

Sharp metastability transition for two-dimensional bootstrap percolation with symmetric isotropic threshold rules

Hugo Duminil-Copin^{1,2} and Ivailo Hartarsky³

¹University of Geneva

²Institut des Hautes Études Scientifiques

³TU Wien, Faculty of Mathematics and Geoinformation, Institute of Statistics and Mathematical Methods in Economics, Research Unit of Mathematical Stochastics, Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria, ivailo.hartarsky@tuwien.ac.at

March 27, 2023

Abstract

We study two-dimensional critical bootstrap percolation models. We establish that a class of these models including all isotropic threshold rules with a convex symmetric neighbourhood, undergoes a sharp metastability transition. This extends previous instances proved for several specific rules. The paper supersedes a draft by Alexander Holroyd and the first author from 2012. While it served a role in the subsequent development of bootstrap percolation universality, we have chosen to adopt a more contemporary viewpoint in its present form.

MSC2020: 60K35; 60C05

Keywords: bootstrap percolation, sharp threshold, metastability

1 Introduction

A threshold bootstrap percolation model is a simple cellular automaton that provides a useful model for studying several phenomena such as metastability, dynamics of glasses or crack formation. A famous example of a threshold model is the *2-neighbour bootstrap percolation* originally introduced by Chalupa, Leath and Reich [7] (also see [19]). In this model, sites of the square lattice \mathbb{Z}^2 are infected or healthy. At time 0, sites are infected with probability p independently of each other (we denote the corresponding measure by

\mathbb{P}_p). At each time step, a site becomes infected if two or more of its nearest neighbours are infected.

The first rigorous result on this model [21], dating back to 1987, established that every site of \mathbb{Z}^2 becomes infected almost surely whenever $p > 0$. This motivates the study of the (random) first time τ at which 0 becomes infected as p goes to 0. In [1], Aizenman and Lebowitz proved that there exist two constants $c, C \in (0, \infty)$ such that

$$\lim_{p \rightarrow 0} \mathbb{P}_p (e^{c/p} \leq \tau \leq e^{C/p}) = 1.$$

We refer to this article for an enlightening exposition of the metastability effects in the model. The question of whether c and C could be chosen arbitrary close to each other was left open for a long time. Finally, a sharp metastability transition was shown to occur in [17]: $p \log \tau$ converges in probability to $\pi^2/18$ as $p \rightarrow 0$. More precise estimates for T were derived later in [16].

Several authors investigated more general growth rules and the right order of magnitude for τ is now known for all rules [4], hence generalising the result of Aizenman and Lebowitz. The sharp metastability transition, though, remained available only for a handful of isolated examples [3, 8, 17, 18]. The goal of this paper is to prove sharp metastability for a wide class of models. In particular, we show that every isotropic symmetric convex threshold bootstrap percolation model exhibits a sharp transition.

1.1 \mathcal{U} -bootstrap percolation

Let $\mathbb{Z}^2 = \{x = (x_1, x_2) : x_1, x_2 \in \mathbb{Z}\}$ be the set of all 2-vectors of integers and $\mathbb{N} = \{0, 1, \dots\}$. Elements of \mathbb{Z}^2 are called *sites*. An *update rule* is any finite non-empty subset of $\mathbb{Z}^2 \setminus \{0\}$. An *update family* is a finite non-empty set of update rules. An update family \mathcal{U} is *symmetric*, if for every $U \in \mathcal{U}$ we have $-U = \{-x : x \in U\} \in \mathcal{U}$. Given an update family \mathcal{U} and a set $A = A_0 \subseteq \mathbb{Z}^2$ of *initial infections*, we recursively define

$$A_{t+1} = A_t \cup \{x \in \mathbb{Z}^2 : \exists U \in \mathcal{U}, \forall u \in U, x + u \in A_t\}$$

to be the set of sites *infected at time t* in the \mathcal{U} -bootstrap percolation process. The set $[A] = \bigcup_{t \geq 0} A_t$ of eventually infected sites is called the *closure* of A . A set $A \subseteq \mathbb{Z}^2$ is called *stable* if $[A] = A$. An observable of particular interest is the infection time of the origin

$$\tau = \inf\{t \in \mathbb{N} : 0 \in A_t\} \in \mathbb{N} \cup \{\infty\}.$$

We will systematically be interested in the asymptotics of τ when each site is initially infected independently with probability $p \rightarrow 0$. We denote the corresponding distribution of A by \mathbb{P}_p .

Among all update families, threshold rules initially received particular attention [9]. They are defined by a finite *neighbourhood* $\mathcal{K} \subset \mathbb{Z}^2$, containing 0, and a positive integer *threshold* θ . Then

$$\mathcal{U}(\mathcal{K}, \theta) = \{U \subseteq \mathcal{K} : |U| = \theta\}$$

is the associated update family. In other words, a site x becomes infected if at least θ of the sites in its neighbourhood $x + \mathcal{K}$ are already infected. A set $\mathcal{K} \subseteq \mathbb{R}^2$ is called

symmetric if $x \in \mathcal{K}$ implies $-x \in \mathcal{K}$ for all $x \in \mathbb{R}^2$. We say that a neighbourhood $\mathcal{K} \subset \mathbb{Z}^2$ is *convex symmetric*, if it is the intersection of a bounded convex symmetric subset of \mathbb{R}^2 with \mathbb{Z}^2 .

We will require a few definitions from the bootstrap percolation universality framework [4, 5, 9, 13]. A *direction* is a unit vector of \mathbb{R}^2 , viewed as an element of the unit circle S^1 . We denote the open half plane with outer normal u by $\mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle u, x \rangle < 0\}$ and its boundary by $l_u = \{x \in \mathbb{Z}^2 : \langle u, x \rangle = 0\}$. A direction $u \in S^1$ is called *stable*, if \mathbb{H}_u is stable. The direction is *unstable* otherwise, which can be reinterpreted as follows: there exists an update rule $U \subset \mathbb{H}_u$. In the case of a threshold rule, unstable directions u are those for which $|\mathbb{H}_u \cap \mathcal{K}| \geq \theta$.

A direction $u \in S^1$ is called *rational* if $\lambda u \in \mathbb{Z}^2 \setminus \{0\}$ for some $\lambda \in \mathbb{R}$. In this case, we denote $\rho_u = \min\{\rho > 0 : \exists x \in \mathbb{Z}^2, \langle u, x \rangle = \rho\}$. Then $u\mathbb{R} \cap \mathbb{Z}^2 = (u/\rho_u)\mathbb{Z}$. Thus, it will be convenient to define $u^\perp = (u_2, -u_1)/\rho_u$, so that $l_u = u^\perp\mathbb{Z}$. We further denote by $l_u(n) = \{x \in \mathbb{Z}^2 : \langle u, x \rangle = n\rho_u\}$ the n -th line perpendicular to u , so that $\mathbb{Z}^2 = \bigsqcup_{n \in \mathbb{Z}} l_u(n)$. Note that for any $n \in \mathbb{Z}$, $l_u(n)$ is a translate of l_u .

In the present work we will only consider models with finitely many stable directions. For an isolated stable or an unstable direction $u \in S^1$, we define its *difficulty*

$$\alpha(u) = \min\{|Z| : Z \subset \mathbb{Z}^2, |[\mathbb{H}_u \cup Z] \setminus \mathbb{H}_u| = \infty\} \in \mathbb{N}. \quad (1.1)$$

That is, the difficulty of u is the minimal number of infected sites needed in addition to the half-plane \mathbb{H}_u , so that infinitely many additional sites become infected. An update family is called *isotropic* if it has a finite but nonzero number of stable directions and each open semicircle of S^1 contains a stable direction of maximal difficulty. For isotropic models we call

$$\alpha = \max_{u \in S^1} \alpha(u)$$

the *difficulty* of the update family. A set Z realising the minimum in Eq. (1.1) is called a *helping set*. A helping set $Z \subset \mathbb{Z}^2$ for u is *voracious* if $[\mathbb{H}_u \cup Z] \supseteq l_u$. The update family is called *voracious* if all helping sets for all directions of difficulty α are voracious.

It was shown in [4] that for every isotropic update family, there exists $C > c > 0$ such that

$$\lim_{p \rightarrow 0} \mathbb{P}_p (e^{c/p^\alpha} < \tau < e^{C/p^\alpha}) = 1.$$

One can check that symmetric threshold models are isotropic if and only if the maximum $\iota(\mathcal{K}) = \max_{u \in S^1} |l_u \cap \mathcal{K}|$ is attained for at least two non-opposite directions u and $|\mathcal{K}| - \iota(\mathcal{K}) < 2\theta < |\mathcal{K}|$. In that case, the difficulty is given by $\alpha = \theta - (|\mathcal{K}| - \iota(\mathcal{K}))/2$ and the difficulty of a direction $u \in S^1$ is $\alpha(u) = \max(0, \theta - |l_u \cap \mathcal{K}|/2)$ (see [9]).

1.2 Main results

Our main result is the following.

Theorem 1.1. *For any symmetric voracious isotropic update family with difficulty α , there exists $\lambda \in (0, \infty)$ such that for all $\varepsilon > 0$,*

$$\lim_{p \rightarrow 0} \mathbb{P}_p (|p^\alpha \log \tau - \lambda| > \varepsilon) = 0.$$

The constant λ is identified as the solution of a variational problem, see Definition 3.8. We note that symmetry will only be used in Section 5, where the lower bound on τ is proved, but not for the upper one.

We further show that voracious models are rather ubiquitous.

Proposition 1.2. *Every isotropic threshold rule with a convex symmetric neighbourhood is voracious.*

In addition to convex symmetric threshold models, to the best of our knowledge, all commonly studied isotropic update families are voracious—the k -cross model, Froböse bootstrap percolation, modified and non-modified 2-neighbour bootstrap percolation. However, for these last examples Theorem 1.1 and more is already known [6, 16]. Therefore, the importance of our result stems from its universality.

It is known that beyond the class of isotropic models, asymptotic behaviours that differ from the one in Theorem 1.1 are displayed [2, 4, 5]. Nevertheless, the result should hold in yet greater generality—for balanced critical models (see [4] for the definition and a weaker result in this direction), but this remains beyond the reach of our techniques.

On the other hand, it should be noted that our techniques in conjunction with those of [12, 14] should lead to sharp threshold results like Theorem 1.1 with λ replaced by 2λ for symmetric voracious isotropic kinetically constrained models.

1.3 Organization of the paper

The rest of the paper is organised as follows. We begin by proving the combinatorial result of Proposition 1.2 in Section 2. We provide the setup for the proof of Theorem 1.1 in Section 3. In particular, we introduce the notions of traversability and droplets, and define the constant λ appearing in Theorem 1.1. Section 4 proves the upper bound of Theorem 1.1, while Section 5 proves the lower one.

2 Convex symmetric threshold rules

In this section, we establish Proposition 1.2 in order to better familiarise ourselves with helping sets. For the rest of the section, we fix a convex symmetric neighbourhood $\mathcal{K} \ni 0$ and threshold θ making the corresponding update family isotropic. We further fix a stable direction u . Thus,

$$\alpha(u) = \theta - |\mathcal{K} \setminus l_u|/2 = \theta - |\mathcal{K} \cap \mathbb{H}_u| > 0.$$

Since $\mathcal{K} \cap l_u \neq \emptyset$, u is necessarily rational.

Lemma 2.1. *We have $l_u(1) \cap \mathcal{K} \neq \emptyset$. Moreover, if $l_u(2) \cap \mathcal{K} \neq \emptyset$, then $|l_u(1) \cap \mathcal{K}| \geq \alpha(u)$.*

Proof. If $\mathcal{K} \subset l_u$, the model would not be isotropic, since all directions $v \in S^1 \setminus \{u, -u\}$ would be unstable, because $\theta < |\mathcal{K}|/2$. Let $x \in \mathcal{K} \cap l_u(n)$ for some $n \geq 2$. If such an x does not exist, by symmetry $\mathcal{K} \subset l_u \cup l_u(-1) \cup l_u(1)$ and we are done.

Since u is stable, $|\mathcal{K} \cap l_u| \geq 3$ by symmetry. Let $y = u^\perp(|\mathcal{K} \cap l_u| - 1)/2 \in \mathcal{K} \cap l_u$. Consider the isosceles triangle $T \subset \mathbb{R}^2$ with vertices $x, y, -y$. By convexity the lattice

sites in it are in \mathcal{K} . But its base length is $(|\mathcal{K} \cap l_u| - 1)/\rho_u$ and its height is $\langle x, u \rangle \geq 2\rho_u$. Therefore, the segment

$$\{t \in T : \langle t, u \rangle = \rho_u\}$$

has length at least $(|\mathcal{K} \cap l_u| - 1)/(2\rho_u) \geq \alpha(u)/\rho_u$. Since $l_u(1)$ is a translate of $(u^\perp)\mathbb{Z}$, it necessarily intersects this segment in $\alpha(u)$ points. \square

Proof of Proposition 1.2. Let H be a helping set for u . That is, a set with $|H| = \alpha(u) = \theta - |\mathcal{K} \setminus l_u|/2$ such that $|(H \cup \mathbb{H}_u) \setminus \mathbb{H}_u| = \infty$. By Lemma 2.1, we know that on the first step of the bootstrap percolation dynamics with initial condition $H \cup \mathbb{H}_u$, only sites in l_u become infected. Indeed, for $x \in l_u(n)$ with $n \geq 1$ we have

$$(x + \mathcal{K}) \cap (H \cup \mathbb{H}_u) \leq |H| + |\mathcal{K} \cap \mathbb{H}_u \setminus l_u(-1)| < \theta - |\mathcal{K} \setminus l_u|/2 + |\mathcal{K} \cap \mathbb{H}_u| = \theta.$$

Assume that $[H \cup \mathbb{H}_u] \setminus (H \cup \mathbb{H}_u) \subseteq l_u$. By symmetry, without loss of generality we may consider a site $y \in l_u \cap [H \cup \mathbb{H}_u]$ such that $\langle y, u^\perp \rangle > \max\langle h + k, u^\perp \rangle$ for all $h \in H$ and $k \in \mathcal{K}$. Further choose y such that no site $z \in l_u$ with $\langle z, u^\perp \rangle > \langle y, u^\perp \rangle$ is infected before y . Then there are at least $\alpha(u)$ infected sites in $y + \mathcal{K} \cap l_u$ before y becomes infected. But then on the next step there are also at least $\alpha(u)$ infected sites in $y + u^\perp + \mathcal{K} \cap l_u$ (including y). Proceeding by induction, we see that for any $m \in \mathbb{Z}$ the site $y + mu^\perp$ becomes infected at most $|m|$ steps after y , which concludes the proof of the voracity of u .

