A random walk approach to high-dimensional critical phenomena

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Abstract

We present a "black box" proof of mean-field near-critical behaviour for a family of functions on $\mathbb{Z}^d$ ($d > 2$) satisfying a short list of assumptions. The functions represent two-point functions of a lattice statistical mechanical model in the subcritical or critical regimes, and are proved to have decay of the form $|x|^{-d+2+\varepsilon}\exp[-c|x|/\xi]$, for any $\varepsilon > 0$. The black box applies to several models for which commonplace methods can be used to verify the assumptions. Applications include models of self-avoiding walk, percolation, spins (Ising, XY, $|\varphi|^4$), and lattice trees, all above their upper critical dimensions. The proof is based on random walk techniques, and provides a new, unified, probabilistic, and relatively simple proof of mean-field near-critical behaviour.

Contents

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1 Introduction and results

1.1 Background and motivation

A central goal in statistical mechanics is to understand the behaviour of lattice models which undergo a *phase transition* at a *critical point.* Of particular interest is the intricate fractal behaviour of the model at and near its critical point. This behaviour can be described by *critical exponents.* To prove the existence of these critical exponents, let alone to compute their values, is in general an open problem. A profound interplay between the dimension of the underlying lattice and the critical behaviour is expected. We focus on the case of models defined on the hypercubic lattice $\mathbb{Z}^d$ in dimensions $d \ge 2$, and particularly on high d.

A striking observation was made in the physics literature around half a century ago [38,89]: above an *upper critical* dimension d_c, the critical behaviour of the model simplifies in the sense that its critical exponents match those obtained when the model is formulated on a tree rather than on $\mathbb{Z}^d$. The regime $d \ge d_c$ constitutes the *mean-field* regime of the model. It is characterised by the emergence of Gaussian features, with logarithmic corrections at the upper critical dimension $d = d_c$.

Our purpose in this paper is to provide a general analysis of dimensions $d > d_c$ for a wide variety of models—a black box for proving mean-field behaviour. Models covered by our method include: the self-avoiding walk, Bernoulli bond percolation, the Ising model, the lattice $|\varphi|^4$ model (with one or two components), the XY model, and lattice trees, all above their upper critical dimension. The self-avoiding walk and spin models are predicted to have $d_c = 4$, for percolation the prediction is $d_c = 6$, and for lattice trees it is $d_c = 8$.

A fundamental object is the *two-point function* $G_\beta(x)$, which for parameter β (e.g., inverse temperature) expresses correlation (e.g., of spins) between the point $x \in \mathbb{Z}^d$ and the origin. The two-point function typically decays exponentially for β below a critical value β_c, and decays algebraically at β_c. The *susceptibilty* $\chi(\beta)$ and *correlation length of order two* $\xi_2(\beta)$ are defined for $\beta < \beta_c$ by

$$\chi(\beta) := \sum_{x\in\mathbb{Z}^d} G_\beta(x), \qquad \xi_2(\beta)^2 := \frac{1}{\chi(\beta)} \sum_{x\in\mathbb{Z}^d} |x|_2^2 G_\beta(x), \tag{1.1}$$

where $|x|_2 := (x_1^2+\ldots+x_d^2)^{1/2}$ is the Euclidean norm. For a *second-order* phase transition, three critical exponents η, γ, ν are associated with these quantities via

$$G_{\beta_c}(x) \approx \frac{1}{|x|_2^{d-2+\eta}}, \qquad \chi(\beta) \approx \frac{1}{(\beta_c-\beta)^\gamma}, \qquad \xi_2(\beta) \approx \frac{1}{(\beta_c-\beta)^\nu}, \tag{1.2}$$

in the limits $|x|_2 \to \infty$ (Euclidean norm) and $\beta \uparrow \beta_c$. For the moment we leave the approximation symbol "$\approx$" undefined, but we will be more careful later. The three exponents are predicted to be related by Fisher's scaling relation: $\gamma = (2-\eta)\nu$. Fisher's relation arises from the ansatz that the near-critical behaviour of the two-point function is given by

$$G_\beta(x) \approx \frac{1}{|x|_2^{d-2+\eta}} g(|x|_2/\xi_2(\beta)), \tag{1.3}$$

for some "nice" function g of rapid decay. The relation (1.3) is meant to capture the joint behaviour of the two-point function for β near β_c and large $|x|_2$. For a fixed $\beta < \beta_c$, as $|x|_2 \to \infty$, it is in many cases proved that there is instead *Ornstein–Zernike decay* [72,91], i.e., with power $|x|_2^{-(d-1)/2}$; the crossover between these two forms of decay is the subject of [66,71]. The mean-field values of the critical exponents, for the aforementioned models except lattice trees, are

$$\eta = 0, \qquad \gamma = 1, \qquad \nu = \frac{1}{2}. \tag{1.4}$$

For lattice trees, instead we have $\eta = 0$, $\gamma = \frac{1}{2}$, $\nu = \frac{1}{4}$.

In this paper, we identify general hypotheses on a class of functions $G_\beta : \mathbb{Z}^d \to [0,\infty]$ that imply an upper bound version of (1.3). Namely, we obtain that for any $\varepsilon > 0$ there exist constants $c, C > 0$ such that for every $\beta \le \beta_c$ and every $x \in \mathbb{Z}^d$,

$$G_\beta(x) \le C\delta_0(x) + \frac{C}{(1\vee|x|_2)^{d-2-\varepsilon}} \exp\left(-c\frac{|x|_2}{\xi_2(\beta)}\right). \tag{1.5}$$

Here δ_0 is the Kronecker function at 0, i.e., $\delta_0(x) = 1$ if $x = 0$ and $\delta_0(x) = 0$ otherwise. Our hypotheses also imply that the susceptibility and correlation length of order 2 obey

$$\frac{c}{\beta_c-\beta} \le \chi(\beta) \le \frac{C}{\beta_c-\beta}, \qquad \frac{c}{(\beta_c-\beta)^{1/2-\delta}} \le \xi_2(\beta) \le \frac{C}{(\beta_c-\beta)^{1/2+\delta}}, \tag{1.6}$$

where δ is a small parameter we require to be present in the model. For example, δ can be associated with the spread parameter of a spread-out model, or the small parameter defining the weakly self-avoiding walk model, etc. The above statements can be interpreted as saying that $\eta \ge -\varepsilon$ for every $\varepsilon > 0$, that $\gamma = 1$, and that $\nu \in [\frac{1}{2}-\delta, \frac{1}{2}+\delta]$. Lattice trees, with their different critical exponents, are handled via a change of variable in order to fit this framework.

These bounds are not new and are not as strong as the best currently available results. However, the novelty lies in our method, which is radically different from previous approaches. Our proof:

- is relatively extremely simple,
- applies generally and essentially simultaneously to many models, and
- is more connected to standard probability theory than previous approaches as it is propelled by elementary random walk theory.

Our hypotheses on the functions G_β involve a finite-difference inequality for the two-point function which is easily verifiable in practice, in any dimension. For percolation it already appeared in [49], and for the Ising model it appeared in [73]. It implies an upper bound on the derivative $\partial_\beta G_\beta(x)$ which has been a standard part of the theory since the 1980s. The finite-difference inequality is supplemented by a lower bound on $\partial_\beta G_\beta$, which has also been a standard part of the theory since the 1980s. The lower bound involves a *diagram* (bubble or triangle or square diagram, depending on the model) which, to be useful, we must prove to be small. This forces the dimension to exceed d_c in order for the diagram to be finite, and also requires a small parameter in the formulation of the model to make the diagram small.

The weakly self-avoiding walk and the weakly-coupled $|\varphi|^4$ model have built-in small parameters. For other models, a small parameter can be introduced by considering a *spread-out* version of the model. For percolation, this was initiated in [44]. In the spread-out models, nearest-neighbour edges are replaced by long-range edges up to a distance of order R, where R is the *spread* parameter. The reciprocal R^{-1} provides the small parameter we require. The spread-out model is still expected to offer a valid description of the critical regime of the nearest-neighbour model thanks to the *universality* hypothesis.

As a brief summary, our method applies to prove (1.5) and (1.6) for the following:

- nearest-neighbour weakly self-avoiding walk and spread-out strictly self-avoiding walk for $d > 4$ (continuous-time models are included),
- spread-out bond percolation for $d > 6$,
- spread-out Ising and XY models for $d > 4$,
- nearest-neighbour weakly-coupled lattice $|\varphi|^4$ model, or spread-out $|\varphi|^4$ model with arbitrary coupling, for $d > 4$ (1-component and 2-component),
- spread-out lattice trees for $d > 8$ (in this case the exponents are different as discussed above).

We discuss applications in more detail in Section 1.3. Precise definitions of the models are given in Section 6.

1.2 Main result: the black box

We now present the model-independent results.

For $x \in \mathbb{R}^d$, we write $|x|_2 := (x_1^2 + \ldots + x_d^2)^{1/2}$ for the Euclidean norm, $|x| = \max_{1 \le i \le d} |x_1|$ for the infinity norm. The two norms are equivalent, but it is convenient to work with both. For $k \geq 1$, we define the box $\Lambda_k := [-k, k]^d \cap \mathbb{Z}^d$, and write $\Lambda_k(x) = \Lambda_k + x$ for the box centred at x. Also, for $G : \mathbb{Z}^d \to [0, \infty)$, we write

$$\|G\|_1 := \sum_{x \in \mathbb{Z}^d} G(x), \qquad \||x|_2^2 \cdot G\|_1 := \sum_{x \in \mathbb{Z}^d} |x|_2^2 G(x). \tag{1.7}$$

The next definition introduces the kernels J which for spin models will be the Hamiltonian's spin-spin coupling.

Definition 1.1. (Admissible kernel.) An *admissible kernel* is a function $J : \mathbb{Z}^d \to [0, \infty)$ with the following properties:

(i) The kernel J satisfies $J_0 = 0$, is normalised in the sense that $\sum_{x \in \mathbb{Z}^d} J_x = 1$, and is $\mathbb{Z}^d$*-symmetric* in the sense that $J_x = J_{x'}$ if x' is obtained by permuting components of x and/or multiplying any component x by ± 1.

(ii) J is *finite-range* in the sense that the *range* R_J obeys

$$R_J = \max\{|x| : J_x > 0\} \in [1, \infty). \tag{1.8}$$

(iii) The *variance* of J is defined by

$$\sigma_J^2 = \sum_{x \in \mathbb{Z}^d} |x|_2^2 J_x. \tag{1.9}$$

By definition, $\sigma_J \leq R_J$. There exists a constant $c_0 \in (0, 1]$ such that, for all $x \in \mathbb{Z}^d$,

$$c_0 R_J \leq \sigma_J, \qquad J_x \leq c_0^{-1} R_J^{-d}. \tag{1.10}$$

Example 1.2. Two important examples of admissible kernels are:

(i) (Nearest-neighbour model). For every $x \in \mathbb{Z}^d$,

$$J_x = \frac{1}{2d} \mathbb{1}_{|x|_2 = 1}. \tag{1.11}$$

For the nearest-neighbour model, $R_J = \sigma_J = 1$ and we may take $c_0 = 1$.

(ii) (Uniform spread-out model). For some $R \geq 1$ and for every $x \in \mathbb{Z}^d$,

$$J_x = \frac{1}{|\Lambda_R| - 1} \mathbb{1}_{0 < |x| \leq R}. \tag{1.12}$$

Here $R_J = R$, $J_x \leq R^{-d}$, and there is a $c_0 > 0$ such that $\sigma_J \geq c_0 R$. More general spread-out examples are given, e.g., in [43, 44, 78].

Next, we introduce the class $\mathcal{G}$ of functions G_β to which our analysis applies. Our motivation stems from statistical mechanics where G_β is the two-point function of some underlying model which undergoes a phase transition. As we will see in Section 6, it is elementary and standard that the two-point functions of all the models mentioned at the end of Section 1.1 do belong to $\mathcal{G}$. In Definition 1.3, we use the notation β_c normally reserved for a critical point. In our applications, β_c will indeed be a (finite) critical point of a statistical mechanical model, but that interpretation is not relevant until we reach Theorem 1.7 below.

Definition 1.3. We define $\mathcal{G}$ to be the family of functions $G_\beta : \mathbb{Z}^d \to [0, \infty]$, indexed by $\beta \in [0, \beta_c]$ for some $\beta_c \in [0, \infty]$, which satisfy the following conditions:

(*i*) (Initial condition.) $G_0 = \delta_0$.

(*ii*) (Regularity.) For every $x \in \mathbb{Z}^d$, the function $\beta \mapsto G_\beta(x)$ is monotone non-decreasing and differentiable on the interval $[0, \beta_c)$.

(*iii*) (Symmetry.) For every $\beta \geq 0$, G_β is $\mathbb{Z}^d$-symmetric.

(*iv*) (Exponential decay.) For every $\beta \in [0, \beta_c)$, the function $x \mapsto G_\beta(x)$ decays exponentially.

(*v*) (Limit as $\beta \nearrow \beta_c$ when $\beta_c < \infty$.) For every $x \in \mathbb{R}^d$, $\lim_{\beta \uparrow \beta_c} G_\beta(x) = G_{\beta_c}(x)$, but we do *not* assume that $G_{\beta_c}(x)$ is finite.

As we discuss in more detail in Remark 1.8 below, the assumption that $G_0 = \delta_0$ can be relaxed to $G_0 = A\delta_0$ with $A > 0$, by a simple change of variables. The more fundamental restrictions appear in the following assumption. Here and later, the convolution of two absolutely summable functions $f, g : \mathbb{Z}^d \to \mathbb{R}$ is defined by $(f * g)(x) := \sum_{y \in \mathbb{Z}^d} f(y)g(x-y)$.

Assumption I. Let J be an admissible kernel and let $G \in \mathcal{G}$. We assume that G satisfies the following:

(*i*) (Finite-difference upper bound.) For every $0 \leq \beta' \leq \beta < \beta_c$,

$$G_\beta \leq G_{\beta'} + (\beta - \beta')(G_{\beta'} * J * G_\beta). \tag{\textbf{I}.\textbf{1}}$$

An immediate and important consequence of (**I**.**1**) is the differential inequality

$$\partial_\beta G_\beta = \lim_{\beta' \uparrow \beta} \frac{G_\beta - G_{\beta'}}{\beta - \beta'} \leq G_\beta * J * G_\beta. \tag{1.13}$$

(*ii*) (Differential lower bound.) For every $\beta < \beta_c$, there exists a function $H_\beta : \mathbb{Z}^d \to [0, \infty)$ (which may depend on J and G) such that

$$\partial_\beta G_\beta \geq G_\beta * (J - H_\beta) * G_\beta. \tag{\textbf{I}.\textbf{2}}$$

In addition, for all $x \in \mathbb{Z}^d$, $H_0(x) = 0$, $H_\beta(x) = H_\beta(-x)$, and $\beta \in [0, \beta_c) \mapsto H_\beta(x)$ is continuous and decays exponentially in x for each fixed β.

The pointwise bound (**I**.**2**) is actually stronger than what we require: our proofs only use the weaker assumption that (**I**.**2**) holds when multiplied by $|x|^p$ for $p = 0, 2$ and then summed over $x \in \mathbb{Z}^d$. In applications, the pointwise bound is verified instead of its weaker summed counterparts, so we have assumed (**I**.**2**) for simplicity.

A basic example satisfying Assumption I is the massive lattice Green function, which is defined as follows. Let $(Y_n)_{n \geq 0}$ denote the random walk on $\mathbb{Z}^d$ started from $Y_0 = 0$ and with transition probabilities $\mathbb{P}_J[Y_{n+1} = y \mid Y_n = x] = J_{y-x}$, where J is an admissible kernel. The *massive lattice Green function* is defined, for $\beta \in [0, 1]$ and $x \in \mathbb{Z}^d$, by

$$\mathbb{C}_\beta(x) = \sum_{n=0}^{\infty} \beta^n \mathbb{P}_J[Y_n = x]. \tag{1.14}$$

It is elementary that $\mathbb{C} = (\mathbb{C}_\beta)_{\beta \geq 0}$ satisfies the requirements of Definition 1.3, with $\beta_c = 1$, and therefore $\mathbb{C} \in \mathcal{G}$. Also, as we verify in detail in Section 2.1, $\mathbb{C}$ satisfies (**I.1**)–(**I.2**) with *equalities* and with $H = 0$:

$$\mathbb{C}_\beta(x) - \mathbb{C}_{\beta'}(x) = (\beta - \beta')(\mathbb{C}_{\beta'} * J * \mathbb{C}_\beta)(x), \tag{1.15}$$

$$\partial_\beta \mathbb{C}_\beta(x) = (\mathbb{C}_\beta * J * \mathbb{C}_\beta)(x). \tag{1.16}$$

Thus, the inequalities of Assumption I specify that the functions G_β obey inequality variants of the equalities obeyed by the massive lattice Green function.

In applications, the upper bound (**I.1**) holds without further assumption in all the models we consider, in all dimensions. The lower bound (**I.2**) also holds in all dimensions, with a model-dependent function H_β. As concrete examples, for self-avoiding walk and percolation we will see in Section 6 that (**I.2**) holds with

$$H_\beta^{\mathrm{SAW}}(x) = \delta_0(x)(G_\beta * J * G_\beta)(0), \qquad H_\beta^{\mathrm{perc}}(x) = G_\beta(x)(G_\beta * J * G_\beta)(x). \tag{1.17}$$

The exponential decay of G_β ensures that each of $H_\beta^{\mathrm{SAW}}(x)$ and $H_\beta^{\mathrm{perc}}(x)$ is not only finite but also summable, for all $\beta < \beta_c$. When summed over $x \in \mathbb{Z}^d$, these quantities are respectively the *open* bubble and *open* triangle diagrams depicted in Figure 1. Summability at β_c will be a consequence of our results, and for this we will need to assume that d is above the upper critical dimension. Furthermore, our method requires $\sum_{x \in \mathbb{Z}^d} H_\beta(x)$ to be not merely finite, but also small, and this forces us to introduce a small parameter into a model's definition.

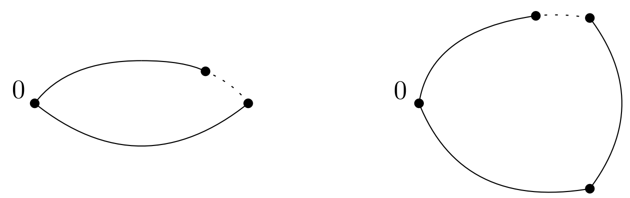

Figure 1: Diagrammatic representations of the *open bubble* (left) and the *open triangle* (right). Unlabelled vertices are summed over $\mathbb{Z}^d$. Solid lines represent a two-point function G_β, and dotted lines represent the kernel J. The diagrams are called "open" due to the dotted line. The J line causes the contribution from all vertices being at the origin to be zero, as opposed to a contribution 1 from the closed diagram without a J factor. For spread-out models, this J is what produces small $\|H_\beta\|_1$ by taking R_J large.

Upper and lower differential inequalities as in (1.13) and (**I.2**) are a well-established part of the mathematical study of phase transitions, going back to the 1980s. A version of the finite-difference upper bound (**I.1**) was used for percolation in [49, Lemma 2.4], and for the Ising model in [73, Proposition 7.3.3]. Here, we use (**I.1**) as a fundamental and general driving principle for our analysis. As noted above, (**I.1**) implies the standard differential upper bound (1.13). The verification of (**I.1**) in applications is sometimes in the spirit of the proof of the Simon–Lieb inequality [60, 80].

Let $F_\beta = J * G_\beta$. By linearity, Assumption I implies that F_β satisfies a version of (**I.1**), (1.13), and (**I.2**), namely, for $0 \leq \beta' \leq \beta < \beta_c$,

$$F_\beta \leq F_{\beta'} + (\beta - \beta')(F_{\beta'} * F_\beta), \tag{1.18}$$

$$\partial_\beta F_\beta \leq F_\beta * F_\beta, \tag{1.19}$$

$$\partial_\beta F_\beta \geq F_\beta * (J - H_\beta) * G_\beta. \tag{1.20}$$

For $\beta < \beta_c$, we define the *susceptibility* χ, and the *correlation length* ξ *of order two* for F_β, by

$$\chi(\beta) = \|G_\beta\|_1 = \|F_\beta\|_1, \tag{1.21}$$

$$\xi(\beta)^2 = \frac{\||x|_2^2 \cdot F_\beta\|_1}{\|F_\beta\|_1} = \frac{\||x|_2^2 \cdot F_\beta\|_1}{\chi(\beta)}. \tag{1.22}$$

By Assumption I, $G \in \mathcal{G}$ and therefore G_β and F_β decay exponentially for $\beta < \beta_c$. It follows that both χ and ξ are finite for $\beta < \beta_c$. It is possible that $\chi(\beta)$ diverges to ∞ as $\beta \uparrow \beta_c$, but we do not assume this until Theorem 1.7. We also define an "error term" by

$$E(\beta) = \sup_{0 \le t \le \beta} \left(\|H_t\|_1 + \frac{\||x|_2^2 \cdot H_t\|_1}{\xi(t)^2} \right). \tag{1.23}$$

The supremum in (1.23) is present solely to ensure that $E(\beta)$ is increasing in β. By definition, $E(0) = 0$. Since each of the three functions inside the supremum in (1.23) is continuous in $t \in [0, \beta_c)$ by the Dominated Convergence Theorem, the supremum $E(\beta)$ is also continuous in $\beta \in [0, \beta_c)$.

Remark 1.4. The conventional notation for the correlation length of order two is ξ_2 (as in (1.1) for G_β), whereas the symbol ξ (without subscript) is typically used for the reciprocal of the exact exponential rate of decay of $G_\beta(x)$ (the *correlation length*) for subcritical β. We do not use this last concept, so in order to lighten the notation, and also to avoid notational clash with (1.1), we write ξ instead of ξ_2 in (1.22).

When $E(\beta)$ is small, the upper and lower bounds of (1.13) and (**I.2**) become closer to equalities. It is thus natural to introduce the following quantity. For $\delta > 0$, let

$$\beta(\delta) := \sup\{\beta \in [0, \beta_c) : E(\beta) < \delta\}. \tag{1.24}$$

Using elementary calculus to integrate (1.13) and (**I.2**), we prove (see Proposition 4.1) that if $\delta < 1$, then for every $0 \le \beta < \beta(\delta)$, we have

$$\beta(\delta) \le (1-\delta)^{-1}, \qquad \left(\frac{\chi(\beta')}{\chi(\beta)}\right)^{\frac{1+\delta}{1-\delta}} \le \left(\frac{\xi(\beta')}{\xi(\beta)}\right)^2 \le \left(\frac{\chi(\beta')}{\chi(\beta)}\right)^{1-2\delta}. \tag{1.25}$$

The second inequality in (1.25) forms a key ingredient in our strategy: the fact that $(\xi(\beta')/\xi(\beta))^2$ remains comparable to $\chi(\beta')/\chi(\beta)$ is used to upgrade bounds that hold at β' to bounds at a larger value β. For the lattice Green function, which has $E = 0$, the two ratios are equal; this is a benefit of defining ξ using F_β rather than G_β.

Since (as we have just observed) $\beta(\delta) < \infty$, it makes sense to evaluate G_β at $\beta = \beta(\delta)$, even if $\beta(\delta) = \beta_c$. Our main theorem provides an upper bound for $G_\beta(x)$ for all $\beta \le \beta(\delta)$, for δ sufficiently small. Recall that σ_J is the variance of J, defined in (1.9), and recall the definition of c_0 from (1.10).

Theorem 1.5. *Let $d > 2$ and $\varepsilon \in (0,1)$. There exist $\mathbf{c}, \mathbf{C} > 0$ depending on (d, c_0), and $\boldsymbol{\delta} \in (0,1)$ depending on (d, c_0, ε), such that, if J and G obey Assumption* I*, then for every $\beta \in [0, \beta(\boldsymbol{\delta})]$ and every $x \in \mathbb{Z}^d$,*

$$G_\beta(x) \le \delta_0(x) + \frac{\mathbf{C}}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2-\varepsilon} \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right). \tag{1.26}$$

It is an interesting question whether Theorem 1.5 remains valid with $\varepsilon = 0$. We do not have an answer for the question.

Theorem 1.5 is general and its proof is independent of the particular choice of J and G. However it fails, on its own, to cover the entire interval $\beta \in [0, \beta_c]$, since it does not assert that $\beta(\boldsymbol{\delta}) = \beta_c$. The remedy for this defect involves the following additional assumption. Its verification exploits the particular model-dependent manner that H_β is expressed in terms of G_β, to extract a bound on the former from a bound on the latter. The explicit forms for H_β given in (1.17) accurately suggest that the verification of Assumption II is in practice a routine task.

Assumption II. The error term obeys $E(\beta(\boldsymbol{\delta})) < \boldsymbol{\delta}$.

Theorem 1.6. *Suppose that J and G obey both of Assumptions* I *and* II. *Then $\beta(\boldsymbol{\delta}) = \beta_c$, so $\beta_c < \infty$ and* (1.26) *holds for every $\beta \le \beta_c$.*

Proof. By Assumption II we have $E(\beta(\boldsymbol{\delta})) < \boldsymbol{\delta}$. If $\beta(\boldsymbol{\delta}) < \beta_c$, then by the continuity of E on $[0, \beta_c)$, there must exist a $\beta^* > \beta(\boldsymbol{\delta})$ such that $E(\beta^*) < \boldsymbol{\delta}$. This contradicts the definition of $\beta(\boldsymbol{\delta})$, so it must be the case that $\beta(\boldsymbol{\delta}) = \beta_c$. This completes the proof. □

The following theorem is an easy consequence of our assumptions, as we will show in Proposition 4.4 via integration of the assumed differential inequalities (1.13) and (**I.2**). Its assumption that $\chi(\beta_c) = \infty$ identifies β_c as the *critical point* in our applications to statistical mechanics.

Theorem 1.7. *Suppose that J and G obey both Assumptions* I *and* II. *Suppose further that $\chi(\beta_c) = \infty$. Then, for every $\beta < \beta_c$, and with $E = E(\beta_c)$ (which is less than $\boldsymbol{\delta}$), we have*

$$1 \le \beta_c \le \frac{1}{1-E}, \tag{1.27}$$

$$\frac{1}{\beta_c - \beta} \le \chi(\beta) \le \frac{1}{1-E}\frac{1}{\beta_c - \beta}, \tag{1.28}$$

$$\left(\frac{1}{\beta_c - \beta}\right)^{1-2E} \le \frac{\xi(\beta)^2}{\sigma_J^2} \le \left(\frac{1}{(1-E)(\beta_c - \beta)}\right)^{\frac{1+E}{1-E}}. \tag{1.29}$$

The linear divergence of the susceptibility in (1.28) is a statement that the critical exponent γ takes its mean-field value $\gamma = 1$, and (1.29) states that the critical exponent ν is within order $E(\beta_c)$ of its mean-field value $\frac{1}{2}$. In the verification of Assumption II for our applications in Section 6, for spread-out models we will find in every case that $E(\beta_c)$ is at most of order σ_J^{-d}, and hence at most of order R^{-d}. This means that the powers in the upper and lower bounds on $(\xi/\sigma_J)^2$ in (1.29) can be brought as close to 1 as desired by taking R large enough. With this estimate on $E(\beta_c)$, it also follows from (1.27) that the critical value obeys $\beta_c = 1 + O(R^{-d})$. This $O(R^{-d})$ is optimal, as previously observed for various models in [48, 55].

Remark 1.8. It is part of Definition 1.3 that $G_0 = \delta_0$. For some models (e.g., the $|\varphi|^4$ model), it is instead the case that $G_0 = A\delta_0$ with $A > 0$ unequal to 1. In this case, the change of variables

$$\widetilde{G}_\beta(x) = \frac{1}{A} G_{\beta/A}(x) \tag{1.30}$$

yields a family of functions $\widetilde{G}_\beta$ which do satisfy Definition 1.3 and Assumption I when G_β satisfies them excepting $G_0 = A\delta_0$. By applying our theory to $\widetilde{G}$ and then undoing the change of variables, we find that (1.26), (1.27) and (1.29) are modified to:

$$G_\beta(x) \le A\delta_0(x) + \frac{A\mathbf{C}}{\sigma_J^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2-\varepsilon} \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right), \tag{1.31}$$

$$\frac{1}{A} \le \beta_c \le \frac{1}{A(1-E)}, \tag{1.32}$$

$$\left(\frac{1}{A(\beta_c-\beta)}\right)^{1-2E} \le \frac{\xi(\beta)^2}{\sigma_J^2} \le \left(\frac{1}{(1-E)A(\beta_c-\beta)}\right)^{\frac{1+E}{1-E}}. \tag{1.33}$$

Equation (1.28) remains unchanged even if $A \neq 1$.

A further consequence of Theorem 1.6 is that, under Assumptions I and II, the p-fold convolution of G_{β_c} satisfies

$$G_{\beta_c}^{*p}(0) < \infty \quad \text{if} \quad d > p(2+\varepsilon). \tag{1.34}$$

The condition for finiteness reduces to $d > 2p$, since d is an integer and ε may be chosen as small as desired. The convolution $G_{\beta_c}^{*p}(0)$ is called the critical *bubble diagram* when $p = 2$, the critical *triangle diagram* when $p = 3$, and the critical *square diagram* when $p = 4$. Thus, the critical bubble, triangle and square diagrams are respectively finite when d is strictly greater than $4, 6, 8$ (for small ε). The closely related open bubble and triangle diagrams (with their extra factors J) are depicted in Figure 1.

In Section 6, we verify that each of the six models mentioned at the end of Section 1.1 satisfies Assumptions I and II. Theorem 1.6 and its consequences therefore apply to all six models.

1.3 Application to statistical mechanical models

1.3.1 Results

We now discuss the application of our black box to specific models. For this, it is sufficient to verify that the model's two-point function obeys Definition 1.3 (possibly with $G_0 = A\delta_0$), Assumption I, and Assumption II. For the models we consider, this is a relatively routine procedure that does not require innovation. Indeed, in all cases the verification of Definition 1.3 and Assumption II is based on theory dating back to the 1980s. Assumption I is a bit more substantial, but also follows a well-trodden path.

We emphasise that for all but the spread-out XY, $|\varphi|^4$, or continuous-time weakly self-avoiding walk models, our results are not as strong as those obtained by previous methods. We do prove new results for these three spread-out models. Notwithstanding, our purpose in this paper is to provide a new, unified, relatively very easy, random walk approach to the subject. We view Theorem 1.6 as a first step for our methods, which upon further development may lead to more refined results for the critical regime.

We summarise our applications here. Precise definitions of the models are deferred to Section 6. In the following, the adjective "nearest-neighbour" indicates that the model is defined with the nearest-neighbour J defined in (1.11), and "spread-out" indicates that J obeys Definition 1.3 with a large range R.

Self-avoiding walk. For precise definitions, see Sections 6.1 and 6.2. Let $d > 4$. Let G_β denote the two-point function for the nearest-neighbour weakly self-avoiding walk with repulsion strength $\lambda \in (0, \lambda_0]$ with λ_0 sufficiently small, or the two-point function of the spread-out strictly self-avoiding walk with range R sufficiently large. In the latter case, $G_\beta(x)$ is a generating function for self-avoiding walks from 0 to x, whose allowed steps are from u to v whenever $J_{v-u} > 0$, and βJ_{v-u} is the weight of a step. Let β_c denote the critical point in either case. Then, $G_\beta(x)$ obeys the near-critical estimate (1.26) for all $\beta \in [0, \beta_c]$. The conclusions of Theorem 1.7 all hold, and the critical bubble diagram $\sum_{x\in\mathbb{Z}^d} G_{\beta_c}(x)^2$ is finite. For the weakly self-avoiding walk the error E is $O(\lambda)$, and for the spread-out strictly self-avoiding walk it is $O(R^{-d})$. Stronger previous results obtained by other methods are discussed in Section 1.3.2. For a continuous-time model of the weakly self-avoiding walk, our results include both weak coupling for the nearest-neighbour model and arbitrary coupling for spread-out models (the latter is a new result).

Bernoulli percolation. For precise definitions, see Section 6.3. Let $d > 6$. Consider spread-out Bernoulli bond percolation on $\mathbb{Z}^d$, with R sufficiently large. The edge set consists of pairs $\{u, v\}$ such that $J_{v-u} > 0$. An edge $\{u, v\}$ is *open*, independently of all other edges, with probability βJ_{v-u} and otherwise is *closed*. Let $G_\beta(x) = \mathbb{P}_\beta[0 \leftrightarrow x]$ denote the probability that 0 and x are connected by a path consisting of open bonds. Let $\beta_c \in (0, \infty)$ denote the critical point. Then, $G_\beta(x)$ obeys the near-critical estimate (1.26) for all $\beta \in [0, \beta_c]$. The conclusions of Theorem 1.7 all hold with $E = O(R^{-d})$, and the critical triangle diagram $\sum_{x,y\in\mathbb{Z}^d} G_{\beta_c}(x)G_{\beta_c}(y-x)G_{\beta_c}(y)$ is finite. In other words, the *triangle condition* of [6] holds. This has important implications for critical exponents, e.g., it is shown in [7] (see also [50]) that the triangle condition implies the existence and mean-field values $\delta = 2$ and $\beta = 1$ for the critical exponents governing the critical cluster-size distribution and the percolation probability (despite the notational clash, these two exponents should not be confused with our different usage of δ, β). Stronger previous results obtained by other methods are discussed in Section 1.3.2.

$|\varphi|^4$ **model.** For precise definitions, see Section 6.4. Let $d > 4$. The nearest-neighbour lattice $|\varphi|^4$ model has single-spin distribution $\exp[-\lambda|\varphi_x|^4 - \mu|\varphi_x|^2]$ with interaction given by $\exp[\beta \sum_{u,v} J_{v-u}(\varphi_u \cdot \varphi_v)]$. We consider spins with either one or two components, i.e., $\varphi_x \in \mathbb{R}$ or $\varphi_x \in \mathbb{R}^2$. The two-point function is again the spin-spin correlation $G_\beta(x) = \langle \varphi_0 \cdot \varphi_x \rangle_\beta$, and the critical point is denoted β_c. We consider both the nearest-neighbour model with $\lambda > 0$ small and any $\mu > 0$[1], and the spread-out model with any $\lambda > 0$ and any $\mu \in \mathbb{R}$. In both cases, we prove that $G_\beta(x)$ obeys the near-critical estimate (1.26) for all $\beta \in [0, \beta_c]$. The conclusions of Theorem 1.7 all hold, with $E = O(\lambda)$ for the weakly-coupled nearest-neighbour model, and with $E = O(R^{-d})$ for the spread-out model. Some stronger previous results obtained by other methods are discussed in Section 1.3.2. Our results for the spread-out 2-component $|\varphi|^4$ model are new.

