\le C_1\exp(-c_1 2^r). \tag{4.22}$$

Using RSW and FKG (similarly to the proof of Lemma 4.6), there exist $C_0,\rho_0>0$ such that for all $m\ge 1$,

$$\phi^0_{\Lambda_{5m}}[\mathcal{V}_{|h|\ge 1}(\Lambda_{2m,m/2})]\le C_0\phi^0_{\Lambda_{10m}}[\exists\text{ circuit of }|h|\ge 1\text{ surrounding }\Lambda_m]^{\rho_0}. \tag{4.23}$$

Now, consider m' to be the smallest integer of the form $20^r m_0$ which is larger than $20m$. Using the FKG inequality for $|h|$, one may combine such circuits in Λ_{20m} to get the existence of C_1,ρ_1 such that

$$\phi^0_{\Lambda_{20m}}[\exists\text{ circuit of }|h|\ge 1\text{ surrounding }\Lambda_m]\le C_1\phi^0_{\Lambda_{5m'}}[\exists\text{ circuit of }|h|\ge 1\text{ surrounding }\Lambda_{2m'}]^{\rho_1}.$$

Conditioning on the exterior-most $\times$-circuit of 1 and using FKG for $|h-1|$ and (4.23), we deduce that there exist $C_3,\rho_3>0$ such that

$$\phi^0_{\Lambda_{5m}}[\mathcal{V}_{|h|\ge 1}(\Lambda_{2m,m/2})]\le C_3 a_{m'}^{\rho_3}.$$

Together with (4.22), we find that there exist $C_2,c_2>0$ such that for every m,

$$\phi^0_{\Lambda_{5m}}[\mathcal{V}_{|h|\ge 1}(\Lambda_{2m,m/2})]\le C_2\exp(-m^{c_2}).$$

Now, the first item of Lemma 2.4 can be trivially adapted to state that there is a crossing of $|h|\ge 1$ from $\partial\Lambda_m$ to $\partial\Lambda_{2m}$ if and only if there is no $\times$-loop of $h=0$ in A_m surrounding 0, so that the $\phi^0_{\Lambda_{5m}}$-probability of this event is bounded by $4\phi^0_{\Lambda_{5m}}[\mathcal{V}_{|h|\ge 1}(\Lambda_{2m,m/2})]\le 4C_2\exp(-m^{c_2})$.

The event that $|h(0)|\ge 2k$ implies that there is no $\times$-loop of 0 (in fact no 0 at all) in A_k surrounding 0. As a consequence, there exists an integer $r\ge 0$ such that there is no $\times$-loop of $h=0$ in $A_{2^r k}$ surrounding 0 but there is one in $A_{2^{r+1}k}$. Conditioning on the exterior-most such loop and using the FKG inequality (to push the zero in) and the estimate above, we deduce that

$$\phi^0_{\Lambda_n}[|h(0)|\ge 4k]\le\sum_{r\ge 0}4C_2\exp(-(2^r k)^{c_2})\le C_3\exp(-k^{c_3}).$$

We now assume that $a_n \geq (2C)^{-1}$ for every n. Note that the first inequality follows trivially from the second one applied to $k+2$ so we only focus on the second inequality. Fix $\varepsilon, \rho > 0$ and k.

First, observe that using loops in successive annuli, there exists a constant $c_0 = c_0(k) > 0$ such that for all n,

$$\phi^0_{\Lambda_{\varepsilon n}}[\exists \times\text{-loop in } A_{\varepsilon 2^{-k-1}n} \text{ of } h \geq k] \geq c_0.$$

Define the annulus $A := \Lambda_{(\rho+\varepsilon)n,n} \setminus \Lambda_{(\rho+\varepsilon/2)n,n/2}$. The FKG inequality and the concatenations of small $\times$-loops with value $h \geq k$ give the existence of $c_1 = c_1(k, \varepsilon, \rho) > 0$ such that

$$\phi^0_D[\exists \times\text{-loop in } A \text{ of } h \geq k] \geq c_1.$$

Remark 4.4 and Lemma 4.6 enable us to fix k_0 sufficiently large that

$$\phi^0_D[\mathcal{E}^\times_{h\geq k_0}(\varepsilon n)] \leq \tfrac{1}{2}c_1.$$

Altogether, we conclude that

$$\phi^0_D[\exists \times\text{-loop in } A \text{ of } h \in [k, k_0)] \geq \tfrac{1}{2}c_1.$$

Condition on the exterior-most such $\times$-loop and let Ω be the domain inside. The boundary conditions induced by the conditioning are between k and k_0. Now, combining small $\times$-loops of $h \leq k$, we may construct a crossing of $\Lambda_{\rho n, n/2}$ with probability $c_2 = c_2(\varepsilon, \rho, k, k_0) > 0$. If this happens, we automatically obtain a $\times$-crossing of R of $h = k$. We deduce that this occurs with probability $\frac{1}{2}c_1c_2$, which is independent of n as desired. $\square$

We now prove logarithmic bounds for the variance of the height function. We first prove them in a box Λ^{even}_n introduced in Section 3.2 (a similar statement can easily be obtained in a generic domain).