Assume, on the contrary, that some site outside l_u becomes infected. This entails $H \cap l_u = \emptyset$ since otherwise there are at most $\alpha(u) - 1 < \theta - |\mathcal{K} \cap \mathbb{H}_u|$ sites outside $\mathbb{H}_u \cup l_u$. We consider two cases.

Firstly, assume that $\mathcal{K} \subset l_u \cup l_u(-1) \cup l_u(1)$ and let $x \in l_u$ be a site infected on the first step. As in the calculation above we need to have $H \subseteq (x + \mathcal{K}) \setminus \mathbb{H}_u$, so $H \subset l_u(1)$. We claim that $x + u^\perp$ becomes infected on the second step or earlier. Indeed, $x \in x + u^\perp + \mathcal{K}$ and $\mathcal{K} \cap l_u(1)$ is a discrete interval, so $|(x + u^\perp + \mathcal{K}) \cap H| \geq |(x + \mathcal{K}) \cap H| - 1 = \alpha(u) - 1 = \theta - |\mathcal{K} \cap \mathbb{H}_u|$. Reasoning similarly by induction, we see that all sites in $(x + \mathcal{K}) \cap l_u$ become infected. However, they are enough to infect l_u on their own, since the first site in $y \in l_u$ outside $x + \mathcal{K}$ has at least $(|\mathcal{K} \cap l_u| - 1)/2 \geq \alpha(u)$ sites in $(x + \mathcal{K}) \cap (y + \mathcal{K})$, which we already established to be infected.

Secondly, assume that $\mathcal{K} \cap l_u(n) \neq \emptyset$ for some $n \geq 2$. Observe that by Lemma 2.1 this implies that $|\mathcal{K} \cap l_u(1)| \geq (|\mathcal{K} \cap l_u| - 1)/2 \geq \alpha(u)$. Consider the first site $x \notin l_u$ which becomes infected and let $m \geq 1$ be such that $x \in l_u(m)$. Then the number of infected sites in $x + \mathcal{K}$ just before x is infected is at most $|H| + |\mathcal{K} \cap \mathbb{H}_u| = \theta$. In order to infect x we need to have equality, so all sites in $(x + \mathcal{K}) \cap l_u(m - 1)$ are infected before x . By our choice of x this means that $m = 1$ and there are at least $\alpha(u)$ consecutive sites infected in l_u . As above, this is enough to infect all of l_u , concluding the proof. \square

3 Setup

3.1 Probabilistic tools

An event $E \subseteq \Omega = \{A : A \subset \mathbb{Z}^2\}$ is *increasing* if $A \in E$ and $A \subseteq A'$ imply $A' \in E$. Two important correlation inequalities related to increasing events will be used in the article.

The first one is the *Harris inequality* [11] stating that for two increasing events E, F ,

$$\mathbb{P}_p(E \cap F) \geq \mathbb{P}_p(E)\mathbb{P}_p(F). \quad (3.1)$$

The second one is the *BK inequality* [20]. For E and F two increasing events, their *disjoint occurrence* $E \circ F$ is defined as follows. A configuration $A \in \Omega$ belongs to $E \circ F$ if there exists a set $B \subseteq A$ such that $B \in E$ and $A \setminus B \in F$. For k increasing events E_1, \dots, E_k , one can define the disjoint occurrence by

$$E_1 \circ \dots \circ E_k = E_1 \circ (E_2 \circ \dots \circ (E_{k-1} \circ E_k)).$$

Then, for any increasing events E_1, \dots, E_k depending on a finite number of sites, the BK inequality reads

$$\mathbb{P}_p(E_1 \circ \dots \circ E_k) \leq \mathbb{P}_p(E_1) \cdots \mathbb{P}_p(E_k). \quad (3.2)$$

We refer the reader to the book [10] for proofs of these two classical inequalities.

3.2 The traversability functions h^u

For the remainder of the paper we fix an isotropic voracious update family \mathcal{U} with difficulty α . For a stable direction u , let \mathcal{H}^u denote the set of helping sets for u . It is known that there exists an integer constant R such that for any (isolated) stable direction u and any helping set $H \in \mathcal{H}^u$ there exists a translation vector $t \in u^\perp \mathbb{Z}$ such that $\max\{\|h\|_\infty : t + h \in H\} < R$ (see [15] for an explicit bound on R).

Definition 3.1 (Occupied lines). A line $l_u(n)$ orthogonal to u is *occupied* in $A \subseteq \mathbb{Z}^2$ if there exist $x \in l_u(n)$ and $H \in \mathcal{H}^u$ such that $x + H \subseteq A$.

This definition is an extension of the definition of occupied rows and columns for the simple bootstrap percolation, see [17].

We call a *rectangle* any translate of the set

$$R^u(m, n) = \{x \in \mathbb{Z}^2 : 0 \leq \langle x, u^\perp \rangle < m/\rho_u^2 \text{ and } 0 \leq \langle x, u \rangle < n\rho_u\}$$

for some $m, n \in \mathbb{N}$. Define the event

$$\mathcal{A}^u(m, n) = \bigcap_{j=0}^n \{l_u(j) \text{ is occupied in } A \cap R^u(m, n + R)\}.$$

Note that this event depends on the state of sites in $R^u(m, n + R)$.

The following proposition studies the behaviour of $\mathbb{P}_p[\mathcal{A}^u(m, n)]$. In particular, we prove that this probability can be expressed in terms of a family of functions h_p^u .

			13	...		
10	11	12			10	11
5	6	7	8	9	5	6
0	1	2	3	4	0	1

Figure 1: The translates of $R^u(2R, R)$ used in the proof of Proposition 3.2(2) in the case $R = 5$, $u = (0, 1)$, $m = 14R$. In each rectangle, we have indicated for which i it is used to occupy the line $l_u(i)$.

Proposition 3.2. *Let u be a stable direction. There exists a family of continuous non-increasing functions $(h_p^u)_{p \in (0,1)} : (0, \infty) \rightarrow (0, \infty)$ such that*

(1) (Link to \mathcal{A}^u) For any $p \in (0, 1)$, m sufficiently large, and $n > 0$,

$$\exp(-h_p^u(p^{\alpha(u)}m)(n+R)) \leq \mathbb{P}_p(\mathcal{A}^u(m, n)) \leq \exp(-h_p^u(p^{\alpha(u)}m)n). \quad (3.3)$$

(2) (Behaviour near 0 and ∞) There exist $p_0, c > 0$ such that for every $p < p_0$ and $x > p^{\alpha(u)}/c$,

$$-c \log(1 - e^{-x/c}) \leq h_p^u(x) \leq -\log(1 - e^{-cx}). \quad (3.4)$$

(3) (Uniform convergence) There exists an integrable function $h^u : (0, \infty) \rightarrow (0, \infty)$ such that, as $p \rightarrow 0$, h_p^u/h^u converges to 1 uniformly on (a, b) for every $a, b > 0$.

In simple cases, the functions h^u could be computed explicitly. The limit h^u corresponds to the functions f and g in [17] and functions g_k in [18]. However, in general, these functions are not explicit. Also note that if m and n are of order $p^{-\alpha(u)}$, then $-p^{\alpha(u)} \log \mathbb{P}_p(\mathcal{A}^u(m, n))$ remains of order 1 when p goes to 0. This is why $p^{-\alpha(u)}$ is the right scale to consider.

Proof of (1). The main ingredient to construct h_p^u is the sub- and super-multiplicativity. Fix m large enough for $\mathbb{P}_p(\mathcal{A}^u(m, n))$ to be non-degenerate for all $n > 0$ and $p \in (0, 1)$. Define $v_{p,m}(n) = \mathbb{P}_p(\mathcal{A}^u(m, n))$. The FKG inequality and the independence imply

$$v_{p,m}(n)v_{p,m}(n') \leq v_{p,m}(n+n') \leq v_{p,m}(n-R)v_{p,m}(n').$$

The sub-additivity lemma implies that there exists $\mu = \mu(u, p, m) \in (0, 1)$ such that $\mu^{n+R} \leq v_{p,m}(n) \leq \mu^n$ for every n . For any $m \in \mathbb{N}$, set $h_p^u(p^{\alpha(u)}m) = -\log \mu$. Extend h_p^u to all $(0, \infty)$ in a piecewise linear way. Note that h_p^u is non-increasing since $\mathcal{A}^u(m, n) \subseteq \mathcal{A}^u(m+1, n)$ for every $m \geq 0$. \square

Proof of (2). In order to upper bound h_p^u , it suffices to consider a particular way of occupying all lines of $R^u(m, n)$ for m large enough. Namely, there exists $c > 0$ such that we can fix a helping set $H \in \mathcal{H}^u$ and, for $0 \leq k < n$, a set $S_k \subset l_u(k)$ with $|S_k| \geq \lfloor m/(2R) \rfloor$

in the following way. We require that for all k and $x \in S_k$ the sets $x + H$ are disjoint and contained in $R^u(m, n + R)$. Indeed, to find such sets it suffices to divide most of the rectangle into disjoint translates of $R^u(2R, R)$ as in Fig. 1 and pick a translate of the helping set in each of the indicated line. Therefore, by independence, for some $c > 0$

$$\mathbb{P}_p(\mathcal{A}^u(m, n)) \geq \left(1 - (1 - p^{\alpha(u)})^{\lfloor m/(2R) \rfloor}\right)^n \geq \exp\left(-\log\left(1 - e^{-cmp^{\alpha(u)}}\right)n\right).$$

The right inequality of (3.4) follows readily for $x = p^{\alpha(u)}m \geq p^{\alpha(u)}/c$.

Turning to the lower bound in (3.4), note that, if $\mathcal{A}^u(m, n)$ occurs, every rectangle of the form $kR\mathbf{u} + R^u(m, R)$ contained in $R^u(m, n)$ must contain a translate of a helping set in \mathcal{H}^u . Since there are at most $(2R)^{2\alpha(u)}$ possibilities for the helping set up to translation, the Harris inequality gives

$$\mathbb{P}_p(\mathcal{A}^u(m, n)) \leq \prod_{j=1}^{\lfloor n/R \rfloor} \left(1 - (1 - p^{\alpha(u)})^{Cm}\right) \leq \left(1 - e^{-2Cmp^{\alpha(u)}}\right)^{\lfloor n/R \rfloor}$$

for an appropriately chosen constant $C > 0$. The left inequality of (3.4) follows by taking the logarithm. \square

Proof of (3). Fix $a < b$. Let us prove that h_p^u converges to some function h^u as $p \rightarrow 0$. The proof of (1) implies that

$$\frac{-\log \mathbb{P}_p(\mathcal{A}^u(xp^{-\alpha(u)}, n))}{n + R} \leq h_p^u(x) \leq \frac{-\log \mathbb{P}_p(\mathcal{A}^u(xp^{-\alpha(u)}, n))}{n},$$

interpolating $\log \mathbb{P}_p[\mathcal{A}^u(xp^{-\alpha(u)}, n)]$ linearly between $x \in p^{\alpha(u)}\mathbb{N}$. It is therefore sufficient to prove that for each fixed $n > 0$, $x \mapsto \mathbb{P}_p(\mathcal{A}^u(xp^{-\alpha(u)}, n))$ converges uniformly on $[a, b]$ as $p \rightarrow 0$ to a limit taking values in $(0, 1)$. The fact that the limit cannot be 0 or 1 and its integrability follow from (2). For any $E \subseteq \{0, \dots, n-1\}$, define $\mathcal{A}^u(m, n, E)$ to be the event that lines $l_u(i)$ for $i \in E$ are not occupied. Via the inclusion-exclusion principle, it is sufficient to show that $x \mapsto \mathbb{P}_p(\mathcal{A}^u(xp^{-\alpha(u)}, n, E))$ converges uniformly on $[a, b]$ for any fixed E .