Ising model. For precise definitions, see Section 6.5. Let $d > 4$. The spread-out Ising model is a ferromagnetic spin system with spins taking values ± 1, defined via a Gibbs measure with Hamiltonian $H(\sigma) = -\sum_{u,v} J_{v-u}\sigma_u\sigma_v$ and inverse temperature β. The two-point function is the spin-spin correlation $G_\beta(x) = \langle \sigma_0\sigma_x \rangle_\beta$. Let β_c denote the critical point. Then, $G_\beta(x)$ obeys the near-critical estimate (1.26) for all $\beta \in [0, \beta_c]$. The

[1] Our analysis of the weakly coupled model perturbs around $\lambda = 0$, and the restriction to positive μ ensures that the model exists when $\lambda = 0$.

conclusions of Theorem 1.7 all hold with $E = O(R^{-d})$. Stronger previous results obtained by other methods are discussed in Section 1.3.2.

XY model. For precise definitions, see Section 6.6. Let $d > 4$. The XY model is a spin model on $\mathbb{Z}^d$ with the single spin distribution uniformly distributed on the unit circle in $\mathbb{R}^2$, with interaction given by $\exp[\beta \sum_{u,v} J_{v-u}(\phi_u \cdot \phi_v)]$. We take J to be the spread-out kernel. The critical point is denoted β_c. Then, for R sufficiently large, $G_\beta(x)$ obeys the near-critical estimate (1.26) for all $\beta \in [0, \beta_c]$. The conclusions of Theorem 1.7 all hold, with $E = O(R^{-d})$. These are new results for the XY model.

Lattice trees. For precise definitions, see Section 6.7. A lattice tree is a finite connected acyclic bond cluster. We consider the spread-out model with J given by the spread-out kernel. Bonds are weighted by a parameter p, and there is a finite critical value p_c. We prove a version of Theorem 1.5 for lattice trees, for $d > 8$ and for R sufficiently large. This is then used to show that the susceptibility diverges as $(p_c - p)^{-1/2}$ and the correlation length of order 2 diverges as $(p_c - p)^{-\frac{1}{4} \pm O(R^{-d})}$. The different critical exponents here, compared with Theorem 1.7, are obtained via a change of variables argument that is not needed for all the other models mentioned above. References to stronger previous results obtained using the lace expansion are given in Section 1.3.2.

1.3.2 Related work

As mentioned above, our results have been obtained earlier by other methods in most cases, via less simple and more model-dependent methods. We discuss this further here.

Reflection positivity. In the context of the study of spin models such as the Ising model, *reflection positivity* and the *infrared bound* [34, 35] provide important tools to understand the mean-field regime of the models in the regime $d \geq d_c$. These tools can be combined with various geometric representations of the models to study the near-critical or critical regime. Differential inequalities, like those in Assumption I, play an important role. The substantial literature on this includes [1, 3–5, 26, 28, 31, 33, 63, 74, 75]. The infrared bound makes it possible to study the nearest-neighbour model without the need to introduce a small parameter. In particular, it was recently proved in [26] that the nearest-neighbour Ising and φ^4 models (without any small coupling assumption) obey, for $d > 4$ and $\beta \leq \beta_c$,

$$\langle \sigma_0 \sigma_x \rangle_\beta \leq \frac{C}{(1 \vee |x|)^{d-2}} \exp\big(- c(\beta_c - \beta)^{1/2} |x| \big). \tag{1.35}$$

This improves our bound (1.26) by including the nearest-neighbour case, by not requiring a small parameter, by omitting the ε from the power law, and by obtaining the sharp power $(\beta_c - \beta)^{1/2}$ in the exponent. On the other hand, reliance on reflection positivity to obtain the infrared bound is a limitation that restricts results to nearest-neighbour interactions.

Lace expansion. The *lace expansion* refers to a collection of expansion methods which provide extremely detailed results concerning mean-field critical phenomena above the upper critical dimension. The first lace expansion was introduced by Brydges and Spencer in 1985 [19] in the context of weakly self-avoiding walk, and was later extended to apply to percolation, lattice trees and lattice animals, oriented percolation, the contact process, the Ising model, and the φ^4 model; see, e.g., [18, 44, 45, 78, 79, 82]. An essential distinction between the lace expansion and our method is that the lace expansion replaces the inequalities (1.13) and (**I.2**) by exact identities. We may think of the inequalities as obtained in practice by a single inclusion-exclusion bound. The lace expansion obtains an identity by performing inclusion-exclusion to *all* orders. This is necessarily a more complicated business which—while leading to very strong results—is much more model-dependent than our approach.

Convergence of the lace expansion requires a small parameter, which can either be proportional to $(d - d_c)^{-1}$ for the nearest-neighbour model, or related to R^{-1} for the spread-out model. Remarkably, with a computer-assisted proof, the nearest-neighbour strictly self-avoiding walk has been proven to have mean-field behaviour (including convergence to Brownian motion) in all dimensions $d \geq 5$ [47]. For spread-out lattice trees in dimensions $d > 8$, extensive results have been obtained, e.g., [20, 22, 45, 46, 65]. For spread-out percolation in dimensions $d > 6$, and for nearest-neighbour percolation in dimensions $d \geq 11$, mean-field behaviour has been proved [32, 44]. A typical result is that the critical two-point function is asymptotically equal to $c|x|^{-(d-2)}(1 + o(1))$ with the amplitude c given by a convergent series. This involves a deconvolution argument that was pioneered in [42, 43], simplified for weakly self-avoiding walk in [13], and reached its simplest and most general form in [64, 67, 83]. An extension of the latter gives the near-critical mean-field bound (1.35) for both weakly [84] and strictly [62] self-avoiding walk above $d_c = 4$. For near-critical percolation (either with $d \geq 11$ for the nearest-neighbour model or with $d > 6$ for the spread-out model), the sharp upper bound (1.35) was proven in [54] (see also [21, 88]).

For the continuous-time weakly self-avoiding walk, a new continuous-time lace expansion was introduced in [18] to prove mean-field behaviour in dimensions $d > 4$ for sufficiently weak interactions. We obtain complementary results here without the lace expansion, including for spread-out models without a weak coupling assumption.

Renormalisation group. The mean-field scaling behaviour of statistical mechanical models at the upper critical dimension d_c is typically conjectured or proved to be modified by logarithmic factors. The computation of these logarithmic corrections is beyond the scope of our paper. The *renormalisation group* [90] refers to a collection of methods which are now part of textbook theoretical physics and which have been used, in particular, to compute these logarithmic corrections. When it applies, it gives very detailed information. Rigorous renormalisation group methods have been developed for the φ^4 model in [30, 36, 41], for the multi-component $|\varphi|^4$ model in [8, 11, 85], and for the continuous-time weakly self-avoiding walk in [9, 10]. The renormalisation group approach involves a multi-scale analysis which is entirely different from all of the other methods we are discussing. However, our approach is philosophically related to the renormalisation group approach: it is multi-scale in the sense that it does involve an induction which advances understanding of the two-point function at a value β' to a larger value β.

Finally, it is worth mentioning that for long-range Bernoulli percolation (for which J has power-law decay), a non-perturbative analysis based on a multi-scale analysis serves to analyse the critical behaviour not only above and at the upper critical dimension, but also below the upper critical dimension [51–53].

The φ_β method. Recently, a novel approach to the study of the mean-field regime has been implemented to study weakly self-avoiding walk for $d > 4$ and spread-out percolation for $d > 6$ [24, 25]. This method can be referred to as the "φ_β method" due to its use of a special quantity—usually denoted by $\varphi_\beta(S)$ for $S \subset \mathbb{Z}^d$—whose relevance in statistical mechanics was observed in [27, 80]. For weakly self-avoiding walk and spread-out percolation, both upper and lower bounds as in (1.35) are obtained. For both models, bounds for half-space two-point functions are obtained too. Unlike in our black-box approach, the extension of the φ_β method from one model to another requires substantial work.

1.4 Guide to the paper

At the core of the proof of Theorem 1.5 lies a family of random walks on $\mathbb{Z}^d$ called the *effective random walk*, whose one-step transition probability is $F_\beta(x)/\chi(\beta)$, with standard deviation $\xi(\beta)$. A key feature of our analysis is the derivation of *uniform* estimates for the effective random walk. These estimates are of two types: Green function estimates derived from a *regularity* property of the effective random walk and employed to analyse the walk at and above its natural scale $\xi(\beta)$; and a *stability* estimate employed to control the walk at smaller scales. The effective random walk and its important properties are presented in Section 2. Proofs are deferred to Appendix A (for the Green function estimates) and Section 4.4 (for the stability estimate).

With these preliminaries at hand, the proof of our main result is entirely contained in Sections 3–5.

In Section 3, we present a *bootstrap* argument, with which we prove Theorem 1.5 conditionally on the results of Section 2 and three additional propositions, namely Propositions 3.5, 3.6, and 3.7. These three propositions are used to complete the bootstrap at different scales. Bootstrap arguments have played a central role in proofs of mean-field behaviour since one was used to prove convergence of the lace expansion in [81].

In Section 4, we first use elementary calculus to integrate the assumed differential inequalities (1.13) and (**I.2**) (Section 4.1). We then complete the proofs of the regularity and stability properties satisfied by the effective random walk (Sections 4.3– 4.4). Interestingly, Section 4.1 is the only place where we use the differential lower bound (**I.2**).

Section 5 provides the proof of the Propositions 3.5, 3.6, 3.7 used in the proof of Theorem 1.5. This relies on elementary convolution estimates whose proofs are deferred to Appendix B.

Finally, Section 6 consists of a detailed verification of Assumptions I and II for the six statistical mechanical models discussed in Section 1.3.1.

2 Random walks: effective and regular

In this section, we first introduce the massive lattice Green function and its important properties in Section 2.1. In Section 2.2, we define a crucial ingredient in our analysis: the *effective random walk*. In Section 2.3, we introduce a second key ingredient: the concept of a *regular random walk*. We state the important fact that the effective random walk is regular, uniformly in β. Also in Section 2.3, we explain that regular walks obey uniform Green function and anti-concentration estimates. Finally, in Section 2.4, we state an essential *stability* estimate.

2.1 Lattice Green function

A simple prototype and point of departure for our theory is the massive lattice Green function. Let $(Y_n)_{n\geq 0}$ denote the random walk on $\mathbb{Z}^d$ started from $Y_0 = 0$ and with transition probabilities $\mathbb{P}_J[Y_{n+1} = y \mid Y_n = x] = J_{y-x}$, where J is an admissible kernel. Recall the definition of the massive Green function $\mathbb{C}_\beta$ from (1.14).

It is elementary that $\mathbb{C} = (\mathbb{C}_\beta)_{\beta \geq 0}$ satisfies the requirements of Definition 1.3, with $\beta_c = 1$, and therefore $\mathbb{C} \in \mathcal{G}$. Also, $\mathbb{C}$ satisfies Assumption I, as claimed in (1.15)–(1.16) and as we verify next. This verification is much simpler than the model-dependent verification of Assumption I for statistical-mechanical models which lack the Markov property, but it is an instructive example.

Let $\beta', \beta \in [0, 1]$. For the finite-difference upper bound (**I.1**), we use the identity $b^n - a^n = (b - a) \sum_{j=0}^{n-1} b^j a^{n-1-j}$ and interchange summations to see that

$$\mathbb{C}_\beta(x) - \mathbb{C}_{\beta'}(x) = (\beta - \beta') \sum_{j=0}^{\infty} \beta^j \sum_{m=j}^{\infty} (\beta')^{m-j} \mathbb{P}_J[Y_{m+1} = x]. \tag{2.1}$$

We then observe that, for $m \geq j$,

$$\mathbb{P}_J[Y_{m+1} = x] = \sum_{y,z \in \mathbb{Z}^d} \mathbb{P}_J[Y_j = y] \mathbb{P}_J[Y_1 = z - y] \mathbb{P}_J[Y_{m-j} = x - z]. \tag{2.2}$$

After insertion of this into (2.1), we find that

$$\mathbb{C}_\beta(x) - \mathbb{C}_{\beta'}(x) = (\beta - \beta')(\mathbb{C}_{\beta'} * J * \mathbb{C}_\beta)(x), \tag{2.3}$$

which is (**I.1**) with equality.

For the lattice Green function, the differential inequality (**I.2**) (also (1.13)) holds as an equality, with $H_\beta = 0$. Indeed, dividing (2.3) by $\beta - \beta'$ and taking the limit $\beta' \to \beta$ gives

$$\partial_\beta \mathbb{C}_\beta(x) = (\mathbb{C}_\beta * J * \mathbb{C}_\beta)(x) \tag{2.4}$$

Since $E(\beta) = 0$ for all β, $\beta(\delta) = \beta_c$ for any $\delta > 0$, and Assumption II is vacuous. We could therefore be tempted to conclude from Theorem 1.5 that the bound (1.26) applies to the lattice Green function. However, that would be circular, as our proof relies crucially on the sharper bound (with no positive ε in (2.6)) stated in the following proposition. Recall that c_0 is the constant occurring in assumed bounds on J in (1.10).

Proposition 2.1 (Anti-concentration and Green function estimates for the J-random walk.)**.** *Let $d > 2$. There exist $\mathsf{c}, \mathsf{C} > 0$, which depend on d and c_0 but not on J, such that for every $\beta \leq 1$, every $m \geq 1$, and every $x \in \mathbb{Z}^d$,*

$$\mathbb{P}_J[X_m = x] \leq \frac{\mathsf{C}}{\sigma_J^d} \frac{1}{m^{d/2}} \exp\left(-\mathsf{c} \frac{|x|}{\sigma_J \sqrt{m}}\right), \tag{2.5}$$

$$\mathbb{C}_\beta(x) \leq \delta_0(x) + \frac{\mathsf{C}}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2} \exp\left(-\mathsf{c}\sqrt{1-\beta} \frac{|x|}{\sigma_J}\right). \tag{2.6}$$

We prove (2.5) and (2.6) for admissible J in Section A.3 using basic random walk theory. To the best of our knowledge, the sharp dependence of these bounds on σ_J has not appeared previously in the literature. It is worth mentioning the well-known fact that, for $d > 2$, at the critical value $\beta_c = 1$ the Green function obeys $\mathbb{C}_1(x) = c\sigma_J^{-2}|x|_2^{-(d-2)} + O(|x|_2^{-d})$ as $|x|_2 \to \infty$, with c depending only on d [57, 87].

When $\beta < 1$, the bound (2.6) implies a bound on G_β of the desired form (1.26), thanks to the finite-difference inequality in Assumption I. Indeed, since $G_0 = \delta_0$ and $F_0 = J$, (**I.1**) applied to $\beta' = 0$ and $\beta \leq 1$ gives

$$G_\beta \leq \delta_0 + \beta(J * G_\beta). \tag{2.7}$$

With $J^{*0} = \delta_0$, and J^{*k} the convolution of J with itself $k-1$ times (for $k \geq 1$), repeated iterations of the above inequality leads, for every $x \in \mathbb{Z}^d$, to the inequality

$$G_\beta(x) \leq \sum_{k \geq 0} \beta^k J^{*k}(x) = \sum_{k \geq 0} \beta^k \mathbb{P}_J[Y_k = x] = \mathbb{C}_\beta(x). \tag{2.8}$$

For $d > 2$, the above infinite series converge when $\beta \leq 1$. We can thus obtain a bound on G_β from (2.6). However, it is not quite the bound we seek in Theorem 1.5: the exponential rate in (2.6) involves $\sqrt{1-\beta}\sigma^{-1} = \sqrt{1-\beta}\xi(0)^{-1}$, and we instead want $\xi(\beta)^{-1}$. The second inequality in (1.25) allows us to make this replacement if $\beta \leq (1-\delta) \wedge \beta(\delta)$ with $\delta \in (0, \frac{1}{2}]$. We prove the following proposition in Section 4.2.

Proposition 2.2 (Bounds on G_β, F_β for $\beta < 1$)**.** *Let $d > 2$. Let c, C be given by Proposition 2.1. Then, for every $\delta \in (0, \frac{1}{2}]$, every $\beta \leq (1-\delta) \wedge \beta(\delta)$, and every $x \in \mathbb{Z}^d$,*

$$G_\beta(x) \leq \delta_0(x) + \frac{2\mathsf{C}}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|} \right)^{d-2} \exp\left(-\frac{\mathsf{c}}{2} \frac{|x|}{\xi(\beta)} \right), \tag{2.9}$$

$$F_\beta(x) \leq \frac{2\mathsf{C}}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|} \right)^{d-2} \exp\left(-\frac{\mathsf{c}}{2} \frac{|x|}{\xi(\beta)} \right). \tag{2.10}$$

We think of Proposition 2.2 as an *initialisation* step. It explains an origin for the constants $\mathbf{c}, \mathbf{C}$ of Theorem 1.5: they must be "worse" than the constants c, C appearing in the bound (2.6) on $\mathbb{C}_\beta$.

The bound (2.8) is useless when $\beta > 1$. To maintain control of G_β in terms of random walk quantities, we replace the random walk of law $\mathbb{P}_J$ with a new random walk which we call the *effective random walk*.

2.2 The effective random walk

We now define the effective random walk of law $\mathbb{P}_\beta$, which is a main actor in our analysis. Throughout the section, we fix an admissible kernel J together with a family of functions $G = (G_\beta)_{\beta \geq 0} \in \mathcal{G}$ and assume they obey Assumption I. Recall that $F_\beta = J * G_\beta$.

Definition 2.3 (Effective random walk)**.** Let $0 \leq \beta' < \beta_c$. We define a random walk $(X_k)_{k \geq 0}$ started at $X_0 = 0$ by the transition probability

$$\mathbb{P}_{\beta'}[X_1 = y] := \frac{F_{\beta'}(y)}{\chi(\beta')} \qquad (y \in \mathbb{Z}^d). \tag{2.11}$$

We denote by $\mathbb{E}_{\beta'}$ the expectation with respect to $\mathbb{P}_{\beta'}$, and we denote the Green function of the effective random walk, for $Z \in [0, 1]$ and $x \in \mathbb{Z}^d$, by

$$\mathbb{G}_{Z,\beta'}(x) = \sum_{k \geq 0} Z^k \mathbb{P}_{\beta'}[X_k = x]. \tag{2.12}$$

Remark 2.4. By definition, $\xi(\beta')^2 = \mathbb{E}_{\beta'}[|X_1|_2^2]$. Also, when $\beta' = 0$, the measure $\mathbb{P}_{\beta'}$ is equal to the random walk measure $\mathbb{P}_J$ with step distribution J.

In terms of the effective random walk, the finite-difference inequality (**I.1**) can be rewritten, for every $x \in \mathbb{Z}^d$, as

$$G_\beta(x) \leq G_{\beta'}(x) + Z_{\beta',\beta} \mathbb{E}_{\beta'}[G_\beta(x - X_1)], \tag{2.13}$$

with

$$Z_{\beta',\beta} := (\beta - \beta')\chi(\beta'). \tag{2.14}$$

Bounds on $Z_{\beta',\beta}$ can be obtained from the differential inequalities for G_β using elementary calculus, as we show in Proposition 4.3.

The following lemma presents useful iterated versions of the finite-difference inequality (**I.1**), expressed in terms of the effective random walk with distribution $\mathbb{P}_{\beta'}$. It will serve as a replacement of (2.8) in the regime $\beta > 1$.

Lemma 2.5. *Let $0 \le \beta' \le \beta < \beta_c$. For every $T \ge 1$ and every $x \in \mathbb{Z}^d$,*

$$G_\beta(x) \le \sum_{k=0}^{T-1} Z_{\beta',\beta}^k \mathbb{E}_{\beta'}[G_{\beta'}(x - X_k)] + Z_{\beta',\beta}^T \mathbb{E}_{\beta'}[G_\beta(x - X_T)]. \tag{2.15}$$

In addition, if $Z_{\beta',\beta} < 1$ then

$$G_\beta(x) \le \sum_{k \ge 0} Z_{\beta',\beta}^k \mathbb{E}_{\beta'}[G_{\beta'}(x - X_k)]. \tag{2.16}$$

The same statements hold when G is replaced by $F = J * G$.

Proof. Let $T \ge 1$ and $x \in \mathbb{Z}^d$. By (2.13),

$$G_\beta(x) \le G_{\beta'}(x) + Z_{\beta',\beta} \mathbb{E}_{\beta'}[G_\beta(x - X_1)]. \tag{2.17}$$

By iteration, we obtain

$$G_\beta(x) \le \sum_{k=0}^{1} Z_{\beta',\beta} \mathbb{E}_{\beta'}[G_{\beta'}(x - X_k)] + Z_{\beta',\beta}^2 \mathbb{E}_{\beta'}[G_\beta(x - X_2)]. \tag{2.18}$$

For any $T \ge 1$, by iterating an appropriate number of times, we obtain (2.15). Then (2.16) follows from (2.15) after taking the limit $T \to \infty$. The hypothesis that $Z_{\beta',\beta} < 1$, together with the uniform boundedness of $G_\beta(\cdot)$, implies that the infinite iteration converges. □

The inequality (2.16) can be rewritten as

$$G_\beta(x) \le \frac{\chi(\beta')}{Z_{\beta',\beta}} \sum_{k \ge 0} Z_{\beta',\beta}^{k+1} \mathbb{P}_{\beta'}[X_{k+1} = x] = \frac{\chi(\beta')}{Z_{\beta',\beta}} \left(\mathbb{G}_{Z_{\beta',\beta},\beta'}(x) - \delta_0(x) \right). \tag{2.19}$$

If an analogue of Proposition 2.1 could be established for the Green function of the effective random walk, we would immediately obtain a bound on $G_\beta(x)$. However, we do not obtain such a result, and our bounds rather involve sums of $\mathbb{G}_{Z_{\beta',\beta},\beta'}$ over sufficiently large boxes (see Theorem 2.10 below).

Lemma 2.5 provides a mechanism with the potential to transfer an estimate valid at β' to an estimate at a larger parameter $\beta \le \beta(\delta)$. This enables an inductive procedure that lies at the heart of the proof. To exploit this mechanism, we will require a *uniform* control over the effective random walk $\mathbb{P}_\beta$ for $\beta \le \beta(\delta)$. We formulate this through two fundamental ingredients: (i) Green function estimates for $\mathbb{P}_\beta$ which hold at scale $\xi(\beta)$; (ii) a stability estimate, which controls $\mathbb{P}_\beta$ below the scale $\xi(\beta)$. We begin with the Green function estimates, which use the concept of a *regular random walk*.

2.3 Regular random walks

A central concept in our analysis is given in the following definition.

Definition 2.6 (Regular random walks)**.** Consider a random walk X on $\mathbb{R}^d$ starting from 0 with law $\mathbb{P}$ and with variance $\sigma^2 := \mathbb{E}[|X_1|_2^2] < \infty$. We assume that the distribution of the random walk is *symmetric* in the sense that $\mathbb{P}[X_1 = x] = \mathbb{P}[X_1 = y]$ if y can be obtained from x by permutation of coordinates and/or replacement of a coordinate by its negative. Let $\mathsf{c}_{\rm reg}, \mathsf{C}_{\rm reg} > 0$. We say that the random walk is $(\mathsf{c}_{\rm reg}, \mathsf{C}_{\rm reg})$-*regular* if it is symmetric and if the moment generating function

$$M(s) = \mathbb{E}\left[\exp\Big(\frac{s}{\sigma}(\mathbf{e}_1 \cdot X_1)\Big)\right] \tag{2.20}$$

satisfies

$$M(\mathsf{c}_{\rm reg}) \le \mathsf{C}_{\rm reg}. \tag{2.21}$$

We say that the random walk is *regular* if it is $(\mathsf{c}_{\rm reg}, \mathsf{C}_{\rm reg})$-regular for some $\mathsf{c}_{\rm reg}, \mathsf{C}_{\rm reg} > 0$. Additionally, a family of random walks is *uniformly regular* if each of the random walks in the family is regular with the same pair $(\mathsf{c}_{\rm reg}, \mathsf{C}_{\rm reg})$.

Remark 2.7. The constant $\mathsf{C}_{\rm reg}$ is somewhat artificial: what regularity requires is that the moment generating function M be analytic for s in some neighbourhood of 0. Once we know the analyticity, since $M(0) = 1$ we can have any $\mathsf{C}_{\rm reg} > 1$ by taking $\mathsf{c}_{\rm reg}$ small enough. Nevertheless, we place stress on $\mathsf{C}_{\rm reg}$ because the constants of Theorem 2.10 below depend only on the dimension and on $(\mathsf{c}_{\rm reg}, \mathsf{C}_{\rm reg})$ and on no other property of the effective random walk.

We will prove the following proposition in Section 4.3.

Proposition 2.8 (Regularity of the effective random walk below $\beta(\delta)$)**.** *Let* $\mathsf{C}_{\rm reg} = 3$. *There exist* $\delta_{\rm reg} > 0$ *(which can be chosen equal to* 2^{-9}*) and* $\mathsf{c}_{\rm reg} > 0$ *(which only depends on* d*) such that for every* $\beta < \beta(\delta_{\rm reg})$*, the effective random walk at parameter* β *is* $(\mathsf{c}_{\rm reg}, \mathsf{C}_{\rm reg})$*-regular.*

The moment generating function of the effective random walk at parameter β is denoted, for $s \in \mathbb{R}$, by

$$M_\beta(s) := \mathbb{E}_\beta\left[\exp\Big(\frac{s}{\xi(\beta)}(\mathbf{e}_1 \cdot X_1)\Big)\right] = \sum_{x\in\mathbb{Z}^d} e^{sx_1/\xi(\beta)}\mathbb{P}_\beta[X_1 = x]. \tag{2.22}$$

A consistency check for why Proposition 2.8 could be true is the following. If we knew that $F_\beta(x) \le C|x|^{-(d-2)} \exp[-c|x|/\xi(\beta)]$ with $c > 0$ independent of β, then we would have, for any $s < c$,

$$M_\beta(s) \le \frac{C}{\chi(\beta)} \sum_x e^{sx_1/\xi(\beta)} \frac{1}{|x|^{d-2}} e^{-c|x|/\xi(\beta)} \lesssim \frac{1}{\chi(\beta)} \int_0^\infty r e^{-(c-s)r/\xi(\beta)} \mathrm{d}r \lesssim \frac{\xi(\beta)^2}{\chi(\beta)}. \tag{2.23}$$

The upper bound on $\xi(\beta)^2/\chi(\beta)$ should in applications be independent of β. However, we cannot implement this simple approach because: (i) we can only prove (later) an upper bound on G (or F) with power $|x|^{-(d-2-\varepsilon)}$, and (ii) we do not prove that $\xi^2 \asymp \chi$ (we can only prove this up to a power δ). So the proof of Proposition 2.8 will be less direct.

An ingredient in the proof is the following corollary of Lemma 2.5. It will allow us to transfer an estimate on $M_{\beta'}(s)$ to $M_\beta(s)$ when $\beta' < \beta$ are close together. We can then proceed inductively to show that the effective random walk is regular all the way to $\beta(\delta)$. This induction proof is given in Section 4.3.

Corollary 2.9. *Let $0 \le \beta' \le \beta < \beta_c$ and let $s \in \mathbb{R}$. If $Z_{\beta',\beta} M_{\beta'}(s\xi(\beta')/\xi(\beta))) < 1$ then*

$$M_\beta(s) \le \frac{\chi(\beta')}{\chi(\beta)} \frac{M_{\beta'}(s\xi(\beta')/\xi(\beta))}{1 - Z_{\beta',\beta} M_{\beta'}(s\xi(\beta')/\xi(\beta))}. \tag{2.24}$$

The inequality (2.24) makes apparent a need to control $\xi(\beta')/\xi(\beta)$ relative to $\chi(\beta')/\chi(\beta)$. This is achieved via the differential lower bound (**I.2**), which provides such a comparison by means of (1.25).

Proof of Corollary 2.9. By definition, by (2.16), and by the factorisation property of the moment generating function of a sum of independent random variables,

$$\begin{aligned} M_\beta(s) &= \frac{1}{\chi(\beta)} \sum_{y \in \mathbb{Z}^d} F_\beta(y) \exp(s(\mathbf{e}_1 \cdot y)/\xi(\beta)) \\ &\le \frac{\chi(\beta')}{\chi(\beta)} \sum_{k \ge 0} Z^k_{\beta',\beta}\, \mathbb{E}_{\beta'}[\exp(s(\mathbf{e}_1 \cdot X_{k+1})/\xi(\beta))] \\ &= \frac{\chi(\beta')}{\chi(\beta)} \sum_{k \ge 0} Z^k_{\beta',\beta}\, [M_{\beta'}(s\xi(\beta')/\xi(\beta))]^{k+1}. \end{aligned} \tag{2.25}$$

By assumption, the geometric series converges and the proof is complete. □

We use the regularity of the effective random walk in Section 3 to prove our main result Theorem 1.5. The proof of Theorem 1.5 relies on the following general theorem about regular random walks, which is of independent interest. Its proof, which uses only classical random walk techniques, is deferred to Appendix A. To the best of our knowledge, Theorem 2.10 has not appeared previously in the literature.

We write the Green function of a generic random walk $(X_n)_{n \ge 0}$ on $\mathbb{R}^d$ with $X_0 = 0$, for $\mu \in [0, 1]$ and $A \subset \mathbb{R}^d$, as

$$\mathbb{G}_\mu(A) := \sum_{m \ge 0} \mu^m \mathbb{P}[X_m \in A]. \tag{2.26}$$

For $x \in \mathbb{R}^d$, recall that $|x| = \max_{1 \le i \le d} |x_i|$. For $k \ge 1$, we define the boxes

$$B_k := [-k, k]^d, \qquad B_k(x) := B_k + x, \tag{2.27}$$

as well as their discrete counterparts,

$$\Lambda_k := [-k, k]^d \cap \mathbb{Z}^d, \qquad \Lambda_k(x) = \Lambda_k + x \tag{2.28}$$

of radius k, centred at 0 and at x respectively.

Theorem 2.10 (Anti-concentration and Green function estimates for regular random walks)**.** *Let $d > 2$. For every $\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}} > 0$, there exist $\mathsf{C}_{\mathrm{RW}} = \mathsf{C}_{\mathrm{RW}}(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}, d) > 0$ and $\mathsf{c}_{\mathrm{RW}} = \mathsf{c}_{\mathrm{RW}}(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}, d) > 0$ such that, for every $(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}})$-regular random walk X (started at 0) on $\mathbb{R}^d$ of law $\mathbb{P}$, Green function $\mathbb{G}$, and variance σ^2, for every $\mu \in [0, 1]$, every $m \ge 1$ and for every $x \in \mathbb{Z}^d$,*

$$\mathbb{P}[X_m \in B_\sigma(x)] \le \frac{\mathsf{C}_{\mathrm{RW}}}{m^{d/2}} \exp\left(-\mathsf{c}_{\mathrm{RW}} \frac{|x|}{\sigma\sqrt{m}}\right), \tag{2.29}$$

$$\mathbb{G}_\mu(B_\sigma(x)) \le \mathsf{C}_{\mathrm{RW}} \left(\frac{\sigma}{\sigma \vee |x|}\right)^{d-2} \exp\left(-\mathsf{c}_{\mathrm{RW}} \sqrt{1-\mu}\frac{|x|}{\sigma}\right). \tag{2.30}$$

Remark 2.11. Theorem 2.10 provides estimates for the Green function averaged over a box whose size is the standard deviation. Pointwise estimates do not generally hold for regular random walks, as is illustrated in Example A.6. In particular, the regularity condition is not strong enough to obtain pointwise bounds on $G_{Z_{\beta',\beta},\beta'}$ in (2.19).

By Proposition 2.8, the effective random walk with $\beta < \beta(\delta_{\mathrm{reg}})$ is regular. It therefore satisfies the anti-concentration estimate (2.29), and its Green function $\mathbb{G}_{Z,\beta}(x)$ (defined in (2.12)) obeys the averaged estimate (2.30).

2.4 Stability below the correlation length

A uniform regularity statement for the effective random walk below $\beta(\delta)$ (for δ small enough) is given in Section 2.3. Proposition 2.8 formulates in a convenient way the fact that—as we increase β to $\beta(\delta)$—we have uniform control over the effective random walk $\mathbb{P}_\beta$ for all scales larger than or equal to $\xi(\beta)$. However, we also require control of the effective random walk at scales below $\xi(\beta)$. The next proposition provides this control at small scales. Proposition 2.12 is a fundamental tool in our analysis.

Proposition 2.12. *Let $d > 2$. Suppose that J and G both obey Assumption* I. *Then there exist $\delta_{\mathrm{stab}} \in (0, \frac{1}{10}]$ and $C_{\mathrm{stab}} > 0$, depending only on d, such that, for every $\beta < \beta(\delta_{\mathrm{stab}})$ and every $\beta' \leq \beta$,*

$$\mathbb{P}_\beta[|X_1| \leq \xi(\beta')] \leq C_{\mathrm{stab}} \frac{\chi(\beta')}{\chi(\beta)}. \tag{2.31}$$

To better appreciate Proposition 2.12, we rewrite its conclusion as follows. Let $\chi_k(\beta) := \sum_{x\in\Lambda_k} F_\beta(x)$. Then, (2.31) is equivalent to

$$\chi_{\xi(\beta')}(\beta) \leq C_{\mathrm{stab}}\chi(\beta'). \tag{2.32}$$

Roughly, this inequality says that if $\beta' \leq \beta$, then the value of $F_\beta(x)$ for $|x| \leq \xi(\beta')$ is comparable—in an averaged sense—to $F_{\beta'}(x)$. To have some intuition for why this might be true, suppose that we knew that $F_\beta(x) \lesssim \sigma_J^{-2}(1\vee|x|)^{-(d-2)}$, and also that $\chi \asymp (\xi/\sigma_J)^2$. So armed, we could conclude that

$$\chi_{\xi(\beta')}(\beta) \lesssim \sum_{|x|\leq\xi(\beta')} \frac{1}{\sigma_J^2(1 \vee |x|)^{d-2}} \lesssim \left(\frac{\xi(\beta')}{\sigma_J}\right)^2 \lesssim \chi(\beta'). \tag{2.33}$$

However, we know neither of the assumptions for the above computation, so we will need a more circuitous inductive argument to conclude. The proof of Proposition 2.12 is given in Section 4.4.

3 Proof of Theorem 1.5

Our main result is Theorem 1.5. Theorem 1.7 is then an elementary consequence of the assumptions, as we show in Proposition 4.4. The proof of Theorem 1.5 relies on a *bootstrap* argument that leverages the Green function and stability estimates for the effective random walk of Definition 2.3.

We now prove Theorem 1.5 conditionally on the results of Section 2 and three additional propositions, namely Propositions 3.5, 3.6, 3.7. The results of Section 2 are proved in Section 4 and the three propositions are proved in Section 5.

3.1 The bootstrap

Let $d > 2$, and fix a kernel J and family of functions $G \in \mathcal{G}$ which satisfy Assumption I. Our goal is to find $\mathbf{c}, \mathbf{C}, \boldsymbol{\delta} > 0$ such that the estimate (1.26) holds for $\beta \leq \beta(\boldsymbol{\delta})$. To prove Theorem 1.5, it suffices to prove the following bound on $F_\beta = J * G_\beta$. Recall the definition of c_0 from (1.10).

