LONG-RANGE MODELS IN 1D REVISITED

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Abstract. In this short note, we revisit a number of classical results on long-range 1D percolation, Ising model and Potts models [FS82, NS86, ACCN88, IN88]. More precisely, we show that for Bernoulli percolation, FK percolation and Potts models, there is symmetry breaking for the $1/r^2$-interaction at large β, and that the phase transition is necessarily discontinuous. We also show, following the notation of [ACCN88] that $\beta^*(q) = 1$ for all $q \geq 1$.

1 Setting

Long-range models on the 1D line have a rich history in physics and mathematical physics. Historically, Dyson, motivated by predictions of Anderson [AYH70] and Thouless [Tho69] as well as connections with the *Kondo problem*, initiated in [Dys69] the rigorous analysis of long-range Ising models on $\mathbb{Z}$ with coupling constants given by $J_{i-j} \sim |i-j|^{-s}$ with $s \in (1,2)$. Fröhlich and Spencer analysed in [FS82] the "scale-invariant" case where the coupling constants are given by $\frac{1}{|i-j|^2}$. Later, Aizenman, Chayes, Chayes and Newman proved in [ACCN88] the discontinuity of the phase transitions which had been anticipated by Thouless for all $q \geq 1$. In this paper, we revisit these classical results.

We begin by treating the case of Bernoulli percolation and then consider the random-cluster model and its applications to the Ising and Potts models. Note that historically, the Ising model was studied before Bernoulli percolation, but our renormalization technique is simpler to present in the Bernoulli percolation setting.

Our notation/setup will follow in part [NS86, ACCN88]. In the whole paper, we consider $J_{i,j} = J(i-j) = 1/|i-j|^2$ for every $i \neq j$. For $\beta, \lambda > 0$, we define

$$p_{i,j}(\beta,\lambda) := \begin{cases} 1 - \exp[-\beta J_{i,j}] & \text{if } |i-j| \geq 2, \\ 1 - \exp[-\lambda] & \text{if } |i-j| = 1. \end{cases} \tag{1}$$

For simplicity, we will only consider the most interesting case where the point-to-point interaction will decay as $1/|i-j|^2$ but our methods easily extends to $J(x) \sim 1/x^2$ for $x \geq 1$, and allows one to recover easily the results for directed percolation for $s \in (1,2)$, see Remark 1.

Let us point out that the renormalization argument in this paper follows the same setup as in our work [DGT20] except that the $1D$ setting here makes things simpler.

2 Long-range Bernoulli percolation

2.1 Statement of the result. Consider the long-range Bernoulli percolation measure $\mathbb{P}_{\beta,\lambda}$ on $\mathbb{Z}$ defined by the property that each unordered pair $\{i,j\}$ (also called edge) is *open* with probability $p_{i,j}(\beta,\lambda)$, and *closed* with probability $1 - p_{i,j}(\beta,\lambda)$, independently for every edge $\{i,j\} \subset \mathbb{Z}$. Let $\theta(\beta,\lambda)$ be the probability that 0 is connected to infinity by a path of open edges.

Theorem 1. *We have the following two properties:*

(i) *For* $\beta > 1$, $\theta(\beta, \lambda) > 0$ *for* $\lambda < \infty$ *large enough.*
(ii) *For every* $\beta, \lambda > 0$, $\theta(\beta, \lambda) > 0$ *implies* $\beta\theta(\beta, \lambda)^2 \geq 1$.

The fact that for large β, (i) is true was obtained by Newman and Schulman in [NS86]. The extension to every $\beta > 1$ was proved in [IN88]. On the other hand, (ii) is the object of [AN86]. Note that (ii) implies that $\theta(\beta, \lambda) = 0$ if $\beta \leq 1$, and that for each fixed $\lambda > 0$, $\beta \mapsto \theta(\beta, \lambda)$ is continuous only if it is identically equal to 0 (hence the conclusion that the phase transition is necessarily discontinuous which it occurs). As such, using the notations in [ACCN88], by combining (i) and (ii), this proves $\beta^*(q = 1) = 1$.

2.2 Notation. In the proofs below, we will use the notion of K-*block*

$$B_K^i = [K(i-1), K(i+1)), \quad i \in \mathbb{Z},\ K \in \mathbb{Z}_+.$$

For simplicity we write B_K instead of B_K^0. Let us point out that in this framework, consecutive blocks B_K^i and B_K^{i+1} are overlapping on half of their length. This will be a key property in the proof below. Let $S \subset \mathbb{Z}$. We call a *cluster in* S a connected component $C \subset S$ of the graph with vertex set S and open edges with *both* endpoints in S.