Consider the rectangle $R^u(m, n)$ and partition it into (translates of) $R^u(k, n)$ (for simplicity, we assume that $k \geq 2R$ divides m). Now, shift the configuration $A \cap R^u(m, n)$ by adding u^\perp to it modulo mu^\perp . This ‘rotation’ can be applied m times. Observe that, if $\mathcal{A}^u(m, n, E)$ does not occur in the original configuration, then $\mathcal{A}^u(k, n, E)$ simultaneously occurs for all rectangles of the partition in at most a Cm/k out of the m possible circular shifts, where $C = C(u, n, E) > 0$ is a constant independent of k . Indeed, a helping set may split across the boundary between two parts of the original rectangle, but is otherwise present in one of the parts. Since the circular shift is measure-preserving, we get that for all n large enough

$$1 \leq \frac{1 - \mathbb{P}_p(\mathcal{A}^u(xp^{-\alpha(u)}, n, E))}{1 - (\mathbb{P}_p(\mathcal{A}^u(k, n, E)))^{xp^{-\alpha(u)}/k}} \leq 1 + \frac{C}{k}.$$

Now, $\mathbb{P}_p(\mathcal{A}^u(k, n, E)) = 1 - C'p^{\alpha(u)} + O(p^{\alpha(u)+1})$, where $C' = C'(u, k, n, E)$ is the number of possible positions of translates of a helping set violating the event. When p goes to 0, this leads to

$$1 - (\mathbb{P}_p(\mathcal{A}^u(k, n, E)))^{xp^{-\alpha(u)/k}} \rightarrow 1 - e^{-xC'/k}.$$

The further quasi-additivity

$$|C'(u, k_1 + k_2, n, E) - C'(u, k_1, n, E) - C'(u, k_2, n, E)| \leq nC''(u)$$

entails that C'/k has a limit $\kappa = \kappa(u, n, E) \in (0, \infty)$ as $k \rightarrow \infty$. Therefore,

$$\mathbb{P}_p(\mathcal{A}^u(xp^{-\alpha(u)}, n, E)) \rightarrow e^{-x\kappa}. \quad \square$$

While the event $\mathcal{A}^u(m, n)$ enjoys good approximate additivity properties, it will be more convenient to work with a slightly more artificial version of it following [12]. To introduce it we will need a few more notions.

One can show [5, Lemma 5.2] that there exists a constant W such that for any $u \in \mathcal{S}$ we have that $\mathbb{H}_u \cup (u^\perp \{1, \dots, W\})$ infects both 0 and $(W+1)u^\perp$ on the first step of the bootstrap percolation dynamics. We will call such a set of W consecutive infections a W -helping set for l_u (and similarly for $l_u(n)$ for $n \neq 0$).

By the fact that there are finitely many stable directions and a finite number of helping sets up to translation, compactness allows us to choose a sufficiently large integer constant $C > 0$ so that the following holds. For any stable direction u and $H \in \mathcal{H}^u$, the bootstrap percolation dynamics with initial condition $H \cup \mathbb{H}_u$ produces a W -helping set for l_u in less than $C \min_{u \in \mathcal{S}} \rho_u / \max_{U \in \mathcal{U}} \max_{x \in U} (2\|x\|)$ steps.

Definition 3.3 (Traversability). Let u be a stable direction, $m > 2C$ and $n > R$. We say that $R^u(m, n)$ is *traversable* in A if the event $\mathcal{A}^u(m - 2C, n - R)$ shifted by Cu^\perp occurs and for all $i \in \{1, \dots, R\}$ there is a W -helping set in $A \cap l_u(n - i) \cap R^u(m, n)$. Let $\mathcal{T}(R^u(m, n))$ be the corresponding event.

If $n \leq R$, we extend the definition by requiring W -helping sets on each line.

In words, we require helping sets to be far from the boundary of the rectangle and further ask for W -helping sets on the last few lines in order not to look at the configuration outside the rectangle. The Harris inequality and Proposition 3.2 yield the following.

Corollary 3.4 (Traversability probability). *For any stable u and m, n large enough*

$$p^{WR} \exp(-h_p^u(p^{\alpha(u)}(m - 2C))n) \leq \mathbb{P}_p(\mathcal{T}(R^u(m, n))) \leq \exp(-h_p^u(p^{\alpha(u)}m)(n - R)).$$

3.3 Droplets

We will need to consider a particular set of directions related to the update family known as *quasi-stable directions* [5]. Namely, let

$$\mathcal{S} = \{u \in S^1 : \exists U \in \mathcal{U}, \exists x \in U : \langle x, u \rangle = 0\}.$$

Note that quasi-stable directions are necessarily rational. We index them $u_1, \dots, u_{|\mathcal{S}|}$ in counterclockwise order and indices are considered modulo $|\mathcal{S}|$. Since we will often consider sequences of numbers indexed by \mathcal{S} , we denote by \mathbf{e}_u the canonical basis of $\mathbb{R}^{\mathcal{S}}$ and use bold letters for vectors in this space. Of particular importance to us will be the set

$$\mathcal{S}_\alpha = \{u \in \mathcal{S} : \alpha(u) = \alpha\} \subseteq \mathcal{S}$$

of stable directions of maximal difficulty. As it will be convenient to work with continuous regions, we further set

$$\mathbb{H}_u(a) = \{x \in \mathbb{R}^2 : \langle x, u \rangle < a\rho_u\}.$$

However, whenever referring to the bootstrap percolation process with an initial condition contained in \mathbb{R}^2 , we will mean its intersection with \mathbb{Z}^2 .

Definition 3.5 (Droplet). A *droplet* D is a non-empty set of the form $D = D[\mathbf{a}] = \bigcap_{u \in \mathcal{S}} \mathbb{H}_u(a_u)$ where $\mathbf{a} \in \mathbb{R}^{\mathcal{S}}$ (see Fig. 2). The *radii* \mathbf{a} are uniquely defined, once we assume that \mathbf{a} is the coordinatewise minimal one whose associated droplet is $D[\mathbf{a}]$. We similarly define \mathcal{S}_α -droplets, replacing \mathcal{S} by \mathcal{S}_α and similarly for all subsequent notions involving droplets.

For $u \in \mathcal{S}$, define the *edge* $E_u(D[\mathbf{a}]) = \{x \in \mathbb{R}^2 : \langle x, u \rangle = a_u, \forall v \in \mathcal{S} \setminus \{u\}, \langle x, v \rangle < a_v\}$. Note that $E_u(D) \cap D = \emptyset$. The *dimension* $\mathbf{m} \in [0, \infty)^{\mathcal{S}}$ of $D[\mathbf{a}]$ is given by $m_u = |E_u(D)|/\rho_u$ for every $u \in \mathcal{S}$, where $|E_u(D)|$ is the Euclidean length of the edge. The *perimeter* $\Phi(D)$ of $D[\mathbf{a}]$ is defined as

$$\Phi(D) = \sum_{u \in \mathcal{S}} m_u.$$

We will require a notion of “circular” droplet. For $k \in [0, \infty)$, let $D[k]$ be the symmetric droplet with dimension (k, \dots, k) . The existence of $D[k]$ is fairly elementary. Indeed, set $x_1 = 0$ and $x_{i+1} = x_i - ku_i^\perp$. Since \mathcal{S} is symmetric, we obtain $x_{|\mathcal{S}|+1} = x_0$ and $D[k]$ is constructed as the polygon with vertices $(x_i)_{i=1}^{|\mathcal{S}|}$ translated appropriately.

The *location* of $D_1[\mathbf{a}] \subseteq D_2[\mathbf{b}]$ is given by $\mathbf{s} = \mathbf{b} - \mathbf{a} \in [0, \infty)^{\mathcal{S}}$. The *total location* $\Psi(D_1, D_2)$ is defined by

$$\Psi(D_1, D_2) = \sum_{u \in \mathcal{S}} s_u.$$

Note that $\Psi(D_1, D_2)$ does not depend on the positions of D_1 and D_2 , but just on their shapes.

Not every $\mathbf{m} \in \mathbb{R}^{\mathcal{S}}$ necessarily corresponds to the dimension of a droplet. Yet it is easy to verify that the condition is additive in the following way: if \mathbf{m} and \mathbf{m}' are the dimensions of two droplets D and D' , then there exists a droplet with dimensions $\mathbf{m} + \mathbf{m}'$. In fact it is given by the Minkowski sum of the droplets

$$D[\mathbf{a}] + D[\mathbf{b}] := D[\mathbf{a} + \mathbf{b}] = \{x + y : x \in D[\mathbf{a}], y \in D[\mathbf{b}]\}. \quad (3.5)$$

For any $z \in \mathbb{R}$ and droplet D we denote $D^z = D + D[z]$. Equation (3.5) immediately entails the following important property of sums that will be used frequently.

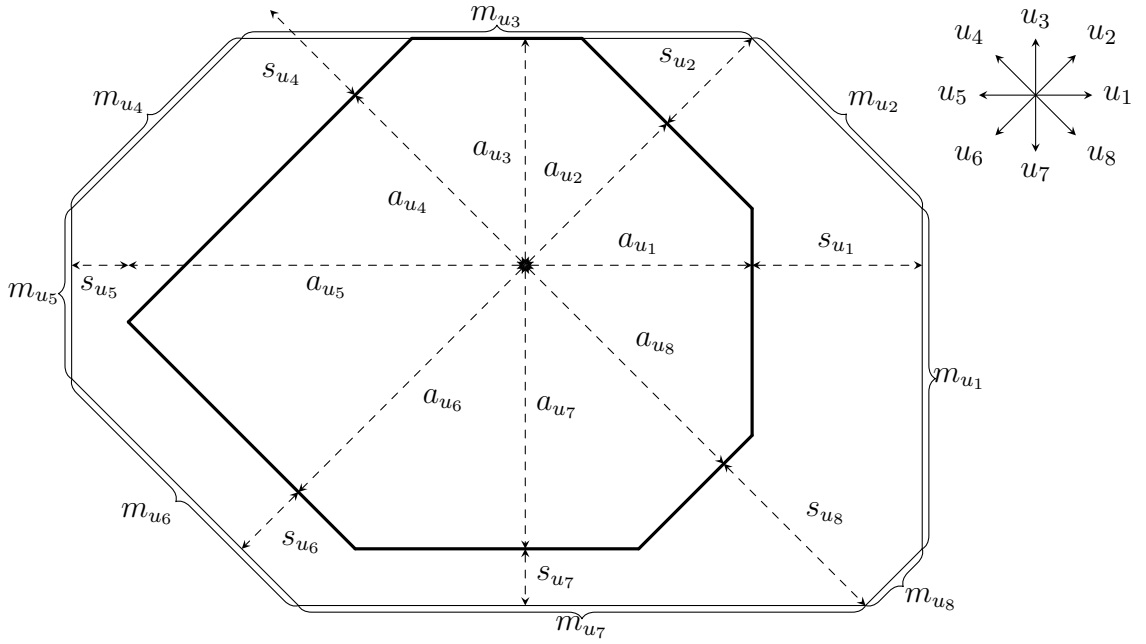


Figure 2: An example of two droplets $D[\mathbf{a}] \subseteq D[\mathbf{b}]$ with $|\mathcal{S}| = 8$. The radii $\mathbf{a} \in \mathbb{R}^{\mathcal{S}}$, the location $\mathbf{s} = \mathbf{b} - \mathbf{a}$ and the dimension \mathbf{m} of $D[\mathbf{b}]$ are indicated. Note that s_{u_3} is not drawn, since it is 0 in this instance. Further note that a_u and s_u are measured in units of ρ_u , while m_u is measured in units of $1/\rho_u$ for every $u \in \mathcal{S}$.

Observation 3.6. Let $D_1 \subseteq D_2$ and D be droplets. The location of $D_1 + D \subseteq D_2 + D$ is equal to the one of $D_1 \subseteq D_2$.

We will require a further operation on droplets.

Definition 3.7 (Span of droplets). The *span* of droplets D_1, \dots, D_k denoted by $D_1 \vee \dots \vee D_k$ is the smallest droplet containing $\bigcup_{i=1}^k D_i$.

The following important property follows directly from Definition 3.7 and Eq. (3.5): one has that $D[\mathbf{a}_1] \vee \dots \vee D[\mathbf{a}_k] = D[\mathbf{a}^{(1)} \vee \dots \vee \mathbf{a}^{(k)}]$ with $\mathbf{a}^{(1)} \vee \dots \vee \mathbf{a}^{(k)} = (\max_{i=1}^k a_u^{(i)})_{u \in \mathcal{S}}$.

3.4 The sharp threshold constant λ

We are now in position to define a functional depending on two droplets, which will quantify the cost of the smaller one growing to become the larger one.

Definition 3.8. For two droplets $D \subseteq D'$ such that the location of D in D' is \mathbf{s} and the

dimension of D is \mathbf{m} , let¹

$$W_p(D, D') = p^\alpha \sum_{u \in \mathcal{S}_\alpha} h_p^u(p^\alpha m_u) s_u,$$

$$W(D, D') = \sum_{u \in \mathcal{S}_\alpha} h^u(m_u) s_u,$$

where h_p^u and h^u are defined in Proposition 3.2. Let \mathfrak{D} is the set of bi-infinite non-decreasing (for inclusion) sequences of droplets $(D_n)_{n \in \mathbb{Z}}$ such that $\bigcap_{n \in \mathbb{Z}} D_n = \{0\}$ and $\bigcup_{n \in \mathbb{Z}} D_n = \mathbb{R}^2$. For a sequence $\mathcal{D} = (D_n)_{n \in \mathbb{Z}} \in \mathfrak{D}$, set

$$\mathcal{W}(\mathcal{D}) = \frac{1}{2} \sum_{n \in \mathbb{Z}} W(D_n, D_{n+1}).$$

Finally, the sharp threshold constant is given by

$$\lambda = \inf_{\mathcal{D} \in \mathfrak{D}} \mathcal{W}(\mathcal{D}).$$

We analogously define \mathfrak{D}_α for \mathcal{S}_α -droplets and set $\lambda_\alpha = \inf_{\mathcal{D} \in \mathfrak{D}_\alpha} \mathcal{W}(\mathcal{D})$.

Let us emphasise that even though droplets are defined with respect to \mathcal{S} , only directions in \mathcal{S}_α are featured in W_p and W . As we will see, this will entail that $\lambda_\alpha = \lambda$.

The definition of λ as the minimizer of some energy is reminiscent of a metastability phenomenon. Since the creation of a droplet of critical size is very unlikely, the procedure to create it tends to minimize the energy. Here, the energy takes the special form of a work along a certain sequence of droplets. The sequence along which the work is minimized is therefore related to the typical shape of a critical droplet.