Theorem 3.1. *Let $d > 2$ and $\varepsilon \in (0,1)$. There exist $\mathbf{c}, \mathbf{C} > 0$ depending on (d, c_0), and $\boldsymbol{\delta} \in (0, \frac{1}{2}]$ depending on (d, c_0, ε), such that, if J and $G \in \mathcal{G}$ obey Assumption* I*, then for every $\beta \in [0, \beta(\boldsymbol{\delta})]$ and for every $x \in \mathbb{Z}^d$,*

$$F_\beta(x) \leq \frac{1}{2} \frac{\mathbf{C}}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|} \right)^{d-2-\varepsilon} \exp\left(-\mathbf{c} \frac{|x|}{\xi(\beta)} \right). \tag{3.1}$$

Proof of Theorem 1.5 *using Theorem* 3.1. We apply (**I.1**) with $\beta' = 0$. This gives, for every $x \in \mathbb{Z}^d$,

$$G_\beta(x) \leq \delta_0(x) + \beta F_\beta(x). \tag{3.2}$$

Since $\boldsymbol{\delta} \leq \frac{1}{2}$, it follows from (1.25) that $\beta(\boldsymbol{\delta}) \leq (1 - \boldsymbol{\delta})^{-1} \leq 2$. This factor 2 cancels the $\frac{1}{2}$ in (3.1), and results in the desired bound (1.26) on $G_\beta(x)$. □

We now focus on the proof of Theorem 3.1. Several important constants which appear throughout the analysis are summarised in the following glossary.

Glossary of important constants. There are four pairs of important constants:

- c_{reg}, C_{reg} are the constants for the moment generating function of a regular random walk; they arise in Definition 2.6 for a generic regular random walk, and are also used to denote the regularity constants for the effective random walk in Proposition 2.8. The constant c_{reg} depends only on d, and we set $\mathsf{C}_{\text{reg}} = 3$.
- c_{RW}, C_{RW} are the decay constants for a general regular random walk; they arise in Theorem 2.10. These constants depend only on d, c_{reg}, and C_{reg}.
- c, C are the constants in the anti-concentration and Green function estimates for the J random walk; they arise in Proposition 2.1. These constants depend only on d and on the constant c_0 which controls the ratio of σ_J and R_J as in (1.10).
- $\mathbf{c}$, $\mathbf{C}$ are the decay constants for G_β in our main result Theorem 1.5, and also in Theorem 3.1. These constants depend only on d and c_0.

To prove Theorem 3.1, we rely on a *bootstrap* argument: we will show that an *a priori* estimate on F_β can be improved by using the regularity of the effective random walk established in Proposition 2.8.

Given $x \in \mathbb{Z}^d$, $\beta \geq 0$, and $\mathbf{c}, \mathbf{C} > 0$, we define the statements $\mathcal{H}_\beta(\mathbf{c}, \mathbf{C}; x)$ and $\mathcal{H}_\beta(\mathbf{c}, \mathbf{C})$ (which at this stage are unverified) by

$$\mathcal{H}_\beta(\mathbf{c}, \mathbf{C}; x): \quad F_\beta(x) \leq \frac{\mathbf{C}}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|} \right)^{d-2-\varepsilon} \exp\left(-\mathbf{c} \frac{|x|}{\xi(\beta)} \right), \tag{3.3}$$

$$\mathcal{H}_\beta(\mathbf{c}, \mathbf{C}): \quad \mathcal{H}_\beta(\mathbf{c}, \mathbf{C}; x) \text{ holds for all } x \in \mathbb{Z}^d. \tag{3.4}$$

For small β, we can restate the bound on F_β of Proposition 2.2 as follows: for $d > 2$, for every $\mathbf{C} \geq 4\mathsf{C}$, every $\mathbf{c} \leq \frac{1}{2}\mathsf{c}$, and every $\delta \in (0, \frac{1}{2}]$,

$$\mathcal{H}_\beta(\mathbf{c}, \tfrac{1}{2}\mathbf{C}) \text{ holds (with } \varepsilon = 0) \text{ for } \beta \leq (1 - \delta) \wedge \beta(\delta). \tag{3.5}$$

The fact that $\mathcal{H}_\beta(\mathbf{c}, \frac{1}{2}\mathbf{C})$ holds for small β serves as the *initialisation step* of the bootstrap argument. The next proposition plays the role of the *contraction step*. To state it, we fix a choice of $(\mathbf{c}, \mathbf{C})$. This choice is explained in detail in Section 3.3. Let

$$\mathbf{c} = \frac{1}{2}\log 2 \wedge \frac{1}{2}\mathsf{c}_{\text{reg}} \wedge \frac{1}{2}\mathsf{c} \wedge \frac{1}{4}\mathsf{c}_{\text{RW}}, \qquad \mathbf{C} = 16\mathsf{C}. \tag{3.6}$$

Proposition 3.2 (Contraction step)**.** *Let $d > 2$ and $\varepsilon \in (0,1)$. Let $(\mathbf{c}, \mathbf{C})$ be as in* (3.6)*. There exists $\delta_1 \in (0, \delta_s \wedge \delta_{\text{reg}} \wedge \frac{1}{2}]$ depending on (d, c_0, ε), such that the following holds. Suppose that J and $G \in \mathcal{G}$ obey Assumption* I*. For every $\beta < \beta(\delta_1)$,*

$$\textit{if } \mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C}) \textit{ holds for all } \beta' \leq \beta \textit{ then } \mathcal{H}_\beta(\mathbf{c}, \tfrac{1}{2}\mathbf{C}) \textit{ holds.} \tag{3.7}$$

Remark 3.3. It follows immediately from Proposition 3.2 that, for every $\beta < \beta(\delta_1)$, if $\mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C})$ holds for every $\beta' \leq \beta$, then $\mathcal{H}_{\beta'}(\mathbf{c}, \frac{1}{2}\mathbf{C})$ holds for every $\beta' \leq \beta$.

We decompose the proof of Proposition 3.2 into *multiscale bounds*. Before introducing the multiscale bounds, we show how (3.5) and Proposition 3.2 can be used to prove Theorem 3.1.

3.2 Proof of Theorem 3.1

We use a *forbidden region* analysis which relies on an *a priori* continuity property. We first establish the relevant continuity property.

As a simple initial observation, let X be any $(\mathsf{c}_{\text{reg}}, \mathsf{C}_{\text{reg}})$-regular random walk with variance σ^2. Then, by symmetry and by Markov's inequality,

$$\mathbb{P}[X_1 = x] \leq 2d \cdot \mathbb{P}[((\mathbf{e}_1 \cdot X_1)/\sigma \geq |x|/\sigma] \leq 2d\mathsf{C}_{\text{reg}} \exp\left(-\mathsf{c}_{\text{reg}}\frac{|x|}{\sigma}\right). \tag{3.8}$$

In particular, an application of (3.8) to the effective random walk shows that, for every $\beta < \beta(\delta_{\text{reg}})$ (given by Proposition 2.8) and every $\mathbf{c} \leq \frac{1}{2}\mathsf{c}_{\text{reg}}$,

$$F_\beta(x) \leq 2d\mathsf{C}_{\text{reg}}\chi(\beta) \exp\left(-2\mathbf{c}\frac{|x|}{\xi(\beta)}\right). \tag{3.9}$$

Let δ_1 be given by Proposition 3.2 and $0 \leq \beta < \beta(\delta_1)$. Given $\mathbf{c} \leq \frac{1}{2}\mathsf{c}_{\text{reg}}$, we define

$$f(\beta; x) = F_\beta(x)\sigma_J^d \left(\frac{\sigma_J \vee |x|}{\sigma_J}\right)^{d-2-\varepsilon} \exp\left(\mathbf{c}\frac{|x|}{\xi(\beta)}\right), \tag{3.10}$$

$$f(\beta) = \sup_{x \in \mathbb{Z}^d} f(\beta; x). \tag{3.11}$$

By definition, $f(\beta; x)$ is continuous in $\beta \in [0, \beta(\delta_1))$. We claim that the supremum $f(\beta)$ is also continuous in $\beta \in [0, \beta(\delta_1))$ (in particular it is finite). To prove the claim, it suffices to prove continuity in $\beta \in [0, \beta_1)$ for every $\beta_1 < \beta(\delta_1)$. For $\beta \in [0, \beta_1)$, (3.9) gives

$$\begin{aligned} f(\beta; x) &\leq 2d\mathsf{C}_{\text{reg}}\chi(\beta) \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right) \sigma_J^d \left(\frac{\sigma_J \vee |x|}{\sigma_J}\right)^{d-2-\varepsilon} \\ &\leq 2d\mathsf{C}_{\text{reg}}\chi(\beta_1) \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta_1)}\right) \sigma_J^d \left(\frac{\sigma_J \vee |x|}{\sigma_J}\right)^{d-2-\varepsilon}, \end{aligned} \tag{3.12}$$

where we used the facts that $\chi(\beta) \leq \chi(\beta_1)$ (by assumption on G) and $\xi(\beta) \leq \xi(\beta_1)$ (by (1.25)). Note that the monotonicity of ξ is not obvious from the definition of ξ. Since the right-hand side of (3.12) goes to zero as $|x| \to \infty$, uniformly in $\beta \le \beta_1$, the supremum over x in (3.11) is attained on a β-independent (but β_1-dependent) finite set of x values. The function f is therefore the supremum of finitely many continuous functions, so it is continuous too.

Proof of Theorem 3.1. Let $(\mathbf{c}, \mathbf{C})$ be as in (3.6), and let $\delta_1 \in (0, \delta_s \wedge \delta_{\rm reg} \wedge \frac{1}{2}]$ be given by Proposition 3.2. We prove that $\mathcal{H}_\beta(\mathbf{c}, \frac{1}{2}\mathbf{C})$ holds, i.e., that $f(\beta) \le \frac{1}{2}\mathbf{C}$, for every $\beta \le \beta(\delta_1)$. We can and do assume that $\frac{1}{2} \le \beta(\delta_1)$, since otherwise (3.5) immediately gives the desired result.

Let $\beta_0 = \frac{1}{2}$. By (3.5) with $\delta = \frac{1}{2}$,

$$f(\beta') \le \frac{1}{2}\mathbf{C} \quad \text{for all } \beta' \le \beta_0. \tag{3.13}$$

We will prove that the combination of Proposition 3.2 and (3.13) shows that values in $(\frac{1}{2}\mathbf{C}, \mathbf{C}]$ are forbidden for $f(\beta)$ for all $\beta < \beta(\delta_1)$.

The proof is by contradiction. Suppose that there exists $\beta_* < \beta(\delta_1)$ such that $f(\beta_*) > \frac{1}{2}\mathbf{C}$. By continuity of f and (3.13), there must be a $\beta_{**} \in (\beta_0, \beta_*]$ such that $f(\beta_{**}) \in (\frac{1}{2}\mathbf{C}, \mathbf{C}]$ and also $f(\beta') \le \mathbf{C}$ for all $\beta' \le \beta_{**}$. But this last condition is the hypothesis in (3.7), and the conclusion of (3.7) therefore implies (see Remark 3.3) that $f(\beta') \le \frac{1}{2}\mathbf{C}$ for all $\beta' \le \beta_{**}$. This contradicts $f(\beta_{**}) > \frac{1}{2}\mathbf{C}$, so β_* cannot exist, and we have therefore proved that $f(\beta) \le \frac{1}{2}\mathbf{C}$ for all $\beta < \beta(\delta_1)$.

To extend the bound $f(\beta) \le \frac{1}{2}\mathbf{C}$ to $\beta = \beta(\delta_1)$, we argue as follows. We know for each x and each $\beta < \beta(\delta_1)$ that

$$F_\beta(x) \le \frac{1}{2}\frac{\mathbf{C}}{\sigma_J^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2-\varepsilon} \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right). \tag{3.14}$$

If $\beta(\delta_1) < \beta_c$, then $\xi(\beta)$ and $F_\beta(x)$ are continuous at $\beta(\delta_1)$ by our assumptions on G_β. Thus, (3.14) extends by continuity to $\beta = \beta(\delta_1)$. Suppose finally that $\beta(\delta_1) = \beta_c$ (the case of primary interest, verified under additional hypothesis in Theorem 1.6). By Definition 1.3(*iii*), $G_{\beta_c}(x)$ is the supremum over $\beta < \beta_c$ of $G_\beta(x)$, and hence the same is true for $F_{\beta_c}(x)$. Consequently, by the monotone convergence theorem, $\lim_{\beta \nearrow \beta_c} \chi(\beta) = \chi(\beta_c)$ and $\lim_{\beta \nearrow \beta_c} \||x|_2^2 F_\beta\|_1 = \||x|_2^2 F_{\beta_c}\|_1$. If $\chi(\beta_c) < \infty$, we additionally obtain that $\lim_{\beta \nearrow \beta_c} \xi(\beta) = \xi(\beta_c)$. On the other hand, if $\chi(\beta_c) = \infty$, then (1.25) implies that $1/\xi(\beta) \to 0$ as $\beta \nearrow \beta_c$. In either case, (3.14) extends to $\beta(\delta_1) = \beta_c$ by continuity. Thus, we conclude that $f(\beta) \le \frac{1}{2}\mathbf{C}$ for all $\beta \le \beta(\delta_1)$. This completes the proof, by setting $\boldsymbol{\delta} := \delta_1$. □

Remark 3.4. In the above proof, we did not really need (3.5). Indeed, it is sufficient to know that f is continuous and that $f(0) = 0$. We have highlighted (3.5) because it plays a role in the proof of Proposition 3.2.

3.3 Reduction of Proposition 3.2 to multiscale bounds

For x with $|x| \le 2\xi(0) = 2\sigma_J$, the exponential factor in (3.1) is unimportant. The following proposition gives a version of (3.1) for these small x. Proposition 3.5 is proved in Section 5.1, as an elementary consequence of our assumptions and the Green function estimate of Proposition 2.1.

Proposition 3.5 (Bounds at small scales)**.** *There exists $\delta_{\rm s} = \delta_{\rm s}(d) \in (0, \frac{1}{2}]$ such that, for every $\beta < \beta(\delta_{\rm s})$, and every $|x| \le 2\xi(0)$,*

$$F_\beta(x) \le \frac{4\mathsf{C}}{\sigma_J^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2}. \tag{3.15}$$

By Proposition 3.5, if $\mathbf{c} \leq \frac{1}{2}\log 2$, $\mathbf{C} \geq 16\mathsf{C}$, and $\beta < \beta(\delta_s)$, then $\mathcal{H}_\beta(\mathbf{c}, \frac{1}{2}\mathbf{C}; x)$ holds for every $|x| \leq 2\xi(0)$. In particular, we do not need to assume the bootstrap hypothesis that $\mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C})$ holds for every $\beta' \leq \beta$ to obtain this result for small x. However, the bootstrap assumption becomes essential in the treatment of the scales $|x| > 2\xi(0)$.

We use two different arguments depending on whether $|x| \gtrsim \xi(\beta)$ or not. This threshold corresponds to the values of $|x|$ for which the exponential term becomes relevant in (3.1).

Proposition 3.6 (Improvement at typical length scales)**.** *Let $d > 2$ and $\varepsilon > 0$. Let $\eta > 0$. Assume that $\mathbf{C} \geq 16\mathsf{C}$ and $\mathbf{c} \leq \frac{1}{2}\mathsf{c} \wedge \frac{1}{4}\mathsf{c}_{\mathrm{RW}} \wedge 1$. There exists $\delta_2 = \delta_2(\eta, \varepsilon, d) \leq \delta_s \wedge \delta_{\mathrm{reg}} \wedge \frac{1}{2}$ such that the following is true for every $\beta < \beta(\delta_2)$: if $\mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C})$ holds for every $\beta' \leq \beta$, then, for every $|x| \geq \eta\xi(\beta) \vee 2\xi(0)$,*

$$F_\beta(x) \leq \frac{1}{8}\frac{\mathbf{C}}{\sigma_J^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2-\varepsilon} \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right). \tag{3.16}$$

Proposition 3.7 (Improvement at intermediate scales)**.** *Let $d > 2$ and $\varepsilon > 0$. Assume that $\mathbf{C} \geq 16\mathsf{C}$ and $\mathbf{c} \leq \frac{1}{2}\mathsf{c} \wedge \frac{1}{4}\mathsf{c}_{\mathrm{RW}} \wedge 1$. There exists $\eta \in (0, 1)$ such that the following holds. Let $\delta_2 = \delta_2(\eta, \varepsilon, d)$ be given by Proposition 3.6. The following is true for every $\beta < \beta(\delta_2)$: if $\mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C})$ holds for every $\beta' \leq \beta$ then for every $2\xi(0) \leq |x| \leq \eta\xi(\beta)$,*

$$F_\beta(x) \leq \frac{1}{6}\frac{\mathbf{C}}{\sigma_J^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2-\varepsilon}. \tag{3.17}$$

It is not hard to deduce Proposition 3.2 from Propositions 3.5, 3.6, and 3.7.

Proof of Proposition 3.2. First, observe that the choice of $(\mathbf{c}, \mathbf{C})$ in (3.6) meets all the requirements of Propositions 3.5, 3.6, and 3.7. Let η, δ_2 be given by Proposition 3.7. Assume that $\beta < \beta(\delta_2)$ and that $\mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C})$ holds for every $\beta' \leq \beta$. To prove that $\mathcal{H}_{\beta'}(\mathbf{c}, \frac{1}{2}\mathbf{C})$ holds, we proceed scale by scale using Propositions 3.5–3.7.

We have already observed that $\mathcal{H}_\beta(\mathbf{c}, \frac{1}{2}\mathbf{C}; x)$ holds for every $|x| \leq 2\xi(0)$, thanks to Proposition 3.5. For typical scales, Proposition 3.6 gives more than we need, as it says that $\mathcal{H}_\beta(\mathbf{c}, \frac{1}{8}\mathbf{C}; x)$ holds for every $|x| \geq \eta\xi(\beta) \vee 2\xi(0)$. Finally, by Proposition 3.7, for the remaining intermediate scale $2\xi(0) \leq |x| \leq \eta\xi(\beta)$, we have

$$F_\beta(x) \leq \frac{\exp(\mathbf{c}\eta)}{3} \cdot \frac{1}{2}\frac{\mathbf{C}}{\sigma_J^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2-\varepsilon} \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right). \tag{3.18}$$

Since $\eta \leq 1$ and $\mathbf{c} \leq \frac{1}{2}\log 2$ by assumption, this proves that $\mathcal{H}_\beta(\mathbf{c}, \frac{1}{3}\mathbf{C}; x)$ holds for the intermediate scale. This concludes the proof, by setting $\delta_1 := \delta_2$. □

4 Preliminaries: integration, regularity, stability

In this section, we develop three preliminary ingredients for the proof of Theorem 3.1. In Section 4.1, we use elementary calculus to derive consequences of the upper and lower differential inequalities (1.13) and (**I.2**) provided by Assumption I. This is the *only* place in the paper where (**I.2**) is used. In Section 4.2, we give a first application of the results of Section 4.1 and prove Proposition 2.2. In Section 4.3, we combine Section 4.1 and Corollary 2.9 to prove the regularity of the effective random walk stated in Proposition 2.8. Finally, in Section 4.4, we prove the stability result stated in Proposition 2.12. It plays an important role in the proofs of Propositions 3.6 and 3.7.

4.1 Integration of differential inequalities

Throughout Section 4.1, we fix $d > 2$ and we assume that J and G obey Assumption I. Recall that $F_\beta = J * G_\beta$ and

$$\chi(\beta) = \sum_{x\in\mathbb{Z}^d} G_\beta(x) = \sum_{x\in\mathbb{Z}^d} F_\beta(x) = \|F_\beta\|_1, \quad E(\beta) = \max_{0\le t\le\beta}\left(\|H_t\|_1 + \frac{\||x|_2^2\cdot H_t\|_1}{\xi(t)^2}\right). \tag{4.1}$$

It is only in this section that we use the definition of $E(\beta)$. Subsequently we will only use the fact that $E(\beta) < \delta$ for every $\beta < \beta(\delta)$, by the definition of $\beta(\delta)$ in (1.24).

Proposition 4.1. *For every* $0 \le \beta' \le \beta < \beta_c$*,*

$$(\beta - \beta')(1 - E(\beta)) \le \frac{1}{\chi(\beta')} - \frac{1}{\chi(\beta)} \le \beta - \beta', \tag{4.2}$$

and, if $E(\beta) < 1$*,*

$$\left(\frac{\chi(\beta')}{\chi(\beta)}\right)^{\frac{1+E(\beta)}{1-E(\beta)}} \le \left(\frac{\xi(\beta')}{\xi(\beta)}\right)^2 \le \left(\frac{\chi(\beta')}{\chi(\beta)}\right)^{1-2E(\beta)}. \tag{4.3}$$

In particular, if $E(\beta) \le \frac{1}{2}$ *then* $\xi(\beta') \le \xi(\beta)$*.*

By taking $\delta \in (0,1)$ and $\beta < \beta(\delta)$, the second inequality of (1.25) is seen to follow immediately from (4.3). By choosing $\beta' = 0$ and $\beta < \beta(\delta)$ in (4.2), we find that $\beta(1-\delta) \le 1/\chi(0) = 1$, which implies that

$$\beta(\delta) \le (1-\delta)^{-1}, \tag{4.4}$$

as stated in the first inequality of (1.25). Also, with $\delta \in (0,1)$, $\beta' = 0$, and $\beta < \beta(\delta) = \beta(\delta) \wedge (1-\delta)^{-1}$, (4.2) rearranges to

$$\frac{1}{1-\beta(1-\delta)} \le \chi(\beta) \le \frac{1}{1-\beta}. \tag{4.5}$$

Proof of Proposition 4.1. We begin with (4.2). It follows by summation of the differential upper bound (1.13) that

$$\partial_\beta \chi(\beta) \le \sum_{x\in\mathbb{Z}^d} (F_\beta * G_\beta)(x) = \chi(\beta)^2. \tag{4.6}$$

Similarly, it follows from summation of the differential lower bound (**I.2**) that

$$\partial_\beta\chi(\beta) \ge \sum_{x\in\mathbb{Z}^d} (F_\beta * [J - H_\beta] * G_\beta)(x) = (1 - \|H_\beta\|_1)\,\chi(\beta)^2 \ge (1 - E(\beta))\chi(\beta)^2. \tag{4.7}$$

The combination of (4.6)–(4.7) gives

$$(1 - E(\beta)) \le -\partial_\beta \chi(\beta)^{-1} \le 1, \tag{4.8}$$

and then integration over the interval $[\beta', \beta]$ gives (4.2). For later use, we also observe that (4.6)–(4.7) yield

$$\partial_\beta \log\chi(\beta) \le \chi(\beta) \le \frac{1}{1-E(\beta)}\partial_\beta \log\chi(\beta). \tag{4.9}$$

To begin the proof of (4.3), we note that by definition and by (1.19),

$$\partial_\beta(\chi(\beta)\xi(\beta)^2) = \sum_{x\in\mathbb{Z}^d} |x|_2^2 \partial_\beta F_\beta(x) \le \sum_{x,y\in\mathbb{Z}^d} |x|_2^2 F_\beta(y) F_\beta(x-y). \tag{4.10}$$

We insert $|x|_2^2 = |y|_2^2 + 2x\cdot(x-y) + |x-y|_2^2$ in the right-hand side, and observe that the cross term vanishes by symmetry of F_β. Therefore,

$$\partial_\beta(\chi(\beta)\xi(\beta)^2) \le 2\chi(\beta)^2\xi(\beta)^2. \tag{4.11}$$

We apply the lower bound of (**I.2**) in a similar manner. With the inequality

$$\||x|^2 G_\beta\|_1 = \||x|^2 F_\beta\|_1 - \sigma_J^2\chi(\beta) \le \||x|^2 F_\beta\|_1, \tag{4.12}$$

and with our assumption that $H_\beta(x) = H_\beta(-x)$, this leads to

$$\begin{aligned}
&\partial_\beta(\chi(\beta)\xi(\beta)^2)\\
&\quad\ge \sum_{x\in\mathbb{Z}^d} |x|_2^2 \left[(F_\beta * F_\beta)(x) - (F_\beta * H_\beta * G_\beta)(x)\right]\\
&\quad= 2\chi(\beta)^2\xi(\beta)^2 - \chi(\beta)^2\||x|_2^2\cdot H_\beta\|_1 - \chi(\beta)^2\|H_\beta\|_1\xi(\beta)^2 - \chi(\beta)\|H_\beta\|_1\cdot\||x|_2^2\cdot G_\beta\|_1\\
&\quad\ge 2\chi(\beta)^2\xi(\beta)^2 - \chi(\beta)^2\||x|_2^2\cdot H_\beta\|_1 - 2\|H_\beta\|_1\chi(\beta)^2\xi(\beta)^2\\
&\quad\ge 2(1-E(\beta))\chi(\beta)^2\xi(\beta)^2.
\end{aligned} \tag{4.13}$$

Together, (4.11) and (4.13) give

$$2(1-E(\beta))\chi(\beta) \le \partial_\beta \log(\chi(\beta)\xi(\beta)^2) \le 2\chi(\beta). \tag{4.14}$$

With (4.9), (4.14) gives

$$2(1-E(\beta))\partial_\beta\log\chi(\beta) \le \partial_\beta\log(\chi(\beta)\xi(\beta)^2) \le \frac{2}{1-E(\beta)}\partial_\beta\log\chi(\beta). \tag{4.15}$$

Then, (4.3) follows after integration over $[\beta',\beta]$. This completes the proof. □

The following corollary of Proposition 4.1 shows that the ratios $\frac{\chi(\beta')}{\chi(\beta)}$ and $(\frac{\xi(\beta')}{\xi(\beta)})^2$ are comparable up to constants, as long as $\beta-\beta'$ is sufficiently small that the ratios $\frac{1}{4}\frac{\chi(\beta')}{\chi(\beta)}$ and $\frac{1}{4}(\frac{\xi(\beta')}{\xi(\beta)})^2$ are not smaller than $E(\beta)$.

Corollary 4.2. *Let $0\le\beta'\le\beta<\beta_c$. If $E(\beta)\le\frac{1}{4}\frac{\chi(\beta')}{\chi(\beta)}\vee\frac{1}{4}(\frac{\xi(\beta')}{\xi(\beta)})^2$ and also $E(\beta)\le\frac{1}{2}$, then*

$$\frac{1}{2}\frac{\chi(\beta')}{\chi(\beta)} \le \left(\frac{\xi(\beta')}{\xi(\beta)}\right)^2 \le 2\frac{\chi(\beta')}{\chi(\beta)}. \tag{4.16}$$

Proof. The claim follows from (4.3) and basic algebra. Let $a = (\frac{\xi(\beta')}{\xi(\beta)})^2$, $b = \frac{\chi(\beta')}{\chi(\beta)}$, and $E = E(\beta)$. Then $0<b\le 1$, and since $E\le\frac{1}{2}$ it follows from Proposition 4.1 that also $0<a\le 1$. We will use the facts that

$$t^t \ge \frac{1}{2} \text{ and } t^{t/2} \ge \frac{1}{2} \text{ for } t>0, \qquad \frac{1}{1-t} \le 1+2t \text{ for } t\in[0,\tfrac{1}{2}]. \tag{4.17}$$

The inequalities in (4.3) can be restated as

$$b\cdot b^{2(\frac{1}{1-E}-1)} \le a \le b\cdot\frac{1}{b^{2E}}. \tag{4.18}$$

Suppose first that $E \le \frac{1}{4}b$. Then, a combination of (4.17) and (4.18) gives the required bounds

$$a \ge b \cdot b^{2(\frac{1}{1-E}-1)} \ge b \cdot b^{4E} \ge b \cdot b^{b} \ge b \cdot \frac{1}{2} \tag{4.19}$$

and

$$a \le b \cdot \frac{1}{b^{2E}} \le b \cdot \frac{1}{b^{b/2}} \le 2b. \tag{4.20}$$

Now suppose that $E \le \frac{1}{4}a$. In this case we restate (4.3) as

$$a \cdot a^{\frac{1}{1-2E}-1} \le b \le a \cdot \frac{1}{a^{\frac{2E}{1+E}}}. \tag{4.21}$$

In the lower bound on b in (4.21), the exponent of the second factor is at most $4E \le a$, so the left-hand side is bounded below by $\frac{1}{2}a$. The right-hand side of (4.21) is increased if we increase the power to $2E \le \frac{1}{2}a$, so the right-hand side is bounded above by $2a$. Together, this gives the desired bounds $\frac{1}{2}b \le a \le 2b$ in this case, and completes the proof. □

Recall that

$$Z_{\beta',\beta} = (\beta - \beta')\chi(\beta'). \tag{4.22}$$

In particular,

$$Z_{0,\beta} = \beta. \tag{4.23}$$

The following proposition shows that $Z_{\beta',\beta}$ is close to and less than 1 as long as $\beta - \beta'$ is sufficiently small that the ratios $\frac{1}{4}\frac{\chi(\beta')}{\chi(\beta)}$ and $\frac{1}{8}(\frac{\xi(\beta')}{\xi(\beta)})^2$ are not smaller than $E(\beta)$. The upper bound on $Z_{\beta',\beta}$ is the useful one in our analysis.

Proposition 4.3 (Bounds on $Z_{\beta',\beta}$)**.** *Let $0 \le \beta' \le \beta < \beta_c$. Then,*

$$1 - \frac{\chi(\beta')}{\chi(\beta)} \le Z_{\beta',\beta} \le \Big(1 - \frac{\chi(\beta')}{\chi(\beta)}\Big)\frac{1}{1 - 1 \wedge E(\beta)}. \tag{4.24}$$

In particular, if $E(\beta) \le \frac{1}{4}\frac{\chi(\beta')}{\chi(\beta)}$, then

$$1 - \frac{\chi(\beta')}{\chi(\beta)} \le Z_{\beta',\beta} \le 1 - \frac{1}{2}\frac{\chi(\beta')}{\chi(\beta)}. \tag{4.25}$$

If we assume instead that $E(\beta) \le \frac{1}{8}(\frac{\xi(\beta')}{\xi(\beta)})^2 \wedge \frac{1}{2}$, then

$$1 - 2\left(\frac{\xi(\beta')}{\xi(\beta)}\right)^2 \le Z_{\beta',\beta} \le 1 - \frac{1}{4}\left(\frac{\xi(\beta')}{\xi(\beta)}\right)^2. \tag{4.26}$$

Proof. The upper bound in (4.2) implies that

$$1 - \frac{\chi(\beta')}{\chi(\beta)} \le (\beta - \beta')\chi(\beta') = Z_{\beta',\beta}, \tag{4.27}$$

which is the lower bound in (4.24). The upper bound in (4.24) is vacuous if $E(\beta) \ge 1$, while for $E(\beta) < 1$ the lower bound in (4.2) yields

$$(1 - E(\beta))Z_{\beta',\beta} \le 1 - \frac{\chi(\beta')}{\chi(\beta)}, \tag{4.28}$$

which is the upper bound in (4.24).

For the upper bound of (4.25), we again use $(1-t)^{-1} \le 1+2t$ for $t \le \frac{1}{2}$, so that (4.24) and the assumption on $E(\beta)$ give

$$Z_{\beta',\beta} \le \left(1 - \frac{\chi(\beta')}{\chi(\beta)}\right)\left(1 + \frac{1}{2}\frac{\chi(\beta')}{\chi(\beta)}\right) \le 1 - \frac{1}{2}\frac{\chi(\beta')}{\chi(\beta)}. \tag{4.29}$$

Finally, for (4.26), we apply Corollary 4.2 to see that $E(\beta) \le \frac{1}{8}(\frac{\xi(\beta')}{\xi(\beta)})^2 \le \frac{1}{4}\frac{\chi(\beta')}{\chi(\beta)}$, and then apply (4.25) and again use the bounds of Corollary 4.2. This completes the proof. □

Theorem 1.7 is contained in the following proposition.

Proposition 4.4. *For $\delta \in (0,1)$, $\beta \le \beta(\delta)$, and $E = E(\beta(\delta))$, we have*

$$\frac{1}{\chi(\beta(\delta))^{-1} + (\beta(\delta) - \beta)} \le \chi(\beta) \le \frac{1}{\chi(\beta(\delta))^{-1} + (\beta(\delta) - \beta)(1-E)}, \tag{4.30}$$

$$\chi(\beta)^{1-2E} \le \frac{\xi(\beta)^2}{\sigma_J^2} \le \chi(\beta)^{\frac{1+E}{1-E}}. \tag{4.31}$$

Assume that $\beta(\delta) = \beta_c$ and $\chi(\beta_c) = \infty$. Then,

$$1 \le \beta_c \le \frac{1}{1-E}, \tag{4.32}$$

$$\frac{1}{\beta_c - \beta} \le \chi(\beta) \le \frac{1}{1-E}\frac{1}{\beta_c - \beta}, \tag{4.33}$$

$$\left(\frac{1}{\beta_c - \beta}\right)^{1-2E} \le \frac{\xi(\beta)^2}{\sigma_J^2} \le \left(\frac{1}{(1-E)(\beta_c - \beta)}\right)^{\frac{1+E}{1-E}}. \tag{4.34}$$

Proof. The inequality (4.30) follows by replacing (β', β) by $(\beta, \beta(\delta))$ in (4.2), and (4.31) follows from (4.3) with $\beta' = 0$. To prove (4.32), we set $\beta' = 0$ and take the limit $\beta \uparrow \beta(\delta) = \beta_c$ in (4.2). The inequalities (4.33)–(4.34) follow by setting $\chi(\beta_c) = \infty$ in (4.30)–(4.31). □

4.2 Proof of Proposition 2.2: bounds on G_β and F_β for $\beta < 1$

We now prove the bounds on G_β and F_β for $\beta < 1$ stated in Proposition 2.2.

Proof of Proposition 2.2. Let $\delta \in (0, \frac{1}{2}]$ and $\beta \le (1-\delta) \wedge \beta(\delta)$. We start with the bound on $G_\beta(x)$. It follows from (2.8) and Proposition 2.1 that, for every $x \in \mathbb{Z}^d$,

$$G_\beta(x) \le \mathbb{C}_\beta(x) \le \delta_0(x) + \frac{\mathsf{C}}{\sigma^d}\left(\frac{\sigma}{\sigma \vee |x|}\right)^{d-2} \exp\left(-\mathsf{c}\sqrt{1-\beta}\frac{|x|}{\xi(0)}\right). \tag{4.35}$$

The second inequality in (4.3) gives that

$$\frac{\xi(\beta)}{\xi(0)} \ge \left(\frac{\chi(\beta)}{\chi(0)}\right)^{\frac{1}{2}-\delta}. \tag{4.36}$$

Combining (4.5) with the inequality $1 - \delta \ge \beta$ yields

$$\frac{\chi(\beta)}{\chi(0)} \ge \frac{1}{1-(1-\delta)\beta} \ge \frac{1}{1-\beta^2}. \tag{4.37}$$

Insertion of (4.37) into (4.36) gives

$$\sqrt{1-\beta}\cdot\frac{\xi(\beta)}{\xi(0)} \geq (1-\beta)^{\delta}\left(\frac{1-\beta}{1-\beta^2}\right)^{\frac{1}{2}-\delta}$$
$$\geq \delta^{\delta}\left(\frac{1}{1+\beta}\right)^{\frac{1}{2}-\delta} \geq \delta^{\delta}\left(\frac{1}{2-\delta}\right)^{\frac{1}{2}-\delta} \geq \frac{1}{2}, \tag{4.38}$$

where last inequality follows from elementary calculus to minimise the penultimate expression. Combining (4.35), and (4.38), we obtain, for every $x \in \mathbb{Z}^d$,

$$G_\beta(x) \leq \mathbb{C}_\beta(x) \leq \delta_0(x) + \frac{\mathsf{C}}{\sigma^d}\left(\frac{\sigma}{\sigma \vee |x|}\right)^{d-2} \exp\left(-\frac{\mathsf{c}}{2}\frac{|x|}{\xi(\beta)}\right). \tag{4.39}$$

This completes the proof of (2.9).