Proposition 4.8. *There exist $c, C > 0$ such that for every n,*

$$c \log n \leq \phi^0_{\Lambda^{\text{even}}_n}[h_0^2] \leq C \log n.$$

Proof. Let $v_n = \phi_{\Lambda^{\text{even}}_n}[h_0^2]$. We start with the lower bound. Let $\mathcal{G}_n$ be the event that there is a $|h| = 2$ $\times$-loop inside A_n which by Corollary 1.3 has $\phi^0_{\Lambda^{\text{even}}_{2n}}$-probability at least c independent of n. On $\mathcal{G}_n$, call the vertices lying inside the outermost $|h| = 2$ $\times$-loop Ω_n. The bound $v_n \geq c \log n$ for some $c > 0$ follows by iterating the inequality

$$\begin{aligned}
v_{2n} &= \phi^0_{\Lambda^{\text{even}}_{2n}}[h_0^2 1_{\mathcal{G}_n}] + \phi^0_{\Lambda^{\text{even}}_{2n}}[h_0^2 1_{\mathcal{G}_n^c}] \\
&= \phi^0_{\Lambda^{\text{even}}_n}\big[\phi^0_{\Omega_n}[(h_0+\xi)^2]1_{\mathcal{G}_n}\big] + \phi^0_{\Lambda^{\text{even}}_{2n}}\big[\phi^0_{\Lambda^{\text{even}}_{2n}}[h_0^2 | h_{|\partial\Lambda^{\text{even}}_n}]1_{\mathcal{G}_n^c}\big] \\
&= \phi^0_{\Lambda^{\text{even}}_n}\big[(\phi^0_{\Omega_n}[h_0^2]+4)1_{\mathcal{G}_n}\big] + \phi^0_{\Lambda^{\text{even}}_{2n}}\big[\phi^0_{\Lambda^{\text{even}}_{2n}}[h_0^2 | h_{|\partial\Lambda^{\text{even}}_n}]1_{\mathcal{G}_n^c}\big] \\
&\geq (\phi^0_{\Lambda^{\text{even}}_n}[h_0^2]+4)\phi^0_{\Lambda^{\text{even}}_{2n}}[\mathcal{G}_n] + \phi_{\Lambda^{\text{even}}_n}[h_0^2]\phi^0_{\Lambda^{\text{even}}_{2n}}[\mathcal{G}_n^c] \\
&= v_n + 4\phi^0_{\Lambda^{\text{even}}_{2n}}[\mathcal{G}_n] \geq v_n + 4c.
\end{aligned}$$

where ξ is a random variable taking values ± 2 with equal probability independent of everything else. The justification of this sequence of inequalities is the following. To see the second equality, note that on the event $\mathcal{G}_n$ we can explore $|h|$ until we discover Ω_n. The third one follows from the spatial Markov property, the independence of h and ξ, and the fact that $\phi^0_{\Omega_n}[h_0] = 0$. The inequality follows from the comparison between boundary conditions and the FKG inequality for $|h|$.

Let us now turn to the upper bound. One can implement a proof which is quite similar to the lower bound here, but we choose a different road which extends trivially to the torus case. Consider ℓ_k to be the outer-most $\times$-loop of $h \geq 2k$ surrounding the origin, if it exists. Also, for each $i \leq \log_2 n$ (here we forget the rounding since it does not impact the rest of the proof), let $\mathbf{N}_i$ be the number of indexes k such that the maximal distance between a vertex in ℓ_k and the origin is between 2^i and 2^{i+1}. Observe that

$$\phi^0_{\Lambda_n}[h_0 \geq N] \leq \sum_{N_1+\cdots+N_{\log_2 n}=N} \phi^0_{\Lambda_n}[\mathbf{N}_i = N_i, \forall i \leq \log_2 n]. \tag{4.24}$$

We claim that for every $\varepsilon > 0$, there exists $C_0 > 0$ such that

$$\phi^0_{\Lambda_n}[\mathbf{N}_i = N_i | \mathbf{N}_1 = N_1, \ldots, \mathbf{N}_{i-1} = N_{i-1}] \leq C_0 \varepsilon^{N_i}. \tag{4.25}$$

Plugging this estimate into (4.24) and using that $\binom{a}{b} \leq (ea/b)^b$ implies that

$$\phi^0_{\Lambda_n}[h_0 \geq N] \leq (1 + \tfrac{\log n}{N})^N C_0^{\log_2 n} (e\varepsilon)^N.$$

Since we may choose ε as small as we wish, this quantity decays exponentially fast in $N \geq C_1 \log n$, thus concluding the proof. We therefore turn to the proof of (4.25).