Proposition 3.9. *The constant λ belongs to $(0, \infty)$.*

Proof. Let us first show that $\lambda > 0$. Observe that $\max_{u \in \mathcal{S}} m_u \leq c \max_{u \in \mathcal{S}_\alpha} a_u$ for some constant $c > 0$, since there are directions of difficulty α in every semicircle. Consider a sequence of droplets $D_n = D[\mathbf{a}^{(n)}]$ as in Definition 3.8. Let n_0 be the smallest integer such that $\max_{u \in \mathcal{S}_\alpha} a_u^{(n_0)} \geq B$ for some fixed constant $B > 0$ and let $u_0 \in \mathcal{S}_\alpha$ be such that $a_{u_0}^{(n_0)} = \max_{\mathcal{S}_\alpha} a_u^{(n_0)}$. Then

$$\sum_{n=-\infty}^{n_0-1} W(D_n, D_{n+1}) \geq h^{u_0}(m_{u_0}^{(n_0-1)}) \sum_{n=-\infty}^{n_0-1} s_{u_0}^{(n)} = h^{u_0}(m_{u_0}^{(n_0-1)}) a_{u_0}^{(n_0)} \geq h^{u_0}(cB)B > 0,$$

since h^{u_0} is non-increasing and positive by Proposition 3.2.

Turning to $\lambda < \infty$, consider the sequence $\mathcal{D} = (D[2^n])_{n \in \mathbb{Z}}$ and let $D[1] = D[\mathbf{a}]$. For some constant $c > 0$, its energy is given by

$$\begin{aligned} \mathcal{W}(\mathcal{D}) &= \sum_{n \in \mathbb{Z}} W(D[2^n], D[2^{n+1}]) = \sum_{u \in \mathcal{S}_\alpha} \sum_{n \in \mathbb{Z}} h^u(2^n) 2^n a_u \\ &\leq \frac{-1}{c} \sum_{n \in \mathbb{Z}} \log(1 - e^{-c2^n}) 2^n < \infty, \end{aligned} \tag{3.6}$$

using Proposition 3.2(2). □

¹Here we make the convention that $h^u(0) = h_p^u(0) = \infty$, but if $m_u = s_u = 0$, then $h^u(m_u) s_u = h_p^u(p^\alpha m_u) s_u = 0$ for any $u \in \mathcal{S}$ and $p \in (0, 1)$.

Proposition 3.10. *We have $\lambda = \lambda_\alpha$.*

Proof. Considering \mathcal{S}_α -droplets as degenerate droplets, it is clear that $\lambda \leq \lambda_\alpha$, so it remains to prove the reverse inequality. Fix $\varepsilon > 0$ and let $\mathcal{D} = (D_n)_{n \in \mathbb{Z}} \in \mathfrak{D}$ be such that $\mathcal{W}(\mathcal{D}) \leq \lambda + \varepsilon$. For each $n \in \mathbb{Z}$, let D'_n be the smallest \mathcal{S}_α -droplet containing D_n . Observe that for each $n \in \mathbb{Z}$ and $u \in \mathcal{S}_\alpha$ we have $m_u^{(n)} \leq m'_u{}^{(n)}$ and $s'_u{}^{(n)} = s_u^{(n)}$, since $\mathcal{S} \supseteq \mathcal{S}_\alpha$, where $\mathbf{m}^{(n)}$ is the dimension of D_n and $\mathbf{s}^{(n)}$ is the location of D_n in D_{n+1} and similarly for $\mathbf{m}'^{(n)}$ and $\mathbf{s}'^{(n)}$. Therefore, setting $\mathcal{D}' = (D'_n)_{n \in \mathbb{Z}}$, we get $\mathcal{W}(\mathcal{D}') \leq \mathcal{W}(\mathcal{D}) = \lambda + \varepsilon$, since the functions h^u are non-increasing. Thus, it remains to check that $\mathcal{D}' \in \mathfrak{D}_\alpha$. But this is clear: $D'_n \supseteq D_n \rightarrow \mathbb{R}^2$ as $n \rightarrow \infty$ and $D'_n \rightarrow \{0\}$ as $n \rightarrow -\infty$ since the same holds for D_n . Hence, $\lambda_\alpha \leq \mathcal{W}(\mathcal{D}') \leq \lambda + \varepsilon$ for any $\varepsilon > 0$ and we are done. \square

3.5 Constants

In the subsequent sections we will require a number of large and small quantities that will depend on each other. In order to simplify statements and for convenience, we gather them here. We will assume that

$$1 \ll C, K \ll \frac{1}{\varepsilon} \ll G \ll B \ll L \ll \frac{1}{Z} \ll \frac{1}{T} \ll \frac{1}{p}.$$

That is to say, C and K are positive numbers chosen large enough, ε is positive small enough depending on C and K , G is positive chosen large enough depending on C , K and ε and so on. Moreover, all these constants are allowed to depend on \mathcal{U} , α , \mathcal{S} , \mathcal{S}_α , as well as c, W, R appearing in Section 3.2 and λ from Section 3.4. When constants are introduced more locally, they may also depend on \mathcal{U} , α , \mathcal{S} , \mathcal{S}_α , W and R , but not on the quantities above, unless otherwise stated.

4 Proof of the upper bound

In this section we focus on the upper bound in Theorem 1.1. Thus, we aim to exhibit a mechanism for infecting large droplets and estimate its probability.

4.1 Lower bound on growth using the functional

For droplets $D_1 \subseteq D_2$, define $\mathcal{I}(D_1, D_2) = \{[(A \cap D_2) \cup D_1] \supseteq D_2\}$ to be the event that D_1 plus the infections present in D_2 are enough to infect D_2 . We now bound the probability of $\mathcal{I}(D_1, D_2)$.

Proposition 4.1. *For any droplets $D_1 \subseteq D_2 \subseteq D[Bp^{-\alpha}]$ satisfying $\Psi(D_1, D_2) \leq Tp^{-\alpha}$, we have*

$$\mathbb{P}_p \left(\mathcal{I} \left(D_1^{Zp^{-\alpha}}, D_2^{Zp^{-\alpha}} \right) \right) \geq p^{-C} \exp \left(-(1 + \varepsilon) \frac{W_p \left(D_1^{Zp^{-\alpha}}, D_2^{Zp^{-\alpha}} \right)}{p^\alpha} \right), \quad (4.1)$$

Proof. Consider two droplets $D_1^{Zp^{-\alpha}} = D[\mathbf{a}] \subseteq D_2^{Zp^{-\alpha}} = D[\mathbf{b}]$ as in the statement. Let $\mathbf{s} = \mathbf{b} - \mathbf{a}$ be the location. We will use the infection mechanism illustrated in Fig. 3. Fix $u \in \mathcal{S}$ and let R^u be the translate of the largest rectangle $R^u(\tilde{m}_u, s_u)$ such that $R^u \subseteq D[\mathbf{a} + \mathbf{e}_u s_u] \setminus D[\mathbf{a}]$, which is the droplet $D_1^{Zp^{-\alpha}}$ extended so that its u -edge is contained in the one of $D_2^{Zp^{-\alpha}}$, while the others contain the corresponding edges of $D_1^{Zp^{-\alpha}}$. Note that

$$m_u \geq \tilde{m}_u \geq m_u - Cs_u \geq m_u - CTp^{-\alpha} \geq m_u(1 - Z), \quad (4.2)$$

where $m_u \geq Zp^{-\alpha}$ is the u -dimension of $D_1^{Zp^{-\alpha}}$, using that $1/T \gg 1/Z \gg C$.

Recalling Definition 3.3, consider the event

$$\mathcal{E} = \bigcap_{u \in \mathcal{S}} \mathcal{T}(R^u).$$

Our first goal is to show that $\mathcal{E} \subseteq \mathcal{I}(D_1^{Zp^{-\alpha}}, D_2^{Zp^{-\alpha}})$. Let us first check that \mathcal{E} implies that $[D_1^{Zp^{-\alpha}} \cup A \cap D_2^{Zp^{-\alpha}}] \supseteq D[\mathbf{a} + \mathbf{e}_u]$ for any $u \in \mathcal{S}$ such that $s_u \geq 1$. Indeed, by our choice of the constant C in Definition 3.3 of traversability, if \mathcal{T} requires a helping set for $l_u(a_u)$, it is contained in $D_2^{Zp^{-\alpha}}$ and it produces a W -helping set in $D[\mathbf{a} + \mathbf{e}_u] \setminus D[\mathbf{a}]$, before seeing the difference between droplet $D_1^{Zp^{-\alpha}}$ and the boundary condition $\mathbb{H}_u(a_u)$. If \mathcal{T} does not require a helping set for $l_u(a_u)$, then traversability asks directly for the W -helping set to be present. In either case, we obtain a W -helping set. However, it is known [5, Lemma 5.4] that this is sufficient to infect $D[\mathbf{a} + \mathbf{e}_u]$ only using $D_1^{Zp^{-\alpha}}$ and this W -helping set. Proceeding by induction on $\Psi(D_1, D_2)$, we obtain the desired conclusion that $\mathcal{E} \subseteq \mathcal{I}(D_1^{Zp^{-\alpha}}, D_2^{Zp^{-\alpha}})$.

Thus, it remains to bound the probability of \mathcal{E} . Corollary 3.4 gives

$$\mathbb{P}_p \left(\mathcal{I} \left(D_1^{Zp^{-\alpha}}, D_2^{Zp^{-\alpha}} \right) \right) \geq \mathbb{P}_p(\mathcal{E}) \geq \prod_{u \in \mathcal{S}} \left(p^{WR} \exp \left(-h_p^u \left(p^{\alpha(u)} (\tilde{m}_u - 2C) \right) s_u \right) \right).$$

Yet, Proposition 3.2 and Eq. (4.2) give

$$h_p^u \left(p^{\alpha(u)} (\tilde{m}_u - 2C) \right) \leq \begin{cases} (1 + \varepsilon) h^u(p^\alpha m_u) & u \in \mathcal{S}_\alpha \\ \exp(-p^{-1/2}) & u \in \mathcal{S} \setminus \mathcal{S}_\alpha, \end{cases}$$

since $p \ll Z \ll \varepsilon$. Putting these bounds together with Definition 3.8, we obtain the desired result. \square

4.2 Proof of the upper bound of Theorem 1.1

We say that a droplet is *D internally filled* if $[D \cap A] \supset D$ and denote the corresponding event by $\mathcal{I}(D)$. Our next goal is to prove the upper bound of our main result. To that end, we prove lower bounds on the probability of internal filling progressively larger droplets. We start by proving that a small droplet is created with fairly good probability.

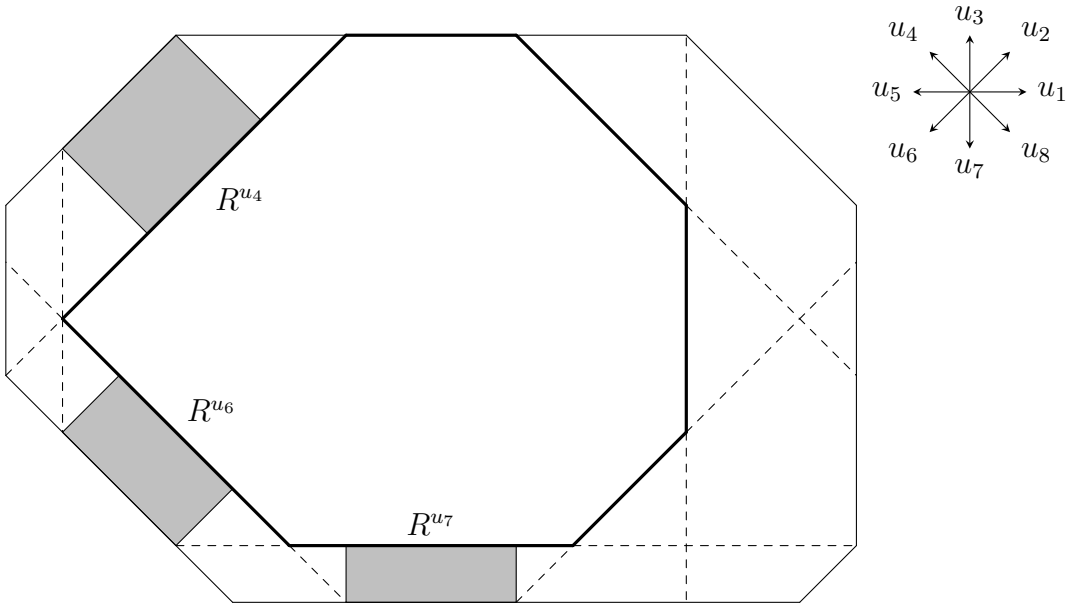


Figure 3: The rectangles R^u used in the proof of Propositions 4.1 and 5.3 for the droplets from Fig. 2. On the picture only 3 of them are non-empty, as it can be seen thanks to the dashed lines. However, in Proposition 4.1 it is not possible for any of them to be empty, since the total location is much smaller than the smallest dimension.

Lemma 4.2 (Subcritical growth). *We have*

$$\mathbb{P}_p(\mathcal{I}(D[1/(Bp^\alpha)])) \geq \exp(-\varepsilon p^{-\alpha}).$$

Proof. To see this, we will proceed similarly to the proof of Proposition 4.1. Fix a constant $c > 1$ close enough to 1. Consider the sequence of droplets $D_n = D[Cc^n]$ for $n \in \{0, \dots, N\}$, where $Cc^N = 1/(Bp^\alpha)$. In order for the final one to be internally filled, it suffices for the first one to be fully infected and all events $\mathcal{I}(D_i, D_{i+1})$ to occur. As in the proof of Proposition 4.1, in order to guarantee the latter, it suffices for suitable translates of the rectangles $R^u(c^i, (c^{i+1} - c^i)a_u)$ to be traversable, where $D_0 = D[\mathbf{a}]$.