Next we consider $F_\beta = J * G_\beta$. By (2.8),

$$F_\beta(x) \leq (J * \mathbb{C}_\beta)(x) = \frac{1}{\beta}(\mathbb{C}_\beta(x) - \delta_0(x)). \tag{4.40}$$

For the case $\beta \geq \frac{1}{2}$, we immediately obtain (2.10) from the bound on $\mathbb{C}_\beta(x)$ in (4.39), with $\mathbf{C}_0 = 2\mathsf{C}$. For the remaining case $\beta < \frac{1}{2}$, it follows from the monotonicity of F_β, (4.40), and the bound on $\mathbb{C}_{1/2}(x)$ from (2.6) that

$$F_\beta(x) \leq F_{1/2}(x) \leq 2(\mathbb{C}_{1/2}(x) - \delta_0(x))$$
$$\leq 2\frac{\mathsf{C}}{\sigma^d}\left(\frac{\sigma}{\sigma \vee |x|}\right)^{d-2} \exp\Big(-\mathsf{c}\frac{1}{\sqrt{2}}\frac{|x|}{\sigma}\Big). \tag{4.41}$$

Since $\xi(\beta) \geq \xi(0) = \sigma$ by (4.36), we then obtain (2.10) with $\mathbf{C}_0 = 2\mathsf{C}$ and $\mathbf{c}_0 = \frac{1}{2}\mathsf{c}$, from the observation that

$$\frac{1}{\sqrt{2}}\frac{1}{\sigma} = \frac{1}{\sqrt{2}}\frac{1}{\xi(0)} \geq \frac{1}{2}\frac{1}{\xi(\beta)}. \tag{4.42}$$

This concludes the proof. □

4.3 Regularity of the effective random walk

We now prove Proposition 2.8, which states that the effective random walk introduced in Definition 2.3 is uniformly regular for $\beta < \beta(\delta_{\mathrm{reg}})$ with δ_{reg} sufficiently small. Consequently, for every $\beta < \beta(\delta_{\mathrm{reg}})$, the effective random walk obeys the anti-concentration and Green function estimates of Theorem 2.10, with uniform constants $(\mathsf{c}_{\mathrm{RW}}, \mathsf{C}_{\mathrm{RW}})$ that depend only on d. The fact that $(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}})$ and $(\mathsf{c}_{\mathrm{RW}}, \mathsf{C}_{\mathrm{RW}})$ do not depend on J or β is essential for our proof.

Proof of Proposition 2.8. The effective random walk X with law $\mathbb{P}_\beta$ is symmetric by definition. Let

$$L := 2^7 = 128. \tag{4.43}$$

Let $\delta > 0$, to be chosen small enough. In particular, we assume that $\delta \leq \frac{1}{4L}$. Fix $\beta < \beta(\delta)$. Then $\chi(\beta) < \infty$. Recall that

$$M_\beta(s) = \mathbb{E}_\beta[\exp(s(\mathbf{e}_1 \cdot X_1)/\xi(\beta))]. \tag{4.44}$$

It suffices to prove that we can choose $\delta, c_1 > 0$ such that $M_\beta(c_1) \leq 3$ for every $\beta < \beta(\delta)$.

We define a finite integer K by

$$K := \min\Big\{k \geq 0 : \text{there exists } \beta_0 < (1-\tfrac{1}{2L}) \text{ such that } \chi(\beta_0) = L^{-k}\chi(\beta)\Big\}. \tag{4.45}$$

With β_0 so defined, for $0 \leq k \leq K$ we introduce β_k by setting

$$\chi(\beta_k) = L^k \chi(\beta_0). \tag{4.46}$$

In particular, $\beta_K = \beta$. We will prove by induction on k that $M_{\beta_k}(c_1) \leq 3$, for appropriate $\delta, c_1 > 0$.

To start the induction, we prove that $M_{\beta_0}(c_1) \leq 3$, with $c_1 > 0$ to be chosen. By Proposition 4.1 and the fact that (by definition of $\beta(\delta)$)

$$E(\beta) \leq \delta \leq \frac{1}{4L} \leq \frac{1}{2}, \tag{4.47}$$

we have $\xi(\beta_0) \geq \xi(0) = \sigma_J$. Let $\lambda_0 = \exp[c_1/c_0]$ where c_0 is given by (1.10). Then $M_0(c_1\sigma_J/\xi(\beta_0)) \leq \exp(c_1 \frac{\sigma_J}{\xi(\beta_0)} \frac{R_J}{\sigma_J}) \leq \lambda_0$. We require that $c_1 = c_1(c_0, L, d) > 0$ be sufficiently small that $\lambda_0 \leq 1 + \frac{1}{4L}$. Then $\beta_0\lambda_0 < (1-\frac{1}{2L})(1+\frac{1}{4L}) < 1$ and hence, by Corollary 2.9,

$$M_{\beta_0}(c_1) \leq \frac{1}{\chi(\beta_0)} \frac{M_0(c_1\sigma_J/\xi(\beta_0))}{1-\beta_0 M_0(c_1\sigma_J/\xi(\beta_0))} \leq \frac{\lambda_0}{\chi(\beta_0)} \frac{1}{1-\beta_0\lambda_0}. \tag{4.48}$$

Also, since $\delta \leq \frac{1}{4L}$, (4.5) implies that

$$\frac{1}{\chi(\beta_0)} = \frac{\chi(0)}{\chi(\beta_0)} \leq 1 - \beta_0(1-\delta) \leq 1 - \beta_0\left(1 - \frac{1}{4L}\right). \tag{4.49}$$

As a result, we have

$$\begin{aligned} M_{\beta_0}(c_1) &\leq \left(1+\frac{1}{4L}\right) \max_{t\in[0,1-\frac{1}{2L}]} \frac{1-(1-\frac{1}{4L})t}{1-(1+\frac{1}{4L})t} \\ &= \left(1+\frac{1}{4L}\right) \frac{1-(1-\frac{1}{4L})(1-\frac{1}{2L})}{1-(1+\frac{1}{4L})(1-\frac{1}{2L})} \leq 3. \end{aligned} \tag{4.50}$$

To advance the induction, we assume that $M_{\beta_k}(c_1) \leq C_1$ and prove that the same bound holds when k is replaced by $k+1$. To abbreviate the notation, we write

$$Z_k = Z_{\beta_k,\beta_{k+1}}. \tag{4.51}$$

Let

$$r_k := \frac{\xi(\beta_k)}{\xi(\beta_{k+1})}. \tag{4.52}$$

By Corollary 2.9, if $Z_k M_{\beta_k}(c_1 r_k) < 1$ then

$$M_{\beta_{k+1}}(c_1) \leq \frac{\chi(\beta_k)}{\chi(\beta_{k+1})} \frac{M_{\beta_k}(c_1 r_k)}{1 - Z_k M_{\beta_k}(c_1 r_k)}. \tag{4.53}$$

To apply this, we must show that $Z_k M_{\beta_k}(c_1 r_k) < 1$. For this, we will make use of the fact that (4.47), Corollary 4.2 and Proposition 4.3 imply the upper bounds

$$r_k^2 \leq 2\frac{\chi(\beta_k)}{\chi(\beta_{k+1})} = \frac{2}{L}, \qquad Z_k \leq 1 - \frac{1}{2}\frac{\chi(\beta_k)}{\chi(\beta_{k+1})} = 1 - \frac{1}{2L}. \tag{4.54}$$

It is elementary that for any $0 \le |u| \le |v|$, we have

$$\cosh u \le 1 + \frac{u^2}{2} + \frac{u^4}{v^4} \cosh v, \tag{4.55}$$

and hence for any real and symmetric random variable U and any $0 \le s \le t$,

$$\mathbb{E}[\exp(sU)] = \mathbb{E}[\cosh sU] \le 1 + \frac{s^2}{2}\mathbb{E}[U^2] + \left(\frac{s}{t}\right)^4 \mathbb{E}[\exp(tU)]. \tag{4.56}$$

We apply (4.56) to $\mathbb{E}_{\beta_k}$ with $s = c_1 r_k$, $t = c_1$, and $U = (\mathbf{e}_1 \cdot X_1)/\xi(\beta_k)$. By symmetry, $\mathbb{E}_{\beta_k}[U^2] = \frac{1}{d}$. Therefore, by (4.54) and the induction hypothesis,

$$M_{\beta_k}(c_1 r_k) \le 1 + \frac{c_1^2}{2d} r_k^2 + 3r_k^4 \le 1 + \frac{c_1^2}{2d}\frac{2}{L} + 3\left(\frac{2}{L}\right)^2 \le 1 + \left(c_1^2 + \frac{12}{L}\right)\frac{1}{L}. \tag{4.57}$$

We choose c_1 small depending on L, so that the above gives

$$M_{\beta_k}(c_1 r_k) \le 1 + \frac{15}{L}\frac{1}{L}. \tag{4.58}$$

Then, by (4.54), and since $\frac{15}{L} = \frac{15}{128} < 1/2$,

$$Z_k\, M_{\beta_k}(c_1 r_k) \le \left(1 - \frac{1}{2L}\right)\left(1 + \frac{15}{L}\frac{1}{L}\right) \le 1 - \left(\frac{1}{2} - \frac{15}{L}\right)\frac{1}{L} < 1. \tag{4.59}$$

This allows us to apply (4.53), to obtain

$$M_{\beta_{k+1}}(c_1) \le \frac{1}{L}\frac{1 + \frac{15}{L}\frac{1}{L}}{(\frac{1}{2} - \frac{15}{L})\frac{1}{L}} = \frac{2 + \frac{30}{L^2}}{1 - \frac{30}{L}}. \tag{4.60}$$

By choice of L, the right-hand side is less than 3. This concludes the induction and the proof.

Something important just happened: we gain by $Z \le 1 - \frac{1}{2L}$ and we lose by $1 + \frac{15}{L^2}$ in (4.57) when c_1 is small enough. When we take the product of these two effects, we still gain. The potentially bad effect of the constant 15 is overcome by a choice of large L .

To summarise the two cases, we have obtained that for $L = 128$ and $\delta \le \frac{1}{4L}$, for every $\beta < \beta(\delta)$ we have

$$M_\beta(c_1) \le 3. \tag{4.61}$$

We now set $\delta_{\mathrm{reg}} := \frac{1}{4L} = 2^{-9}$ and $(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}) = (c_1, 3)$. This completes the proof. □

4.4 Stability of the finite-volume susceptibility

We now prove the stability estimate stated in Proposition 2.12 and reformulated in (2.32). For $k \ge 0$, the finite-volume susceptibility is defined by

$$\chi_k(\beta) = \sum_{x \in \Lambda_k} F_\beta(x). \tag{4.62}$$

Proposition 4.5 (Stability estimate)**.** *Let $d > 2$. Suppose that J and G both obey Assumption* I. *There exist $\delta_{\mathrm{stab}} \in (0, \frac{1}{10}]$ and $C_{\mathrm{stab}} > 0$, depending only on d, such that, for every $\beta < \beta(\delta_{\mathrm{stab}})$,*

$$\chi_{\xi(\beta')}(\beta) \le C_{\mathrm{stab}}\chi(\beta'), \qquad \forall \beta' \le \beta. \tag{4.63}$$

Remark 4.6. Let $(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}} = 3)$ be the regularity constants of the effective random walk from Proposition 2.8, and let $\mathsf{C}_{\mathrm{RW}} = \mathsf{C}_{\mathrm{RW}}(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}) > 0$ be the constants of Theorem 2.10. The proof of Proposition 4.5 shows that the constant C_{stab} can be chosen as $C_{\mathrm{stab}} = 32\mathsf{C}_{\mathrm{RW}}$.

The proof of Proposition 4.5 builds on the following three elementary lemmas. The first lemma is a straightforward extension of Lemma 2.5. Its bound on $F_\beta(x)$ serves as the starting point for bounding $\chi_{\xi(\beta')}(\beta)$ in the proof of Proposition 4.5.

Lemma 4.7. *Let $0 \le \beta' \le \beta < \beta_c$. For every $T \ge 1$ and every $x \in \mathbb{Z}^d$,*

$$F_\beta(x) \le (Z_{\beta',\beta} \vee 1)^T \Big(\chi(\beta')(\mathbb{G}_{1,\beta'}(x) - \delta_0(x)) + \sum_{z\in\mathbb{Z}^d} F_\beta(z)\mathbb{P}_{\beta'}[X_T = x - z]\Big). \tag{4.64}$$

Proof. Let $T \ge 1$ and $x \in \mathbb{Z}^d$. Recall from (2.15) (with G replaced by F) that

$$F_\beta(x) \le \sum_{k=0}^{T-1} Z^k_{\beta',\beta}\mathbb{E}_{\beta'}[F_{\beta'}(x - X_k)] + Z^T_{\beta',\beta}\mathbb{E}_{\beta'}[F_\beta(x - X_T)]. \tag{4.65}$$

By (4.65), and since $Z_{\beta',\beta} \le (Z_{\beta',\beta} \vee 1)$, we obtain

$$\begin{aligned} F_\beta(x) &\le (Z_{\beta',\beta} \vee 1)^{T-1}\chi(\beta') \sum_{k=0}^{T-1} \mathbb{P}_{\beta'}[X_{k+1} = x] + (Z_{\beta',\beta} \vee 1)^T \sum_{z\in\mathbb{Z}^d} \mathbb{P}_{\beta'}[X_T = z]F_\beta(x - z) \\ &\le (Z_{\beta',\beta} \vee 1)\Big(\chi(\beta')(\mathbb{G}_{1,\beta'}(x) - \delta_0(x)) + \sum_{z\in\mathbb{Z}^d} F_\beta(z)\mathbb{P}_{\beta'}[X_T = x - z]\Big). \end{aligned} \tag{4.66}$$

This completes the proof. □

The second lemma provides a means, given a bound on a finite-volume susceptibility for one volume, to extract a bound for a larger volume. As is clear from its proof, the exponent $\frac{5}{2}$ in (4.68) could be reduced to any exponent $a > 2$ by choosing δ appropriately small. We use $\frac{5}{2}$ as a concrete choice; later we only need that $d > a$, which is satisfied for every $d > 2$.

Lemma 4.8. *Let $\delta \le \frac{1}{10}$ and $0 < \beta' < \beta < \beta(\delta)$. Assume that there exists $C_1 \ge 1$ such that, for every $\beta'' < \beta$ with $\frac{\xi(\beta'')}{\xi(\beta')} \ge 2$, the inequality*

$$\chi_{\xi(\beta'')}(\beta) \le C_1\chi(\beta'') \tag{4.67}$$

holds. Then, for every $k \ge 2$,

$$\chi_{k\xi(\beta')}(\beta) \le C_1k^{5/2}\chi(\beta'). \tag{4.68}$$

Proof. Fix $\delta \le \frac{1}{10}$ and $0 < \beta' < \beta < \beta(\delta)$. Let $k \ge 2$. We consider two cases. First, suppose that $k\xi(\beta') \ge \xi(\beta)$. By (4.3) and the fact that $\frac{2}{1-2\delta} \le \frac{5}{2}$,

$$\chi_{k\xi(\beta')}(\beta) \le \chi(\beta) = \frac{\chi(\beta)}{\chi(\beta')}\chi(\beta') \le \left(\frac{\xi(\beta)}{\xi(\beta')}\right)^{5/2} \chi(\beta') \le k^{5/2}\chi(\beta'). \tag{4.69}$$

If instead $k\xi(\beta') < \xi(\beta)$, then we define $\beta'' < \beta$ by the requirement that $\xi(\beta'') = k\xi(\beta')$. Since $k \ge 2$, we have $\frac{\xi(\beta'')}{\xi(\beta')} \ge 2$. It then follows from (4.67) that

$$\chi_{k\xi(\beta')}(\beta) = \chi_{\xi(\beta'')}(\beta) \le C_1\chi(\beta'') \le C_1 \left(\frac{\xi(\beta'')}{\xi(\beta')}\right)^{5/2} \chi(\beta') = C_1k^{5/2}\chi(\beta'). \tag{4.70}$$

This concludes the proof. □

The third and final lemma is an elementary estimate. It will be used in conjunction with Lemma 4.8 to handle the convolution appearing in the last term of (4.64).

Lemma 4.9. *Let $\alpha, K > 0$ and $\xi \geq 1$. Suppose that $f : \mathbb{Z}^d \to [0,\infty)$ satisfies, for every $m \geq 2$,*

$$\sum_{x \in \Lambda_{m\xi}} f(y) \leq K m^{\alpha}. \tag{4.71}$$

Then, for every $\kappa > 0$, there exists $C_0 = C_0(\kappa, \alpha) > 0$ such that, for every $T \geq 4$,

$$\sum_{x \in \mathbb{Z}^d} f(x) \exp\left(-\kappa \frac{|x|}{\xi\sqrt{T}}\right) \leq C_0 K T^{\alpha/2}. \tag{4.72}$$

Proof. We decompose the sum into annuli centred at the origin. For this, we define radii $u_0 := 0$, $u_1 := \xi\sqrt{T}$, and $u_{k+1} := 2u_k$ for $k \geq 1$. We also let $\varphi_T(x) = \exp(-\kappa \frac{|x|}{\xi\sqrt{T}})$. Then, as $\sqrt{T} \geq 2$, we can apply our hypothesis on f to obtain

$$\begin{aligned}
\sum_{x \in \mathbb{Z}^d} f(x)\varphi_T(x) &= \sum_{k \geq 0} \sum_{u_k \leq |x| < u_{k+1}} f(x)\varphi_T(x) \\
&\leq \sum_{x \in \Lambda_{\xi\sqrt{T}}} f(x) + \sum_{m \geq 1} e^{-\kappa 2^{m-1}} \sum_{x \in \Lambda_{2^m \xi\sqrt{T}}} f(x) \\
&\leq K T^{\alpha/2} + \sum_{m \geq 1} e^{-\kappa 2^{m-1}} K (2^m \sqrt{T})^{\alpha}.
\end{aligned} \tag{4.73}$$

The right-hand side is bounded by a constant multiple of $KT^{\alpha/2}$, with a constant depending only on κ and α. □

With Lemmas 4.7–4.9, we are now in a position to prove Proposition 4.5.

Proof of Proposition 4.5. Let $\delta \in (0, \frac{1}{10}]$ to be chosen small enough, and C_{stab} to be chosen large enough. Fix $\beta < \beta(\delta)$. We recursively define a decreasing sequence $(\beta_\ell)_{\ell \geq 0}$ as follows:

- Set $\beta_0 := \beta$.
- Assume that $\beta_0, \ldots, \beta_\ell$ have been constructed. If $\xi(\beta_\ell) \in [\xi(0), 2\xi(0))$, we stop the construction. Otherwise, we define $\beta_{\ell+1} \in [0, \beta_\ell)$ by $\xi(\beta_{\ell+1}) = \frac{1}{2}\xi(\beta_\ell)$.
- Let $M \geq 0$ be the largest ℓ such that β_ℓ has been constructed, and let $\beta_{M+1} = 0$.

We now assume that $\delta \leq \delta_{\text{reg}}$, with δ_{reg} given by Proposition 2.8. For such δ, the effective random walk is uniformly $(\mathsf{c}_{\text{reg}}, 3)$-regular for some $\mathsf{c}_{\text{reg}} = \mathsf{c}_{\text{reg}}(d) > 0$. The bounds for regular random walks in Theorem 2.10 therefore apply. Let $\mathsf{C}_{\text{RW}} \geq 1$ be given by Theorem 2.10 for the pair $(\mathsf{c}_{\text{reg}}, 3)$.

We will prove that if δ is small enough, then for every $0 \leq \ell \leq M$, we have

$$\chi_{\xi(\beta_\ell)}(\beta) \leq 4\mathsf{C}_{\text{RW}}\chi(\beta_\ell). \tag{4.74}$$

We claim that the above is sufficient to conclude the proof, with $C_{\text{stab}} := 32\mathsf{C}_{\text{RW}}$. Indeed, let $\beta' < \beta$ (there is nothing to prove if $\beta' = \beta$). There exists a unique $0 \leq \ell \leq M$ such that $\beta' \in [\beta_{\ell+1}, \beta_\ell)$. Therefore,

$$\chi_{\xi(\beta')}(\beta) \leq \chi_{\xi(\beta_\ell)}(\beta) \leq 4\mathsf{C}_{\text{RW}}\chi(\beta_\ell) = 4\mathsf{C}_{\text{RW}}\frac{\chi(\beta_\ell)}{\chi(\beta')}\chi(\beta') \leq 32\mathsf{C}_{\text{RW}}\chi(\beta'), \tag{4.75}$$

where in the first inequality we used $\delta \le \frac{1}{2}$ and Proposition 4.1 to see that $\xi(\beta') \le \xi(\beta_\ell)$, in the second we used (4.74), and in the last inequality we used (4.3) and the assumption that $\delta \le \frac{1}{10}$ to obtain

$$\frac{\chi(\beta_\ell)}{\chi(\beta')} \le \left(\frac{\xi(\beta_\ell)}{\xi(\beta')}\right)^{\frac{2}{1-2\delta}} \le 2^3 = 8. \tag{4.76}$$

We prove (4.74) by induction on ℓ. First, observe that (4.74) for $\ell = 0$ follows from the fact that $4\mathsf{C}_{\mathrm{RW}} \ge 4 \ge 1$.

To advance the induction, we fix $1 \le \ell \le M$ and assume that (4.74) holds for all values $m \le \ell - 1$. Our goal is to prove (4.74) at ℓ. Let $\beta' = \beta_\ell$, $n := \xi(\beta')$, $Z := Z_{\beta',\beta}$, and $\mathbb{P} = \mathbb{P}_{\beta'}$. By (4.64), for every $T \ge 1$, and every $x \in \mathbb{Z}^d$,

$$F_\beta(x) \le (Z \vee 1)^T \Big(\chi(\beta')\mathbb{G}(x) + \sum_{z\in\mathbb{Z}^d} F_\beta(z)\mathbb{P}[X_T = x - z]\Big), \tag{4.77}$$

where $\mathbb{G}$ is the Green function associated with the effective random walk at β' at $Z = 1$. Summation of (4.77) over $x \in \Lambda_n$ gives

$$\chi_n(\beta) \le (Z \vee 1)^T \left(\mathbb{G}(\Lambda_n)\chi(\beta') + \sum_{z\in\mathbb{Z}^d} F_\beta(z)\mathbb{P}[X_T \in \Lambda_n(z)]\right). \tag{4.78}$$

Since it follows from Theorem 2.10 that $\mathbb{G}(\Lambda_n) \le \mathsf{C}_{\mathrm{RW}}$, the first term on the right-hand side of (4.78) obeys

$$(Z \vee 1)^T \mathbb{G}(\Lambda_n)\chi(\beta') \le (Z \vee 1)^T \mathsf{C}_{\mathrm{RW}}\chi(\beta'). \tag{4.79}$$

For the last term in (4.78), the anti-concentration estimate (2.29) gives, for every $z \in \mathbb{Z}^d$

$$\mathbb{P}[X_T \in \Lambda_n(z)] \le \frac{\mathsf{C}_{\mathrm{RW}}}{T^{d/2}} \exp\left(-\mathsf{c}_{\mathrm{RW}}\frac{|z|}{n\sqrt{T}}\right). \tag{4.80}$$

In order to apply Lemma 4.9, we need to show that there exists $C_2 = C_2(d) > 0$ such that, for every $k \ge 2$,

$$\sum_{x\in\Lambda_{kn}} F_\beta(x) = \chi_{kn}(\beta) \le C_2\chi(\beta')k^{5/2}. \tag{4.81}$$

By Lemma 4.8, (4.81) will follow if we show the existence of $C_3 = C_3(d) > 0$ such that for any $\beta'' < \beta$ with $\frac{\xi(\beta'')}{\xi(\beta')} \ge 2$,

$$\chi_{\xi(\beta'')}(\beta) \le C_3\chi(\beta''). \tag{4.82}$$

However, if β'' is as above, then $\beta'' \ge \beta_{\ell-1}$. Therefore, by the induction hypothesis and the same computation as in (4.75), (4.82) is satisfied with $C_3 = 32\mathsf{C}_{\mathrm{RW}}$. Hence, the conclusion of Lemma 4.8 holds and so does (4.81) for some $C_2 = C_2(d) > 0$. We can now take $T \ge 4$ and apply Lemma 4.9 with $\alpha = \frac{5}{2}$ and $K = C_2\chi(\beta')$ to find $C_4 = C_4(d) > 0$ such that

$$\sum_{x\in\mathbb{Z}^d} F_\beta(x)\varphi_T(x) \le C_4\chi(\beta')T^{5/4}. \tag{4.83}$$

The combination of (4.80), (4.82) and (4.83) gives the existence of $C_5 = C_5(d) > 0$ such that

$$\sum_{z\in\mathbb{Z}^d} F_\beta(z)\mathbb{P}[X_T \in \Lambda_n(z)] \le \frac{C_5}{T^{\frac{1}{2}(d-\frac{5}{2})}}\chi(\beta'). \tag{4.84}$$

Since $d \geq 3$, we may choose T large enough (depending only on d) that

$$\frac{C_5}{T^{\frac{1}{2}(d-\frac{5}{2})}} \leq \mathsf{C}_{\mathrm{RW}}. \tag{4.85}$$

This shows that

$$(Z \vee 1)^T \sum_{z \in \mathbb{Z}^d} F_\beta(z) \mathbb{P}[X_T \in \Lambda_n(z)] \leq (Z \vee 1)^T \mathsf{C}_{\mathrm{RW}} \chi(\beta'). \tag{4.86}$$

We now insert the bounds (4.79) and (4.86) into (4.78), and obtain

$$\chi_n(\beta) \leq 2(Z \vee 1)^T \mathsf{C}_{\mathrm{RW}} \chi(\beta'). \tag{4.87}$$

Finally, it follows from (4.24) that $(Z \vee 1) \leq (1-\delta)^{-1}$. We choose δ small enough (depending on d) such that $(1-\delta)^{-T} \leq 2$. For this choice, (4.87) gives $\chi_n(\beta) \leq 4\mathsf{C}_{\mathrm{RW}} \chi(\beta')$, and the induction step is complete. This concludes the proof. □

5 Proof of the multiscale bounds

We now turn to the proofs of the three remaining propositions, Propositions 3.5–3.7. Throughout the section, we fix $d > 2$ and $\varepsilon > 0$. We also suppose that J, G satisfy Assumption I. For convenience, we drop the subscript J in the notations and write simply σ instead of σ_J. We always assume that $\delta \leq \delta_{\mathrm{reg}} \wedge \delta_{\mathrm{stab}}$ so that Propositions 2.8 and 4.5 hold. As a result, by Proposition 2.8, we know that the effective random walk is uniformly $(\mathsf{c}_{\mathrm{reg}}, 3)$-regular for every $\beta < \beta(\delta)$. This allows us to apply Theorem 2.10 with constants $(\mathsf{c}_{\mathrm{RW}}, \mathsf{C}_{\mathrm{RW}})$ which depend on $(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}} = 3, d)$.

5.1 Proof of Proposition 3.5: bound on $F_\beta(x)$ for $|x| \leq 2\sigma$

Recall that the constant C, which appears in (3.15), is the constant arising in the estimate for the Green function $\mathbb{C}$ of the random walk $\mathbb{P}_J$, in Proposition 2.1.

Proof of Proposition 3.5. We will prove that there exists $\delta_s \in (0, \frac{1}{2}]$ such that, for every $\beta < \beta(\delta_s)$ and every $|x| \leq 2\xi(0)$,

$$F_\beta(x) \leq 2(\mathbb{C}_1(x) - \delta_0(x)) + 2\mathsf{C}\left(\frac{1}{2\sigma}\right)^d. \tag{5.1}$$

We can bound the right-hand side using Proposition 2.1, which together with $|x| \leq 2\sigma$ gives

$$(\mathbb{C}_1(x) - \delta_0(x)) + \mathsf{C}\left(\frac{1}{2\sigma}\right)^d \leq \frac{\mathsf{C}}{\sigma^d}\left(\frac{\sigma}{\sigma \vee |x|}\right)^{d-2} + \frac{\mathsf{C}}{\sigma^d}\left(\frac{\sigma}{\sigma \vee |x|}\right)^{d-2}. \tag{5.2}$$

With (5.1), this proves our goal (3.15).

It remains to prove (5.1). Let $\delta \in (0, \frac{1}{2}]$ to be taken small enough, let $\beta < \beta(\delta)$, and set $\beta' = 0$. Then $\chi(\beta') = 1$, $\mathbb{P}_{\beta'} = \mathbb{P}_J$, and $\mathbb{G}_{1,\beta'} = \mathbb{C}_1$. We write $Z := Z_{\beta',\beta}$. By (4.64), for every $T \geq 1$ and every $x \in \mathbb{Z}^d$, we have

$$F_\beta(x) \leq (Z \vee 1)^T \Big((\mathbb{C}_1(x) - \delta_0(x)) + \sum_{z \in \mathbb{Z}^d} F_\beta(z) \mathbb{P}_J[X_T = x - z] \Big). \tag{5.3}$$

Our main effort is to bound the second term on the right-hand side of (5.3). We argue as in the proof of Proposition 4.5. The anti-concentration estimate (2.5) asserts that there exists (c, C) (depending on d and c_0 from (1.10)) such that, for every $z \in \mathbb{Z}^d$,

$$\mathbb{P}_J[X_T = z] \le \frac{\mathsf{C}}{\sigma^d T^{d/2}} \exp\left(-\mathsf{c}\frac{|z|}{\sigma\sqrt{T}}\right). \tag{5.4}$$

Since $|x| \le 2\xi(0)$, the choice $T \ge 4$ in (5.4) gives, for every $z \in \mathbb{Z}^d$,

$$\begin{aligned}\mathbb{P}_J[X_T = x - z] &\le \frac{\mathsf{C}}{\sigma^d T^{d/2}} \exp\left(-\mathsf{c}\frac{|z|}{\sigma\sqrt{T}}\right) \exp\left(\mathsf{c}\frac{2}{\sqrt{T}}\right) \\ &\le \frac{\mathsf{C}e^{\mathsf{c}}}{\sigma^d T^{d/2}} \exp\left(-\mathsf{c}\frac{|z|}{\sigma\sqrt{T}}\right).\end{aligned} \tag{5.5}$$

Proposition 4.5 and Lemma 4.8 then imply that, if $\delta \le \delta_{\mathrm{stab}}$, then for every $k \ge 2$,

$$\sum_{z \in \Lambda_{k\sigma}} F_\beta(z) \le C_{\mathrm{stab}} k^{5/2} \chi(\beta') = C_{\mathrm{stab}} k^{5/2}. \tag{5.6}$$

We apply Lemma 4.9 with $\alpha = \frac{5}{2}$, $f = F_\beta$, and $K = C_{\mathrm{stab}}$, to obtain $C_1 = C_1(\mathsf{c}, d) > 0$ such that

$$\sum_{z \in \mathbb{Z}^d} F_\beta(z) \exp\left(-\mathsf{c}\frac{|z|}{\sigma\sqrt{T}}\right) \le C_1 T^{5/4}. \tag{5.7}$$

The combination of (5.5) and (5.7) gives $C_2 = C_2(\mathsf{c}, d) > 0$ such that, for every $T \ge 4$ and every $|x| \le 2\xi(0)$,

$$\sum_{z \in \mathbb{Z}^d} F_\beta(z) \mathbb{P}_J[X_T = x - z] \le \frac{C_2}{\sigma^d T^{\frac{1}{2}(d - \frac{5}{2})}}. \tag{5.8}$$

Since $d > 2$, we may now choose T large enough so that

$$\frac{C_2}{T^{\frac{1}{2}(d - \frac{5}{2})}} \le \frac{\mathsf{C}}{2^d}. \tag{5.9}$$

Finally, we pick δ sufficiently small so that the prefactor on the right-hand side of (5.3) satisfies $(Z \vee 1)^T \le (1-\delta)^{-T} \le 2$. With (5.7)–(5.8), we see that the second term on the right-hand side of (5.3) is at most $2\mathsf{C}(2\sigma)^{-d}$. This proves (5.1) and completes the proof. □

5.2 Proof of Proposition 3.6: contraction step for typical scales

Sections 5.2 and 5.3 are dedicated to the proof of the contraction step of the bootstrap argument, namely Propositions 3.6 and 3.7. We now prove Proposition 3.6.

Recall that, for every $x \in \mathbb{Z}^d$, for every $\beta \ge 0$, and for $\mathbf{c}, \mathbf{C} > 0$, $\mathcal{H}_\beta(\mathbf{c}, \mathbf{C}; x)$ and $\mathcal{H}_\beta(\mathbf{c}, \mathbf{C})$ are the statements:

$$\mathcal{H}_\beta(\mathbf{c}, \mathbf{C}; x): \quad F_\beta(x) \le \frac{\mathbf{C}}{\sigma^d} \left(\frac{\sigma}{\sigma \vee |x|}\right)^{d-2-\varepsilon} \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right), \tag{5.10}$$

$$\mathcal{H}_\beta(\mathbf{c}, \mathbf{C}): \quad \mathcal{H}_\beta(\mathbf{c}, \mathbf{C}; x) \text{ holds for all } x \in \mathbb{Z}^d. \tag{5.11}$$

Proposition 3.6 can be reformulated as follows: if $\eta > 0$, $\mathbf{C} \ge 16\mathsf{C}$, $\mathbf{c} \le \frac{1}{2}\mathsf{c} \wedge \frac{1}{4}\mathsf{c}_{\mathrm{RW}} \wedge 1$, then there exists δ_2 small enough such that for every $\beta < \beta(\delta_2)$, if $\mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C})$ holds for $\beta' \le \beta$, then $\mathcal{H}_\beta(\mathbf{c}, \frac{\mathbf{C}}{8}; x)$ holds for every $|x| \ge \eta\xi(\beta) \vee 2\xi(0)$. The improved bound will be obtained by relying on the finite-difference inequality (**I.1**).