A K-block B_K^i is is said to be θ-*good* if there exists a cluster in it of cardinality at least $2\theta K$. When a block is not θ-good, we call it θ-*bad* and we define

$$p_{\beta,\lambda}(K, \theta) := \mathbb{P}_{\beta,\lambda}[B_K \text{ is } \theta\text{-bad}].$$

2.3 Proof of Theorem 1(i). The proof of Theorem 1(i) relies on the idea that clusters at scale K and local density θ will merge and with high probability create new clusters at scale CK of local density $\theta' = \theta - O(1/C)$ slightly smaller than θ (this slight loss of density allows us to lose a few clusters at scale K in the process). More precisely, we prove the following renormalization inequality.

Lemma 2. *Let* $\beta > 1$ *and* $\theta_\infty \in (\frac{3}{4}, 1)$ *satisfying* $\theta_\infty^2 \beta > 1$. *There exist* $C_0 \geq 1$ *large enough (depending on* θ_∞, β*) such that the following holds. For every* $\lambda > 0$, $\theta \geq \theta_\infty$, *and for every integers* $C \geq C_0$ *and* $K \geq 2$,

$$p_{\beta,\lambda}(CK, \theta - C_0/C) \leq \tfrac{1}{100}\, p_{\beta,\lambda}(K, \theta) + 2C^2\, p_{\beta,\lambda}(K, \theta)^2. \tag{2}$$

Proof. In the proof, we focus on the K-blocks included in B_{CK}. Given such a block B_K^i, we write $\mathbf{C}(B_K^i)$ for the largest cluster in B_K^i. Notice that $\mathbf{C}(B_K^i)$ has size at least $2\theta K$ when the block is θ-good, and it is the unique cluster in B_K^i with this property when $\theta > 3/4$.

Let $C_0 > 0$ to be a large constant to be chosen later (more precisely, the constant C_0 will be chosen in such a way that the second inequality of (10) holds, and this choice depends on β and θ_∞ only) and set $\theta' := \theta - C_0/C$. For $|i| \leq C$, let E_i be the event that B_K^i is θ-bad and all the blocks B_K^j are θ-good for $j \in [-C+1, C-1] \setminus \{i-1, i, i+1\}$, and set

$$F_i = E_i \cap \{B_{CK} \text{ is } \theta'\text{-bad}\}. \tag{3}$$

Observe that if all K-blocks B_K^j, $-C+1 \leq j \leq C-1$, are θ-good, then the assumption that $\theta > 3/4$ and the overlapping property between subsequent K-blocks guarantees that all the clusters $\mathbf{C}(B_K^j)$ are connected together in B_{CK}, which implies the existence of a cluster in B_{CK} with cardinality larger than $2\theta CK$. In particular, if B_{CK} is θ'-bad, then either there exist (at least) two disjoint θ-bad

K-blocks, or there exists i such that F_i occurs. The union bound implies

$$p_{\beta,\lambda}(CK,\theta') \le \sum_{i=-C+1}^{C-1} \mathbb{P}_{\beta,\lambda}[F_i] + \mathbb{P}_{\beta,\lambda}[\text{there are two } \textit{disjoint } \theta\text{-bad } K\text{-blocks}]. \quad (4)$$

By independence and the union bound, we have

$$\mathbb{P}_{\beta,\lambda}[\text{there are two } \textit{disjoint } \theta\text{-bad } K\text{-blocks}] \le \binom{2C-1}{2} p_{\beta,\lambda}(K,\theta)^2. \quad (5)$$

It remains to bound the first term in (4), which is the object of the end of the proof. If all K-blocks B_K^j with $|j| \le C - C_0$ are θ-good, the same argument as above implies that B_{CK} is θ'-good, therefore $F_i = \emptyset$ whenever $|i| \le C - C_0$.