Fix $r > 0$. Let Ω_k be the domain enclosed by ℓ_k and set $k_i := N_1 + \cdots + N_i$. Pave the annulus A_{2^i} by balls of size 2^{i-r} centred at $x_0, \ldots, x_R$. Let M_k be the number of such balls that are intersecting Ω_k. We claim that there exists a constant $c_0 > 0$ such that for every $k \in (k_{i-1}, k_i)$,

$$\phi^0_{\Lambda_n}[M_{k+1} = M_k > 0 | \ell_1, \ldots, \ell_k] \leq (1 - c_0)^r \qquad \text{a.s.}. \tag{4.26}$$

Indeed, the conditional measure is, up to a sign, equal to $\phi^{2k}_{\Omega_k}$. Now, since $M_k > 0$ and ℓ_k intersects the annulus, one may choose x_k such that the ball of radius 2^{i-r} around it intersects ℓ_k. Note that $M_{k+1} = M_k > 0$ imposes the occurrence, for every $1 \leq s \leq r$, of the event $\mathcal{E}_s$ that there exists a $\times$-crossing of $h = 2k + 2$ in the annulus A_s around x_k of inner and outer radii 2^{i-s-1} and 2^{i-s}. Using the FKG for $|h - 2k - 2|$, we therefore deduce that

$$\phi^0_{\Lambda_n}[M_{k+1} = M_k > 0 | \ell_1, \ldots, \ell_k] \leq \phi^{2k}_{\Omega_k}\big[\bigcap_{s=1}^r \mathcal{E}_s\big] \leq \prod_{s=1}^r \phi^{\kappa_s}_{A_s \cap \Omega_k}[\mathcal{E}_s], \tag{4.27}$$

where κ_s is the boundary conditions equal to $h = 2k + 2$ on the inner and outer boundaries of A_s, and $h = 2k$ on the rest of the boundary. Since the existence of the $\times$-crossing of $h = 2k + 2$ from inside to outside is the complement under these boundary conditions of the existence of a $*$-path of $2k$ from ℓ_k to itself, we may use the FKG inequality for $|h - 2k|$ and the shifting of the height-function down by $2k$, to bound the probability of the event $\mathcal{E}_s$ by the event that there exists a $*$-circuit of 0 surrounding the origin in an annulus with boundary conditions 2. This probability is bounded by $1 - c_0$ using Corollary 1.3, and we therefore obtain (4.26).

Now, if one finds N_i loops with radius between 2^i and 2^{i+1}, there must be at least $N_i - C_2$ indexes $k \in (k_{i-1}, k_i)$ for which $M_{k+1} = M_k > 0$, where C_2 is a function of r only. We deduce that

$$\phi^0_{\Lambda_n}[\mathbf{N}_i = N_i | \mathbf{N}_1 = N_1, \ldots, \mathbf{N}_{i-1} = N_{i-1}] \leq N_i^{C_2} e^{-r(N_i - C_2)}$$

which implies (4.25) with ε and a constant $C_0 = C_0(\varepsilon) > 0$ provided that we select r large enough. $\square$

We conclude this article with the proof of Theorem 1.1.

Proof of Theorem 1.1. Fix a representative of the equivalence class of each homomorphism by setting $h(x) = 0$. Using FKG for $|h|$, we deduce that

$$\phi_{\mathbb{T}_n}[(h(y) - h(x))^2] = \phi_{\mathbb{T}_n}^{\{x\},0}[h(y)^2] \geq \phi^0_{\Lambda_{|x-y|}(y)}[h(y)^2] \geq c \log |x - y|.$$

The upper bound can be deduced by an argument similar to the one developed in the last proof (defining the circuits starting from 0).

□

References

[1] I. Benjamini, O. Häggström, and E. Mossel. On random graph homomorphisms into $\mathbb{Z}$. *Journal of Combinatorial Theory, Series B*, 78(1):86–114, 2000.

[2] I. Benjamini and Y. Peres. Tree-indexed random walks on groups and first passage percolation. *Probab. Theory Related Fields*, 98(1):91–112, 1994.