Therefore, the independence of these events, Corollary 3.4 and Proposition 3.2 give

$$\begin{aligned} \mathbb{P}_p[\mathcal{I}(D[1/(Bp^\alpha)])] &\geq p^{C^3 + NWR|\mathcal{S}|} \prod_{i=0}^{N-1} \prod_{u \in \mathcal{S}} \exp\left(\left(c^{i+1} - c^i\right) a_u \log\left(1 - e^{-p^\alpha c^i}\right)\right) \\ &\geq e^{C \log^2(1/p)} \exp\left(\sum_{i=0}^{N-1} \sum_{u \in \mathcal{S}} C c^i \log\left(1 - e^{-p^\alpha c^i}\right)\right). \end{aligned}$$

The terms corresponding to $u \in \mathcal{S} \setminus \mathcal{S}_\alpha$ contribute a negligible factor $\exp[-C^2 p^{-\alpha(u)}]$. On the other hand, terms with $u \in \mathcal{S}_\alpha$ can be bounded by

$$\exp\left(-p^{-\alpha} \varepsilon / (2|\mathcal{S}_\alpha|)\right),$$

since B is large enough depending on C and ε . Putting these bounds together, we obtain the desired result. \square

Before we turn to ‘critical’ droplet sizes, which is the most important scale, we will need a truncation and refinement statement for the threshold constant λ from Definition 3.8.

Lemma 4.3. *There exists a sequence of droplets $(D_n)_{n \leq N}$ such that:*

- $D_0^Z \subseteq D[1/B]$,
- $D[B] \subseteq D_N^Z \subseteq D[L/2]$,
- $\Psi(D_n^Z, D_{n+1}^Z) \leq T$ for every $0 \leq n \leq N - 1$,
- $\sum_{n=0}^{N-1} W(D_n^Z, D_{n+1}^Z) \leq 2\lambda + \varepsilon$.

Proof. In order to deduce the existence of $(D_n)_{n \leq N}$ from Definition 3.8, we proceed as follows. We start with a sequence $\mathcal{D} \in \mathfrak{D}$ such that $\mathcal{W}(\mathcal{D}) \leq \lambda + \varepsilon/3$, so that \mathcal{D} does not depend on B , but only on ε . Note that along this sequence if there is a dimension $m_u^{(n)} = 0$ for $u \in \mathcal{S}_\alpha$, then $a_u^{(n+1)} - a_u^{(n)} = 0$, since otherwise $\mathcal{W}(\mathcal{D})$ would be infinite.¹ We truncate and index the sequence so that its first term is $D_0 \subseteq D[1/(2B)]$ and its last one is $D_N \supseteq D[B]$. Since L can be chosen large enough depending on \mathcal{D} , we can ensure that $D_N^Z \subseteq D[L/2]$ and that $m_u^{(i)} \geq 1/L$ for all $u \in \mathcal{S}_\alpha$ and $i \in \{0, \dots, N - 1\}$ such that $a_u^{(n+1)} - a_u^{(n)} \neq \emptyset$. Note that since $Z < 1/L$, we have

$$0 \leq \sum_{n=0}^{N-1} (W(D_n, D_{n+1}) - W(D_n^Z, D_{n+1}^Z)) \leq \omega(Z) |\mathcal{S}_\alpha| \max_{u \in \mathcal{S}} a_u L,$$

where ω is the maximum of the moduli of continuity of all h^u over the compact set $[1/L, L]$ and $D[1] = D[\mathbf{a}]$. The right-hand side above goes to 0 uniformly in the choice of the sequence as $Z \rightarrow 0$ with L fixed.

It therefore remains to show that we can refine the sequence in order to have $\Psi(D_n, D_{n+1}) \leq T$ (recall Observation 3.6). Let $D_n = D[\mathbf{a}^{(n)}]$ and $D_{n+1} = D[\mathbf{a}^{(n+1)}]$. We create a sequence of intermediate droplets from D_n to D_{n+1} as follows. At each step, let u be an arbitrarily chosen direction such that the quantity m_u for the current droplet is larger than $m_u^{(n)}$. Increase the radius a_u of the current droplet by $\min(T, a_u^{(n+1)} - a_u)$. In doing this, it is clear that $h^u(m_u) \leq h^u(m_u^{(n)})$ for all m_u appearing in the energy \mathcal{W} of this sequence. Thus, the existence of the sequence claimed is established. \square

Equipped with Lemma 4.3, we are ready to prove a bound on the critical growth probability.

Proposition 4.4 (Critical growth). *There exists a droplet $D[Bp^{-\alpha}] \subseteq D_p \subseteq D[Lp^{-\alpha}]$ with*

$$\mathbb{P}_p(\mathcal{I}(D_p)) \geq \exp(-(2\lambda + C\varepsilon)/p^\alpha).$$

Proof. We first construct rescaled droplets $(D_n^Z)_p = D_n^{Zp^{-\alpha}}[\mathbf{a}^{(n)}p^{-\alpha}]$ with $\lfloor \mathbf{a}^{(n)}p^{-\alpha} \rfloor = (\lfloor a_u^{(n)}p^{-\alpha} \rfloor)_{u \in \mathcal{S}}$ and $\mathbf{a}^{(n)}$ are the radii of D_n provided by Lemma 4.3. We obtain

$$\begin{aligned} \mathbb{P}_p \left(\mathcal{I} \left((D_N^Z)_p \right) \right) &\geq \mathbb{P}_p \left(\mathcal{I} \left(D \left[1/(Bp^\alpha) \right] \right) \right) \prod_{n=0}^{N-1} \mathbb{P}_p \left(\mathcal{I} \left((D_n^Z)_p, (D_{n+1}^Z)_p \right) \right) \\ &\geq \exp(-\varepsilon/p^\alpha) \prod_{n=0}^{N-1} p^{-C} \exp \left(-(1+\varepsilon) \frac{W_p((D_n^Z)_p, (D_{n+1}^Z)_p)}{p^\alpha} \right) \\ &\geq \exp(-2\varepsilon/p^\alpha) \prod_{n=0}^{N-1} \exp \left(-(1+\varepsilon)^2 \frac{W(D_n^Z, D_{n+1}^Z)}{p^\alpha} \right) \\ &\geq \exp \left(-\frac{2\varepsilon + (1+\varepsilon)^2(2\lambda + \varepsilon)}{p^\alpha} \right). \end{aligned}$$

In the first inequality, we used the Harris inequality and the fact that $(D_0^Z)_p$ is contained in a translate of $D(Bp^{-\alpha})$. In the second, we used Proposition 4.1 and Lemma 4.2. In the third, we used that h_p^u converges uniformly to h^u and N does not depend on p . In the last, we harnessed the fourth property of the sequence (D_n) . The claim follows since C is large and ε small enough. \square

Once we are past the critical scale, growth becomes easy, as shown by the following result.

Corollary 4.5 (Supercritical growth). *We have*

$$\mathbb{P}_p \left(\mathcal{I} \left(D \left[p^{-3W} \right] \right) \right) \geq \exp(-(2\lambda + 2C\varepsilon)/p^\alpha).$$

Proof. Proposition 4.4 implies that there exists a droplet $D_p = D[\mathbf{a}] \supseteq D(Bp^{-\alpha})$ and

$$\mathbb{P}_p(\mathcal{I}(D_p)) \geq \exp(-(2\lambda + C\varepsilon)/p^\alpha).$$

We may then proceed as in the proof of Lemma 4.2, growing the droplet dimensions exponentially. This leads to

$$\begin{aligned} &\mathbb{P}_p \left(\mathcal{I} \left(D \left[p^{-3\alpha} \right] \right) \right) \\ &\geq \exp \left(-\frac{2\lambda + C\varepsilon}{p^\alpha} \right) p^{NW_R|\mathcal{S}|} \prod_{i=0}^{N-1} \exp \left(\frac{|\mathcal{S}|(c^{i+1} - c^i)CB}{p^\alpha} \log \left(1 - e^{-c^i B/C} \right) \right), \end{aligned}$$

where $c > 1$ is a constant close enough to 1 and we assumed for simplicity that $p^{-3W} = c^N B p^{-\alpha}$ for some integer N . Taking B large the above product can be made larger than $\exp[-\varepsilon/p^\alpha]$ and we have that N is logarithmic in $1/p$, so the conclusion follows. \square

Finally, we can conclude the proof of the upper bound of Theorem 1.1 in the usual way following [1].

Proof of the upper bound in Theorem 1.1. Fix $\Lambda = \exp((\lambda + 2C\varepsilon)/p^\alpha)$. Let \mathcal{E} be the event that for all $u \in \mathcal{S}$, every translate of the rectangle $R^u(p^{-3W}, 1)$ included in $D[\Lambda]$ contains a W -helping set. The probability of this event can be bounded from below by

$$\mathbb{P}_p(\mathcal{E}) \geq \left(1 - (1 - p^W)^{\lfloor p^{-3W}/W \rfloor}\right)^{|\mathcal{S}| \cdot |D[\Lambda] \cap \mathbb{Z}^2|} \rightarrow 1.$$

Denote by \mathcal{F} the event that there exists a translate of $D[p^{-3W}]$ included in $D[\Lambda]$ which is internally filled. Applying Corollary 4.5 and fitting $(\Lambda p^{3W})^2/C$ disjoint translates of $D[p^{-3W}]$ into $D[\Lambda]$, one obtains

$$\mathbb{P}_p(\mathcal{F}) \geq 1 - (1 - \exp[-(2\lambda + 2C\varepsilon)/p^\alpha])^{(\Lambda p^{3W})^2/C} \rightarrow 1.$$

Moreover, the simultaneous occurrence of \mathcal{E} and \mathcal{F} implies that $p^\alpha \log \tau \leq \lambda + 3C\varepsilon$ for p small enough. Indeed, each site in the internally filled translate of $D[p^{-3W}]$ granted by \mathcal{F} becomes occupied in time at most $|D[p^{-3W}] \cap \mathbb{Z}^2|$, since at least one new site becomes infected at each step. After the creation of this supercritical droplet, it only takes a time of order $p^{-3W}\Lambda$ to progress and reach 0, thanks to the event \mathcal{E} . More precisely, growing one of the radii of our droplet by 1 only requires a time of order p^{-3W} regardless of its size, since each W -helping set grows linearly along its edge. The Harris inequality yields

$$\mathbb{P}_p(p^\alpha \log T \leq \lambda + 3C\varepsilon) \geq \mathbb{P}_p(\mathcal{E} \cap \mathcal{F}) \geq \mathbb{P}_p(\mathcal{E})\mathbb{P}_p(\mathcal{F}) \rightarrow 1$$

which concludes the proof of the upper bound of Theorem 1.1, since $C\varepsilon \ll 1$. \square

5 Proof of the lower bound

We next turn to the lower bound in Theorem 1.1, which is harder, since we need to control all possible ways of creating large droplets.

5.1 Upper bound on growth using the functional

Since the process is not obliged to form droplets, but could instead use more complicated shapes, we will need some further notions to suitably reduce them to droplets.

Definition 5.1 (Δ -connected). Given $\Delta > 0$, we say that a set $X \subseteq \mathbb{Z}^2$ is Δ -connected if it is connected in the graph $\Gamma = (\mathbb{Z}^2, \{\{x, y\} : \|x - y\| \leq \Delta\})$.

It is known that there exists a constant $K = K(\mathcal{U}) > 0$ such that for all stable directions u and all sets $S \subset \mathbb{Z}^d$ such that $S \notin \mathcal{H}^u$ and $|S| \leq \alpha(u)$, we have

$$\max\{d(x, S) : x \in [S \cup \mathbb{H}_u] \setminus \mathbb{H}_u\} < K/3 \quad (5.1)$$

(see [15] for an explicit bound on K). In particular, applying this to both u and $-u$, we see that for any $S \subset \mathbb{Z}^d$ such that $|S| < \alpha(u)$ we have

$$\max\{d(x, S) : x \in [S]\} < K/3. \quad (5.2)$$

We further assume K large enough so that for any stable u and any $S \in \mathcal{H}^u$ we have $\text{diam}(S) < K/3$ and $\max\{\|x\| : x \in \bigcup_{U \in \mathcal{U}} U\} < K/3$.

Definition 5.2 (Spanning). For two \mathcal{S}_α -droplets $D_1 \subseteq D_2$, let $\mathcal{E}(D_1, D_2)$ be the event that there exists a K -connected set $X \subseteq [(A \cap D_2) \cup D_1]$ such that every \mathcal{S}_α -droplet containing X also contains D_2 .

We further write $\mathcal{E}(D) = \mathcal{E}(\emptyset, D)$ for any \mathcal{S}_α -droplet D and say that D is *spanned* when $\mathcal{E}(D)$ occurs.

Spanning events \mathcal{E} will play a similar role to the filling events \mathcal{I} used for the upper bound in Section 4, so our first step is again to link them to the function W .

Proposition 5.3. *For any \mathcal{S}_α -droplets $D_1 \subseteq D_2$ satisfying $\Phi(D_2) \leq CBp^{-\alpha}$ and $\Psi(D_1, D_2) \leq Tp^{-\alpha}$, we have*

$$\mathbb{P}_p(\mathcal{E}(D_1, D_2)) \leq C \exp\left(- (1 - \varepsilon)^2 \frac{W_p(D_1^{Zp^{-\alpha}}, D_2^{Zp^{-\alpha}})}{p^\alpha}\right). \quad (5.3)$$

Before turning to the proof of (5.3), let us discuss a lemma first. For any $m, n \in \mathbb{N}$, define the *strip*

$$S^u(n) = \{x \in \mathbb{Z}^2 : 0 \leq \langle x, u \rangle < n\rho_u\} = \bigcup_{i=0}^{n-1} l^u(i).$$

Also, consider the events

$$\begin{aligned} & \mathcal{C}^u(m, n, E) \\ &= \{l^u(0) \text{ and } l^u(n) \text{ } K\text{-connected in } [(A \cap R^u(m, n)) \cup (\mathbb{Z}^2 \setminus S^u(n)) \cup E]\}, \end{aligned} \quad (5.4)$$

where $E \subseteq \mathbb{Z}^2 \setminus R^u(m, n)$ is viewed as a ‘‘boundary condition’’. For such a set E , define s_E to be the number of $j \in \{0, \dots, n-1\}$ such that $l_u(j)$ is at distance at most $3K$ from a $3K$ -connected set of cardinality $\alpha(u)$ in E .