Now, let $|i| \le C - C_0$. Since $\mathbb{P}_{\beta,\lambda}[E_i] \le p_{\beta,\lambda}(K,\theta)$, we deduce that

$$\mathbb{P}_{\beta,\lambda}[F_i] \le p_{\beta,\lambda}(K,\theta) \cdot \mathbb{P}_{\beta,\lambda}[B_{CK} \text{ is } \theta'\text{-bad} \mid E_i]. \quad (6)$$

In order to bound the conditional probability above, define $\mathbf{C}^-$ (resp. $\mathbf{C}^+$) as the union of all the clusters $\mathbf{C}(B^j)$, $j \le i-2$ (resp. $j \ge i+2$). Let us examine the properties of these two sets when the event E_i occurs. First, the goodness property of the K-blocks at the left of B_K^{i-1} and the right of B_K^{i+1} imply that $\mathbf{C}^-$ and $\mathbf{C}^+$ are two connected sets. Second, writing $Ki > x_1 > x_2 > \cdots$ (resp. $Ki < y_1 < y_2 < \cdots$) for the ordered elements of $\mathbf{C}^-$ (resp. $\mathbf{C}^+$), the θ-density in each K-block implies that for every $a, b \ge 1$,

$$x_a \ge Ki - 3K - \frac{a-1}{\theta} \quad \text{and} \quad y_b \le Ki + 3K + \frac{b-1}{\theta}. \quad (7)$$

Furthermore, if B_{CK} is θ'-bad, then $\mathbf{C}^-$ and $\mathbf{C}^+$ cannot be connected together. Conditioning on $\mathbf{C}^-$ and $\mathbf{C}^+$ provides no information on edges $\{x,y\}$ with $x \in \mathbf{C}^-$ and $y \in \mathbf{C}^+$ since the definition of $\mathbf{C}^-$ and $\mathbf{C}^+$ involves only edges with endpoints within a distance $2K$ of each other. Therefore, the conditional probability that $\mathbf{C}^-$ and $\mathbf{C}^+$ are *not* connected by an edge is equal to

$$P(\mathbf{C}^-,\mathbf{C}^+) := \prod_{x\in\mathbf{C}^-}\prod_{y\in\mathbf{C}^+} e^{-\beta J_{x,y}} = \exp\Big[-\beta \sum_{a=1}^{|\mathbf{C}^-|}\sum_{b=1}^{|\mathbf{C}^+|} J_{x_a,y_b}\Big].$$

Using that $|\mathbf{C}^-|, |\mathbf{C}^+| \ge A := \theta K(C - |i| - 2)$ together with (7), we find that

$$\begin{aligned} P(\mathbf{C}^-,\mathbf{C}^+) &\le \exp\Big[-\beta \sum_{1\le a,b\le A} \frac{1}{(6K + (a+b-2)/\theta)^2}\Big] \\ &\le \exp\Big[-\beta\theta^2 \int_{0\le x,y\le A-1} \frac{dx\,dy}{(6K\theta^2 + x + y)^2}\Big] \\ &\le \Big(\frac{12}{C-|i|}\Big)^{\beta\theta^2}. \end{aligned} \quad (8)$$

Integrating over all possible choices of $\mathbf{C}^-$ and $\mathbf{C}^+$, we get

$$\mathbb{P}_{\beta,\lambda}[B_{CK} \text{ is } \theta'\text{-bad} \mid E_i] \le \Big(\frac{12}{C-|i|}\Big)^{\beta\theta^2}. \quad (9)$$

Plugging this estimate in (6), we finally obtain that provided C_0 is large enough,

$$\sum_{i=-C}^{C} \mathbb{P}_{\beta,\lambda}[F_i] \le p_{\beta,\lambda}(K,\theta) \sum_{i=C_0-C}^{C-C_0} \Big(\frac{12}{C-|i|}\Big)^{\beta\theta^2} \le \tfrac{1}{100}\, p_{\beta,\lambda}(K,\theta). \quad (10)$$

Plugging (10) and (5) in (4) concludes the proof. □

Proof of Theorem 1(i). Fix $\beta > 1$. Let $\theta_\infty \in (\frac{3}{4}, 1)$ such that $\beta\theta_\infty^2 > 1$. Choose $\theta_1 < 1$ and $C_1 \geq C_0(\beta, \theta_\infty)$ (where C_0 is provided by Lemma 2) such that the sequences

$$\begin{cases} C_{n+1} = (n+1)^3 C_1, \\ \theta_{n+1} := \theta_n - \frac{C_0}{C_{n+1}}, \end{cases} \qquad \text{for } n \geq 1$$

satisfy $\theta_n \geq \theta_\infty$ for every $n \geq 1$. Now, set $\lambda > 0$ so large that

$$p_{\beta,\lambda}(C_1, \theta_1) \leq \mathbb{P}_{\beta,\lambda}[\exists \{x, x+1\} \subset B_{C_1} \text{ closed}] \leq C_1 e^{-\lambda} \leq \frac{1}{400 C_1^2}.$$

and consider the sequence of scales defined by

$$\begin{cases} K_1 = C_1, \\ K_{n+1} = C_{n+1} K_n \quad n \geq 1. \end{cases} \tag{11}$$