We stress that the proof of Proposition 3.6 is the only place in the entire paper where the assumption that $\varepsilon > 0$ is used. We begin by stating two simple lemmas. The first one, an elementary consequence of (**I.1**), is the starting point of the proof. Recall from (2.12) that the Green function of the effective random walk is defined, for $Z \in [0,1]$ and $x \in \mathbb{Z}^d$, by

$$\mathbb{G}_{Z,\beta'}(x) = \sum_{k\geq 0} Z^k \mathbb{P}_{\beta'}[X_k = x]. \tag{5.12}$$

Lemma 5.1. *Let $0 \leq \beta' \leq \beta < \beta_c$ be such that $Z_{\beta',\beta} < 1$. For every $m > 0$ and every $x \in \mathbb{Z}^d$ with $|x| > m$, we have*

$$F_\beta(x) \leq \frac{\chi_m(\beta')}{\chi(\beta')} \sup_{y\in\Lambda_m(0)} F_\beta(x-y) + \sum_{y\notin\Lambda_m(0)} F_{\beta'}(y)\mathbb{G}_{Z_{\beta',\beta},\beta'}(x-y). \tag{5.13}$$

Proof. By the finite-difference upper bound in (**I.1**),

$$F_\beta(x) \leq F_{\beta'}(x) + (\beta-\beta') \sum_{y\in\mathbb{Z}^d} F_{\beta'}(y)F_\beta(x-y). \tag{5.14}$$

The contribution to the convolution sum due to $y \in \Lambda_m(0)$ can be rewritten and bounded (since $Z_{\beta',\beta} < 1$) by

$$\frac{Z_{\beta',\beta}}{\chi(\beta')} \sum_{y\in\Lambda_m(0)} F_{\beta'}(y)F_\beta(x-y) \leq \frac{\chi_m(\beta')}{\chi(\beta')} \sup_{y\in\Lambda_m(0)} F_\beta(x-y). \tag{5.15}$$

This gives the first term on the right-hand side of (5.13).

From (2.16) with G replaced by F, we see that

$$(\beta-\beta')F_\beta(x) \leq Z_{\beta',\beta} \sum_{k\geq 0} Z^k_{\beta',\beta}\mathbb{P}_{\beta'}[X_{k+1} = x] = \mathbb{G}_{Z_{\beta',\beta},\beta'}(x) - \delta_0(x). \tag{5.16}$$

By (5.16), the remaining part of the right-hand side of (5.14) is therefore at most

$$F_{\beta'}(x) + \sum_{y\notin\Lambda_m(0)} F_{\beta'}(y)\big[\mathbb{G}_{Z_{\beta',\beta},\beta'}(x-y) - \delta_0(x-y)\big]. \tag{5.17}$$

By our assumption that $|x| > m$, the first term is cancelled by the Kronecker delta. This completes the proof. □

To bound the last term on the right-hand side of (5.13), we will use the following convolution lemma. We defer the elementary proof to Appendix B.

Lemma 5.2. *Let $a, b, c_1, c_2, \sigma, \xi > 0$, $\mu > 0$ and $\varepsilon \in [0,1]$. Suppose that the functions $f, g : \mathbb{Z}^d \to [0,\infty)$ satisfy*

$$f(x) \leq c_1 \frac{1}{\sigma^d}\left(\frac{\sigma}{\sigma\vee|x|}\right)^{d-2-\varepsilon} e^{-a|x|/\xi}, \tag{5.18}$$

$$g(\Lambda_\xi(x)) \leq c_2 \left(\frac{\xi}{\xi\vee|x|}\right)^{d-2} e^{-b|x|/\xi}. \tag{5.19}$$

Then, there exists $C_{a,\mu} > 0$ (depending on a, μ, d) such that for every $|x| \geq 2(\sigma\vee\xi)$,

$$\sum_{y\notin\Lambda_{\mu\xi}(0)} f(y)g(x-y) \leq \frac{c_1}{\sigma^2|x|^{d-2}}\left(\frac{|x|}{\sigma}\right)^\varepsilon \left(2^d\|g\|_1 e^{-a|x|/2\xi} + c_2 C_{a,\mu}\left(\frac{\xi}{|x|}\right)^\varepsilon e^{-b|x|/2\xi}\right). \tag{5.20}$$

We are now in a position to prove Proposition 3.6. Let c_{RW} and C_{RW} be given by Theorem 2.10, for the pair $(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}) = (\mathsf{c}_{\mathrm{reg}}, 3)$ provided by Proposition 2.8.

Proof of Proposition 3.6. Let $d > 2$, $\varepsilon > 0$ and $\eta > 0$. Let $\mathbf{C} \geq 16\mathsf{C}$ and $\mathbf{c} \leq \frac{1}{2}\mathsf{c} \wedge \frac{1}{4}\mathsf{c}_{\mathrm{RW}} \wedge 1$. Our goal is to prove that there exists $\delta_2 = \delta_2(\eta, \varepsilon, d) \leq \delta_s \wedge \delta_{\mathrm{reg}} \wedge \frac{1}{2}$ such that, for every $\beta < \beta(\delta_2)$, if $\mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C})$ holds for every $\beta' \leq \beta$, then, for every $|x| \geq \eta\xi(\beta) \vee 2\xi(0)$,

$$F_\beta(x) \leq \frac{\mathbf{C}}{8}\left(\frac{|x|}{\sigma}\right)^{\varepsilon} \frac{1}{\sigma^2|x|^{d-2}} \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right). \tag{5.21}$$

On the right-hand side, we used our assumption that $|x| \geq 2\xi(0) = 2\sigma$.

Let $\delta \leq \frac{1}{4} \wedge \delta_{\mathrm{reg}} \wedge \delta_{\mathrm{stab}}$, where δ_{reg} and δ_{stab} are respectively given by Propositions 2.8 and 4.5. If $\beta \leq \beta_0 := (1-\delta) \wedge \beta(\delta)$, the desired bound (5.21) is already known from (2.10). We can therefore assume that $\beta \in (\beta_0, \beta(\delta))$. Fix such a β. We wish to apply Lemma 5.1, which states that if $0 \leq \beta' \leq \beta$ are such that $Z_{\beta',\beta} < 1$, and if $|x| > m > 0$, then

$$F_\beta(x) \leq \frac{\chi_m(\beta')}{\chi(\beta')} \sup_{y \in \Lambda_m(0)} F_\beta(x-y) + \sum_{y \notin \Lambda_m(0)} F_{\beta'}(y)\mathbb{G}_{Z_{\beta',\beta},\beta'}(x-y). \tag{5.22}$$

Later in the proof, we write the two terms on the right-hand side of (5.22) as (I) and (II).

To apply (5.22), we need to pick β' and m. These choices are made in terms of $\xi(\beta)$, as follows. First, given $0 < \beta'' < \beta' < \beta$, we set

$$m = \xi(\beta''), \qquad n = \xi(\beta'), \qquad N = \xi(\beta). \tag{5.23}$$

Given β'' and β', we use $\mu, \nu \in (0,1]$ to denote the ratios

$$\mu = \frac{m}{n} = \frac{\xi(\beta'')}{\xi(\beta')}, \qquad \nu = \frac{n}{N} = \frac{\xi(\beta')}{\xi(\beta)}. \tag{5.24}$$

We will want to take μ, ν to be small, but we must have $m = \mu\nu N \geq \xi(0)$ in order to have the existence of β'' with $m = \xi(\beta'')$. This can be accomplished as follows. First, if $\delta \leq \frac{1}{4}$ and $\beta < \beta(\delta)$, (4.3) and (4.5) yield

$$\frac{\xi(\beta)}{\xi(0)} \geq \left(\frac{\chi(\beta)}{\chi(0)}\right)^{\frac{1}{2}-\delta} \geq \left(\frac{\chi(\beta)}{\chi(0)}\right)^{1/4} \geq \frac{1}{(1-\beta(1-\delta))^{1/4}}, \tag{5.25}$$

so,

$$\frac{\xi(\beta_0)}{\xi(0)} \geq \frac{1}{(1-(1-\delta)^2)^{1/4}} \geq \frac{1}{(2\delta)^{1/4}}. \tag{5.26}$$

Therefore, given any $\mu, \nu \leq 1$, we can choose $\delta(\mu, \nu)$ to be small enough that $m = \mu\nu\xi(\beta) \geq \mu\nu\xi(\beta_0) \geq (2\delta)^{1/4}\xi(\beta_0) \geq \xi(0)$.

For (5.22), we also need $|x| > m = \mu\nu\xi(\beta)$. With later needs in mind, we require that $\nu \leq \frac{1}{2}\eta$. Then, since we assume that $|x| \geq \eta\xi(\beta)$, we (more than) ensure that $|x| > m$ since $m = \mu\nu N \leq \nu N \leq \frac{1}{2}\eta N$. In the following, we will first fix μ, and then choose a ν satisfying $\nu \leq \frac{1}{2}\eta$ (with further restriction below). This adds η-dependence to δ. We also require that $\delta \leq \frac{1}{8}(\mu \wedge \nu)^2$ so that the conclusions of Corollary 4.2 and Proposition 4.3 hold with the pairs (β'', β') and (β', β).

By Proposition 4.5 and (4.16), the ratio in the first term on the right-hand side of (5.22) obeys

$$\frac{\chi_m(\beta')}{\chi(\beta')} = \frac{\chi_{\xi(\beta'')}(\beta')}{\chi(\beta'')} \frac{\chi(\beta'')}{\chi(\beta')} \leq C_{\mathrm{stab}} \cdot 2\left(\frac{\xi(\beta'')}{\xi(\beta')}\right)^2 = C_{\mathrm{stab}} \cdot 2\mu^2. \tag{5.27}$$

By the assumption that $\mathcal{H}_\beta(\mathbf{c}, \mathbf{C})$ holds, and by using $\mu n/N = \mu\nu \le 1$ and our assumption $\mathbf{c} \le 1$ in the exponent, we see that

$$\sup_{y \in \Lambda_m(0)} F_\beta(x-y) \le e \cdot \frac{\mathbf{C}}{\sigma^d} \left(\frac{\sigma}{\sigma \vee (|x| - \mu n)} \right)^{d-2-\varepsilon} \exp\left(-\mathbf{c} \frac{|x|}{\xi(\beta)} \right). \tag{5.28}$$

Since $|x| \ge \eta N$, and since $\mu\nu \le \nu \le \frac{1}{2}\eta$, we have

$$|x| - \mu n \ge |x| \left(1 - \frac{\mu n}{\eta N} \right) = |x| \left(1 - \frac{\mu\nu}{\eta} \right) \ge \frac{1}{2}|x|. \tag{5.29}$$

Therefore, with the choice

$$\mu^2 = \frac{1}{16} \frac{1}{e 2^{d-2-\varepsilon} C_{\mathrm{stab}} 2}, \tag{5.30}$$

the first term of (5.22) obeys

$$(\mathrm{I}) \le \frac{1}{16} \frac{\mathbf{C}}{\sigma^d} \left(\frac{\sigma}{\sigma \vee |x|} \right)^{d-2-\varepsilon} \exp\left(-\mathbf{c} \frac{|x|}{\xi(\beta)} \right). \tag{5.31}$$

To handle the second term on the right-hand side of (5.22), we apply Lemma 5.2 with $f = F_{\beta'}$, $g = \mathbb{G}_{Z_{\beta',\beta},\beta'}$, $\xi = \xi(\beta')$, and with the μ we chose in (5.30). The hypothesis on f is verified by $\mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C})$ with $c_1 = \mathbf{C}$, $a = \mathbf{c}$, $\xi = n$, and the hypothesis on g is verified by Theorem 2.10 with $c_2 = \mathsf{C}_{\mathrm{RW}}$, $b = \mathsf{c}_{\mathrm{RW}} \sqrt{1 - Z_{\beta',\beta}}$, $\xi = n$. It therefore follows from Lemma 5.2, since $|x| \ge \eta N \ge 2n$ (as $\nu \le \frac{1}{2}\eta$) and $\|\mathbb{G}_{Z_{\beta',\beta},\beta'}\|_1 = (1 - Z_{\beta',\beta})^{-1}$, that

$$(\mathrm{II}) \le \frac{\mathbf{C}}{\sigma^2 |x|^{d-2}} \left(\frac{|x|}{\sigma} \right)^{\varepsilon} \left(\frac{2^d}{1 - Z} e^{-\mathbf{c}|x|/2n} + \mathsf{C}_{\mathrm{RW}} C_{\mathbf{c},\mu} \left(\frac{n}{|x|} \right)^{\varepsilon} e^{-\mathsf{c}_{\mathrm{RW}} \sqrt{1-Z}|x|/2n} \right), \tag{5.32}$$

where $Z = Z_{\beta',\beta}$. By (4.26) applied to (β', β), we have

$$Z \le 1 - \frac{1}{4} \Big(\frac{n}{N} \Big)^2. \tag{5.33}$$

Since $\mathbf{c} \le \frac{1}{4} \mathsf{c}_{\mathrm{RW}}$, we also find

$$\mathsf{c}_{\mathrm{RW}} \frac{\sqrt{1-Z}}{2n} \ge 4\mathbf{c} \frac{1}{4N} = \frac{\mathbf{c}}{N}. \tag{5.34}$$

We use the above, together with $|x| \ge \eta N$, to conclude that there is a constant $C_1 = C_1(d, \eta, \mathbf{c}, \mu) > 0$ such that

$$(\mathrm{II}) \le C_1 \Big(\frac{n}{N} \Big)^{\varepsilon} \frac{\mathbf{C}}{\sigma^2 |x|^{d-2}} \left(\frac{|x|}{\sigma} \right)^{\varepsilon} \left(\left(\frac{N}{n} \right)^{2+\varepsilon} e^{-\mathbf{c}|x|/2n} + e^{-\mathbf{c}|x|/N} \right). \tag{5.35}$$

Again using $|x| \ge \eta N$, we see that

$$\frac{|x|}{2n} = \frac{|x|}{N} + \Big(\frac{N}{2n} - 1 \Big) \frac{|x|}{N} \ge \frac{|x|}{N} + \Big(\frac{N}{2n} - 1 \Big) \eta, \tag{5.36}$$

and hence

$$\left(\frac{N}{n} \right)^{2+\varepsilon} e^{-\mathbf{c}|x|/2n} \le e^{-\mathbf{c}|x|/N} e^{\mathbf{c}\eta} \left(\frac{N}{n} \right)^{2+\varepsilon} e^{-\frac{1}{2}\mathbf{c}\eta(N/n)} \le C_2(\mathbf{c}, \eta, \varepsilon) e^{-\mathbf{c}|x|/N}. \tag{5.37}$$

Therefore, by (5.35) and (5.37), there is a constant $C_3 = C_3(d, \eta, \mathbf{c}, \mu, \varepsilon) > 0$ such that

$$(\mathrm{II}) \le C_3 \nu^\varepsilon \frac{\mathbf{C}}{\sigma^2 |x|^{d-2}} \left(\frac{|x|}{\sigma}\right)^\varepsilon \exp\left(-\mathbf{c}\frac{|x|}{N}\right). \tag{5.38}$$

The inequality (5.38) holds for any ν such that $\nu \le \frac{1}{2}\eta$. In terms of our fixed choice of μ in (5.30), we can now choose ν small enough (which entails taking δ small) so that $C_3 \nu^\varepsilon \le \frac{1}{16}$.

From the bound on (I) in (5.31), together with the bound on (II) in (5.38) with our choice of ν, we obtain the desired result that

$$F_\beta(x) \le \left(\frac{1}{16} + \frac{1}{16}\right) \frac{\mathbf{C}}{\sigma^2 |x|^{d-2}} \left(\frac{|x|}{\sigma}\right)^\varepsilon \exp\left(-\mathbf{c}\frac{|x|}{N}\right). \tag{5.39}$$

This completes the proof. □

Remark 5.3. The need for $\varepsilon > 0$ occurs only near the end of the previous proof, to achieve $C_3 \nu^\varepsilon \le \frac{1}{16}$. This is the essential element of the contraction proof for typical scales. There is no additional need for ε elsewhere in the entire paper.

5.3 Proof of Proposition 3.7: contraction step for intermediate scales

The proof of Proposition 3.7 uses the following elementary convolution lemma, whose proof is deferred to Appendix B.

Lemma 5.4. *Let $p, a > 0$. For $i = 1, 2$, suppose that $f_i \in \ell^1(\mathbb{Z}^d)$ satisfy $0 \le f_i(x) \le a(1 \vee |x|)^{-p}$ for every $x \in \mathbb{Z}^d$. Let $k \ge 1$. Then, for every $x \in \mathbb{Z}^d$,*

$$(f_1 * f_2)(x) \le \frac{a}{(1 \vee |x|)^p} \left(\frac{1}{k^p} \|f_1\|_1 + 2^p \sum_{y \in \Lambda_{k|x|}(0)} (f_1(y) + f_2(y)) \right). \tag{5.40}$$

Proposition 3.7 establishes an improvement on the bound on F_β for the *intermediate* scales $2\xi(0) \le |x| \le \eta\xi(\beta)$. We do not use $\varepsilon > 0$ in the proof of Proposition 3.7—the parameter ε is merely a spectator.

Proof of Proposition 3.7. Let $d > 2$, $\varepsilon > 0$, $\mathbf{C} \ge 16\mathsf{C}$, $\mathbf{c} \le \frac{1}{2}\mathsf{c} \wedge \frac{1}{4}\mathsf{c}_{\mathrm{RW}} \wedge 1$ and let $\delta_2 = \delta_2(\eta, \varepsilon, d)$ be given by Proposition 3.6. Our goal is to prove that there exists $\eta \in (0, 1)$ such that the following is true for every $\beta < \beta(\delta_2)$: If $\mathcal{H}_{\beta'}(\mathbf{c}, \mathbf{C})$ holds for every $\beta' \le \beta$ then, for every $2\xi(0) \le |x| \le \eta\xi(\beta)$,

$$F_\beta(x) \le \frac{1}{6} \frac{\mathbf{C}}{\sigma^d} \left(\frac{\sigma}{\sigma \vee |x|}\right)^{d-2-\varepsilon}. \tag{5.41}$$

Since $|x| \ge 2\xi(0) = 2\sigma$, the above is equivalent to

$$F_\beta(x) \le \frac{1}{6} \mathbf{C} \left(\frac{|x|}{\sigma}\right)^\varepsilon \frac{1}{\sigma^2 |x|^{d-2}}. \tag{5.42}$$

Let $\eta \in (0, 1)$ to be chosen small enough. Let $\beta < \beta(\delta_2)$. Fix x with $2\xi(0) \le |x| \le \eta\xi(\beta)$. We may assume that $2\xi(0) \le \eta\xi(\beta)$, since otherwise there is nothing to prove. Then, since $2/\eta \ge 1$,

$$\xi(0) \le \frac{|x|}{\eta} \le \xi(\beta). \tag{5.43}$$

Since $\xi(\beta)$ is monotone increasing in $\beta < \beta(\delta_2)$ (by Proposition 4.1), we can choose $0 < \beta' \le \beta$ such that $|x| = \eta\xi(\beta')$. With this choice, we apply (**I.1**) to obtain

$$F_\beta(x) \le F_{\beta'}(x) + (\beta - \beta')(F_{\beta'} * F_\beta)(x). \tag{5.44}$$

For the first term on the right-hand side, we apply Proposition 3.6 at the parameter β' to get

$$F_{\beta'}(x) \le \frac{1}{8}\mathbf{C}\left(\frac{|x|}{\sigma}\right)^\varepsilon \frac{1}{\sigma^2|x|^{d-2}}. \tag{5.45}$$

The second term on the right-hand side of (5.44) is

$$\frac{Z_{\beta',\beta}}{\chi(\beta')}(F_{\beta'} * F_\beta)(x) \le \frac{2}{\chi(\beta')}(F_{\beta'} * F_\beta)(x) \tag{5.46}$$

because, by (4.24), $Z_{\beta',\beta} \le \frac{1}{1-\delta_2} \le 2$ since $\delta_2 \le \frac{1}{2}$. As a result, to prove (5.42) it suffices to show that, if η is small enough, then

$$\frac{1}{\chi(\beta')}(F_{\beta'} * F_\beta)(x) \le \frac{1}{48}\mathbf{C}\left(\frac{|x|}{\sigma}\right)^\varepsilon \frac{1}{\sigma^2|x|^{d-2}}. \tag{5.47}$$

Given that $\mathcal{H}_{\beta'}(\mathbf{c},\mathbf{C})$ and $\mathcal{H}_\beta(\mathbf{c},\mathbf{C})$ hold, the functions $f_1 = F_{\beta'}$ and $f_2 = F_\beta$ obey the hypothesis of Lemma 5.4 with $p = d-2-\varepsilon$ and $a = \mathbf{C}\sigma^{-2-\varepsilon}$. Since $f_1 \le f_2$, this gives, for every $k \ge 1$,

$$\frac{1}{\chi(\beta')}(F_{\beta'} * F_\beta)(x) \le \mathbf{C}\left(\frac{|x|}{\sigma}\right)^\varepsilon \frac{1}{\sigma^2|x|^{d-2}}\left(\frac{1}{k^{d-2-\varepsilon}} + 2^{d-2-\varepsilon}\frac{\chi_{k|x|}(\beta)}{\chi(\beta')}\right). \tag{5.48}$$

We choose $k = \eta^{-1/2} \ge 1$, so that $k|x| = k\eta\xi(\beta') = \eta^{1/2}\xi(\beta')$. Since

$$\xi(0) \le \eta^{-1/2}|x| = \eta^{1/2}\xi(\beta') \le \xi(\beta'), \tag{5.49}$$

there is a $\beta'' \le \beta'$ for which $\eta^{1/2}\xi(\beta') = \xi(\beta'')$. Then, the ratio in the last term on the right-hand side of (5.48) satisfies

$$\frac{\chi_{k|x|}(\beta)}{\chi(\beta')} = \frac{\chi_{\xi(\beta'')}(\beta)}{\chi(\beta'')}\frac{\chi(\beta'')}{\chi(\beta')} \le C_{\rm stab}\left(\frac{\xi(\beta'')}{\xi(\beta')}\right)^{2\frac{1-\delta_2}{1+\delta_2}} \le C_{\rm stab}\eta^{1/3}, \tag{5.50}$$

where we successively used Proposition 4.5, (4.3), and the fact that the exponent in the third member is at least $\frac{2}{3}$ because $\delta_2 \le \frac{1}{2}$. We now choose $\eta \in (0,1)$ small enough that

$$2^{d-2-\varepsilon}(\eta^{\frac{1}{2}(d-2-\varepsilon)} + \eta^{1/3}) \le \frac{1}{48}. \tag{5.51}$$

With (5.48), this completes the proof of (5.47), and concludes the proof of the proposition. □

Remark 5.5. In the proof of Proposition 3.7, we showed a weak form of pointwise stability: the second term in the right-hand side of (5.44) is much smaller than *any* input bound on $F_\beta(x)$. More precisely, we proved that, if $\kappa > 0$, for $2\xi(0) \le |x| \le \eta\xi(\beta')$, if η (and hence δ_2) is sufficiently small, we can ensure that

$$F_\beta(x) \le F_{\beta'}(x) + \frac{\kappa}{\sigma^2|x|^{d-2}}\left(\frac{|x|}{\sigma}\right)^\varepsilon. \tag{5.52}$$

This is the essential element in the contraction proof for intermediate scales.

6 Applications

In this section, we show that our results apply to several statistical mechanical models above their upper critical dimensions. The applications are to: self-avoiding walk for $d > 4$, percolation for $d > 6$, spin models (Ising, XY, 1- and 2-component $|\varphi|^4$) for $d > 4$, and lattice trees for $d > 8$. For each of these we: (1) define the model, (2) verify that its two-point function satisfies Definition 1.3, (3) verify Assumption I, and (4) verify Assumption II. Throughout Section 6, the kernel J is any admissible kernel as in Definition 1.1(*i*)–(*ii*). Definition 1.1(*iii*) does not play a role in Section 6, except for lattice trees in Section 6.7 (it is also used indirectly in Section 6.8 via application of Theorem 1.5).

Most of the analysis in Section 6 is well-known from literature going back to the 1980s. We include it for clarity and completeness, and to illustrate the parallels between the various models. Although we work on $\mathbb{Z}^d$ because that is the setting of our results, the verification of the bounds of Assumptions I and II presented here apply to general transitive graphs. If the model has a small parameter as part of its definition, then the nearest-neighbour J of (1.11) can be used. If there is no small parameter in the definition of the model, then the verification of Assumption II requires us to use spread-out models with sufficiently large σ_J.

The verification of Assumption II is deferred to Section 6.8 for the sake of efficiency, since it is similar for all models under consideration.

6.1 Self-avoiding walk

6.1.1 The model

Let $d \geq 1$. Detailed introductions to the self-avoiding walk on $\mathbb{Z}^d$ can be found in [12, 68]. To define the model, we introduce the *repulsion parameter* $\lambda \in [0,1]$. Since λ is fixed, we omit it from the notation. For $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if $J_{y-x} > 0$. An *n-step walk* from x to y is a sequence $(\gamma(i))_{0\leq i\leq n}$ with $\gamma(0) = x$, $\gamma(n) = y$, and $\gamma(i) \sim \gamma(i+1)$ for all $0 \leq i \leq n-1$. Let $\mathcal{W}_0$ be the set of all walks in $\mathbb{Z}^d$ with $x_0 = 0$. We denote the length of an n-step walk γ by $|\gamma| = n$.

For $\gamma \in \mathcal{W}_0$, $0 \leq s < t \leq |\gamma|$, and $\beta \geq 0$, we define

$$U_{s,t}(\gamma) := -\lambda \mathbb{1}_{\gamma(s)=\gamma(t)}, \tag{6.1}$$

$$\rho(\gamma) := \prod_{0\leq s<t\leq|\gamma|} (1 + U_{s,t}(\gamma)), \tag{6.2}$$

$$(\beta J)^\gamma := \prod_{0\leq i\leq|\gamma|-1} \beta J_{\gamma(i+1)-\gamma(i)}. \tag{6.3}$$

The *two-point function* is defined, for every $\beta \geq 0$ and every $x \in \mathbb{Z}^d$ by

$$G_\beta(x) := \sum_{\gamma:0\to x} (\beta J)^\gamma \rho(\gamma). \tag{6.4}$$

When $\lambda = 0$, we recover the Green function of the random walk with step distribution J. The case $\lambda = 1$ is the *strictly* self-avoiding walk model. When $\lambda \in (0,1)$, we have the *weakly* self-avoiding walk (also known as the *Domb–Joyce model*).

We define $c_n := \sum_{\gamma\in\mathcal{W}_0:|\gamma|=n} J^\gamma \rho(\gamma)$. This sequence satisfies the inequality $c_{m+n} \leq c_m c_n$, from which we conclude by Fekete's Lemma (see [68, Lemma 1.2.2]) that $c_n^{1/n}$ approaches a limit

$$\lim_{n\to\infty} c_n^{1/n} = \inf_{n\geq1} c_n^{1/n}. \tag{6.5}$$

This limit can be shown to lie in $(0,\infty)$. Its reciprocal is defined to be the *critical point* β_c. The *susceptibility* is defined by

$$\chi(\beta) = \sum_{x\in\mathbb{Z}^d} G_\beta(x) = \sum_{n=0}^{\infty} c_n \beta^n. \tag{6.6}$$

By (6.5), β_c is the radius of convergence of χ. In particular, $G_\beta(x)$ is finite for $\beta \in [0,\beta_c)$. Also, since $c_n \geq \beta_c^{-n}$, we have $\chi(\beta) \geq (1-\beta/\beta_c)^{-1}$ for $\beta \leq \beta_c$.

6.1.2 Verification of Definition 1.3

We verify the conditions imposed by Definition 1.3:

(i) (Initial condition.) It is immediate from the definition that $G_0 = \delta_0$.

(ii) (Regularity.) For every $x \in \mathbb{Z}^d$, the function $\beta \in [0,\beta_c) \mapsto G_\beta(x)$ is a power series in β with positive coefficients, so it is monotone and differentiable.

(iii) (Symmetry.) Since J is $\mathbb{Z}^d$-symmetric, so is G_β.

(iv) (Exponential decay.) Fix $\beta \in [0,\beta_c)$ and let $\beta_1 = \frac{1}{2}(\beta+\beta_c)$. A walk from 0 to x must take at least $|x|/R_j$ steps, so

$$G_\beta(x) \leq \sum_{n=|x|/R_j} c_n\beta^n \leq \Big(\frac{\beta}{\beta_1}\Big)^{|x|/R_J} \chi(\beta_1). \tag{6.7}$$

Since $\beta < \beta_1$, the right-hand side decays exponentially in $|x|$.

(v) (Limit as $\beta \nearrow \beta_c$ when $\beta_c < \infty$.) By the Monotone Convergence Theorem, $\lim_{\beta\uparrow\beta_c} G_\beta(x) = G_{\beta_c}(x)$.

6.1.3 Verification of Assumption I

Proof of (**I.1**). For $a, b \in \mathbb{R}$, we have $b^n - a^n = (b-a)\sum_{k=0}^{n-1} b^k a^{n-1-k}$. As a result,

$$\begin{aligned} G_\beta(x) - G_{\beta'}(x) &= \sum_{\gamma:0\to x} \big((\beta J)^\gamma - (\beta' J)^\gamma\big)\rho(\gamma) \\ &= \sum_{\gamma:0\to x}\left((\beta-\beta')\sum_{k=0}^{|\gamma|-1}(\beta' J)^{\gamma[0:k]} J_{\gamma(k+1)-\gamma(k)}(\beta J)^{\gamma[k+1:|\gamma|]}\right)\rho(\gamma) \\ &= (\beta-\beta')\sum_{u,v\in\mathbb{Z}^d}\sum_{\substack{\gamma_1:0\to u\\ \gamma_2:v\to x}}(\beta' J)^{\gamma_1} J_{v-u}(\beta J)^{\gamma_2}\rho(\gamma_1\circ(uv)\circ\gamma_2). \end{aligned} \tag{6.8}$$

By definition, $\rho(\gamma_1\circ(uv)\circ\gamma_2) \leq \rho(\gamma_1)\rho(\gamma_2)$, so

$$\sum_{\gamma_2:v\to x}(\beta J)^{\gamma_2}\rho(\gamma_1\circ(uv)\circ\gamma_2) \leq \rho(\gamma_1)G_\beta(x-v), \tag{6.9}$$

and therefore

$$\begin{aligned} G_\beta(x) - G_{\beta'}(x) &\leq (\beta-\beta')\sum_{\substack{u,v\in\mathbb{Z}^d\\ \gamma:0\to u}}(\beta' J)^\gamma J_{v-u}\rho(\gamma)G_\beta(x-v) \\ &= (\beta-\beta')(G_{\beta'} * J * G_\beta)(x). \end{aligned} \tag{6.10}$$

This proves (**I.1**). Figure 2 illustrates the upper bound in (6.10). □

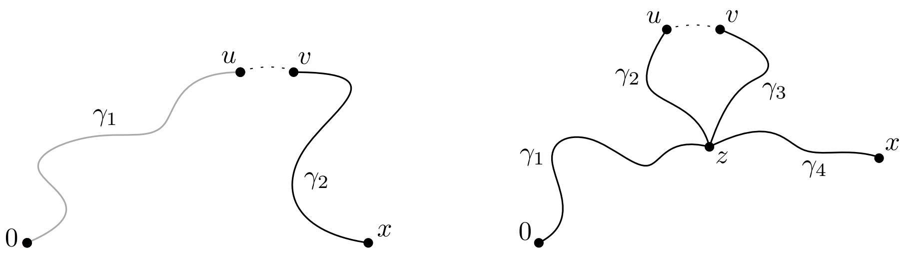

Figure 2: Diagrammatic representations for (6.10) (left) and (6.17) (right). On the left, the grey path has parameter β' and the black has β. The dotted lines represent a single-step walk.

We define the *open bubble diagram*

$$B^{\mathrm{o}}(\beta) := (G_\beta * J * G_\beta)(0). \tag{6.11}$$

and verify the lower differential inequality (**I.2**) with

$$H_\beta(x) = \lambda B^{\mathrm{o}}(\beta)\delta_0(x). \tag{6.12}$$

By definition, $H_0 = 0$, $H_\beta(x) = H_\beta(-x)$, and H_β is increasing and continuous on $[0, \beta_c)$, as required by Assumption I. Also, since H_β is supported on $\{0\}$, it certainly decays exponentially.

Since we have verified (**I.1**), it follows from (1.13) that

$$\partial_\beta G_\beta \le G_\beta * J * G_\beta. \tag{6.13}$$

We will use this in the proof of the complementary lower bound

$$\partial_\beta G_\beta \ge G_\beta * J * G_\beta - G_\beta * H_\beta * G_\beta, \tag{6.14}$$

which is (**I.2**).

Proof of (**I.2**). By writing $|\gamma| = \sum_{z\in\mathbb{Z}^d}\sum_{i=1}^{|\gamma|}\mathbb{1}_{\gamma(i)=z}$, we see that

$$\begin{aligned} \partial_\beta G_\beta(x) &= \sum_{v\in\mathbb{Z}^d}\sum_{\substack{\gamma:0\to x\\ |\gamma|\ge 1}} \beta^{|\gamma|-1}J^\gamma\rho(\gamma)\sum_{i=1}^{|\gamma|}\mathbb{1}_{\gamma(i)=v} \\ &= \sum_{u,v\in\mathbb{Z}^d}\sum_{\substack{\gamma_1:0\to u\\ \gamma_2:v\to x}} (\beta J)^{\gamma_1}J_{v-u}(\beta J)^{\gamma_2}\rho(\gamma_1\circ(uv)\circ\gamma_2). \end{aligned} \tag{6.15}$$

We apply the inequality

$$\rho(\gamma_1\circ(uv)\circ\gamma_2) \ge \rho(\gamma_1)\rho(\gamma_2) - \lambda\rho(\gamma_1)\rho(\gamma_2)\sum_{\substack{0\le i\le|\gamma_1|\\ 0\le j\le|\gamma_2|}}\mathbb{1}_{\gamma_1(i)=\gamma_2(j)}. \tag{6.16}$$

Insertion of the first term on the right-hand side into (6.15) gives the first term $(G_\beta * J * G_\beta)(x)$ in our desired bound (6.14).