Lemma 5.4. *Let u be a stable direction. For $m \in [Tp^{-\alpha(u)}, CBp^{-\alpha(u)}]$, $n \geq 1/T$ and $E \subseteq \mathbb{Z}^2 \setminus R^u(m, n)$, we have*

$$\mathbb{P}_p(\mathcal{C}^u(m, n, E)) \leq \exp\left(- (1 - \varepsilon) h_p^u(p^{\alpha(u)} m) (n - Ls_E)\right).$$

Proof. We prove the result by slicing the rectangle into rectangles of fixed (but large) height $k = L/3$. Let us first prove that for any E with $s_E = 0$,

$$\mathbb{P}_p(\mathcal{C}^u(m, k, E)) \leq \exp\left(- (1 - 2\varepsilon) h_p^u(p^{\alpha(u)} m) k\right). \quad (5.5)$$

Let $\mathcal{E}^u(m, k)$ be the event that $A \cap R^u(m, k)$ contains a $3K$ -connected set of size $\alpha(u) + 1$ or there is a site $a \in A \cap R^u(m, k)$ such that $\langle a, u^\perp \rangle \in [0, 3K] \cup [m, m - 3K]$. The number of possible such sets included in $R^u(m, k)$ is bounded by Mkm for some universal constant $M = M(K) > 0$. Therefore, Proposition 3.2(2) implies that for p small enough depending on C, B, M, k , it holds that

$$\begin{aligned} \mathbb{P}_p(\mathcal{E}^u(m, k)) &\leq Mkm p^{\alpha(u)+1} + Mp \leq M(kCB + 1)p \\ &\leq \exp\left(- h_p^u(T)k\right) \leq \exp\left(- h_p^u(p^{\alpha(u)} m) k\right), \end{aligned} \quad (5.6)$$

since $CB \geq p^{\alpha(u)}m \geq T$.

Note that $k > 3K(\alpha(u) + 1)$. Let us assume in the following that $\mathcal{E}^u(m, k)$ does not occur. Therefore, $A' = (A \cap R^u(m, k)) \cup E$ consists of $3K$ -connected components of size at most $\alpha(u)$ contained entirely in $R^u(m, k)$ and $3K$ -connected components of size at most $\alpha(u) - 1$ contained entirely in E (since $s_E = 0$). Make the further assumption that neither $l^u(0)$ nor $l^{-u}(-k + 1)$ is occupied, using the notation $l^{-u}(i)$ in order to specify that the line must be occupied in direction $-u$. Consider one of the $3K$ -connected components discussed above. By Eqs. (5.1) and (5.2), in the process with initial condition $A' \cup (\mathbb{Z}^2 \setminus S^u)$ each component grows at most by a distance K , which is insufficient for different components to start interacting or reach the opposite boundary of S^u , if they are close to one. Thus, each K -connected component of $[A' \cup (\mathbb{Z}^2 \setminus S^u)] \setminus S^u$ is generated by a single $3K$ -connected component of A' , so it cannot K -connect $l^u(0)$ to $l^u(k)$. In conclusion, if $\mathcal{E}^u(m, k)$ does not occur, $l^u(0)$ or $l^{-u}(-k + 1)$ must be occupied.

By induction, we deduce that for $N = 3K(\alpha(u) + 1)$ there exists k' between 0 and k such that $l^u(0), \dots, l^u(k' - 1)$ and $l^{-u}(-k' + N), \dots, l^{-u}(-k + 1)$ are occupied. Set $\mathbb{P}_p(\mathcal{A}^u(m, k)) = 1$ for $k < 0$. We find

$$\begin{aligned} \mathbb{P}_p(\mathcal{C}^u(m, k, E)) &\leq \mathbb{P}_p(\mathcal{E}^u(m, k)) + \sum_{k'=0}^k \mathbb{P}_p(\mathcal{A}^u(m, k')) \mathbb{P}_p(\mathcal{A}^{-u}(m, k - k' - N)) \\ &\leq (k + 2) \exp(-h_p^u(p^{\alpha(u)}m)(k - N)), \end{aligned}$$

where we used the fact that lines at a distance greater than K are independently occupied for the first inequality, and Proposition 3.2(1), Eq. (5.6) and symmetry for the second one. Using the fact that $h_p^u(p^{\alpha(u)}m) \leq h_p^u(CB)$, $k + 2 \leq \exp[h_p^u(CB)(k - N)\varepsilon/3]$, and $k - N \geq (1 - \varepsilon/3)k$, we deduce

$$\mathbb{P}_p(\mathcal{C}^u(m, k, E)) \leq \exp(-(1 - 2\varepsilon/3)h_p^u(p^{\alpha(u)}m)k).$$

Now, the rectangle $R^u(m, n)$ can be divided into $\lfloor n/k \rfloor$ translates of $R^u(m, k)$. Then, at least $\lfloor n/k \rfloor - 2s_E$ of these translated rectangles satisfy the condition of (5.5). We thus obtain

$$\begin{aligned} \mathbb{P}_p(\mathcal{C}^u(m, n, E)) &\leq (\mathbb{P}_p(\mathcal{C}^u(m, k, E)))^{\lfloor n/k \rfloor - 2s_E} \\ &\leq \exp(-(1 - 2\varepsilon/3)h_p^u(p^{\alpha(u)}m)(k\lfloor n/k \rfloor - 2ks_E)) \\ &\leq \exp(-(1 - \varepsilon)h_p^u(p^{\alpha(u)}m)(n - 3ks_E)) \end{aligned}$$

for $n \geq 1/T$. This concludes the proof. \square

Remark 5.5. As it is clear from the proof, Lemma 5.4 applies equally well to parallelograms

$$P^{u,v}(m, n) = \{x \in \mathbb{Z}^2 : 0 \leq \langle x, v \rangle < m\langle u^\perp, v \rangle, 0 \leq \langle x, u \rangle < n\rho_u\} \quad (5.7)$$

instead of rectangles $R^u(m, n)$, where $v \in \mathcal{S} \setminus \{u, -u\}$. The definition of $\mathcal{C}^{u,v}(m, n, E)$ is Eq. (5.4) with $P^{u,v}$ instead of R^u and the definition of s_E remains unchanged.

We are now in a position to prove Proposition 5.3.

Proof of Proposition 5.3. Consider two droplets $D_1[\mathbf{a}] \subseteq D_2[\mathbf{b}]$ satisfying $\Psi(D_1, D_2) \leq Tp^{-\alpha}$. Let \mathbf{m} be the dimension of D_1 and \mathbf{s} be its location in D_2 . For each $u \in \mathcal{S}_\alpha$ we define R^u as in the proof of Proposition 4.1, namely, let R^u be the translate of the largest rectangle $R^u(\tilde{m}_u, s_u)$ such that $R^u \subseteq D[\mathbf{a} + \mathbf{e}_u s_u] \setminus D[\mathbf{a}]$ (recall Fig. 3). Let $x_u \in \mathbb{R}^2$ be such that $R^u = x_u + R^u(\tilde{m}_u, s_u)$. We set $\bar{R}^u = R^u$ if $\tilde{m}_u \geq Tp^{-\alpha}$ and $\bar{R}^u = \emptyset$ otherwise and let $\bar{m}_u = \tilde{m}_u$ if $\tilde{m}_u \geq Tp^{-\alpha}$ and $\bar{m}_u = 0$ otherwise. Let

$$X = D_2 \setminus \left(D_1 \cup \bigcup_{u \in \mathcal{S}_\alpha} \bar{R}^u \right)$$

be the *leftover* region (see Fig. 3), which may have a rather complicated shape, but is, crucially, small. Conditioning on $A \cap X$ and recalling Eq. (5.4), we get

$$\mathbb{P}_p(\mathcal{E}(D_1, D_2)) \leq \mathbb{E}_p \left[\prod_{u \in \mathcal{S}_\alpha} \mathbb{P}_p((A - x_u) \in \mathcal{C}^u(\bar{m}_u, s_u, (A \cap X) - x_u) | A \cap X) \right].$$

In words, each rectangle \bar{R}^u is crossed with boundary condition given by the infections in the leftover region. Note that this event is simply an indicator function for u such that $\tilde{m}_u < Tp^{-\alpha}$, since it is measurable with respect to the conditioning.

Following Lemma 5.4, for each $u \in \mathcal{S}_\alpha$ let $s_{A \cap X}^u$ be the number of $j \in \{0, \dots, s_u - 1\}$ such that $l_u(j) + x_u$ is at a distance at most $3K$ from a $3K$ -connected set of cardinality α in $A \cap X$. Then, Lemma 5.4 gives

$$\mathbb{P}_p(\mathcal{E}(D_1, D_2)) \leq \mathbb{E}_p \left[\prod_{\substack{u \in \mathcal{S}_\alpha \\ \bar{m}_u = 0}} \mathbb{1}_{s_{A \cap X}^u = s_u} \exp \left(- (1 - \varepsilon) \sum_{\substack{u \in \mathcal{S}_\alpha \\ \bar{m}_u \neq 0}} h_p^u(p^\alpha \bar{m}_u) (s_u - L s_{A \cap X}^u) \right) \right], \quad (5.8)$$

which becomes an expectation just over the $(s_{A \cap X}^u)_{u \in \mathcal{S}_\alpha}$.

We argue that for each u either s^u is small enough not to perturb s_u much or it is large, which is unlikely by itself. Indeed, denoting by \mathbf{m}^Z the dimension of $D_1^{Zp^{-\alpha}}$, we can bound Eq. (5.8) from above by

$$\sum_{V \subset \{u \in \mathcal{S}_\alpha : \bar{m}_u \neq 0\}} \mathbb{P}_p(\forall u \in \mathcal{S}_\alpha \setminus V, s_{A \cap X}^u > \varepsilon s_u / L) \exp \left(- (1 - \varepsilon)^2 \sum_{u \in V} h_p^u(p^\alpha m_u^Z) s_u \right),$$

noting that $m_u^Z \geq m_u \geq \bar{m}_u$ for all $u \in \mathcal{S}_\alpha$. Thus, it only remains to prove that for any $V \subset \mathcal{S}_\alpha$ such that $V \supset \{u \in \mathcal{S}_\alpha : \bar{m}_u = 0\}$, we have

$$\mathbb{P}_p(\forall u \in V, s_{A \cap X}^u > \varepsilon s_u / L) \leq \exp \left(- \sum_{u \in V} h_p^u(m_u^Z p^\alpha) s_u \right).$$

Fix $u \in V$ such that s_u is maximal. Since $u \in V$, there exist at least $\varepsilon s_u / (CKL)$ disjoint $3K$ -connected sets of α infections in $X \setminus \mathbb{H}_u(a_u)$. But by construction $|X| \leq$

$Cs_u T p^{-\alpha}$, so the union bound gives

$$\begin{aligned}
\mathbb{P}_p(s_{A \cap X}^u > \varepsilon s_u / L) &\leq p^{\alpha \varepsilon s_u / (CKL)} \binom{K^C s_u T p^{-\alpha}}{\varepsilon s_u / (CKL)} \\
&\leq p^{\alpha \varepsilon s_u / (CKL)} \left(\frac{e K^C s_u T p^{-\alpha}}{\varepsilon s_u / (CKL)} \right)^{\varepsilon s_u / (CKL)} \\
&\leq (K^{2C} L T / \varepsilon)^{\varepsilon s_u / (CKL)} \leq \exp(-L s_u) \\
&\leq \exp\left(-\sum_{v \in \mathcal{S}_\alpha \setminus V} h_p^v(m_v^Z p^\alpha) s_v\right),
\end{aligned}$$

since T is chosen small enough depending on ε, C, K, Z, L and $h_p^v(m_v^Z p^\alpha) \leq h_p^v(Z) < L/|\mathcal{S}_\alpha|$, since L is chosen large enough depending on Z . \square

5.2 Hierarchies

We next introduce the notion of hierarchies we will use, following [17], where this method was introduced.

Definition 5.6 (Hierarchy). Let D be a nonempty \mathcal{S}_α -droplet. A *hierarchy* $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$ for D is an oriented rooted tree with edges pointing away from the root and the following additional structure. Each vertex v is labelled by a non-empty \mathcal{S}_α -droplet D_v . Let $N(v)$ denote the out-neighbourhood of v . We require the following conditions to hold.

- (1) The label of the root is D .
- (2) For any $v \in V_{\mathcal{H}}$, $|N(v)| \leq 2$.
- (3) For any $v \in V_{\mathcal{H}}$ and $u \in N(v)$, $D_u \subseteq D_v$.
- (4) If $v \in V_{\mathcal{H}}$ and $N(v) = \{u, w\}$, then $D_u \cup D_w$ is K -connected and $D_v = D_u \vee D_w$.

Vertices of $v \in V_{\mathcal{H}}$ are called *seeds*, *normal vertices* and *splitters* if $|N(v)| = 0, 1, 2$ respectively.

Definition 5.7 (Precision of a hierarchy). Let $z \geq |\mathcal{S}_\alpha|$ and $t > 0$. A *hierarchy of precision* (t, z) is a hierarchy \mathcal{H} such that the following hold.