(note that it gives $K_n = (n!)^3 C_1^n$ for all $n \geq 1$). Applying Lemma 2 to $(\lambda, \beta, \theta_n, C_n, K_n)$, we see that the sequence $u_n := p_{\beta,\lambda}(K_n, \theta_n)$ satisfies

$$\forall n \geq 1, \qquad u_{n+1} \leq \tfrac{1}{100} u_n + 2 C_{n+1}^2 u_n^2.$$

By induction, we obtain that $u_n \leq \frac{1}{400} C_n^{-2}$ for every $n \geq 1$, and therefore,

$$\mathbb{P}_{\beta,\lambda}[B_{K_n} \ \theta_n\text{-good}] \geq 1 - \tfrac{1}{400} C_n^{-2} \geq \tfrac{1}{2}.$$

First using the estimate above and then translation invariance, we get that for every $n \geq 1$,

$$\frac{3}{4} K_n \leq \mathbb{E}[|\mathbf{C}(B_{K_n})| \cdot \mathbf{1}_{\{B_{K_n} \text{ is } \frac{3}{4}\text{-good}\}}] \leq 2K_n \mathbb{P}_{\beta,\lambda}[0 \text{ is in a cluster of size at least } \tfrac{3}{2} K_n]. \tag{12}$$

Dividing both sides by $2K_n$, we obtain

$$\mathbb{P}_{\beta,\lambda}[0 \text{ is in a cluster of size at least } \tfrac{3}{2} K_n] \geq \tfrac{3}{8},$$

which by measurability implies that the probability that 0 is connected to infinity is larger than or equal to $\frac{3}{8}$. □

Remark 1. When considering $J_{i,j} = 1/|i-j|^s$ with $s \in (1,2)$, the estimate in (8) becomes of the order of $\exp[-c(\beta,\theta)((C-|i|)K)^{2-s}]$ and one can easily deduce the existence, for every $\beta > 0$, of $\lambda = \lambda(\beta) > 0$ large enough so that percolation occurs. Let us remark that in this case $p_{\beta,\lambda}(K_n, \theta_n)$ decays stretched-exponentially fast in K_n (while it decays polynomially fast in the case of $s = 2$).

2.4 Proof of Theorem 1(ii). We say that a block B_{3K}^i is K-*crossed* if there exist $x < 3Ki - 3K$ and $y \geq 3Ki + 3K$ such that x is connected to y using open edges of length at most K. Introduce

$$\overline{p}_{\beta,\lambda}(K) = 1 - \mathbb{P}[B_{3K} \text{ is } K\text{-crossed}].$$

The proof is based on the following inequality (a similar inequality was obtained for another quantity in [DMT20]).

Lemma 3. *Let $\beta, \lambda, \theta > 0$ such that $\beta\theta^2 < 1$ and $\theta(\beta, \lambda) < \theta$. Then, there exists $C_0 = C_0(\beta, \lambda, \theta)$ such that for every integers $C, K \geq C_0$,*

$$\overline{p}_{\beta,\lambda}(CK) \geq \frac{C^{1-\beta\theta^2}}{9e} \min\{\overline{p}_{\beta,\lambda}(K), C^{-1}\}. \tag{13}$$

Proof. Let us fix β, λ, θ and drop them from the notation. Fix $R \geq 1$ and $K \geq 2R$ such that

$$\mathbb{P}[0 \leftrightarrow \mathbb{Z} \setminus B_R]^2 + |B_R|^2 e^{-\beta(K-2R)} \leq \theta^2. \tag{14}$$

Call an edge $\{x,y\}$ a *bridge* if it is open and in $\omega \setminus \{x,y\}$, both x and y are connected to distance R. Call a $3K$-block B^i_{3K} *bridged* if there is bridge $\{x,y\}$ with $K < y-x \le CK$ and either $x < 3K(i+1)$ or $y \ge 3K(i-1)$ (otherwise it is said to be *unbridged*). Let $\mathbf{B} = \mathbf{B}(\omega)$ be the set of unbridged $3K$-blocks $B^i_{3K} \subset B_{CK}$ with *i divisible by* 3 (note that the blocks are subsets of the box B_{CK} and not B_{3CK}).