To arrive at the remaining term in (6.14), we decompose according to the common value z of $\gamma_1(i)$ and $\gamma_2(j)$. The subtracted term in (6.16) becomes

$$\begin{aligned}
&\lambda \sum_{u,v\in\mathbb{Z}^d} \sum_{\substack{\gamma_1:0\to u\\ \gamma_2:v\to x}} (\beta J)^{\gamma_1} J_{v-u} (\beta J)^{\gamma_2} \rho(\gamma_1)\rho(\gamma_2) \sum_{\substack{0\le i\le|\gamma_1|\\ 0\le j\le|\gamma_2|}} \mathbb{1}_{\gamma_1(i)=\gamma_2(j)} \\
&\le \lambda \sum_{u,v,z\in\mathbb{Z}^d} \sum_{\substack{\gamma_1:0\to z\\ \gamma_2:z\to u\\ \gamma_3:v\to z\\ \gamma_4:z\to x}} (\beta J)^{\gamma_1\circ\gamma_2} J_{v-u} (\beta J)^{\gamma_3\circ\gamma_4} \rho(\gamma_1\circ\gamma_2)\rho(\gamma_3\circ\gamma_4) \\
&\le \lambda \sum_{u,v,z\in\mathbb{Z}^d} G_\beta(z) G_\beta(u-z) J_{v-u} G_\beta(z-v) G_\beta(x-z) \\
&= \lambda B^{\circ}(\beta) \sum_{z\in\mathbb{Z}^d} G_\beta(z) G_\beta(x-z) = \lambda B^{\circ}(\beta)(G_\beta * G_\beta)(x). \qquad (6.17)
\end{aligned}$$

The right-hand side is exactly $(G_\beta * H_\beta * G_\beta)(x)$ with H_β given by (6.12). □

6.2 Continuous-time weakly self-avoiding walk

We now verify Assumption I for the continuous-time weakly self-avoiding walk. Since the argument closely follows that of the (discrete-time) weakly self-avoiding walk, we only sketch the details.

Let $d \ge 1$. The study of the continuous-time weakly self-avoiding walk on $\mathbb{Z}^d$ (also known as the *discrete Edwards model*) goes back to [14]. The model is interesting even for $d = 1$, e.g., [61].

To define the model, we first consider the continuous-time random walk X on $\mathbb{Z}^d$ which takes steps according to an admissible kernel J, with Exp(1) holding times. In other words, the random time spent at a vertex before making a next step has density $e^{-t}\mathrm{d}t$, and the holding times at each step are independent of each other and of the choice of next vertex. For a walk with trace γ taking $|\gamma|$ steps, let $T_0, \dots, T_{|\gamma|}$ denote the independent Exp(1) holding times, and let $T = \sum_{i=0}^{|\gamma|} T_i$ denote the total time of γ.

The *local time* at a vertex y visited by γ is the random variable

$$L_{y,\gamma} = \sum_{i=0}^{|\gamma|} T_i \mathbb{1}_{\gamma(i)=y} = \int_0^T \mathbb{1}_{X(t)=y}\mathrm{d}t. \qquad (6.18)$$

The *intersection local time*

$$I(\gamma) := \sum_{y\in\mathbb{Z}^d} L_{y,\gamma}^2 = \int_0^T \int_0^T \mathbb{1}_{X(s)=X(t)}\mathrm{d}s\mathrm{d}t \qquad (6.19)$$

gives an indication of the total time that the walk γ spends intersecting itself. Given $\lambda > 0$, we define a weight $\rho(\gamma)$ by

$$\rho(\gamma) := \mathbb{E}\,(e^{-\lambda I(\gamma)}), \qquad (6.20)$$

where $\mathbb{E}$ denotes the expectation over the holding times.

When y is visited n_y times by the walk γ, the local time $L_{y,\gamma}$ has a Gamma$(n_y, 1)$ distribution. Therefore, in terms of the measure on $(\mathbb{R}^+)^{\mathbb{Z}^d}$ defined by

$$\mathrm{d}\nu_\gamma(\mathbf{t}) = \prod_{v\in\mathbb{Z}^d} \left(1_{n_v(\gamma)=0}\delta_0(t_v)\mathrm{d}t_v + 1_{n_v(\gamma)\ge 1} \frac{t_v^{n_v(\gamma)-1}}{(n_v(\gamma)-1)!}\mathrm{d}t_v \right), \qquad (6.21)$$

the weight of γ is equal to

$$\rho(\gamma) = \mathbb{E}\left(e^{-\lambda I(\gamma)}\right) = \int e^{-\sum_y t_y} e^{-\lambda \sum_y t_y^2} \mathrm{d}\nu_\gamma(\mathbf{t}). \tag{6.22}$$

The *two-point function* is defined by

$$G_\beta(x) := \sum_{\gamma:0\to x} (\beta J)^\gamma \rho(\gamma). \tag{6.23}$$

Verification of Definition 1.3. This is identical to the verification for the self-avoiding walk model in Section 6.1.2. Indeed, in Section 6.1.2, we did not use the precise form of $\rho(\gamma)$, but only the fact that it does not depend on β. This remains true in (6.23).

Proof of (**I.1**). Suppose that $\gamma = \gamma_1 \circ (uv) \circ \gamma_2$. The intersection local time obeys the inequality

$$I(\gamma) \geq I(\gamma_1) + I(\gamma_2). \tag{6.24}$$

Since the holding times of γ_1 and γ_2 are independent, so are the random variables $I(\gamma_1)$ and $I(\gamma_2)$. This leads to the inequality $\rho(\gamma_1 \circ (uv) \circ \gamma_2) \leq \rho(\gamma_1)\rho(\gamma_2)$ that was used to verify Assumption I in Section 6.1.3. Then (**I.1**) follows exactly as in Section 6.1.3. □

Proof of (**I.2**). We start from (6.15), with the new interpretation of the weight ρ. Let $\gamma = \gamma_1 \circ (uv) \circ \gamma_2$. The error in the upper bound $\rho(\gamma) \leq \rho(\gamma_1)\rho(\gamma_2)$ used to prove (**I.1**) arises from

$$\begin{aligned} e^{-\lambda I(\gamma_1)} e^{-\lambda I(\gamma_2)} - e^{-\lambda I(\gamma)} &= e^{-\lambda I(\gamma_1)} e^{-\lambda I(\gamma_2)} \Big[1 - e^{-\lambda[I(\gamma) - (I(\gamma_1)+I(\gamma_2))]}\Big] \\ &\leq e^{-\lambda I(\gamma_1)} e^{-\lambda I(\gamma_2)} \lambda[I(\gamma) - (I(\gamma_1) + I(\gamma_2))]. \end{aligned} \tag{6.25}$$

By definition, $L_{z,\gamma} = L_{z,\gamma_1} + L_{z,\gamma_2}$. Therefore,

$$I(\gamma) - (I(\gamma_1) + I(\gamma_2)) = \sum_{z\in\mathbb{Z}^d} (L_{z,\gamma_1} + L_{z,\gamma_2})^2 - \Big(L_{z,\gamma_1}^2 + L_{z,\gamma_2}^2\Big) = 2\sum_{z\in\mathbb{Z}^d} L_{z,\gamma_1} L_{z,\gamma_2}. \tag{6.26}$$

As a consequence,

$$\rho(\gamma_1)\rho(\gamma_2) - \rho(\gamma) \leq 2\lambda \sum_{z\in\mathbb{Z}^d} \Big(\mathbb{E}\big(e^{-\lambda I(\gamma_1)} L_{z,\gamma_1}\big)\Big)\Big(\mathbb{E}\big(e^{-\lambda I(\gamma_2)} L_{z,\gamma_2}\big)\Big). \tag{6.27}$$

Now, we sum over γ as in (6.15). The result is

$$\begin{aligned} &(G_\beta * J * G_\beta)(x) - \partial_\beta G_\beta(x) \\ &\quad\leq 2\lambda \sum_{u,v,z\in\mathbb{Z}^d} \sum_{\gamma_1:0\to u} (\beta J)^{\gamma_1} \Big(\mathbb{E}\big(e^{-\lambda I(\gamma_1)} L_{z,\gamma_1}\big)\Big) J_{v-u} \sum_{\gamma_2:v\to x} (\beta J)^{\gamma_2} \Big(\mathbb{E}\big(e^{-\lambda I(\gamma_2)} L_{z,\gamma_2}\big)\Big). \end{aligned} \tag{6.28}$$

We appeal to [17, Lemma 2.1] to see that

$$\sum_{\gamma_1:0\to u} (\beta J)^{\gamma_1} \Big(\mathbb{E}\big(e^{-\lambda I(\gamma_1)} L_{z,\gamma_1}\big)\Big) \leq G_\beta(z) G_\beta(u-z), \tag{6.29}$$

and similarly for the sum over γ_2. We therefore obtain

$$\begin{aligned} &(G_\beta * J * G_\beta)(x) - \partial_\beta G_\beta(x) \\ &\quad\leq 2\lambda \sum_{z\in\mathbb{Z}^d} G_\beta(z) G_\beta(x-z) \sum_{u,v\in\mathbb{Z}^d} G_\beta(u-z) J_{v-u} G_\beta(z-v). \end{aligned} \tag{6.30}$$

By replacing u, v by $u - x, v - z$, this leads to

$$(G_\beta * J * G_\beta)(x) - \partial_\beta G_\beta(x) \le 2\lambda (G_\beta * H_\beta * G_\beta)(x) \tag{6.31}$$

with

$$H_\beta(x) = 2\lambda \delta_0(x)(G_\beta * J * G_\beta)(0). \tag{6.32}$$

This completes the proof. □

6.3 Bernoulli percolation

6.3.1 The model

For an introduction to percolation theory, see [40]. Let $d \ge 2$. We consider Bernoulli bond percolation on the infinite graph whose vertex set is $\mathbb{Z}^d$ and whose edge set $\mathcal{E} = \mathcal{E}_J$ consists of pairs $\{x, y\}$ with $J_{y-x} > 0$. Let $\beta \in [0, (\max_{x\in\mathbb{Z}^d} J_x)^{-1}]$. Edges are independently open with probability βJ_{y-x} and otherwise are closed. We write $\{x \leftrightarrow y\}$ for the event that x and y are connected by a path consisting of open bonds, and define the *two-point function*

$$G_\beta(x, y) = G_\beta(y - x) := \mathbb{P}_\beta[x \leftrightarrow y]. \tag{6.33}$$

6.3.2 Verification of Definition 1.3

For $d \ge 2$, the properties listed in Definition 1.3 are standard facts about percolation [40]. The critical value β_c separates the subcritical regime, where G_β decays exponentially, from the supercritical regime, where there is a positive probability for the existence of an infinite connected cluster [2, 70]. We assume familiarity with two basic techniques: the BK inequality (van den Berg–Kesten) and Russo's formula [40, Chapter 2].

6.3.3 Verification of Assumption I

Proof of (**I.1**). A version of (**I.1**) appeared in [49, Lemma 2.4]. We use the standard increasing coupling $\mathbb{P}$, as follows. First, we assign independent uniform random variables $\eta_{u,v}$ in $[0, 1]$ to each edge $\{u, v\}$. Given a realisation of these random variables, and given $\beta \in [0, (\max_{x\in\mathbb{Z}^d} J_x)^{-1}]$, we define a percolation configuration $\omega_\beta \in \{0, 1\}^{\mathcal{E}}$ as follows: $\omega_\beta(\{u, v\}) = 1$ if $\eta_{u,v} < \beta J_{v-u}$ (the bond $\{u, v\}$ is *open*), and otherwise $\omega_\beta(\{u, v\}) = 0$. Below, we also view ω_β as a subgraph of $(\mathbb{Z}^d, \mathcal{E})$ of vertex set $\mathbb{Z}^d$ and edge set $\{\{u, v\} \in \mathcal{E} : \omega_\beta(\{u, v\}) = 1\}$. We also define $\{x \overset{A}{\longleftrightarrow} y\}$ to be the event that x is connected to y by a path which does not pass through any vertex in A^c. By definition of the coupling, if $\beta < \beta'$ then $\omega_{\beta'}$ is a subgraph of ω_β, and, for every $x \in \mathbb{Z}^d$,

$$G_\beta(x) - G_{\beta'}(x) = \mathbb{P}[\{0 \overset{\omega_\beta}{\longleftrightarrow} x\} \setminus \{0 \overset{\omega_{\beta'}}{\longleftrightarrow} x\}]. \tag{6.34}$$

For $\beta \in [0, (\max_{x\in\mathbb{Z}^d} J_x)^{-1}]$ and $z \in \mathbb{Z}^d$, we define $\mathcal{C}_\beta(z)$ to be the cluster of z in ω_β. For $\{u, v\} \in \mathcal{E}$, we define $\omega_\beta^{\{u,v\}}$ to be the percolation configuration obtained from ω_β by setting $\omega_\beta(\{u, v\}) = 0$, and we define $\mathcal{C}_\beta^{\{u,v\}}(z)$ to be the cluster of z in $\omega_\beta^{\{u,v\}}$. For fixed $\beta' < \beta$ and $x \in \mathbb{Z}^d$, we claim that the event on the right-hand side of (6.34) satisfies

$$\begin{aligned} &\{0 \overset{\omega_\beta}{\longleftrightarrow} x\} \setminus \{0 \overset{\omega_{\beta'}}{\longleftrightarrow} x\} \\ &\quad \subset \bigcup_{\{u,v\}\in\mathcal{E}} \Big\{u \in \mathcal{C}_{\beta'}(0)\Big\} \cap \{\omega_{\beta'}(\{u, v\}) = 0,\ \omega_\beta(\{u, v\}) = 1\} \cap \Big\{v \xleftrightarrow{\omega_\beta \setminus \mathcal{C}_{\beta'}(0)} x\Big\}. \end{aligned} \tag{6.35}$$

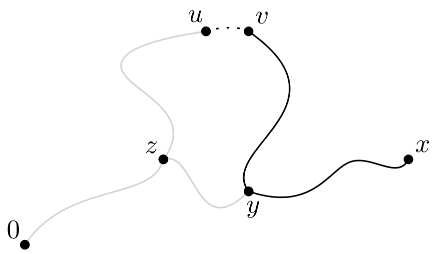

Figure 3: Depiction of $(G_\beta * H_\beta * G_\beta)(x)$ for percolation. Lines represent two-point functions and the gap between u, v represents a factor J_{v-u}. The vertices u, v, y, z are summed over $\mathbb{Z}^d$.

The claim is justified as follows. For every configuration in $\{0 \xleftrightarrow{\omega_\beta} x\} \setminus \{0 \xleftrightarrow{\omega_{\beta'}} x\}$, there must exist $u \in \mathcal{C}_{\beta'}(0)$ and $v \notin \mathcal{C}_{\beta'}(0)$ such that $\omega_\beta(\{u,v\}) = 1$ and v is connected to x in ω_β without using the vertices in $\mathcal{C}_{\beta'}(0)$. This can be seen by exploring an open self-avoiding path from 0 to x in ω_β and marking its edge $\{u,v\}$ for which u is the last vertex of $\mathcal{C}_{\beta'}(0)$ visited by this path.

Since $\omega_{\beta'}(\{u,v\}) = 0$, we can replace $\mathcal{C}_{\beta'}(0)$ by $\mathcal{C}_{\beta'}^{\{u,v\}}(0)$ in (6.35). A union bound then gives

$$
\begin{aligned}
&G_\beta(x) - G_{\beta'}(x) \\
&\leq \sum_{\{u,v\} \in \mathcal{E}} \mathbb{P}\Big[\Big\{u \in \mathcal{C}_{\beta'}^{\{u,v\}}(0)\Big\} \cap \{\omega_{\beta'}(\{u,v\}) = 0,\, \omega_\beta(\{u,v\}) = 1\} \cap \Big\{v \xleftrightarrow{\omega_\beta \setminus \mathcal{C}_{\beta'}^{\{u,v\}}(0)} x\Big\}\Big].
\end{aligned} \tag{6.36}
$$

Given $\{u,v\} \in \mathcal{E}$, we condition on the cluster $\mathcal{C}_{\beta'}^{\{u,v\}}(0)$. This gives

$$
\begin{aligned}
&\mathbb{P}\Big[\Big\{u \in \mathcal{C}_{\beta'}^{\{u,v\}}(0)\Big\} \cap \{\omega_{\beta'}(\{u,v\}) = 0,\, \omega_\beta(\{u,v\}) = 1\} \cap \Big\{v \xleftrightarrow{\omega_\beta \setminus \mathcal{C}_{\beta'}^{\{u,v\}}(0)} x\Big\}\Big] \\
&\qquad = \sum_{C \ni 0,u} \mathbb{P}[\mathcal{C}_{\beta'}^{\{u,v\}}(0) = C]\mathbb{P}[\omega_{\beta'}(\{u,v\}) = 0,\, \omega_\beta(\{u,v\}) = 1]\mathbb{P}[v \xleftrightarrow{\omega_\beta \setminus C} x], \quad (6.37)
\end{aligned}
$$

because the three events on the right hand side of (6.37) depend on disjoint sets of edges and are therefore independent. By definition of the coupling, $\mathbb{P}[\omega_{\beta'}(\{u,v\}) = 0,\, \omega_\beta(\{u,v\}) = 1] = \mathbb{P}[\eta_{u,v} \in [\beta' J_{v-u}, \beta J_{v-u})] = \beta - \beta'$, and by inclusion of events,

$$
\mathbb{P}[v \xleftrightarrow{\omega_\beta \setminus C} x] \leq \mathbb{P}[v \xleftrightarrow{\omega_\beta} x] = G_\beta(x - v), \tag{6.38}
$$

$$
\sum_{C \ni 0,u} \mathbb{P}[\mathcal{C}_{\beta'}^{\{u,v\}}(0) = C] = \mathbb{P}[u \in \mathcal{C}_{\beta'}^{\{u,v\}}(0)] \leq G_{\beta'}(u). \tag{6.39}
$$

The last three observations, combined with (6.36)–(6.37), complete the proof. □

We will prove the lower differential inequality (**I.2**) with

$$
H_\beta := (G_\beta * J * G_\beta) \cdot G_\beta. \tag{6.40}
$$

By definition, $H_0(x) = J_x \delta_0(x) = 0$, and $H_\beta(-x) = H_\beta(x)$ for every $x \in \mathbb{Z}^d$ and all $\beta \in [0, \beta_c)$. Since G_β decays exponentially for $\beta \in (0, \beta_c)$, the same is true for H_β.

Proof of (**I.2**). The lower differential inequality (**I.2**), with H_β given by (6.40), is a minor modification of a differential inequality for the susceptibility that was first proved in [6] before the advent of the BK inequality. For a more modern proof, see [82, Proposition 9.11]. Although we have no new insights for the proof, we provide the argument for the sake of completeness. Let $0 \le \beta < \beta_c$. By Russo's formula,

$$\begin{aligned}\partial_\beta G_\beta(x) &= \sum_{u,v\in\mathbb{Z}^d} J_{v-u}\mathbb{P}_\beta[\{u,v\} \text{ closed and pivotal for } 0 \leftrightarrow x] \\ &= \sum_{u,v\in\mathbb{Z}^d} J_{v-u}\mathbb{P}_\beta[\{0\leftrightarrow u\}\cap\{v\leftrightarrow x\}\cap\{u\leftrightarrow v\}^c] \end{aligned} \tag{6.41}$$

(to focus on the key ideas, we do not address the subtlety that Russo's formula initially applies only to events depending on finitely many bonds). It suffices to prove that

$$\begin{aligned}\mathbb{P}_\beta[\{0\leftrightarrow u\}\cap\{v\leftrightarrow x\}\cap\{u\leftrightarrow v\}^c] &\ge G_\beta(u)G_\beta(x-v) \\ &- \sum_{y,z\in\mathbb{Z}^d} G_\beta(z)G_\beta(u-z)G_\beta(y-z)G_\beta(y-v)G_\beta(x-y), \end{aligned} \tag{6.42}$$

since this yields

$$\partial_\beta G_\beta(x) \ge (G_\beta * (J - H_\beta) * G_\beta)(x). \tag{6.43}$$

Connections arising in the proof of (6.42) are illustrated in Figure 3.

Given a set A of vertices, we define $\{x \leftrightarrow y \text{ using } A\}$ to be the event that x is connected to y and that every path that realises the connection must contain a vertex in A.

To prove (6.42), we condition on the cluster $\mathcal{C}(u)$ of u. This gives,

$$\mathbb{P}_\beta[\{0\leftrightarrow u\}\cap\{v\leftrightarrow x\}\cap\{u\leftrightarrow v\}^c] = \sum_{C\ni 0,u} \mathbb{P}_\beta[\mathcal{C}(u)=C]\mathbb{P}_\beta[v \xleftrightarrow{C^c} x], \tag{6.44}$$

where we used the facts that $\mathbb{P}_\beta[v \xleftrightarrow{C^c} x] = 0$ if $v \in C$, and that the events $\{\mathcal{C}(u) = C\}$ and $\{v \xleftrightarrow{C^c} x\}$ are independent. It follows by definition of the events that

$$\mathbb{P}_\beta[v \xleftrightarrow{C^c} x] = G_\beta(x-v) - \mathbb{P}[v \leftrightarrow x \text{ using } C]. \tag{6.45}$$

If $\{v \leftrightarrow x \text{ using } C\}$ occurs, there must exist $y \in C$ such that $\{v\leftrightarrow y\}\circ\{y\leftrightarrow x\}$ occurs. Therefore, by the BK inequality,

$$\mathbb{P}_\beta[v \xleftrightarrow{C^c} x] \ge G_\beta(x-v) - \sum_{y\in\mathbb{Z}^d} \mathbb{1}_{y\in C} G_\beta(y-v)G_\beta(x-y). \tag{6.46}$$

We insert (6.46) into (6.44) and obtain

$$\begin{aligned}\mathbb{P}_\beta[\{0\leftrightarrow u\}\cap\{v\leftrightarrow x\}\cap\{u\leftrightarrow v\}^c] &\ge G_\beta(u)G_\beta(x-v) - \\ &\sum_{y\in\mathbb{Z}^d} G_\beta(y-v)G_\beta(x-y) \sum_{C\ni 0,u,y} \mathbb{P}_\beta[\mathcal{C}(u)=C]. \end{aligned} \tag{6.47}$$

Finally, we observe that $\sum_{C\ni 0,u,y}\mathbb{P}_\beta[\mathcal{C}(u) = C] = \mathbb{P}_\beta[0,u,y \text{ lie in the same cluster}]$. If this event occurs, then there must be $z \in \mathbb{Z}^d$ such that $\{0\leftrightarrow z\}\circ\{z\leftrightarrow u\}\circ\{z\leftrightarrow y\}$. Combining this observation with the BK inequality gives

$$\mathbb{P}_\beta[0,u,y \text{ lie in the same cluster}] \le \sum_{z\in\mathbb{Z}^d} G_\beta(z)G_\beta(u-z)G_\beta(y-z). \tag{6.48}$$

The combination of (6.48) and (6.47) completes the proof. □

6.4 1- and 2-component $|\varphi|^4$ models

6.4.1 The model

Let $d \geq 2$. We define the 1- and 2-component $|\varphi|^4$ models on $\mathbb{Z}^d$, as follows. Let $n \in \{1, 2\}$ and let $x \cdot y$ denote the dot product of $x, y \in \mathbb{R}^n$. We denote the Euclidean norm on $\mathbb{R}^n$ by $|\cdot|_2$; context will distinguish this from the norm on $\mathbb{Z}^d$ which is denoted in the same way. Let $\Lambda \subset \mathbb{Z}^d$ be finite, $\beta \geq 0$, J an admissible interaction, and $F : (\mathbb{R}^n)^\Lambda \to \mathbb{R}$. The n-component φ^4 model on Λ is the measure $\langle \cdot \rangle_{\Lambda,\beta}$ on $(\mathbb{R}^n)^\Lambda$ given by:

$$\langle F(\varphi) \rangle_{\Lambda,\beta} := \frac{1}{Z_{\Lambda,\beta}} \int_{(\mathbb{R}^n)^\Lambda} F(\varphi) \exp(-\beta H_\Lambda(\varphi)) \mathrm{d}s_\Lambda(\varphi), \tag{6.49}$$

where

$$H_\Lambda(\varphi) := -\frac{1}{2} \sum_{x,y \in \Lambda} J_{y-x}(\varphi_x \cdot \varphi_y), \qquad Z_{\Lambda,\beta} := \int_{(\mathbb{R}^n)^\Lambda} \exp(-\beta H_\Lambda(\varphi)) \mathrm{d}s_\Lambda(\varphi), \tag{6.50}$$

and for $\lambda > 0$ and $\mu \in \mathbb{R}$,

$$\mathrm{d}s_\Lambda(\varphi) := \prod_{x \in \Lambda} g(|\varphi_x|_2^2) \mathrm{d}\varphi_x \qquad \text{with} \qquad g(t) := \exp\left(-\frac{1}{4}\lambda t^2 - \frac{1}{2}\mu t\right). \tag{6.51}$$

Additionally, we denote by $\mathrm{d}m_{\lambda,\mu}(\varphi_x)$ the probability distribution on $\mathbb{R}^n$ with density proportional to $g(|\varphi_x|_2^2)\mathrm{d}\varphi_x$. It is a classical consequence of Griffiths' [39] (for $n = 1$) or Ginibre's [37] (for $n = 2$) inequalities that the sequence of measures $\langle \cdot \rangle_{\Lambda,\beta}$ admits a weak limit as $\Lambda \nearrow \mathbb{Z}^d$. We denote the limiting measure by $\langle \cdot \rangle_\beta$.

The *two-point function* is defined for $\beta \geq 0$ and $x \in \mathbb{Z}^d$ by

$$G_\beta(x) := \langle \varphi_0^1 \varphi_x^1 \rangle_\beta, \tag{6.52}$$

where $\varphi^1 = \varphi$ for $n = 1$ and $\varphi = (\varphi^1, \varphi^2)$ for $n = 2$. By definition, $G_0 = A\delta_0$ with

$$A = A(\lambda, \mu) = \int_{\varphi_0 \in \mathbb{R}^n} (\varphi_0^1)^2 \mathrm{d}m_{\lambda,\mu}(\varphi_0). \tag{6.53}$$

The fact that A is not necessarily equal to 1 violates Definition 1.3(i) but this is not a problem, as explained in Remark 1.8. As usual, we set $F_\beta = J * G_\beta$.

6.4.2 The Brydges–Fröhlich–Spencer random walk representation

To prepare for the verification of the finite-difference upper bound (**I**.**1**) of Assumption I, we recall the Brydges–Fröhlich–Spencer (BFS) random walk expansion [17] (see also [56] for a recent alternative perspective). To state the expansion, we introduce the following definitions.

First, given $\mathbf{t} = (t_x)_{x \in \Lambda} \in \mathbb{R}_+^\Lambda$, we define a measure on $(\mathbb{R}^n)^\Lambda$ and its associated partition function by

$$\mathrm{d}s_{\Lambda,\mathbf{t}}(\varphi) := \prod_{x \in \Lambda} g(|\varphi_x|_2^2 + 2t_x) \mathrm{d}\varphi_x, \tag{6.54}$$

$$Z_{\Lambda,\beta}(\mathbf{t}) := \int_{(\mathbb{R}^n)^\Lambda} \exp(-\beta H_\Lambda(\varphi)) \mathrm{d}s_{\Lambda,\mathbf{t}}(\varphi). \tag{6.55}$$

We also define a normalised version of $Z_{\Lambda,\beta,J}(\mathbf{t})$ by

$$z_{\Lambda,\beta}(\mathbf{t}) := \frac{Z_{\Lambda,\beta}(\mathbf{t})}{Z_{\Lambda,\beta}}. \tag{6.56}$$

As in Section 6.1.1, a *walk* $\gamma = (\gamma(0), \dots, \gamma(|\gamma|))$ is a sequence of points in $\mathbb{Z}^d$ satisfying $J_{\gamma(i+1)-\gamma(i)} > 0$. The *local time* of γ at $v \in \Lambda$ is defined by $\ell_v(\gamma) = \sum_{i=0}^{|\gamma|} \mathbb{1}_{\gamma(i)=v}$. As in (6.21), we define a measure on $(\mathbb{R}^+)^\Lambda$ by

$$\mathrm{d}\nu_{\Lambda,\gamma}(\mathbf{t}) := \prod_{v\in\Lambda} \left(1_{\ell_v(\gamma)=0}\delta_0(t_v)\mathrm{d}t_v + 1_{\ell_v(\gamma)\geq 1} \frac{t_v^{\ell_v(\gamma)-1}}{(\ell_v(\gamma)-1)!}\mathrm{d}t_v \right). \tag{6.57}$$

Proposition 6.1 (The BFS expansion [17])**.** *Let $\Lambda \subset \mathbb{Z}^d$ be finite and $\beta \geq 0$. Then*

$$\langle \varphi_x^1 \varphi_y^1 \rangle_{\Lambda,\beta} = \sum_{\gamma: x\to y} (\beta J)^\gamma \int z_{\Lambda,\beta}(\mathbf{t}) \mathrm{d}\nu_{\Lambda,\gamma}(\mathbf{t}), \tag{6.58}$$

where the sum runs overs paths $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|})$, and, as usual, $(\beta J)^\gamma = \prod_{i=1}^{|\gamma|-1} \beta J_{\gamma_{i+1}-\gamma_i}$.

The BFS expansion matches the formula (6.23) for the two-point function of the continuous-time weakly self-avoiding walk if we define the weight

$$\rho(\gamma) = \int z_{\Lambda,\beta}(\mathbf{t}) \mathrm{d}\nu_{\Lambda,\gamma}(\mathbf{t}). \tag{6.59}$$

However, the fact that this ρ depends on β makes the arguments of Section 6.1 inapplicable, and we must proceed more delicately.

Our verification of Assumption I combines the BFS expansion with the following two monotonicity properties. They use the measure $\langle\cdot\rangle_{\Lambda,\beta,\mathbf{t}}$ on $(\mathbb{R}^n)^\Lambda$ defined by replacing $\mathrm{d}s_\Lambda$ by $\mathrm{d}s_{\Lambda,\mathbf{t}}$ in (6.49).

Lemma 6.2. *Let $\Lambda \subset \mathbb{Z}^d$ be finite and $0 \leq \beta' \leq \beta$. Then, for every $\mathbf{t} \in (\mathbb{R}^+)^\Lambda$,*

$$z_{\Lambda,\beta}(\mathbf{t}) \leq z_{\Lambda,\beta'}(\mathbf{t}). \tag{6.60}$$

Proof. We first observe that

$$\prod_{x\in\Lambda} g(|\varphi_x|_2^2) = f(\mathbf{t}) F_{\mathbf{t}}(\varphi) \prod_{x\in\Lambda} g(|\varphi_x|_2^2 + 2\mathbf{t}_x), \tag{6.61}$$

with

$$F_{\mathbf{t}}(\varphi) := \exp\Big(\sum_{x\in\Lambda} \lambda t_x |\varphi_x|_2^2 \Big), \qquad f(\mathbf{t}) = \exp\Big(\sum_{x\in\Lambda} \lambda t_x^2 + \mu t_x \Big). \tag{6.62}$$

It follows that

$$Z_{\Lambda,\beta} = f(\mathbf{t}) \int_{(\mathbb{R}^n)^\Lambda} F_{\mathbf{t}}(\varphi) \exp(-\beta H_\Lambda(\varphi)) \mathrm{d}s_{\Lambda,\mathbf{t}}(\varphi), \tag{6.63}$$

and therefore

$$z_{\Lambda,\beta}(\mathbf{t}) = \frac{Z_{\Lambda,\beta}(\mathbf{t})}{Z_{\Lambda,\beta}} = \frac{1}{f(\mathbf{t})} \frac{1}{\langle F_{\mathbf{t}}(\varphi) \rangle_{\Lambda,\beta,\mathbf{t}}}. \tag{6.64}$$

By Griffiths' (for $n = 1$) or Ginibre's (for $n = 2$) inequality,

$$\langle F_{\mathbf{t}}(\varphi) \rangle_{\Lambda,\beta,\mathbf{t}} \geq \langle F_{\mathbf{t}}(\varphi) \rangle_{\Lambda,\beta',\mathbf{t}}. \tag{6.65}$$

This implies the desired inequality. □

Lemma 6.3. *Let $\Lambda \subset \mathbb{Z}^d$ be finite and $\beta \geq 0$. Then, for every $\mathbf{t} \in (\mathbb{R}^+)^\Lambda$ and every $x, y \in \Lambda$,*

$$\langle \varphi_x^1 \varphi_y^1 \rangle_{\Lambda,\beta,\mathbf{t}} \leq \langle \varphi_x^1 \varphi_y^1 \rangle_{\Lambda,\beta}. \tag{6.66}$$

Proof. As in (6.63), with $F_{\mathbf{t}}$ defined by (6.62),

$$\langle \varphi_x^1 \varphi_y^1 \rangle_{\Lambda,\beta} = \frac{\langle \varphi_x^1 \varphi_y^1 F_{\mathbf{t}}(\varphi) \rangle_{\Lambda,\beta,\mathbf{t}}}{\langle F_{\mathbf{t}}(\varphi) \rangle_{\Lambda,\beta,\mathbf{t}}}. \tag{6.67}$$

Again, by Griffiths' (for $n = 1$) or Ginibre's (for $n = 2$) inequality,

$$\frac{\langle \varphi_x^1 \varphi_y^1 F_{\mathbf{t}}(\varphi) \rangle_{\Lambda,\beta,\mathbf{t}}}{\langle F_{\mathbf{t}}(\varphi) \rangle_{\Lambda,\beta,\mathbf{t}}} \geq \langle \varphi_x^1 \varphi_y^1 \rangle_{\Lambda,\beta,\mathbf{t}}. \tag{6.68}$$

This completes the proof. □

6.4.3 Verification of Definition 1.3

Let $d \geq 2$. Although G does not satisfy Definition 1.3(i), this is harmless, as explained in Remark 1.8. The remaining properties listed in Definition 1.3 are classical facts about the 1- and 2-component $|\varphi|^4$ model. The monotonicity in (ii) is a consequence of Griffiths' or Ginibre's inequality, and the differentiability is proved in [58] for the Ising model and follows more generally for 1- and 2-component models using the extension of the Lebowitz inequality in [17]. Property (iii) is inherited from J. Property (iv) follows by defining

$$\beta_c := \inf \Big\{ \beta \geq 0 : \chi(\beta) = \sum_{x \in \mathbb{Z}^d} G_\beta(x) = \infty \Big\}, \tag{6.69}$$

and using the Simon–Lieb [60, 80] (for $n = 1$) or Rivasseau [77] (for $n = 2$) inequality as in [27, Section 2.5]. From [27], we additionally obtain that $\{\beta \geq 0 : \chi(\beta) = \infty\}$ is a closed set, which forces

$$\chi(\beta_c) = \infty. \tag{6.70}$$

Finally, (v) holds as a consequence of the aforementioned correlation inequalities, which imply left-continuity[2] of the map $\beta \mapsto G_\beta(x)$ on $[0, \beta_c]$ for every $x \in \mathbb{Z}^d$.

6.4.4 Verification of Assumption I

The verification of Assumption I proceeds by taking the infinite-volume limit of finite-volume versions of the two inequalities. We first state two lemmas with the finite-volume inequalities, then verify Assumption I, and finally prove the two finite-volume lemmas.