[3] V. Beffara and H. Duminil-Copin. The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$. *Probab. Theory Related Fields*, 153(3-4):511-542, 2012.

[4] I. Benjamini and G. Schechtman. Upper bounds on the height difference of the Gaussian random field and the range of random graph homomorphisms into $\mathbb{Z}$. *Random Structures Algorithms*, 17(1):20–10, 2000.

[5] I. Benjamini, A. Yadin, and A. Yehudayoff. Random graph-homomorphisms and logarithmic degree. *Electron. J. Probab.*, 12(32):926–920, 2007.

[6] B. Bollobás and O. Riordan. A short proof of the Harris-Kesten theorem. *Bull. London Math. Soc.*, 38(3):470–484, 2006.

[7] G. Ray and Y. Spinka. A short proof of the discontinuity of phase transition in the planar random-cluster model with $q > 4$. *arXiv:1904.10557*, 2019.

[8] B. Bollobás and O. Riordan. Percolation on self-dual polygon configurations, *An irregular mind, Bolyai Soc. Math. Stud.,* 7:131–217, 2010.

[9] A. Bufetov and A. Knizel. Asymptotics of random domino tilings of rectangular Aztec diamonds. *Ann. of IHP*, 54(3):150–1290, 2018.

[10] N. Chandgotia, R. Peled, S. Sheffield, and M. Tassy. Delocalization of uniform graph homomorphisms from $\mathbb{Z}^2$ to $\mathbb{Z}$. *arXiv:1810.10124*, 2018.

[11] H. Duminil-Copin, V. Sidoravicius, and V. Tassion. Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$. *Communications in Mathematical Physics*, 349(1):47–107, 2017.

[12] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu and V. Tassion. Discontinuity of the phase transition for the planar random-cluster and Potts models with $q > 4$. *arXiv:1611.09877*, 2016.

[13] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu and V. Tassion. The Bethe ansatz for the six-vertex and XXZ models: an exposition *Prob. Surveys*, 15:102–130, 2018.

[14] H. Duminil-Copin, A. Glazman, R. Peled and Y. Spinka. Macroscopic loops in the loop $O(n)$ model at Nienhuis' critical point. *arXiv:1707:09335*, 2017.

[15] H. Duminil-Copin, C. Hongler, and P. Nolin. Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model. *Comm. Pure Appl. Math.* 64(9):1165–1198, 2011.

[16] H. Duminil-Copin and V. Tassion. Renormalization of crossing probabilities in the planar random-cluster model. *arXiv:1901:08294*, 2019.

[17] A. Erschler. Random mappings of scaled graphs. *Probab. Theory Related Fields*, 144(3-4):543–579, 2009.

[18] J. Fröhlich and T. Spencer. The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas. *Comm. Math. Phys.*, 81(4):527–602, 1981.

[19] D. Galvin. On homomorphisms from the Hamming cube to $\mathbb{Z}$. *Israel J. Math.*, 138(1):189–213, 2003.

[20] A. Glazman and I. Manolescu. Uniform Lipschitz functions on the triangular lattice have logarithmic variations. *arXiv:1810.05592*, 2018.

[21] G. Grimmett. *Percolation*, volume 37 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.

[22] J. Kahn. Range of cube-indexed random walk. *Israel J. Math.*, 124:189–201, 2001.

[23] E.H. Lieb. Exact Solution of the Two-Dimensional Slater KDP Model of a Ferroelectric. *Physical Review*, 19(3):108–110, 1967.

[24] E.H. Lieb. Residual entropy of square ice. *Physical Review*, 162(1):162, 1967.

[25] E.H. Lieb. Exact solution of the F model of an antiferroelectric. *Condensed Matter Physics and Exactly Soluble Models*, 453-455, 1967.

[26] M. Loebl, J. Nešetřil, and B. Reed. A note on random homomorphism from arbitrary graphs to $\mathbb{Z}$. *Discrete Math.*, 273(1-3):173–181, 2003.

[27] R. Peled. High-dimensional Lipschitz functions are typically flat. *Ann. Probab.*, 45(3):1351–1447, 2017.

[28] L. Russo. A note on percolation. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 43(1):39–48, 1978.

[29] P. D. Seymour and D. J. A. Welsh. Percolation probabilities on the square lattice. *Ann. Discrete Math.*, 3:227–245, 1978.

[30] S. Sheffield. Random surfaces. *Astérisque. Société mathématique de France*, 2005.

[31] V. Tassion. Crossing probabilities for Voronoi percolation. *Annals of Probability*, 44(5):3385–3398, 2016.