Assume that the block B_{3CK} is CK-crossed, then all of the $3K$-blocks in B_{CK} must be either bridged or K-crossed. In particular, all the $3K$-blocks $B \in \mathbf{B}$ must be K-crossed. This implies that

$$1 - \overline{p}_{\beta,\lambda}(CK) \le \mathbb{P}_{\beta,\lambda}\big[\forall B \in \mathbf{B},\ B \text{ is } K\text{-crossed}\big]. \tag{15}$$

Now, we consider a random variable $X = (\eta, \mathcal{E})$ defined as follows.

- $\eta = \omega_{|\{\{x,y\}\,:\,K<y-x\le CK\}}$ is the configuration restricted to all the edges of length between $K+1$ and CK; write $V(\eta)$ for the set of endpoints of these edges that are open;
- for each $u \in V(\eta)$, consider the cluster $\mathbf{C}(u,\omega)$ of u in $[u-R, u+R)$ and let now $\mathcal{E}$ be the set of all the edges with one endpoint in one of the $\mathbf{C}(u,\omega)$ for some $u \in V(\eta)$.

First, observe that the set $\mathbf{B}$ of unbridged blocks is measurable with respect to X, and conditionally on X, whether a block $B \in \mathbf{B}$ is K-crossed or not is independent of the other blocks in $\mathbf{B}$ (since they are at a distance at least K of each other). Therefore, the right hand side in (15) is equal to

$$\mathbb{E}_{\beta,\lambda}\Big[\prod_{B\in\mathbf{B}} \mathbb{P}_{\beta,\lambda}[B \text{ is } K\text{-crossed} \mid X]\Big]. \tag{16}$$

Also, the conditioning on X only brings negative information on the fact that a $3K$-block $B \in \mathbf{B}$ is K-crossed (since $K > 2R$). Hence, each term in the product above is smaller that $1 - \overline{p}_{\beta,\lambda}(K)$, and we get that

$$\overline{p}_{\beta,\lambda}(CK) \ge 1 - \mathbb{E}[(1-\overline{p}_{\beta,\lambda}(K))^{|\mathbf{B}|}]. \tag{17}$$

Setting $t = \min\{\overline{p}_{\beta,\lambda}(K), C^{-1}\}$, we have

$$(1-\overline{p}_{\beta,\lambda}(K))^{|\mathbf{B}|} \le (1-t)^{|\mathbf{B}|} \le e^{-t|\mathbf{B}|} \le 1 - e^{-1}t|\mathbf{B}|, \tag{18}$$

where the last inequality uses $t|\mathbf{B}| \le 1$. Taking the expectation and plugging it in (17), we deduce that

$$\overline{p}_{\beta,\lambda}(CK) \ge e^{-1}\mathbb{E}[|\mathbf{B}|]\min\{\overline{p}_{\beta,\lambda}(K), C^{-1}\}. \tag{19}$$

To conclude, it remains to bound $\mathbb{E}[|\mathbf{B}|]$ from below. We do it by summing on i the following estimate for $3K$-blocks $B^i_{3K} \subset B_{CK}$,

$$\mathbb{P}[B^i_K \text{ unbridged}] \ge \prod_{\substack{x<K(i+1),\\ y\ge K(i-1),\\ K<y-x\le CK}} \mathbb{P}[\{x,y\} \text{ not a bridge}] \ge \tfrac{1}{2}C^{-\beta\theta^2}, \tag{20}$$

where the first inequality is due to the FKG inequality, and the second to a sum-integral comparisons (together with the assumption that C is large enough) using the following estimate

$$\begin{aligned}\mathbb{P}[\{x,y\} \text{ is a bridge}] &\le (1-e^{-\beta J_{x,y}})(\mathbb{P}[0 \leftrightarrow \mathbb{Z}\setminus B_R]^2 + |B_R|^2 e^{-\beta(K-2R)})\\ &\le (1-e^{-\beta J_{x,y}})\theta^2. \end{aligned} \tag{21}$$

The first inequality is due to the fact that either there is an open edge in $\omega \setminus \{x,y\}$ between $[x-R, x+R)$ and $[y-R, y+R)$, or the two events are independent.The second is due to the choice of R given by (14). □

Proof of Theorem 1(ii). Fix $\beta, \lambda, \theta > 0$ such that $\theta(\beta, \lambda) < \theta$ and $\beta\theta^2 < 1$. Let C_0 as in Lemma 3, and pick $C \geq C_0$ such that

$$\frac{C^{1-\beta\theta^2}}{9e} \geq 1 \quad \text{and} \quad p_{\beta,\lambda}(C_0) \geq C^{-1}.$$

Setting $K_n := C^n C_0$ $(n \geq 0)$, (13) applied to $K = K_n$ and C implies that for every $n \geq 0$,

$$\overline{p}_{\beta,\lambda}(K_{n+1}) \geq \min(\overline{p}_{\beta,\lambda}(K_n), C^{-1}). \tag{22}$$

By induction, we deduce that $p_{\beta,\lambda}(K_n) \geq C^{-1}$ for every $n \geq 1$.