For $\Lambda \subset \mathbb{Z}^d$ and $x, y \in \mathbb{Z}^d$, we write $G_{\Lambda,\beta}(x, y) := \langle \varphi_x \varphi_y \rangle_{\Lambda,\beta}$. By definition, $G_{\Lambda,\beta}(x, y) = 0$ if $x \notin \Lambda$ or $y \notin \Lambda$. We also define

$$F_{\Lambda,\beta}(x, y) = \Big(\sum_{z \in \mathbb{Z}^d} G_{\Lambda,\beta}(x, z) J_{y-z} \Big) \vee \Big(\sum_{z \in \mathbb{Z}^d} G_{\Lambda,\beta}(y, z) J_{x-z} \Big), \tag{6.71}$$

so that $F_{\Lambda,\beta}(x, y) = F_{\Lambda,\beta}(y, x)$. When $\Lambda = \mathbb{Z}^d$, we have $F_\Lambda = F$.

The following lemma is a finite-volume version of (**I.1**).

[2] Here we use the fact that $G_\beta(x)$ is the increasing limit of the increasing continuous functions $\beta \mapsto \langle \varphi_0^1 \varphi_x^1 \rangle_{\Lambda,\beta,J}$ as Λ approximates $\mathbb{Z}^d$.

Lemma 6.4. *Let $\Lambda \subset \mathbb{Z}^d$ be finite. For every $0 \le \beta' \le \beta$ and every $x \in \mathbb{Z}^d$, we have*

$$G_{\Lambda,\beta}(0,x) \le G_{\Lambda,\beta'}(0,x) + (\beta - \beta') \sum_{u,v\in\mathbb{Z}^d} G_{\Lambda,\beta}(0,u) J_{v-u} G_{\Lambda,\beta}(v,x). \tag{6.72}$$

For the finite-volume version of (**I.2**), we define

$$K_{\Lambda,\beta}(x,y) = G_{\Lambda,\beta}(x,y) + \beta F_{\Lambda,\beta}(x,y), \tag{6.73}$$

$$S^{(1)}_{\Lambda,\beta}(0,x) = 6\lambda \sum_{u,v\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} G_{\Lambda,\beta}(0,z) G_{\Lambda,\beta}(z,u) J_{v-u} G_{\Lambda,\beta}(v,z) G_{\Lambda,\beta}(z,x), \tag{6.74}$$

$$S^{(2)}_{\Lambda,\beta}(0,x) = 3\max\left(\frac{1}{A^2},1\right) \sum_{u,v\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} K_{\Lambda,\beta}(0,z) K_{\Lambda,\beta}(z,u) J_{v-u} K_{\Lambda,\beta}(v,z) K_{\Lambda,\beta}(z,x), \tag{6.75}$$

and

$$S_{\Lambda,\beta}(0,x) = S^{(1)}_{\Lambda,\beta}(0,x) \wedge S^{(2)}_{\Lambda,\beta}(0,x). \tag{6.76}$$

Lemma 6.5. *Let $\Lambda \subset \mathbb{Z}^d$ be finite. For every $\beta \ge 0$ and every $x \in \mathbb{Z}^d$, we have*

$$\partial_\beta G_{\Lambda,\beta}(0,x) \ge \sum_{u,v\in\mathbb{Z}^d} G_{\Lambda,\beta}(0,u) J_{v-u} G_{\Lambda,\beta}(v,x) - S_{\Lambda,\beta}(0,x). \tag{6.77}$$

The choice $S^{(1)}$ for S in Lemma 6.5 comes with a prefactor λ, which is perfect for our application to weakly-coupled and spread-out $|\varphi|^4$ models. However, we also wish to derive the counterpart of Lemma 6.5 for Ising and XY models in Sections 6.5–6.6, by taking a limit $\lambda \to \infty$. For this, we will use the choice $S^{(2)}$ instead.

It is straightforward to deduce Assumption I from Lemmas 6.4–6.5. The lower differential inequality is in terms of

$$H_\beta := H^{(1)}_\beta \wedge H^{(2)}_\beta, \tag{6.78}$$

$$H^{(1)}_\beta(x) = 6\lambda\delta_0(x)(F_\beta * G_\beta)(x), \tag{6.79}$$

$$H^{(2)}_\beta(x) = 3\max\left(\frac{1}{A^2},1\right) \left[(\delta_0 + \beta J) * (\delta_0 + \beta J)\right](x) (K_\beta * J * K_\beta)(0), \tag{6.80}$$

with

$$K_\beta = G_\beta + \beta F_\beta = G_\beta * (\delta_0 + \beta J). \tag{6.81}$$

Proof of (**I.1**). Let $0 \le \beta' \le \beta < \beta_c$ and $x \in \mathbb{Z}^d$. By Griffiths' or Ginibre's inequality, for every $u,v \in \mathbb{Z}^d$, the sequence $G_{\Lambda_k,\beta'}(0,u) J_{v-u} G_{\Lambda_k,\beta}(v,x)$ increases monotonically to the limit $G_\beta(u) J_{v-u} G_\beta(x-v)$. By applying Lemma 6.4 to the sequence of boxes $(\Lambda_k)_{k\ge1}$, in conjunction with the monotone convergence theorem, we conclude that (**I.1**) holds. □

Proof of (**I.2**). Let $\beta < \beta_c$ and $x \in \mathbb{Z}^d$. For the infinite-volume limit of the right-hand side of (6.77), by monotonicity we have

$$\lim_{k\to\infty} \sum_{u,v\in\mathbb{Z}^d} G_{\Lambda_k,\beta}(0,u) J_{v-u} G_{\Lambda_k,\beta}(v,x) = (G_\beta * J * G_\beta)(x), \tag{6.82}$$

$$\begin{aligned}\lim_{k\to\infty} S^{(1)}_{\Lambda_k,\beta}(0,x) &= 6\lambda \sum_{u,v,z\in\mathbb{Z}^d} G_\beta(z) G_\beta(u-z) J_{v-u} G_\beta(z-v) G_\beta(x-z) \\ &= (G_\beta * H^{(1)}_\beta * G_\beta)(x),\end{aligned} \tag{6.83}$$

and

$$\lim_{k\to\infty} S^{(2)}_{\Lambda_k,\beta}(0,x) = 3\max\left(\frac{1}{A^2},1\right) \sum_{u,v,z\in\mathbb{Z}^d} K_\beta(z)K_\beta(u-z)J_{v-u}K_\beta(z-v)K_\beta(x-z)$$
$$= (G_\beta * H^{(2)}_\beta * G_\beta)(x). \tag{6.84}$$

The function $H_\beta = H^{(1)}_\beta \wedge H^{(2)}_\beta$ satisfies all the hypotheses stated in Assumption I. To complete the proof, it suffices to know that the finite-volume derivative on the left-hand side of (6.77) converges to the infinite-volume derivative. This latter fact is a consequence of correlation inequalities; see [58] for the Ising model and [17] for the extension of the Lebowitz inequality to more general 1- and 2-component spins. This completes the proof. □

Proof of Lemma 6.4. Let $\Lambda \subset \mathbb{Z}^d$ be finite, $0 \le \beta' \le \beta$, and $x \in \mathbb{Z}^d$. By Proposition 6.1,

$$G_{\Lambda,\beta}(0,x) = \langle \varphi^1_0\varphi^1_x\rangle_{\Lambda,\beta} = \sum_{\gamma:0\to x} (\beta J)^\gamma \int z_{\Lambda,\beta}(\mathbf{t}) d\nu_{\Lambda,\gamma}(\mathbf{t}). \tag{6.85}$$

We apply the identity $b^n - a^n = (b-a)\sum_{k=0}^{n-1} b^k a^{n-1-k}$ with $b=\beta$ and $a=\beta'$. This gives

$$G_{\Lambda,\beta}(0,x) = \sum_{\gamma:0\to x} (\beta' J)^\gamma \int z_{\Lambda,\beta}(\mathbf{t}) d\nu_{\Lambda,\gamma}(\mathbf{t}) \tag{6.86}$$
$$+ (\beta-\beta') \sum_{u,v\in\mathbb{Z}^d} \sum_{\substack{\gamma_1:0\to u\\ \gamma_2:v\to x}} (\beta' J)^{\gamma_1} J_{v-u} (\beta J)^{\gamma_2} \int z_{\Lambda,\beta}(\mathbf{t}) d\nu_{\Lambda,\gamma_1\circ(uv)\circ\gamma_2}(\mathbf{t}),$$

where $\gamma_1 \circ (uv) \circ \gamma_2$ denotes the concatenation of γ_1, the step uv, and γ_2. By Lemma 6.2 and Proposition 6.1, the first term on the right-hand side of (6.86) is bounded above by the term $G_{\Lambda,\beta'}(0,x)$ appearing in (6.72).

For the second term on the right-hand side of (6.86), we use the fact that[3]

$$\int z_{\Lambda,\beta}(\mathbf{t}) d\nu_{\Lambda,\gamma_1\circ(uv)\circ\gamma_2}(\mathbf{t}) = \int z_{\Lambda,\beta}(\mathbf{t}_1+\mathbf{t}_2) d\nu_{\Lambda,\gamma_1}(\mathbf{t}_1) d\nu_{\Lambda,\gamma_2}(\mathbf{t}_2)$$
$$= \int d\nu_{\Lambda,\gamma_1}(\mathbf{t}_1) z_{\Lambda,\beta}(\mathbf{t}_1) \int d\nu_{\Lambda,\gamma_2}(\mathbf{t}_2) \frac{z_{\Lambda,\beta}(\mathbf{t}_1+\mathbf{t}_2)}{z_{\Lambda,\beta}(\mathbf{t}_1)}. \tag{6.87}$$

By Lemma 6.3, for every fixed $\mathbf{t}_1$,

$$\sum_{\gamma_2:v\to x} (\beta J)^{\gamma_2} \int \frac{z_{\Lambda,\beta}(\mathbf{t}_1+\mathbf{t}_2)}{z_{\Lambda,\beta}(\mathbf{t}_1)} d\nu_{\Lambda,\gamma_2}(\mathbf{t}_2) = \langle \varphi^1_v\varphi^1_x\rangle_{\Lambda,\beta,\mathbf{t}_1} \le G_{\Lambda,\beta}(v,x). \tag{6.88}$$

We use (6.87)–(6.88) in (6.86) and perform the sum over γ_1 to obtain the second term on the right-hand side of (6.72). This completes the proof. □

Proof of Lemma 6.5. Let $\Lambda \subset \mathbb{Z}^d$ be a finite set containing 0. Let $\beta \ge 0$ and $x \in \mathbb{Z}^d$. We consider the 1- and 2-component models simultaneously, by writing $\varphi = (\varphi^1, \varphi^2)$ with $\varphi^2 = 0$ when $n = 1$. To simplify the notation, throughout the proof we write simply $\langle\cdot\rangle = \langle\cdot\rangle_{\Lambda,\beta}$ and generally omit these subscripts.

We use Ursell's four-point functions:

$$U^{(4)}(0,x,u,v) = \langle \varphi^1_0\varphi^1_x\varphi^1_u\varphi^1_v\rangle - \langle \varphi^1_0\varphi^1_x\rangle\langle \varphi_u\varphi^1_v\rangle - \langle \varphi^1_0\varphi^1_u\rangle\langle \varphi^1_v\varphi^1_x\rangle - \langle \varphi^1_0\varphi^1_v\rangle\langle \varphi^1_u\varphi^1_x\rangle, \tag{6.89}$$

[3]The first equality can be seen as a kind of binomial theorem for the measures $d\rho_{\Lambda,\gamma}$ on the occupation time $\mathbf{t}$.

and

$$\tilde{U}^{(4)}(0,x,u,v) = \langle \varphi_0^1 \varphi_x^1 \varphi_u^2 \varphi_v^2 \rangle - \langle \varphi_0^1 \varphi_x^1 \rangle \langle \varphi_u^2 \varphi_v^2 \rangle. \tag{6.90}$$

From (6.49), we express the derivative in β as:

$$\partial_\beta \langle \varphi_0^1 \varphi_x^1 \rangle = \frac{1}{2} \sum_{u,v \in \Lambda} J_{v-u} \Big(U^{(4)}(0,x,u,v) + \tilde{U}^{(4)}(0,x,u,v) + \langle \varphi_0^1 \varphi_u^1 \rangle \langle \varphi_v^1 \varphi_x^1 \rangle + \langle \varphi_0^1 \varphi_v^1 \rangle \langle \varphi_u^1 \varphi_x^1 \rangle \Big). \tag{6.91}$$

By definition,

$$\frac{1}{2} \sum_{u,v \in \Lambda} J_{v-u} \Big(\langle \varphi_0^1 \varphi_u^1 \rangle \langle \varphi_v^1 \varphi_x^1 \rangle + \langle \varphi_0^1 \varphi_v^1 \rangle \langle \varphi_u^1 \varphi_x^1 \rangle \Big) = \sum_{u,v \in \mathbb{Z}^d} G_{\Lambda,\beta}(0,u) J_{v-u} G_{\Lambda,\beta}(v,x), \tag{6.92}$$

which is the first term on the right-hand side of (6.77).

The subtracted term on the right-hand side of (6.77) arises from an upper bound on

$$-U^{(4)}(0,u,v,x) = |U^{(4)}(0,x,u,v)| \quad \text{and} \quad -\tilde{U}^{(4)}(0,x,u,v) = |\tilde{U}^{(4)}(0,x,u,v)| \tag{6.93}$$

(the equalities follow from Lebowitz's inequality [59]). We follow [33, (37)] and use

$$|U^{(4)}(0,u,v,x)| \le \sum_P \sum_{\gamma_1,\gamma_2}^{P} (\beta J)^{\gamma_1} (\beta J)^{\gamma_2} \int \mathrm{d}\nu_{\gamma_1}(\mathbf{t}^1) \mathrm{d}\nu_{\gamma_2}(\mathbf{t}^2) z(\mathbf{t}^1) z(\mathbf{t}^2) f_\lambda(\mathbf{t}^1,\mathbf{t}^2), \tag{6.94}$$

where $\sum_P$ is a sum over partitions in pairs (or *pairings*) of $\{0,u,v,x\}$, $\sum_{\gamma_1,\gamma_2}^P$ is a sum over paths γ_1, γ_2 which respect the partition P (i.e.,, γ_1 connects the elements of the first pair of the partition, and γ_2 the elements of the second pair), and

$$f_\lambda(\mathbf{t}^1,\mathbf{t}^2) = 1 - \exp\Big(-2\lambda \sum_{z \in \mathbb{Z}^d} \mathbf{t}_z^1 \mathbf{t}_z^2 \Big). \tag{6.95}$$

We use the elementary bound

$$f_\lambda(\mathbf{t}^1,\mathbf{t}^2) \le \Big(2\lambda \sum_{z \in \mathbb{Z}^d} \mathbf{t}_z^1 \mathbf{t}_z^2 \Big) \wedge \sum_{z \in \mathbb{Z}^d} \mathbb{1}_{z \in \gamma_1 \cap \gamma_2}. \tag{6.96}$$

The term $|\tilde{U}^{(4)}(0,x,u,v)|$ satisfies a bound like (6.94), where instead of a sum over all pairings P, we only have the term corresponding to the pairing $P = \{\{0,x\},\{u,v\}\}$.

Consider the first alternative on the right-hand side of (6.96). It follows from [16, (3.13)] that, for a fixed $z \in \mathbb{Z}^d$, and $\gamma : a \to b$,

$$\sum_{\gamma : a \to b} (\beta J)^\gamma \int \mathrm{d}\nu_\gamma(\mathbf{t}) z(\mathbf{t}) \mathbf{t}_z \le \langle \varphi_a^1 \varphi_z^1 \rangle \langle \varphi_z^1 \varphi_b^1 \rangle. \tag{6.97}$$

Since there are three pairings of $\{0,u,v,x\}$, this implies that

$$2\lambda \sum_{z \in \mathbb{Z}^d} \sum_P \sum_{\gamma_1,\gamma_2}^{P} (\beta J)^{\gamma_1} (\beta J)^{\gamma_2} \int \mathrm{d}\nu_{\gamma_1}(\mathbf{t}^1) \mathrm{d}\nu_{\gamma_2}(\mathbf{t}^2) z(\mathbf{t}^1) z(\mathbf{t}^2) \mathbf{t}_z^1 \mathbf{t}_z^2$$
$$\le 6\lambda \sum_{z \in \mathbb{Z}^d} \langle \varphi_0^1 \varphi_z^1 \rangle \langle \varphi_u^1 \varphi_z^1 \rangle \langle \varphi_v^1 \varphi_z^1 \rangle \langle \varphi_x^1 \varphi_z^1 \rangle. \tag{6.98}$$

The combination of (6.94) and (6.98) leads to

$$\frac{1}{2}\sum_{u,v\in\Lambda} J_{v-u}\Big(U^{(4)}(0,x,u,v)+\tilde{U}^{(4)}(0,x,u,v)\Big) \geq \\ -6\lambda \sum_{u,v\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} G(0,z)G(z,u)J_{v-u}G(v,z)G(z,x). \quad (6.99)$$

This gives the first half of Lemma 6.5, i.e., the lower bound with $S^{(1)}(0,x)$.

To obtain the lower bound with $S^{(2)}(0,x)$, we bound f_λ by the second alternative on the right-hand side of (6.96). Recall that $K(s,t)=G(s,t)+\beta F(s,t)$. We will prove that

$$\sum_{z\in\mathbb{Z}^d}\sum_{P}\sum_{\gamma_1,\gamma_2}^{P}(\beta J)^{\gamma_1}(\beta J)^{\gamma_2}\int \mathrm{d}\nu_{\gamma_1}(\mathbf{t}^1)\mathrm{d}\nu_{\gamma_2}(\mathbf{t}^2)z(\mathbf{t}^1)z(\mathbf{t}^2)\mathbb{1}_{z\in\gamma_1\cap\gamma_2} \\ \leq 3\max\left(\frac{1}{A^2},1\right)\sum_{z\in\mathbb{Z}^d}K(0,z)K(z,u)K(v,z)K(z,x). \quad (6.100)$$

By (6.94), this suffices, since it gives (recall that a bound on $|U^{(4)}|$ also bounds $|\tilde{U}^{(4)}|$)

$$\frac{1}{2}\sum_{u,v\in\Lambda} J_{v-u}\Big(U^{(4)}(0,x,u,v)+\tilde{U}^{(4)}(0,x,u,v)\Big) \geq \\ -3\max\left(\frac{1}{A^2},1\right)\sum_{u,v\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}K(0,z)K(z,u)J_{v-u}K(v,z)K(z,x), \quad (6.101)$$

which is the desired estimate with $S^{(2)}$.

It remains only to prove (6.100). For this, it suffices to show that, for every a,b,

$$\sum_{\gamma:a\to b}(\beta J)^{\gamma}\int \mathrm{d}\nu_\gamma(\mathbf{t})\mathrm{d}z(\mathbf{t})\mathbb{1}_{z\in\gamma}\leq \max\left(\frac{1}{A},1\right)K(a,z)K(z,b). \quad (6.102)$$

We assume that $a,b,z\in\Lambda$ since otherwise the sum on the left-hand side of (6.102) is equal to 0. If $a=b=z$, then

$$\sum_{\gamma:a\to b}(\beta J)^{\gamma}\int \mathrm{d}\nu_\gamma(\mathbf{t})\mathrm{d}z(\mathbf{t})\mathbb{1}_{z\in\gamma}=G(a,a)=\frac{1}{A}G(a,a)G_{\Lambda,0}(a,a) \\ \leq \frac{1}{A}G(a,a)G(a,a)=\frac{1}{A}G(a,z)G(z,b).$$

We may therefore assume that a,b,z are not all equal. In this case, if $z\in\gamma$ then at least one of the following holds true: (i) there exists z' with $J_{z'-z}>0$, $\gamma_1:a\to z$, and $\gamma_2:z'\to b$ such that $\gamma=\gamma_1\circ(zz')\circ\gamma_2$ (where (zz') denotes the one-step walk from z to z'); (ii) there exists z' with $J_{z-z'}>0$, $\gamma_1:a\to z'$, and $\gamma_2:z\to b$ such that $\gamma=\gamma_1\circ(z'z)\circ\gamma_2$. The contributions of these two scenarios to the left-hand side of (6.102) are respectively denoted (I) and (II). The term (I) is given by

$$(\mathrm{I})=\sum_{z'\in\mathbb{Z}^d}\sum_{\substack{\gamma_1:a\to z\\ \gamma_2:z'\to x}}(\beta J)^{\gamma_1}\beta J_{z'-z}(\beta J)^{\gamma_2}\int z(\mathbf{t})\mathrm{d}\nu_{\gamma_1\circ(z,z')\circ\gamma_2}(\mathbf{t}). \quad (6.103)$$

Proceeding as in the proof of Lemma 6.4, and using $J_u=J_{-u}$, we obtain

$$(\mathrm{I})\leq\sum_{z'\in\mathbb{Z}^d}G(a,z)\beta J_{z'-z}G(z',b)=\beta G(a,z)\sum_{z'\in\mathbb{Z}^d}G(b,z')J_{z-z'}\leq\beta G(a,z)F(z,b). \quad (6.104)$$

Similarly,

$$(\mathrm{II}) \le \beta F(a,z)G(z,b). \tag{6.105}$$

The bounds (6.104)–(6.105) give (6.102), which we have seen to be sufficient. This completes the proof. □

6.5 The Ising model

To verify Assumption I for the Ising model, we combine Lemmas 6.4–6.5 with Proposition 6.6 below to transfer the desired properties from $|\varphi|^4$ to Ising. Direct proofs can also be obtained via the random current representation [1, 23] of the Ising model.

6.5.1 The model

Let $d \ge 2$. Let $\Lambda \subset \mathbb{Z}^d$ be finite, $\beta \ge 0$, J an admissible interaction , and $F : \{-1,1\}^\Lambda \to \mathbb{R}$. The Ising model on Λ is the measure $\langle\cdot\rangle^{\mathrm{Ising}}_{\Lambda,\beta}$ on $\{-1,1\}^\Lambda$ given by

$$\langle F(\sigma)\rangle^{\mathrm{Ising}}_{\Lambda,\beta} := \frac{1}{Z^{\mathrm{Ising}}_{\Lambda,\beta}} \sum_{\sigma\in\{-1,1\}^\Lambda} F(\sigma)\exp(-\beta H_\Lambda(\sigma)), \tag{6.106}$$

where

$$H_\Lambda(\sigma) = -\frac{1}{2}\sum_{x,y\in\Lambda} J_{y-x}\sigma_x\sigma_y, \qquad Z^{\mathrm{Ising}}_{\Lambda,\beta} = \sum_{\sigma\in\{-1,1\}^\Lambda} \exp(-\beta H_\Lambda(\sigma)). \tag{6.107}$$

It is again a consequence of Griffiths' inequalities that the sequence of measures $\langle\cdot\rangle^{\mathrm{Ising}}_{\Lambda,\beta}$ admits a weak limit as $\Lambda \nearrow \mathbb{Z}^d$. We denote the limit by $\langle\cdot\rangle^{\mathrm{Ising}}_{\beta}$. The two-point function is defined for $\beta \ge 0$ and $x \in \mathbb{Z}^d$ by

$$G_\beta(x) := \langle\sigma_0\sigma_x\rangle^{\mathrm{Ising}}_{\beta}. \tag{6.108}$$

The 1-component φ^4 model and the Ising model are expected to lie in the same universality class. Support for this conjecture is provided by the fact that the latter is a limit of the former. Indeed, the *normalised* single-site distribution with $\mu = -\lambda$ obeys (in the sense of weak convergence)

$$\frac{1}{z_\lambda}\exp\Big(-\frac{\lambda}{4}(\varphi_0^2-1)^2\Big)\mathrm{d}\varphi_0 \underset{\lambda\to\infty}{\longrightarrow} \frac{\delta_{-1}+\delta_1}{2}. \tag{6.109}$$

Easy consequences of (6.109) are formulated in the next proposition. Recall the definition of $\langle\cdot\rangle_{\Lambda,\beta,J} = \langle\cdot\rangle_{\Lambda,\beta,J,\lambda,\mu}$ from (6.49), and of $A(\lambda,\mu)$ from (6.53).

Proposition 6.6. *Let $\Lambda \subset \mathbb{Z}^d$ be finite and $\beta \ge 0$. Then, for every $x, y \in \Lambda$,*

$$\lim_{\lambda\to\infty}\langle\varphi_x\varphi_y\rangle_{\Lambda,\beta,J,\lambda,-\lambda} = \langle\sigma_x\sigma_y\rangle^{\mathrm{Ising}}_{\Lambda,\beta,J}, \tag{6.110}$$

$$\lim_{\lambda\to\infty}\partial_\beta\langle\varphi_x\varphi_y\rangle_{\Lambda,\beta,J,\lambda,-\lambda} = \partial_\beta\langle\sigma_x\sigma_y\rangle^{\mathrm{Ising}}_{\Lambda,\beta,J}, \tag{6.111}$$

$$\lim_{\lambda\to\infty} A(\lambda,-\lambda) = 1. \tag{6.112}$$

6.5.2 Verification of Definition 1.3

For $d \ge 2$, the properties of the Ising model listed in Definition 1.3 are classical facts that can be derived using the same arguments as for the 1-component φ^4 model. We omit further details. We stress that we define β_c as

$$\beta_c := \inf\Big\{\beta \ge 0 : \chi(\beta) = \sum_{x\in\mathbb{Z}^d} G_\beta(x) = \infty\Big\}. \tag{6.113}$$

6.5.3 Verification of Assumption I

Proof of (**I.1**). Let $\Lambda \subset \mathbb{Z}^d$ be finite, $0 \le \beta' \le \beta < \beta_c$, and $x \in \mathbb{Z}^d$. The combination of Lemma 6.4 and Proposition 6.6 leads to

$$\langle \sigma_0 \sigma_x \rangle^{\mathrm{Ising}}_{\Lambda,\beta} \le \langle \sigma_0 \sigma_x \rangle^{\mathrm{Ising}}_{\Lambda,\beta'} + (\beta - \beta') \sum_{u,v \in \mathbb{Z}^d} \langle \sigma_0 \sigma_u \rangle^{\mathrm{Ising}}_{\Lambda,\beta} J_{v-u} \langle \sigma_v \sigma_x \rangle^{\mathrm{Ising}}_{\Lambda,\beta}. \tag{6.114}$$

We can then pass to the limit $\Lambda \nearrow \mathbb{Z}^d$ by using Griffiths' inequalities in the same manner as for the 1-component φ^4 model. This concludes the proof. □

We will prove the differential lower bound with

$$H_\beta(x) = 3\big[(\delta_0 + \beta J) * (\delta_0 + \beta J)\big](x)(K_\beta * J * K_\beta)(0), \tag{6.115}$$

where $K_\beta = G_\beta + \beta F_\beta$. By definition, H_β satisfies all the properties listed below (**I.2**).

Proof of (**I.2**). Let $\Lambda \subset \mathbb{Z}^d$ be finite, $0 \le \beta < \beta_c$, and $x \in \mathbb{Z}^d$. The combination of Lemma 6.5 (with $S^{(2)}$) and Proposition 6.6 (with the contribution from $\tilde{U}^{(4)}$ omitted) leads to

$$\begin{aligned} \partial_\beta \langle \sigma_0 \sigma_x \rangle^{\mathrm{Ising}}_{\Lambda,\beta} \ge & \sum_{u,v \in \mathbb{Z}^d} \langle \sigma_0 \sigma_u \rangle^{\mathrm{Ising}}_{\Lambda,\beta} J_{v-u} \langle \sigma_v \sigma_x \rangle^{\mathrm{Ising}}_{\Lambda,\beta} \\ & - 3 \sum_{u,v \in \mathbb{Z}^d} \sum_{z \in \mathbb{Z}^d} K^{\mathrm{Ising}}_{\Lambda,\beta}(0,z) K^{\mathrm{Ising}}_{\Lambda,\beta}(z,u) J_{v-u} K^{\mathrm{Ising}}_{\Lambda,\beta}(v,z) K^{\mathrm{Ising}}_{\Lambda,\beta}(z,x), \end{aligned} \tag{6.116}$$

where $K^{\mathrm{Ising}}_{\Lambda,\beta}$ is defined similarly as for the φ^4 model. We can again take the infinite-volume limit by appealing to [58]. This gives

$$\partial_\beta G_\beta(x) \ge (G_\beta * (J - H_\beta) * G_\beta)(x), \tag{6.117}$$

and completes the proof. □

6.6 The XY model

Assumption I can be verified for the XY model by combining Lemmas 6.4–6.5 with Proposition 6.7 below to transfer inequalities from the 2-component $|\varphi|^4$ model to the XY model. Direct proofs can be obtained using the BFS expansion of the XY model (see [56]).

6.6.1 The model

Let $d \ge 2$. Let $\Lambda \subset \mathbb{Z}^d$ be finite, $\beta \ge 0$, and J an admissible interaction. Let $\mathbb{S}^1$ denote the unit circle in $\mathbb{R}^2$, and let $\mathrm{d}\sigma_x$ denote the Haar measure on $\mathbb{S}^1$. Let $F : (\mathbb{S}^1)^\Lambda \to \mathbb{R}$. The XY model on Λ is the measure $\langle \cdot \rangle^{\mathrm{XY}}_{\Lambda,\beta}$ on $(\mathbb{S}^1)^\Lambda$ given by

$$\langle F(\sigma) \rangle^{\mathrm{XY}}_{\Lambda,\beta} := \frac{1}{Z^{\mathrm{XY}}_{\Lambda,\beta}} \int_{(\mathbb{S}^1)^\Lambda} F(\sigma) \exp(-\beta H_\Lambda(\sigma)) \prod_{x \in \Lambda} \mathrm{d}\sigma_x, \tag{6.118}$$

where

$$H_\Lambda(\sigma) = - \sum_{\{x,y\} \subset \Lambda} J_{y-x} (\sigma_x \cdot \sigma_y), \qquad Z^{\mathrm{XY}}_{\Lambda,\beta} = \int_{(\mathbb{S}^1)^\Lambda} \exp(-\beta H_\Lambda(\sigma)) \prod_{x \in \Lambda} \mathrm{d}\sigma_x. \tag{6.119}$$

By Ginibre's inequalities, the sequence of measures $\langle\cdot\rangle^{\mathrm{XY}}_{\Lambda,\beta}$ admits a weak limit as $\Lambda \nearrow \mathbb{Z}^d$. We denote it by $\langle\cdot\rangle^{\mathrm{XY}}_{\beta}$. The two-point function is defined for $\beta \geq 0$ and $x \in \mathbb{Z}^d$ by

$$G_\beta(x) := \langle \sigma^1_0 \sigma^1_x \rangle^{\mathrm{XY}}_{\beta}. \tag{6.120}$$

The 2-component $|\varphi|^4$ model and the XY model are expected to lie in the same universality class. As in (6.109), there is convergence of the $|\varphi|^4$ single-spin distribution:

$$\frac{1}{z_\lambda} \exp\Big(- \frac{\lambda}{4}(|\varphi_0|_2^2 - 1)^2\Big) \mathrm{d}\varphi_0 \underset{\lambda\to\infty}{\longrightarrow} \mathrm{d}\sigma_0. \tag{6.121}$$

Easy consequences of (6.121) are formulated in the next proposition. Recall the definition of $\langle\cdot\rangle_{\Lambda,\beta,J} = \langle\cdot\rangle_{\Lambda,\beta,J,\lambda,\mu}$ from (6.49), and of $A(\lambda,\mu)$ from (6.53).

Proposition 6.7. *Let $\Lambda \subset \mathbb{Z}^d$ be finite and $\beta \geq 0$. Then, for every $x, y \in \Lambda$,*

$$\lim_{\lambda\to\infty} \langle \varphi^1_x \varphi^1_y \rangle_{\Lambda,\beta,\lambda,-\lambda} = \langle \sigma^1_x \sigma^1_y \rangle^{\mathrm{XY}}_{\Lambda,\beta}, \tag{6.122}$$

$$\lim_{\lambda\to\infty} \partial_\beta \langle \varphi^1_x \varphi^1_y \rangle_{\Lambda,\beta,\lambda,-\lambda} = \partial_\beta \langle \sigma^1_x \sigma^1_y \rangle^{\mathrm{XY}}_{\Lambda,\beta}, \tag{6.123}$$

$$\lim_{\lambda\to\infty} A(\lambda,-\lambda) = \frac{1}{2}. \tag{6.124}$$

6.6.2 Verification of Definition 1.3

For $d \geq 2$, the properties of the XY model listed in Definition 1.3 are classical facts that can be derived using the same arguments as for the 2-component $|\varphi|^4$ model. We omit the details. Again we define β_c as

$$\beta_c := \inf\Big\{\beta \geq 0 : \chi(\beta) = \sum_{x\in\mathbb{Z}^d} G_\beta(x) = \infty\Big\}. \tag{6.125}$$

6.6.3 Verification of Assumption I

The infinite-volume limit of the derivative is again justified using correlation inequalities [17, 58]. We omit the details, which proceed as they do for the Ising model. The limiting value of A in (6.124) gives rise to an extra factor 4, with the result that

$$H_\beta(x) = 12\big[(\delta_0 + \beta J) * (\delta_0 + \beta J)\big](x)(K_\beta * J * K_\beta)(0), \tag{6.126}$$

with $K_\beta = G_\beta + \beta F_\beta$.

6.7 Lattice trees

6.7.1 The model

Let $d > 2$, J be an admissible kernel, and let $E = \{\{x,y\} : J_{y-x} > 0\}$. A *lattice tree* is a finite acyclic subgraph of the infinite graph $(\mathbb{Z}^d, E)$. We write $|T|$ for the number of bonds (also called edges) in T. For $p \geq 0$ and for a lattice tree T, let

$$(pJ)^T = \prod_{\{u,v\}\in T} pJ_{v-u}. \tag{6.127}$$

Let $\mathcal{T}_{0,x}$ denote the set of lattice trees containing the vertices 0 and x, and let $\mathcal{T}_0 = \mathcal{T}_{0,0}$. The *two-point function*, *one-point function*, and *susceptibility* are defined by

$$\rho_p(x) = \sum_{T\in\mathcal{T}_{0,x}} (pJ)^T, \qquad g_p = \sum_{T\in\mathcal{T}_0} (pJ)^T = \rho_p(0), \qquad \hat{\chi}(p) = \sum_{x\in\mathbb{Z}^d} \rho_p(x). \tag{6.128}$$

We also define

$$\hat{\xi}(p)^2 = \frac{1}{\hat{\chi}(p)} \sum_{x\in\mathbb{Z}^d} |x|_2^2 (J * \rho_p)(x). \tag{6.129}$$

A standard subadditivity argument [15, 45] shows that the susceptibility is finite below a critical point $p_c \in (0, \infty)$, and that it diverges at p_c at least as fast as

$$\hat{\chi}(p) \gtrsim (p_c - p)^{-1/2}. \tag{6.130}$$

Since a lattice tree T has $|T| + 1$ vertices, g_p and $\hat{\chi}(p)$ are related by

$$\frac{\mathrm{d}(pg_p)}{\mathrm{d}p} = \sum_{T\in\mathcal{T}_0} (|T| + 1)(pJ)^T = \sum_{T\in\mathcal{T}_0} \sum_{x\in T} (pJ)^T = \hat{\chi}(p). \tag{6.131}$$

In dimensions $d > 8$, lattice trees should have critical exponents $\gamma = \frac{1}{2}$ and $\nu = \frac{1}{4}$ [45, 46], so this does not fit immediately into our black box which is designed for situations where $\gamma = 1$ and $\nu = \frac{1}{2}$. To fix this mismatch, we make the following change of variables. For $p \in [0, p_c)$, let

$$\beta = \beta(p) = pg_p. \tag{6.132}$$

We denote the inverse of the strictly increasing function $p \mapsto \beta(p)$ by $p = p(\beta)$.