- (1) A vertex $v \in V_{\mathcal{H}}$ is a seed if and only if $\Phi(D_v) \leq z$.
- (2) If $N(u) = \{v\}$, then $\Psi(D_v, D_u) \leq t$.
- (3) If $v \in N(u)$ and either u is a splitter or v is a normal vertex, then $\Psi(D_v, D_u) > t/2$.

We now relate the concept of hierarchy to our study.

Definition 5.8 (Occurrence of a hierarchy). A hierarchy *occurs* if the following disjoint occurrence event holds (recall Section 3.1):

$$\mathcal{E}(\mathcal{H}) = \bigcirc_{\substack{u \in V_{\mathcal{H}}, \\ N(u) = \emptyset}} \mathcal{E}(D_u) \circ \bigcirc_{\substack{u, v \in V_{\mathcal{H}}, \\ N(u) = \{v\}}} \mathcal{E}(D_v, D_u).$$

The proof of the following key deterministic result is omitted, as it is identical to [4, Lemma 8.7].

Proposition 5.9 (Existence of a hierarchy). *Let $z \geq |\mathcal{S}_\alpha|$, $t > 0$ and D be a non-empty \mathcal{S}_α -droplet. If D is spanned, then there exists a hierarchy of precision (t, z) for D that occurs.*

The next lemma allows us to bound the number of hierarchies in order to use the union bound on their occurrence. For the purposes of counting, we identify \mathcal{S}_α -droplets with their intersection with \mathbb{Z}^d .

Lemma 5.10 (Number of hierarchies). *Fix $a > 0$. Let $t > 0$ and $z \geq |\mathcal{S}_\alpha|$. Let D be a \mathcal{S}_α -droplet such that $\Phi(D)/t \leq a$. Then, there exists a constant $c(a) > 0$ such that the number of hierarchies for D of precision (t, z) is at most $c(a)\Phi(D)^{c(a)}$.*

Proof. The definition of the hierarchy of precision (t, z) implies that every two steps away from the root, the absolute location of droplets decreases by at least $t/2$. Therefore, the height of a hierarchy with root label $D = D[\mathbf{a}]$ is at most $4 \sum_{u \in \mathcal{S}_\alpha} a_u/t \leq C\Phi(D)/t$ for a suitably large $C > 0$. In particular, there is a bounded number of possible tree structures for \mathcal{H} (without the labels). Moreover, for each label the number of possibilities is at most $C\Phi(D)^{|\mathcal{S}_\alpha|}$, since C is large enough. Indeed, for each $u \in \mathcal{S}_\alpha$ the number of n such that $l_u(n) \cap D \neq \emptyset$ is at most of order $\Phi(D)$ and those are the possible choices of a_u in the vector \mathbf{a} defining the given labelling droplet $D[\mathbf{a}]$. \square

5.3 The probability of occurrence of a hierarchy

In order to use a union bound on hierarchies, we will need to estimate $\mathbb{P}_p(\mathcal{E}(\mathcal{H}))$ for a given hierarchy \mathcal{H} . If \mathcal{H} involves no splitters, this is straightforward, as one can directly apply Proposition 5.3. Even though this is the dominant scenario, we will need to account for all other possibilities as well. Naturally, the main issue are hierarchies with many splitters and, therefore, many seeds. It is therefore natural to introduce the following quantity, still following [17].

Definition 5.11 (Pod of a hierarchy). The *pod* of a hierarchy \mathcal{H} , denoted by $\text{Pod}(\mathcal{H})$, is defined by

$$\text{Pod}(\mathcal{H}) = \sum_{\substack{u \in V_{\mathcal{H}}, \\ N(u) = \emptyset}} \Phi(D_u).$$

Before dealing with an entire hierarchy, we first bound the probability of a single seed. Let us note that a more general statement can be found in [13, Corollary A.11], but in the symmetric setting we are dealing with one has an easier way to achieve the following.

Lemma 5.12 (Seed bound). *If D is a \mathcal{S}_α -droplet such that $\Phi(D) \leq CB/p^\alpha$, then*

$$\mathbb{P}_p(\mathcal{E}(D)) \leq \exp \left(- \min_{u \in \mathcal{S}_\alpha} h_p^u \left(\max \left(Z, C \min_{v \in \mathcal{S}_\alpha} (a_v - a_{-v}) \right) \right) \Phi(D)/C \right).$$

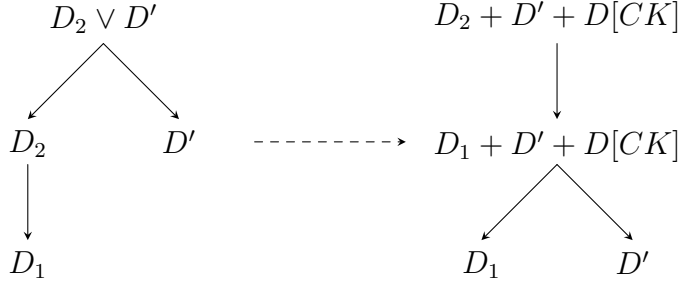


Figure 4: The operation on hierarchies provided by Lemmas 5.14 and 5.15. However, since D_1 and D' have no reason to be K -connected, the result on the right is no longer a hierarchy.

Proof. Let $D = D[\mathbf{a}]$ for $\mathbf{a} \in \mathbb{R}^{\mathcal{S}_\alpha}$. Fix $u, v \in \mathcal{S}_\alpha$ such that $a_u - a_{-u} = \max_{w \in \mathcal{S}_\alpha} (a_w - a_{-w})$, $a_v - a_{-v} = \min_{w \in \mathcal{S}_\alpha} (a_w - a_{-w})$ and $v \notin \{u, -u\}$. Up to translating, we may assume that D is contained in the parallelogram $P^{u,v}(m, n)$ (recall Eq. (5.7)) with $n = a_u - a_{-u} \geq 2\Phi(D)/C$ and $m = C(a_v - a_{-v})$. Finally, observe that the event $\mathcal{E}(D)$ implies that $\mathcal{C}^{u,v}(m, n, \emptyset)$ from Eq. (5.4) also occurs (recall Remark 5.5). Then, we are done by Lemma 5.4 and Remark 5.5. \square

Applying the BK inequality Eq. (3.2) to Lemma 5.12 and recalling that $h_p^u(x) \rightarrow \infty$ as $x \rightarrow 0$ for all $u \in \mathcal{S}_\alpha$, we immediately obtain the following.

Corollary 5.13. *Let \mathcal{H} be a hierarchy for D of precision $(T/p^\alpha, Z/p^\alpha)$. Then*

$$\mathbb{P}_p \left(\bigcirc_{\substack{u \in V_{\mathcal{H}}, \\ N(u) = \emptyset}} \mathcal{E}(D_u) \right) \leq \exp(-L \text{Pod}(\mathcal{H})).$$

If $\text{Pod}(\mathcal{H}) \geq 2\lambda/(Lp^\alpha)$, Corollary 5.13 will be sufficient to conclude. In order to deal with the more relevant hierarchies with smaller pods, we will need a more precise bound.

The goal of the next two lemmas is, roughly speaking, to transform a hierarchy with a splitter root into one with a normal root, as depicted in Fig. 4. The first lemma is essentially [4, Eq. (16)], so we omit the proof.

Lemma 5.14 (Subadditivity of the span). *Assume D_1, D_2, D are \mathcal{S}_α -droplets such that $D_1 \cup D_2$ is K -connected. Then some translate of $D_1 + D_2 + D[CK]$ contains $D_1 \vee D_2$.*

Lemma 5.15. *Let $D_1 \subseteq D_2$ and D' be three \mathcal{S}_α -droplets. We have*

$$W_p(D_1, D_2) \geq W_p(D_1 + D', D_2 + D').$$

Proof. This follows from the fact that h_p^u is non-decreasing and Observation 3.6. \square

Lemma 5.15 is the main reason why the occupied sites form droplets. It is always more efficient for the infections to appear near existing infected droplets. Hence, the dynamics has a tendency to create large droplets.

As a result of the operation from Fig. 4 and Proposition 5.3, we obtain the following bound.

Proposition 5.16. *Let D be a \mathcal{S}_α -droplet with $\Phi(D) \leq CBp^{-\alpha}$. For any hierarchy \mathcal{H} of precision $(Tp^{-\alpha}, Zp^{-\alpha})$ for D with $N - 1$ normal vertices and S splitters, there exists a non-decreasing sequence of \mathcal{S}_α -droplets $D_1 \subseteq \dots \subseteq D_N$ satisfying*

- $\Phi(D_1) \leq BS + \text{Pod}(\mathcal{H})$,
- either $Bp^{-\alpha} \leq \Phi(D_N) \leq CBp^{-\alpha}$, or both $\Phi(D_N) < Bp^{-\alpha}$ and $D_N \supseteq D$,
- $\mathbb{P}_p(\mathcal{E}(\mathcal{H})) \leq C^N \exp(-(1 - \varepsilon)^2 \sum_{i=1}^{N-1} W_p(D_i^{Zp^{-\alpha}}, D_{i+1}^{Zp^{-\alpha}})/p^\alpha)$.

Proof. We proceed by induction on hierarchies. Let D_r be the label of the root of \mathcal{H} .

Assume the root r is a seed. Then \mathcal{H} is a singleton, $N = 1$ and it is sufficient to set $D_1 = D_r$.

Assume the root r is a normal vertex. Let $N(r) = \{u\}$. The induction hypothesis for the hierarchy with r removed yields a sequence $D_1 \subseteq \dots \subseteq D_{N-1}$ of \mathcal{S}_α -droplets. If $Bp^{-\alpha} \leq \Phi(D_{N-1}) \leq CBp^{-\alpha}$, we set $D_N = D_{N-1}$ and we are done. Assume that, on the contrary, $D_{N-1} \supseteq D_u$ and $\Phi(D_{N-1}) < Bp^{-\alpha}$. In this case we set $D_N = D_r \vee D_{N-1}$. The resulting sequence clearly satisfies the first condition. Since r is a normal vertex, by Definition 5.7 we have $\Psi(D_u, D_r) \leq Tp^{-\alpha}$, so $D_N \subseteq D_{N-1} + D[CTp^{-\alpha}]$. Therefore, $\Phi(D_N) \leq CBp^{-\alpha}$ and $D_N \supseteq D$, so the second condition is also satisfied. The third one follows from

$$\mathbb{P}_p(\mathcal{E}(D_u, D_r)) \leq \mathbb{P}_p(\mathcal{E}(D_{N-1}, D_N)) \leq C \exp\left(- (1 - \varepsilon)^2 W_p\left(D_{N-1}^{Zp^{-\alpha}}, D_N^{Zp^{-\alpha}}\right) / p^\alpha\right),$$

using $\mathcal{E}(D_u, D_r) \subseteq \mathcal{E}(D_{N-1}, D_N)$ for the first inequality and Proposition 5.3 for the second one. Note that here we use that $\Phi(D_N) \leq CB/p^{-\alpha}$.

Finally, assume the root r is a splitter. Denote $N(r) = \{u, v\}$ and let $D_1^u, \dots, D_{N^u}^u$ and $D_1^v, \dots, D_{N^v}^v$ be the sequences yielded by the induction hypothesis for the sub-hierarchies $\mathcal{H}^u, \mathcal{H}^v$ with roots u and v respectively. Without loss of generality, assume $\Phi(D_{N^u}^u) \geq \Phi(D_{N^v}^v)$. If $\Phi(D_{N^u}^u) \geq Bp^{-\alpha}$, then the sequence

$$D_i = \begin{cases} D_i^u & i \in \{1, \dots, N^u\}, \\ D_{N^u}^u & i \in \{N^u + 1, \dots, N^u + N^v - 1\} \end{cases}$$

clearly satisfies the desired properties. Assume that, on the contrary, $\Phi(D_{N^u}^u) < Bp^{-\alpha}$. In this case, we define

$$D_i = \begin{cases} D[CK] + D_1^u + D_i^v & i \in \{1, \dots, N^v\}, \\ D[CK] + D_{i-N^v+1}^u + D_{N^v}^v & i \in \{N^v + 1, \dots, N^v + N^u - 1\}. \end{cases}$$

Since the perimeter is additive, we have

$$\Phi(D_1) = \Phi(D_1^u) + \Phi(D_1^v) + \Phi(D[CK]),$$

so the first condition is met, using the induction hypothesis. We have $D_{N^u+N^v-1} \supseteq D_r$ by Lemma 5.14 up to translating the sequence $(D_i)_{i=1}^{N^u+N^v-1}$ appropriately. Moreover,

$$\Phi(D_{N^u+N^v-1}) = \Phi(D_{N^u}^u) + \Phi(D_{N^v}^v) + CK|\mathcal{S}_\alpha| \leq 2Bp^{-\alpha} + B < CBp^{-\alpha},$$

so the second condition is also verified. Finally, the BK inequality and the induction hypothesis give

$$\begin{aligned} \mathbb{P}_p(\mathcal{E}(\mathcal{H})) &\leq \mathbb{P}_p(\mathcal{E}(\mathcal{H}^u))\mathbb{P}_p(\mathcal{E}(\mathcal{H}^v)) \leq C^{N^u+N^v} \exp\left(-\frac{(1-\varepsilon)^2}{p^\alpha}\right) \\ &\quad \times \left(\sum_{i=1}^{N^u-1} W_p\left((D_i^u)^{Zp^{-\alpha}}, (D_{i+1}^u)^{Zp^{-\alpha}}\right) + \sum_{i=1}^{N^v-1} W_p\left((D_i^v)^{Zp^{-\alpha}}, (D_{i+1}^v)^{Zp^{-\alpha}}\right)\right), \end{aligned}$$

which is enough to conclude, using Lemma 5.15. \square

5.4 Truncating λ_α

In order to relate the bound from Proposition 5.16 to the constant λ_α from Definition 3.8, we will need to truncate our bi-infinite sequences of droplets. We start by showing that it is always cheap to extend sequences to $+\infty$.