Now, let $A(K_n)$ be the event that there exists $x \in B_{3K_n}$ connected to $y \notin B_{9K_n}$, and $B(K_n)$ be the event that all the edges of length strictly larger than K_n with one endpoint in B_{9K_n} are closed. Notice that if $B(K_n)$ occurs and neither $B^{-2}_{3K_n}$ nor $B^2_{3K_n}$ is K_n-crossed, then $A(K_n)$ does not occur. Hence, by independence, we have that

$$\mathbb{P}_{\beta,\lambda}[B(K_n)]\overline{p}_{\beta,\lambda}(K_n)^2 \leq 1 - \mathbb{P}[A(K_n)]. \tag{23}$$

Since $\mathbb{P}_{\beta,\lambda}[B(K_n)] \geq c_1(\beta) > 0$ (by a computation very similar to (20)), we deduce that for every $n \geq 0$,

$$\mathbb{P}_{\beta,\lambda}[A(K_n)] \leq 1 - c_1(\beta)/C^2.$$

We obtained the above estimate by assuming $\theta(\beta, \lambda) < \theta$ with $\beta\theta^2 < 1$. We see from this estimate that it is not possible to also have $\theta(\beta, \lambda) > 0$. Indeed, otherwise, this would contradict measurability since the probability that there exists $x \in B_{K_n}$ connected to infinity, which is itself included in $A(K_n)$, would have a probability tending to 1 in this case. □

3 Long-range Fortuin-Kasteleyn percolation and its applications to the Ising and Potts models

3.1 Statement of the results. Here, we define the Fortuin-Kasteleyn percolation [For71, FK72] (we also refer to [Gri06] for a manuscript and [Dum17] for recent results on (finite-range) FK percolation). Let $S \subset T$ be two finite subsets of $\mathbb{Z}$, let ξ be a partition of the vertices $T \setminus S$. The FK percolation measure on edges included in T with at least one endpoint in S, with boundary conditions (b.c.) ξ, is defined by the formula

$$\mathbb{P}^{\xi}_{S,T,\beta,\lambda,q}[\omega] = \frac{q^{k(\omega^\xi)}}{Z} \prod_{\{i,j\}\subset T:\{i,j\}\cap S\neq\emptyset} p_{i,j}(\beta,\lambda)^{\omega_{i,j}}(1-p_{i,j}(\beta,\lambda))^{1-\omega_{i,j}},$$

where $\omega_{i,j} = 1$ if $\{i,j\}$ is open and 0 is it is closed, ω^ξ is the graph obtained from ω by wiring all the vertices outside S belonging to the same element of the partition ξ. Let $\xi = 1$ (resp. $\xi = 0$) be the wired (resp. free) boundary conditions corresponding to the partitions equal to $\{T \setminus S\}$ (resp. only singletons).

Given a partition ξ of $\mathbb{Z} \setminus S$, define $\mathbb{P}^{\xi}_{S,\beta,\lambda,q}$ as the limit of the measure $\mathbb{P}^{\xi_K}_{S,B_K,\beta,\lambda,q}$ as K tends to infinity, where ξ_K denotes the partition induced by ξ on $B_K \setminus S$. Let $\mathbb{P}^1_{\beta,\lambda,q}$ be the measure on $\mathbb{Z}$ defined as the limit as K tends to infinity of the measures $\mathbb{P}^1_{B_K,\beta,\lambda,q}$ and $\theta(q, \beta, \lambda)$ be the $\mathbb{P}^1_{\beta,\lambda,q}$-probability that 0 is connected to infinity by a path of open edges[1].

Theorem 4. *For $q \geq 1$,*

(i) *For $\beta > 1$, there exists $\lambda < \infty$ large enough so that $\theta(q, \beta, \lambda) > 0$.*
(ii) *For $\beta, \lambda > 0$, $\theta(q, \beta, \lambda) > 0$ implies $\beta\theta(q, \beta, \lambda)^2 \geq 1$.*

[1]The proof that these limits exist for any partition ξ of $\mathbb{Z}$ proceeds as usual by monotony.

The result above covers a certain number of results, including [NS86] for (i) and Aizenman, Chayes, Chayes and Newman [ACCN88] for (ii). In its current form, the result (i) corresponds to a paper of Imbrie and Newman [IN88].

Using the coupling between FK percolation and Potts models (see e.g. [Gri06]), the previous theorem has the following corollary for the long-range 1D Ising and Potts models (see the beginning of the paper for the corresponding history and references). We do not define the models there and simply introduce the parameter $m(q,\beta,\lambda)$ corresponding to the magnetization of the Potts model.

Corollary 5. *Fix an integer $q \geq 2$,*

(i) *For $\beta > 1$, there exists $\lambda < \infty$ large enough so that $m(q,\beta,\lambda) > 0$.*
(ii) *For $\beta,\lambda > 0$, $m(q,\beta,\lambda) > 0$ implies $\beta m(q,\beta,\lambda)^2 \geq 1$.*

3.2 Proof of Theorem 4(i). The proof is very similar to the proof of Theorem 1(ii) and we simply explain how the proof is modified. Define

$$p_{q,\beta,\lambda}(K,\theta) := \max_{\xi \text{ b.c. on } \mathbb{Z}\setminus B_K} \mathbb{P}^{\xi}_{K,\beta,\lambda,q}[B_K \text{ is } \theta\text{-bad}]$$

(note that by monotonicity it is achieved for free boundary conditions $\xi = 0$ but we will not use this fact). In the proof of Lemma 2, all the deterministic observations are the same. Also, (5) can be obtained in the same way as before using the spatial Markov property (since $p_{q,\beta,\lambda}(K,\theta)$ is expressed in terms of the maximum over boundary conditions).

The only step that requires care is the proof of (8). Indeed, the first difference is that the states of the edges $\{x,y\}$ with $x \in \mathbf{C}^-$ and $y \in \mathbf{C}^+$ are not independent of the conditioning on $\mathbf{C}^-$ and $\mathbf{C}^+$, and second that the state of edges are not independent of each other. Still, we now show that the estimate (8) holds for every $q > 0$.

Condition on $\mathbf{C}^-$, $\mathbf{C}^+$, and every edge that is not linking $x \in \mathbf{C}^-$ and $y \in \mathbf{C}^+$. Consider the graph G composed of vertices in $\mathbf{C}^- \cup \mathbf{C}^+$ and edges between $x \in \mathbf{C}^-$ and $y \in \mathbf{C}^+$ and observe that the previous conditioning does not reveal the state of these edges. Let ξ be the boundary condition induced by this conditioning and let $\mathbb{P}^{\xi}_G$ be the associated FK percolation on G. Note that all the vertices in $\mathbf{C}^-$ (resp. $\mathbf{C}^+$) are wired together.

At this stage it is unclear whether the vertices in $\mathbf{C}^-$ are wired to those in $\mathbf{C}^+$ or not by ξ. If they are, then the probability that each edge $\{x,y\}$ is closed is $e^{-\beta J_{x,y}}$ independently of the other edges and the same computation as in the Bernoulli case holds true. Otherwise, consider an intermediate constant $R \in [1,C)$ and define A (resp. B) as the event that all the edges in G with $\{x,y\} \subset B_{RK}$ are closed (resp. all the remaining edges of G).

Notice that

$$\mathbb{P}^{\xi}_G[A \cap B] \leq \frac{\mathbb{P}^{\xi}_G[B|A^c]}{\mathbb{P}^{\xi}_G[A^c|B]}\,. \tag{24}$$

Let us analyze first the numerator $P^{\xi}_G[B|A^c]$. Thanks to the conditioning on A^c, we may work with products over edges of $e^{-\beta J_{x,y}}$ instead of the less convenient quantity

$$e^{-\beta J_{x,y}} \vee \frac{q\, e^{-\beta J_{x,y}}}{1 - e^{-\beta J_{x,y}} + q\, e^{-\beta J_{x,y}}} \tag{25}$$