Lemma 5.17 (Extension at $+\infty$). *For any \mathcal{S}_α -droplet D with $\Phi(D) \geq G$, there exists a sequence of \mathcal{S}_α -droplets $D = D_0 \subseteq D_1 \subseteq \dots$ such that $\bigcup_{i \geq 0} D_i = \mathbb{R}^2$ and $\sum_{i=0}^{\infty} W(D_i, D_{i+1}) \leq \varepsilon$.*

Proof. After translating, we may assume that for some sufficiently large k depending on ε we have that $D \subseteq D[2^k]$, but D is not contained in any translate of $D[2^{k-1}]$. As we saw in Eq. (3.6), taking k large we can ensure that $\sum_{i \geq k} W(D[2^i], D[2^{i+1}]) \leq \varepsilon/2$. Therefore it suffices to find $D = D_0 \subseteq \dots \subseteq D_N = D[2^k]$ such that $\sum_{i=0}^{N-1} W(D_i, D_{i+1}) \leq \varepsilon/2$.

Since T is small enough, all dimensions of D are much larger than T . Set $D = D[\mathbf{a}^{(0)}]$ and $D[2^k] = D[\mathbf{a}^{(\infty)}]$. We define $\mathbf{a}^{(i)}$ by induction as follows, set $D_i = D[\mathbf{a}^{(i)}]$ and denote by $\mathbf{m}^{(i)}$ the dimension of D_i . Further let $u_i \in \mathcal{S}_\alpha$ be such that $m_{u_i}^{(i)} = \max\{m_u^{(i)} : u \in \mathcal{S}_\alpha, a_u^{(i)} \neq a_u^{(\infty)}\}$. As long as $D_i \neq D[2^k]$ (at which point the construction is done), we set

$$\mathbf{a}^{(i+1)} = \mathbf{a}^{(i)} + \mathbf{e}_{u_i} \min(T, a_{u_i}^{(\infty)} - a_{u_i}^{(i)}).$$

This procedure clearly yields $D_N = D[2^k]$ for some finite N . Further observe that $m_{u_i}^{(i)} \geq 2^k/C$ for all $i \in \{0, \dots, N-1\}$ and $C > 0$ large enough. That is, the largest edge that has not yet reached its final position is always big. Indeed, every two edges of D_i that have reached the final value for their radius are necessarily far apart, so there has to be a large side between them. Using this property, we have that

$$\sum_{i=0}^{N-1} W(D_i, D_{i+1}) \leq \sum_{u \in \mathcal{S}_\alpha} h^u (2^k/C) a_u^{(\infty)} \leq \varepsilon/2$$

for k large enough, using Proposition 3.2(2). \square

Unfortunately, the analogous statement for extending sequences to $-\infty$ is not true, since arbitrarily small droplets have a divergent cost to produce if they are too elongated (see Lemma 5.12). Nevertheless, we are able to obtain the following.

Lemma 5.18 (Truncating λ_α). *Let $D_1 \subseteq \dots \subseteq D_N$ be a sequence of \mathcal{S}_α -droplets such that $\Phi(D_N) \geq Gp^{-\alpha}$ and $\Phi(D_1) \leq 1/(Gp^\alpha)$. Then*

$$\sum_{i=1}^{N-1} W(D_i, D_{i+1}) \geq 2\lambda_\alpha - 2\varepsilon.$$

Proof. Set $D = D[\Phi(D_1)]$ and set $D'_i = D + D_i$ for all $i \in \{1, \dots, N\}$. By Lemma 5.15 we have

$$\sum_{i=1}^{N-1} W(D_i, D_{i+1}) \geq \sum_{i=1}^{N-1} W(D'_i, D'_{i+1}).$$

We further use Lemma 5.17 applied to D'_N to define D'_i for all $i > N$ in such a way that $\sum_{i \geq N} W(D'_i, D'_{i+1}) < \varepsilon$. However, now we have ensured that D'_1 is roughly circular. Using this fact, up to translation, we can assume that $D[2^{-k-C}] \subseteq D'_1 \subseteq D[2^{-k}]$ with $k > 0$ large enough depending on ε . We then proceed as in the proof of Lemma 5.17 to define droplets $D[2^{-k-C}] = D'_{-N'} \subseteq \dots \subseteq D'_1$ for some $N' \geq 0$ in such a way that

$$\sum_{i=-N'}^0 W(D'_i, D'_{i+1}) \leq \varepsilon/2.$$

Here, we crucially use that the dimensions of all D'_i for $i \in \{-N', \dots, 0\}$ are at least $2^{-k-C}/C$, but the proof is the same as for Lemma 5.17. Finally, recalling Eq. (3.6), we may set $D'_i = D[2^{-k-C+i+N'}]$ for $i < -N'$ to obtain

$$\sum_{i \in \mathbb{Z}} W(D'_i, D'_{i+1}) \leq 2\varepsilon + \sum_{i=1}^{N-1} W(D_i, D_{i+1}).$$

Since $(D'_i)_{i \in \mathbb{Z}} \in \mathfrak{D}_\alpha$, we are done by the definition of λ_α . □

5.5 Proof of the lower bound of Theorem 1.1

We are ready to upper bound the probability that a droplet of size $Bp^{-\alpha}$ is spanned, using Proposition 5.16. Once that is done, Theorem 1.1 will follow immediately.

Proposition 5.19 (Critical spanning bound). *For any \mathcal{S}_α -droplet D satisfying $Bp^{-\alpha} \leq \Phi(D) \leq CBp^{-\alpha}$ we have*

$$\mathbb{P}_p(\mathcal{E}(D)) \leq \exp(-(2\lambda - C\varepsilon)/p^\alpha).$$

Proof. Proposition 5.9 gives that if $\mathcal{E}(D)$ occurs, then $\mathcal{E}(\mathcal{H})$ does for some hierarchy of precision $(Tp^{-\alpha}, Zp^{-\alpha})$ for D denoted \mathcal{H} . Using Lemma 5.10, we obtain that for some $c(T) > 0$ large enough

$$\mathbb{P}_p(\mathcal{E}(D)) \leq c(T)\Phi(D)^{c(T)} \cdot \max_{\mathcal{H}} \mathbb{P}_p(\mathcal{E}(\mathcal{H})) \leq \exp(\varepsilon p^{-\alpha}) \max_{\mathcal{H}} \mathbb{P}_p(\mathcal{E}(\mathcal{H})).$$

It is thus sufficient to prove that for any \mathcal{H}

$$\mathbb{P}_p(\mathcal{E}(\mathcal{H})) \leq \exp\left(- (2\lambda - (C - 1)\varepsilon)p^{-\alpha}\right).$$

If $\text{Pod}(\mathcal{H}) \geq 2\lambda/(Lp^\alpha)$, we are done by Corollary 5.13. We therefore assume that $\text{Pod}(\mathcal{H}) \leq 2\lambda/(Lp^\alpha)$. Proposition 5.16 yields the existence of a sequence $D_1 \subseteq \dots \subseteq D_N$ with $\Phi(D_1) < 1/(Bp^\alpha)$ and $\Phi(D_N) \geq Gp^{-\alpha}$ satisfying

$$\mathbb{P}_p(\mathcal{E}(\mathcal{H})) \leq C^N \exp\left(- (1 - \varepsilon)^2 p^{-\alpha} \sum_{n=1}^{N-1} W_p\left(D_n^{Zp^{-\alpha}}, D_{n+1}^{Zp^{-\alpha}}\right)\right).$$

However, by Lemma 5.18 and Proposition 3.10, we have

$$\sum_{n=1}^{N-1} W_p\left(D_n^{Zp^{-\alpha}}, D_{n+1}^{Zp^{-\alpha}}\right) \geq 2\lambda - 2\varepsilon. \quad (5.9)$$

Thus,

$$\mathbb{P}_p(\mathcal{E}(\mathcal{H})) \leq C^N \exp\left(- (1 - 2\varepsilon)(2\lambda - 2\varepsilon)/p^\alpha\right).$$

Since N and C do not depend on p , this concludes the proof. \square

Concluding the proof of Theorem 1.1 from Proposition 5.19 is very standard, the argument dating back to [1].

Proof of the lower bound in Theorem 1.1. Let $\Lambda = \exp(\lambda - C\varepsilon)/p^\alpha$. Let \mathcal{E} be the event that $0 \in [A \cap [-\Lambda, \Lambda]^2]$. We claim that $\mathbb{P}_p(\mathcal{E}) \rightarrow 0$ as $p \rightarrow 0$. Indeed, if \mathcal{E} occurs, then the origin belongs to a spanned \mathcal{S}_α -droplet D with $1 \leq \Phi(D) \leq C\Lambda$. There are at most $(C\Phi(D))^{|S_\alpha|}$ possible choices for this droplet. If $\Phi(D) \leq CB/p^\alpha$, we are done by a union bound and Lemma 5.12.

On the other hand, if $\Phi(D) > CB/p^\alpha$, the Aizenman–Lebowitz lemma [4, Lemma 6.18] (see also [13, Lemma A.9]) allows us to extract a \mathcal{S}_α -droplet $D' \subseteq D$ such that $\mathcal{E}(D')$ occurs and $B/p^\alpha \leq \Phi(D') \leq CB/p^\alpha$. We can then conclude by another union bound and Proposition 5.19. \square

Acknowledgements

This project was supported by the Austria Science Fund (FWF): P35428-N. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 757296).

References

- [1] M. Aizenman and J. L. Lebowitz, *Metastability effects in bootstrap percolation*, J. Phys. A **21** (1988), no. 19, 3801–3813 pp. MR968311

- [2] P. Balister, B. Bollobás, M. Przykucki, and P. Smith, *Subcritical U -bootstrap percolation models have non-trivial phase transitions*, Trans. Amer. Math. Soc. **368** (2016), no. 10, 7385–7411 pp. MR3471095
- [3] B. Bollobás, H. Duminil-Copin, R. Morris, and P. Smith, *The sharp threshold for the Duarte model*, Ann. Probab. **45** (2017), no. 6B, 4222–4272 pp. MR3737910
- [4] B. Bollobás, H. Duminil-Copin, R. Morris, and P. Smith, *Universality for two-dimensional critical cellular automata*, Proc. Lond. Math. Soc. (3) **126** (2023), no. 2, 620–703 pp. MR4550150
- [5] B. Bollobás, P. Smith, and A. Uzzell, *Monotone cellular automata in a random environment*, Combin. Probab. Comput. **24** (2015), no. 4, 687–722 pp. MR3350030
- [6] K. Bringmann and K. Mahlburg, *Improved bounds on metastability thresholds and probabilities for generalized bootstrap percolation*, Trans. Amer. Math. Soc. **364** (2012), no. 7, 3829–3859 pp. MR2901236
- [7] J. Chalupa, P. L. Leath, and G. R. Reich, *Bootstrap percolation on a Bethe lattice*, J. Phys. C **12** (1979), no. 1, L31–L35 pp.
- [8] H. Duminil-Copin and A. C. D. van Enter, *Sharp metastability threshold for an anisotropic bootstrap percolation model*, Ann. Probab. **41** (2013), no. 3A, 1218–1242 pp. MR3098677
- [9] J. Gravner and D. Griffeath, *Scaling laws for a class of critical cellular automaton growth rules*, Random walks (Budapest, 1998), 1999, 167–186 pp. MR1752894
- [10] G. Grimmett, *Percolation*, Second edition, Grundlehren der mathematischen Wissenschaften, Springer, Berlin, Heidelberg, 1999. Originally published by Springer, New York (1989). MR1707339
- [11] T. E. Harris, *A lower bound for the critical probability in a certain percolation process*, Math. Proc. Camb. Phil. Soc. **56** (1960), no. 1, 13–20 pp. MR115221
- [12] I. Hartarsky, *Refined universality for critical KCM: upper bounds*, arXiv e-prints (2021), available at arXiv:2104.02329.
- [13] I. Hartarsky and L. Marêché, *Refined universality for critical KCM: lower bounds*, Combin. Probab. Comput. **31** (2022), no. 5, 879–906 pp. MR4472293
- [14] I. Hartarsky, F. Martinelli, and C. Toninelli, *Sharp threshold for the FA-2f kinetically constrained model*, Probab. Theory Related Fields **185** (2023), no. 3, 993–1037 pp. MR4556287
- [15] I. Hartarsky and T. R. Mezei, *Complexity of two-dimensional bootstrap percolation difficulty: algorithm and NP-hardness*, SIAM J. Discrete Math. **34** (2020), no. 2, 1444–1459 pp. MR4117299
- [16] I. Hartarsky and R. Morris, *The second term for two-neighbour bootstrap percolation in two dimensions*, Trans. Amer. Math. Soc. **372** (2019), no. 9, 6465–6505 pp. MR4024528
- [17] A. E. Holroyd, *Sharp metastability threshold for two-dimensional bootstrap percolation*, Probab. Theory Related Fields **125** (2003), no. 2, 195–224 pp. MR1961342
- [18] A. E. Holroyd, T. M. Liggett, and D. Romik, *Integrals, partitions, and cellular automata*, Trans. Amer. Math. Soc. **356** (2003), no. 8, 3349–3368 pp. MR2052953
- [19] P. M. Kogut and P. L. Leath, *Bootstrap percolation transitions on real lattices*, J. Phys. C **14** (1981), no. 22, 3187–3194 pp.
- [20] J. van den Berg and H. Kesten, *Inequalities with applications to percolation and reliability*, J. Appl. Probab. **22** (1985), no. 3, 556–569 pp. MR799280
- [21] A. C. D. van Enter, *Proof of Straley’s argument for bootstrap percolation*, J. Stat. Phys. **48** (1987), no. 3-4, 943–945 pp. MR914911