which would arise from the probability $\frac{1-p}{p+(1-p)q}$ that an edge $e = \{x,y\}$ is closed knowing that its endpoints are not connected in $\omega_{|G\setminus e}$. This is the reason why we introduced the event A on the mesoscopic scale $RK \ll CK$. Recall that the argument assumes that the K-blocks $B^j_k, j \notin \{i-1,i,i+1\}$ are *θ-good* which implies that the intensity of $\mathbf{C}^-$ and $\mathbf{C}^+$ are at least θ on both sides. The above observation that the weight on each edge is $e^{-\beta J_{x,y}}$ knowing A^c now implies (provided R is chosen large enough, and then C even larger), by following the same sum/integral analysis as for the estimate (8), that

$$P_G^\xi[B|A^c] \le O(1)\exp\Big[-\beta\theta^2 \sum_{r,s=RK}^{CK} \frac{1}{(r+s-1)^2}\Big] \le O(1)\Big(\frac{R}{C}\Big)^{\beta\theta^2}. \tag{26}$$

Now, for the denominator $\mathbb{P}_G^\xi[A^c|B]$, start by noticing that if R is large enough and if $q \ge 1$, then the FK weight in (25) for any $e = \{x,y\}$ in G is at most

$$\begin{aligned}\frac{q\,e^{-\beta J_{x,y}}}{1-e^{-\beta J_{x,y}}+q\,e^{-\beta J_{x,y}}} &= e^{-\beta J_{x,y}}\frac{1}{1-\frac{q-1}{q}(1-e^{-\beta J_{x,y}})}\\ &\le e^{-\beta J_{x,y}}\Big(1+\tfrac{q-2/3}{q}(1-e^{-\beta J_{x,y}})\Big)\\ &\le e^{-\beta J_{x,y}}e^{\frac{q-1/2}{q}\beta J_{x,y}}\\ &= e^{-\frac{\beta}{2q}J_{x,y}}.\end{aligned}$$

If on the other hand $0 < q \le 1$, this is simpler as the weights in (25) are then smaller than $e^{-\beta J_{x,y}}$.

As such, modulo the same analysis as for estimate (8), this gives us an upper bound for $\mathbb{P}_G^\xi[A|B]$ of order

$$\begin{aligned}\mathbb{P}_G^\xi[A|B] &\le O(1)\exp\Big[-\beta\theta^2(1\wedge\frac{1}{2q})\sum_{r,s=K}^{RK}\frac{1}{(r+s-1)^2}\Big]\\ &\le O(1)\exp(-\Omega(1)\log R) \le \frac{1}{2},\end{aligned}$$

if R is chosen large enough. By plugging this estimate together with (26) into (24) and choosing $R = C^\delta$ with C large enough, this gives us

$$\mathbb{P}_G^\xi[A\cap B] \le O(1)C^{-\beta\theta^2(1-\delta)},$$

for any small exponent δ. This ends the proof of the analog of estimate (8) for FK percolation. (Note that we have written the proof for the central block B_K^0, but the same analysis would give an upper bound of $O(1)\,(C-|i|)^{-\beta\theta^2(1-\delta)}$ for the block B_k^i).

Remark 2. Notice that remarkably, the proof of Theorem 4(i) works for every $q > 0$.

3.3 Proof of Theorem 4(ii).

Define

$$\overline{p}_{q,\beta,\lambda}(K,\theta) := 1 - \max_{\substack{S\subset B_{3K}\\ \xi \text{ b.c. on } \mathbb{Z}\setminus B_{3K}}} \mathbb{P}^\xi_{S,\beta,\lambda,q}[B_{3K}\text{ is }K\text{-crossed}].$$

(the maximum is achieved for $S = B_{3K}$ and for wired boundary conditions $\xi = 1$ but we will not use this fact here). Then, the proof of Lemma 3 is the same except in two places. First, the states of edges in different unbridged boxes are not independent anymore so to derive (17), we replace independence by the spatial Markov property and the fact that $\overline{p}_{q,\beta,\lambda}(K,\theta)$ is defined as a minimum over boundary conditions.

The rest of the proof is the same (we can use the FKG inequality since $q \ge 1$), except in the proof of (21). There, we use that conditioned on the configuration outside of $\{x,y\}$, the probability that $\{x,y\}$ is open is smaller than $1-e^{-\beta J_{x,y}}$ (since $q \ge 1$), as well as the observation that we can pick R such that

$$\max_\xi \mathbb{P}^\xi_{B_R,\beta,\lambda,q}[0\leftrightarrow \mathbb{Z}\setminus B_R]^2 + |B_R|^2 e^{-\beta(K-2R)/q} \le \theta^2$$

since the maximum is reacher for $\xi = 1$ and that the quantity converges to $\theta(q, \beta, \lambda)$ as R tends to infinity.

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