The critical point is defined by $\beta_c = \sup_{p\le p_c} \beta(p)$. A priori, it is possible that $\beta_c = \infty$ because we do not initially know that the critical one-point function g_{p_c} is finite. The possibility that $\beta_c = \infty$ is allowed in Assumption I. A posteriori, we do learn that β_c is finite (when $d > 8$ and σ_J is large).

Let $\deg_T(x)$ denote the number of bonds in the lattice tree T that are incident to the vertex $x \in T$. We consider the following reduced set of trees:

$$\mathcal{T}^1_{0,0} = \{0\}, \qquad \mathcal{T}^1_{0,x} = \{T \in \mathcal{T}_{0,x} \,|\, \deg_T(x) = 1\} \text{ if } x \neq 0, \tag{6.133}$$

and define a modified *two-point function* by

$$G_\beta(0) = 1, \qquad G_\beta(x) = \sum_{T\in\mathcal{T}^1_{0,x}} (p(\beta)J)^{|T|} \text{ if } x \neq 0. \tag{6.134}$$

We will show that G_β satisfies Assumption I, and satisfies Assumption II when $d > 8$ and σ_J is sufficiently large. This implies that the conclusion of Theorem 1.6 applies to G_β with

$$\chi(\beta) = \sum_{x\in\mathbb{Z}^d} G_\beta(x), \qquad \xi(\beta)^2 = \frac{1}{\chi(\beta)} \sum_{x\in\mathbb{Z}^d} |x|_2^2 G_\beta(x). \tag{6.135}$$

In particular, $\beta_c < \infty$. We also show that $\chi(\beta_c) = \infty$, so the conclusions of Theorem 1.7 also apply to G_β.

Those conclusions do not immediately refer to the original parametrisation of the model in terms of p, rather than β. Therefore, before verifying Assumptions I and II, we show how to draw conclusions for $\rho(p)$, g_p, $\hat{\chi}(p)$ and $\hat{\xi}(p)$.

Theorem 6.8. *Let $d > 8$. For every admissible J with R_J sufficiently large, for every $p < p_c$, and with $E = O(R_J^{-d})$,*

$$\rho_p(x) \lesssim \delta_0(x) + \frac{1}{\sigma_J^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2-\varepsilon} \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right), \tag{6.136}$$

$$\hat{\chi}(p) \asymp \frac{1}{(p_c - p)^{1/2}}, \tag{6.137}$$

$$\frac{1}{(p_c - p)^{\frac{1}{4}-E}} \lesssim \hat{\xi}(p) \lesssim \frac{1}{(p_c - p)^{\frac{1}{4}+E}}, \tag{6.138}$$

$$g_{p_c} - g_p \asymp (p_c - p)^{1/2}. \tag{6.139}$$

Remark 6.9. It is a consequence of (1.27) that $1 \le \beta_c \le 1 + O(E)$. In fact, this occurs due to a cancellation in $p_c g_{p_c}$, because $p_c \sim e^{-1}$ and $g_{p_c} \sim e$ under the hypotheses of Theorem 6.8, and also for the nearest-neighbour model for d sufficiently large [55, 69, 76].

Proof of Theorem 6.8. We assume that G_β has been proven to satisfy Assumptions I and II for kernels J with range R_J sufficiently large, so that the conclusion of Theorem 1.6 applies to G_β.

A tree $T \ni x$ can be decomposed into at most $(2R_J+1)^d$ trees in $\mathcal{T}^1_{x,x}$, since the degree of x in $(\mathbb{Z}^d, E)$ is $(2R_J+1)^d - 1$. Therefore, together with a trivial lower bound, we have

$$G_{pg_p}(x) \le \rho_p(x) \le G_{pg_p}(x)\Big(1 + \max_{z:J_z>0} G_{pg_p}(z)\Big)^{(2R_J+1)^d}. \tag{6.140}$$

By Theorem 1.6, $G_{pg_p}(z) \le \mathbf{C}\sigma_J^{-d}$. Since $R_J \asymp \sigma_J$ for every admissible kernel (here we use Definition 1.1(*iii*)), it follows that

$$G_{pg_p}(x) \le \rho_p(x) \lesssim G_{pg_p}(x) \le \delta_0(x) + \frac{\mathbf{C}}{\sigma_J^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2-\varepsilon} \exp\left(-\mathbf{c}\frac{|x|}{\xi(\beta)}\right). \tag{6.141}$$

This proves (6.136).

Equation (6.140) also shows that

$$\chi(pg_p) \le \hat{\chi}(p) \lesssim \chi(pg_p), \qquad \hat{\xi}(p) \asymp \xi(pg_p). \tag{6.142}$$

In particular, the relation between $\hat{\chi}$ and χ implies that $\beta_c = p_c g_{p_c}$, and then (6.130) implies that $\chi(\beta_c) = \infty$. This tells us that the conclusions of Theorem 1.7 apply to $\chi(\beta)$, $\xi(\beta)$, and β_c. In particular, with $E = O(R_J^{-d})$ small, we see that

$$\frac{1}{p_c g_{p_c} - pg_p} \le \hat{\chi}(p) \lesssim \frac{1}{p_c g_{p_c} - pg_p}, \tag{6.143}$$

$$\left(\frac{1}{p_c g_{p_c} - pg_p}\right)^{1-E} \lesssim \frac{\hat{\xi}(p)^2}{\sigma_J^2} \lesssim \left(\frac{1}{p_c g_{p_c} - pg_p}\right)^{1+E}. \tag{6.144}$$

To obtain the bounds (6.137)–(6.139) from the above, we need to compare $p_c g_{p_c} - pg_p$ and $p_c - p$. For this, we apply (6.131) to obtain

$$\frac{\mathrm{d}p(\beta)}{\mathrm{d}\beta} = \frac{1}{\hat{\chi}(p(\beta))}. \tag{6.145}$$

With (6.143), this gives

$$\beta_c - \beta \lesssim \frac{\mathrm{d}p(\beta)}{\mathrm{d}\beta} \le \beta_c - \beta, \tag{6.146}$$

and then integration over the interval $[\beta, \beta_c]$ gives

$$(\beta_c - \beta)^2 \lesssim p(\beta_c) - p(\beta) \leq \frac{1}{2}(\beta_c - \beta)^2. \tag{6.147}$$

Since $\beta = pg_p$, this implies that

$$\sqrt{2}(p_c - p)^{1/2} \leq p_c g_{p_c} - pg_p \lesssim (p_c - p)^{1/2}. \tag{6.148}$$

With (6.143)–(6.144), this proves (6.137)–(6.138), and hence that the critical exponents for the original problem are $\gamma = \frac{1}{2}$ and $\nu = \frac{1}{4} \pm O(E)$. It also follows from (6.148) that

$$\frac{\sqrt{2}}{p_c}(p_c - p)^{1/2}\Big[1 - \frac{g_p}{\sqrt{2}}(p_c - p)^{1/2}\Big] \leq g_{p_c} - g_p \lesssim \frac{1}{p_c}(p_c - p)^{1/2}, \tag{6.149}$$

which proves (6.139). This completes the proof. □

6.7.2 Verification of Definition 1.3

It follows from its definition that G_β satisfies the following properties: $G_0 = \delta_0$, G_β is monotone and differentiable (by the inverse function theorem) for $\beta \in [0, \beta_c)$, and G_β is $\mathbb{Z}^d$-symmetric.

To see that the function $x \mapsto G_\beta(x)$ decays exponentially for each fixed $\beta \in [0, \beta_c)$, we observe that a tree containing $0, x$ must contain at least $|x|/R_J$ bonds. Given $p < p_c$, let $q = \frac{1}{2}(p + p_c)$. Then $q < p_c$ and

$$G_\beta(x) \leq \rho_{p(\beta)}(x) \leq \sum_{\substack{T \in \mathcal{T}_{0,x} \\ |T| \geq |x|/R_J}} (pJ)^T \leq \left(\frac{p}{q}\right)^{|x|/R_J} \sum_{T \in \mathcal{T}_{0,x}} (qJ)^T \leq \left(\frac{p}{q}\right)^{|x|/R_J} \hat{\chi}(q). \tag{6.150}$$

Since $\hat{\chi}(q) < \infty$, this proves the desired exponential decay.

Finally, the limit $\lim_{\beta \uparrow \beta_c} G_\beta(x) = G_{\beta_c}(x)$ exists by monotone convergence.

6.7.3 Verification of Assumption I

Since $G_\beta(0) = G_{\beta'}(0) = 1$, there is nothing to prove for $x = 0$. We therefore assume that $x \neq 0$. We begin with a decomposition of a tree $T \in \mathcal{T}^1_{0,x}$ into subtrees. The *backbone* $\Gamma_{0,x}$ of T is the unique path in T between 0 and x. In particular, $\Gamma_{0,0} = \{0\}$. We order the backbone vertices by setting $y < z$ for $y, z \in \Gamma_{0,x}$ if y is closer to 0 than z (for the graph metric on T). Similarly, we order the backbone bonds: the least bond is adjacent to 0 and the last bond is adjacent to x.

Deletion of the backbone bonds from T leaves behind non-intersecting connected subgraphs of T which we refer to as *ribs* $R_0, \ldots, R_x$. Given a directed bond $e = (u, v) \in \Gamma_{0,x}$ with $u \leq v$, we define R_e to be the union of the rib R_u containing u and the bond $\{u, v\}$. With these definitions, T is the bond-disjoint union of the R_e, as e ranges over the backbone bonds of T. The requirement that $T \in \mathcal{T}^1_{0,x}$ ensures that when e_n is its last edge, the tree R_{e_n} includes the vertex x which is the entire rib R_x, so nothing is omitted from T in the union of all R_e. Thus, we have

$$T = \bigsqcup_{e \in \Gamma_{0,x}} R_e, \qquad |T| = \sum_{e \in \Gamma_{0,x}} |R_e| \tag{6.151}$$

For an illustration, see Figure 4.

The verification of Assumption I is a small variant of the skeleton inequality arguments of [15, 45, 86], as follows.

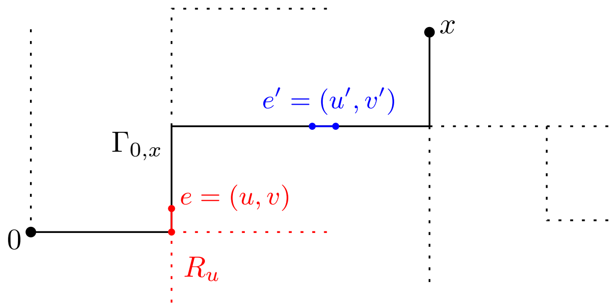

Figure 4: An example of a tree $T \in \mathcal{T}^1_{0,x}$ with its backbone $\Gamma_{0,x}$ in black, and its ribs R_y ($y \in \Gamma_{0,x}$) in dotted lines. The tree R_e is coloured red. The tree $R_{e'}$ is coloured blue; in this case $R_{u'}$ is the single vertex $\{u'\}$.

Proof of (**I**.1). As mentioned above, we assume that $x \neq 0$. For $0 \leq \beta' < \beta < \beta_c$, let $p = p(\beta)$ and $p' = p(\beta')$. The identity

$$\prod_{i=1}^n a_i - \prod_{i=1}^n b_i = \sum_{i=1}^n \Big(\prod_{\ell<i} b_\ell\Big)\Big(\prod_{j>i} a_j\Big)(a_i - b_i) \tag{6.152}$$

holds for any real numbers a_i, b_i (an empty product equals 1). In particular, by (6.151),

$$(pJ)^T - (p'J)^T = \sum_{e\in\Gamma_{0,x}} (p'J)^{\sqcup_{f<e} R_f} (pJ)^{\sqcup_{g>e} R_g} \left((pJ)^{R_e} - (p'J)^{R_e}\right). \tag{6.153}$$

Given a (directed) backbone bond $e = (u, v)$, we define trees $T_e^- \in \mathcal{T}^1_{0,u}$ and $T_e^+ \in \mathcal{T}^1_{v,x}$ by

$$T_e^- = \bigsqcup_{f<e} R_f, \qquad T_e^+ = \bigsqcup_{g>e} R_g. \tag{6.154}$$

An example of this decomposition is illustrated in Figure 5. With this notation, we have

$$(pJ)^T - (p'J)^T = \sum_{e\in\Gamma_{0,x}} (p'J)^{T_e^-} \left((pJ)^{R_e} - (p'J)^{R_e}\right) (pJ)^{T_e^+}. \tag{6.155}$$

After summation over all $T \in \mathcal{T}^1_{0,x}$, this gives

$$G_\beta(x) - G_{\beta'}(x) = \sum_{T\in\mathcal{T}^1_{0,x}} \sum_{e\in\Gamma_{0,x}} (p'J)^{T_e^-} \left((pJ)^{R_e} - (p'J)^{R_e}\right) (pJ)^{T_e^+}. \tag{6.156}$$

Fix $e = (u, v)$ with $u < v$. In the summation on the right-hand side of (6.156) there are avoidance constraints between T_e^-, the rib R_u, the directed edge (u, v), and $T^+_{(u,v)}$. More precisely, it is required that they are bond disjoint and that their union forms a lattice tree. We write $\mathbb{1}_C(T^-, R, (u, v), T^+)$ for the indicator that these constraints hold. We can then reorganise the summation in (6.156) as

$$\begin{aligned} &G_\beta(x) - G_{\beta'}(x) \\ &\quad= \sum_{u,v\in\mathbb{Z}^d} \sum_{\substack{T^-\in\mathcal{T}^1_{0,u} \\ R\in\mathcal{T}_{u,u} \\ T^+\in\mathcal{T}^1_{v,x}}} (p'J)^{T^-} J_{v-u} \left(p(pJ)^R - p'(p'J)^R\right) (pJ)^{T^+} \mathbb{1}_C(T^-, R, (u, v), T^+). \end{aligned} \tag{6.157}$$

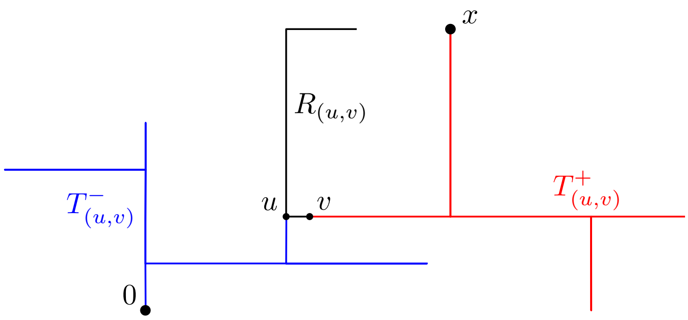

Figure 5: An example of a tree $T \in \mathcal{T}^1_{0,x}$, backbone bond $e = (u, v)$, and the decomposition into T_e^-, R_e, T_e^+. If $u = 0$ then $T^-_{(u,v)}$ is empty, and if $v = x$ then $T^+_{(u,v)}$ is empty.

To prove (**I.1**), we simply bound the indicator function by 1 and obtain

$$\begin{aligned} G_\beta(x) - G_{\beta'}(x) &\le (G_{\beta'} * (pg_p - p'g_{p'})J * G_\beta)(x) \\ &= (\beta - \beta')(G_{\beta'} * J * G_\beta)(x). \end{aligned} \tag{6.158}$$

This completes the proof of (**I.1**). □

Proof of (**I.2**). We divide (6.158) by $p - p'$ and then take the limit $p' \to p$. By the chain rule and the fact that $\frac{\mathrm{d}}{\mathrm{d}p} p(pJ)^R = (1 + |R|)(pJ)^R$, we obtain

$$\frac{\partial G_\beta(x)}{\partial \beta} \hat{\chi}(p) = \sum_{u,v \in \mathbb{Z}^d} \sum_{\substack{T^- \in \mathcal{T}^1_{0,u} \\ R \in \mathcal{T}_{u,u} \\ T^+ \in \mathcal{T}^1_{v,x}}} (pJ)^{T^-} J_{v-u} (1 + |R|)(pJ)^R (pJ)^{T^+} \mathbb{1}_C(T^+, T^-, R, (u, v)). \tag{6.159}$$

Since the number of vertices in R is $1 + |R|$ (i.e., $1 + |R| = \sum_{y \in \mathbb{Z}^d} \mathbb{1}_{y \in R}$), this gives

$$\frac{\partial G_\beta(x)}{\partial \beta} = \frac{1}{\hat{\chi}(p)} \sum_{y \in \mathbb{Z}^d} \sum_{u,v \in \mathbb{Z}^d} \sum_{\substack{T^- \in \mathcal{T}^1_{0,u} \\ R \in \mathcal{T}_{u,y} \\ T^+ \in \mathcal{T}^1_{v,x}}} (pJ)^{T^-} J_{v-u} (pJ)^R (pJ)^{T^+} \mathbb{1}_C(T^+, T^-, R, (u, v)). \tag{6.160}$$

We then write $\mathbb{1}_C$ as $1 - (1 - \mathbb{1}_C)$. As in the derivation of (**I.1**) above, the contribution involving "1" in (6.160) is equal to

$$\frac{1}{\hat{\chi}(p)} \hat{\chi}(p)(G_\beta * J * G_\beta)(x) = (G_\beta * J * G_\beta)(x), \tag{6.161}$$

which is the leading term in (**I.2**).

The factor $1 - \mathbb{1}_C$ enforces certain intersections to occur. There are three possible (and non-exclusive) types of intersections: T^- and R intersect, R and T^+ intersect, or T^- and T^+ intersect. See Figure 6 for an illustration. Therefore, the contribution coming from the $1 - \mathbb{1}_C$ term in (6.160) is bounded by (I) + (II) + (III), where

$$(\mathrm{I}) = \frac{1}{\hat{\chi}(p)} \sum_{y\in\mathbb{Z}^d} \sum_{u,v\in\mathbb{Z}^d} \sum_{\substack{w\in\mathbb{Z}^d \\ w\neq u}} \sum_{\substack{T^-\in\mathcal{T}^1_{0,u}:\, T^-\ni w \\ R\in\mathcal{T}_{u,y}:\, R\ni w \\ T^+\in\mathcal{T}^1_{v,x}}} (pJ)^{T^-} J_{v-u} (pJ)^R (pJ)^{T^+}, \tag{6.162}$$

$$(\mathrm{II}) = \frac{1}{\hat{\chi}(p)} \sum_{y\in\mathbb{Z}^d} \sum_{u,v\in\mathbb{Z}^d} \sum_{w\in\mathbb{Z}^d} \sum_{\substack{T^-\in\mathcal{T}^1_{0,u} \\ R\in\mathcal{T}_{u,y}:\, R\ni w \\ T^+\in\mathcal{T}^1_{v,x}:\, T^+\ni w}} (pJ)^{T^-} J_{v-u} (pJ)^R (pJ)^{T^+}, \tag{6.163}$$

$$(\mathrm{III}) = \frac{1}{\hat{\chi}(p)} \sum_{y\in\mathbb{Z}^d} \sum_{u,v\in\mathbb{Z}^d} \sum_{w\in\mathbb{Z}^d} \sum_{\substack{T^-\in\mathcal{T}^1_{0,u}:\, w\in T^- \\ R\in\mathcal{T}_{u,y} \\ T^+\in\mathcal{T}^1_{v,x}:\, w\in T^+}} (pJ)^{T^-} J_{v-u} (pJ)^R (pJ)^{T^+}. \tag{6.164}$$

The restriction $w \neq u$ in (I) occurs because the intersection between T^- and R must be in addition to the common point u that both of these trees are required to contain simply by definition.

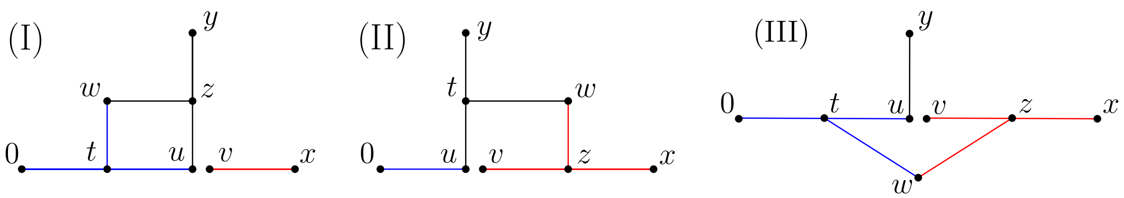

Figure 6: An illustration of the error terms (I), (II), (III). For clarity, branches not contributing to intersections are not shown.

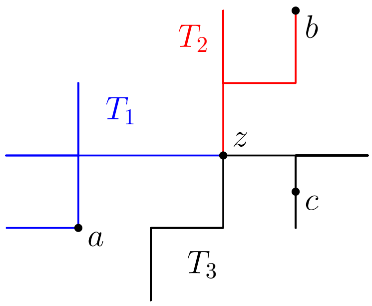

Figure 7: An example of a decomposition for the tree graph inequality.

To estimate these terms, we use a tree-graph inequality. If a tree T contains three points a, b, c, then there exists a unique point z such that the three points are bond-disjointly connected to z in T. We can therefore divide T into three subtrees: $T_1 \in \mathcal{T}^1_{a,z}$, $T_2 \in \mathcal{T}^1_{b,z}$ and $T_3 \in \mathcal{T}_{c,z}$. An example illustrating this decomposition is given in Figure 7. After removing the avoidance constraint between the subtrees, we find that the sums over

T^- and T^+ in (6.164) can be bounded above as

$$\sum_{T^- \in \mathcal{T}^1_{0,u} : w \in T^-} (pJ)^{T^-} \le \sum_{z\in\mathbb{Z}^d} \sum_{\substack{T_1 \in \mathcal{T}^1_{0,z} \\ T_2 \in \mathcal{T}^1_{w,z} \\ T_3 \in \mathcal{T}^1_{z,u}}} (pJ)^{T_1}(pJ)^{T_2}(pJ)^{T_3} \le \sum_{z\in\mathbb{Z}^d} G_\beta(z) G_\beta(z-w) G_\beta(u-z), \tag{6.165}$$

$$\sum_{T^+ \in \mathcal{T}^1_{v,x} : w \in T^+} (pJ)^{T^+} \le \sum_{t\in\mathbb{Z}^d} \sum_{\substack{T_1 \in \mathcal{T}^1_{v,t} \\ T_2 \in \mathcal{T}^1_{w,t} \\ T_3 \in \mathcal{T}^1_{t,x}}} (pJ)^{T_1}(pJ)^{T_2}(pJ)^{T_3} \le \sum_{t\in\mathbb{Z}^d} G_\beta(t-v) G_\beta(t-w) G_\beta(x-t). \tag{6.166}$$

The factor $1/\hat{\chi}(p)$ in (III) is cancelled by the sums over R and y. Inserting the above bounds into (III) yields

$$\text{(III)} \le \Big(G_\beta * \Big[[G_\beta * J * G_\beta] \cdot [G_\beta * G_\beta]\Big] * G_\beta\Big)(x). \tag{6.167}$$

An almost identical calculation for the terms (I) and (II) shows that the sum (I)+(II)+(III) is bounded by $(G_\beta * H_\beta * G_\beta)(x)$, where

$$H_\beta = \Big([G_\beta \cdot [G_\beta * G_\beta * G_\beta] - \delta_0] * J + [J * G_\beta] \cdot [G_\beta * G_\beta * G_\beta] + [G_\beta * J * G_\beta] \cdot [G_\beta * G_\beta]\Big). \tag{6.168}$$

A detail for the bound on (I) is that the square in its depiction in Figure 6 cannot have all four of its vertices be identical in the intersection of T^- and R, since the sum is constrained by $w \neq u$. The subtracted delta function eliminates the main contribution to this scenario. This concludes the proof. □

6.8 Verification of Assumption II for each application

We now verify Assumption II for each of the models discussed in Sections 6.1–6.7. Recall from (1.23) the definition

$$E(\beta) = \sup_{0\le t\le\beta} \left(\|H_t\|_1 + \frac{\||x|_2^2 \cdot H_t\|_1}{\xi(t)^2} \right), \tag{6.169}$$

where H_β is the model-dependent function that occurs in Assumption I. Assumption II asserts that

$$E(\beta(\boldsymbol{\delta})) < \boldsymbol{\delta}, \tag{6.170}$$

where $\boldsymbol{\delta}$ is given by Theorem 1.5. By definition,

$$E(\beta) \le E_0(\beta) + E_2(\beta) \tag{6.171}$$

with

$$E_0(\beta) = \sup_{0\le t\le\beta} \|H_t\|_1, \quad E_2(\beta) = \sup_{0\le t\le\beta} \frac{\||x|_2^2 \cdot H_t\|_1}{\xi(t)^2}. \tag{6.172}$$

The function H_β is identified for self-avoiding walk in (6.12), for continuous-time self-avoiding walk in (6.32), for percolation in (6.40), for $|\varphi|^4$ in (6.79), for the Ising model in (6.115), for the XY model in (6.126), and for lattice trees in (6.168). Explicitly, with $K_\beta = G_\beta + \beta F_\beta$,

$$H_\beta^{\mathrm{SAW}} = \lambda \delta_0 (G_\beta * F_\beta), \tag{6.173}$$

$$H_\beta^{\mathrm{CTWSAW}} = 2\lambda \delta_0 (G_\beta * F_\beta), \tag{6.174}$$

$$H_\beta^{|\varphi|^4} = 6\lambda \delta_0 (F_\beta * G_\beta), \tag{6.175}$$

$$H_\beta^{\mathrm{Ising}} = 3\big[(\delta_0 + \beta J) * (\delta_0 + \beta J)\big](K_\beta * J * K_\beta)(0), \tag{6.176}$$

$$H_\beta^{\mathrm{XY}} = 12\big[(\delta_0 + \beta J) * (\delta_0 + \beta J)\big](K_\beta * J * K_\beta)(0), \tag{6.177}$$

$$H_\beta^{\mathrm{perc}} = G_\beta \cdot (G_\beta * F_\beta), \tag{6.178}$$

$$H_\beta^{\mathrm{LT}} = (G_\beta \cdot [G_\beta * G_\beta * G_\beta] - \delta_0) * J + F_\beta \cdot (G_\beta * G_\beta * G_\beta) + (G_\beta * F_\beta) \cdot (G_\beta * G_\beta). \tag{6.179}$$

By (4.4) and the fact that $\boldsymbol{\delta} < \frac{1}{2}$, we have $\beta(\boldsymbol{\delta}) < 2$, so factors of β are not problematic. In all cases, $E_0(\beta) = \|H_\beta\|_1$, but we do not know that the ratio in the definition of $E_2(\beta)$ in (6.172) is monotone in t, so we cannot ignore the supremum for E_2.

Since we have verified Assumption I, Theorem 1.5 tells us that, for every $x \in \mathbb{Z}^d$ and every $\beta \in [0, \beta(\boldsymbol{\delta})]$, we have the bound

$$G_\beta(x) \le \delta_0(x) + \frac{\mathbf{C}}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|} \right)^{d-2-\varepsilon} \exp\left(-\mathbf{c} \frac{|x|}{\xi(\beta)} \right). \tag{6.180}$$

Under Assumption I, we also have the bound (3.1), which states that for every $x \in \mathbb{Z}^d$ and every $\beta \in [0, \beta(\boldsymbol{\delta})]$,

$$F_\beta(x) \le \frac{1}{2} \frac{\mathbf{C}}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|} \right)^{d-2-\varepsilon} \exp\left(-\mathbf{c} \frac{|x|}{\xi(\beta)} \right). \tag{6.181}$$

The constants $\mathbf{c}, \mathbf{C}, \boldsymbol{\delta}$ depend only on c_0, ε, d. In particular, they do not depend on the admissible kernel J nor on any particular model in our applications. For nearest-neighbour models (for which $\sigma_J = 1$) with an inherent small parameter λ, we will prove that $E(\beta(\boldsymbol{\delta})) < O(\lambda)$. This establishes (6.170) once we take λ sufficiently small. For spread-out models (which have large σ_J), we will instead prove that (6.180) implies that $E(\beta(\boldsymbol{\delta})) < O(\sigma_J^{-d})$. This establishes (6.170) once we take σ_J sufficiently large.

The L^1 norm of H_β for self-avoiding walk and $|\varphi|^4$ is an open bubble diagram, and for Ising and XY something quite similar is true. The L^1 norm of H_β^{perc} is the open triangle diagram, and the L^1 norm of H_β^{LT} involves the open square diagram. The early mathematical development of the theory of high-dimensional critical phenomena was based on the observation that when the critical exponent η is equal to its mean-field value $\eta = 0$, the critical bubble, triangle and square diagrams are respectively finite above dimensions 4, 6 and 8. This was an important step in the understanding, in the 1980s, that 4, 6 and 8 are the upper critical dimensions for the models [1,6,15,33]. The finiteness of the diagrams above the upper critical dimension was first proved in various contexts for self-avoiding walk [19, 47], spin systems [1, 33], percolation [44], and lattice trees [45]. As outlined in Section 1.3.2, our contribution is to give a new and unified approach to the subject.

6.8.1 Self-avoiding walk and $|\varphi|^4$

Fix any $\beta \in [0, \beta(\boldsymbol{\delta})]$. By (6.173)–(6.175), it suffices to consider

$$H_\beta(x) = \lambda (G_\beta * F_\beta)(0) \delta_0(x), \tag{6.182}$$

since the factors 2 or 6 are of no significance. In this case, $E_2(\beta) = 0$ and $E(\beta) = \lambda(G_\beta * F_\beta)(0)$.

Let $\bar{G}_\beta = G_\beta - \delta_0$. Then

$$G_\beta * F_\beta = F_\beta + \bar{G}_\beta * F_\beta. \tag{6.183}$$

By (6.181), $F_\beta(0) \lesssim \sigma_J^{-d}$. Also, by (6.180) and (6.181),

$$(\bar{G}_\beta * F_\beta)(0) \lesssim \frac{1}{\sigma_J^d}\frac{1}{\sigma_J^d}\sum_{x\in\mathbb{Z}^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{2d-4-2\varepsilon}. \tag{6.184}$$

For dimensions $d > 4+2\varepsilon$, the sum is bounded for the nearest-neighbour J (for which $\sigma_J = 1$), so $E(\beta) = \lambda(G_\beta * F_\beta)(0) = O(\lambda)$. This bound is uniform in β, and in particular it holds at $\beta(\boldsymbol{\delta})$.

For the spread-out case, by Riemann sum approximation, if $d > 4+2\varepsilon$ then

$$(\bar{G}_\beta * F_\beta)(0) \lesssim \frac{1}{\sigma_J^d}\int_{\mathbb{R}^d}\frac{1}{1\vee |u|^{2d-4-2\varepsilon}}\mathrm{d}u \lesssim \frac{1}{\sigma_J^d}, \tag{6.185}$$

so $E(\beta) = O(\sigma_J^{-d})$. Again, the bound holds at $\beta(\boldsymbol{\delta})$.

6.8.2 Ising and XY models

Let $t \le \beta(\boldsymbol{\delta}) \le 2$. By (6.176)–(6.177), it suffices to consider $H_t = \big[(\delta_0 + tJ) * (\delta_0 + tJ)\big](K_t * J * K_t)(0)$. Its zeroth and second moments satisfy

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d} H_t(x) &= \sum_{x\in\mathbb{Z}^d}\big[\delta_0(x) + 2tJ_x + t^2(J*J)(x)\big](K_t * J * K_t)(0)\\ &\le \big[1+4+4\big](K_t * J * K_t)(0), &(6.186)\\ \frac{1}{\xi(t)^2}\sum_{x\in\mathbb{Z}^d}|x|_2^2 H_t(x) &= \frac{1}{\xi(t)^2}\sum_{x\in\mathbb{Z}^d}|x|_2^2\big[\delta_0(x) + 2tJ_x + t^2(J*J)(x)\big](K_t * J * K_t)(0)\\ &\le \frac{1}{\xi(t)^2}\big[1+4\sigma_J^2+8\sigma_J^2\big](K_t * J * K_t)(0). &(6.187)\end{aligned}$$

By Proposition 4.1, $\xi(t) \ge \xi(0) = \sigma_J$. Also, by definition of K_t,

$$(K_t * J * K_t)(0) = (G_t * F_t)(0) + 2t(F_t * F_t)(0) + t^2(F_t * J * F_t)(0). \tag{6.188}$$

It therefore suffices to prove that the right-hand side of (6.188) is bounded by a multiple of σ_J^{-d}. The first two terms are indeed bounded by a multiple of σ_J^{-d}, exactly as in Section 6.8.1.

For the last term, it suffices to show that

$$(J * F_t)(x) \lesssim \frac{1}{\sigma_J^d}\left(\frac{\sigma_J}{\sigma_J \vee |x|}\right)^{d-2-\varepsilon}, \tag{6.189}$$

since then the computations of Section 6.8.1 can be applied. By (6.181),

$$(J * F_t)(x) \lesssim \frac{1}{\sigma_J^d}\sum_{y\in\mathbb{Z}^d} J_y\left(\frac{\sigma_J}{\sigma_J \vee |x-y|}\right)^{d-2-\varepsilon}. \tag{6.190}$$

We divide the sum according to whether (i) $|y| \le |x|/2$, or (ii) $|y| \ge |x|/2$. In case (i), up to a constant we may replace $|x-y|$ by $|x|$ and then bound the resulting sum over y by

1, so this contribution does satisfy the bound (6.189). In case (ii), since J has finite range R_J, the summand is nonzero only if $|x| \le 2R_J \le 2c_0^{-1}\sigma_J$, where $c_0 \in (0,1]$ is given by Definition 1.1. The contribution to the sum (6.190) from case (ii) is therefore bounded by

$$\frac{1}{\sigma_J^d} \cdot 1 = \frac{1}{\sigma_J^d} \left(\frac{2c_0^{-1}\sigma_J}{2c_0^{-1}\sigma_J \vee |x|} \right)^{d-2-\epsilon} \le \frac{1}{\sigma_J^d} \left(\frac{2c_0^{-1}\sigma_J}{\sigma_J \vee |x|} \right)^{d-2-\epsilon}. \tag{6.191}$$

This completes the proof.

6.8.3 Bernoulli percolation

For $E_0(t)$, we use

$$\|H_t\|_1 = (G_t * G_t * F_t)(0) = F_t(0) + 2(\bar{G}_t * F)(0) + (\bar{G}_t * \bar{G}_t * F_t)(0). \tag{6.192}$$

The first term on the right-hand side is $O(\sigma^{-d})$ by (6.181), and the second term is $O(\sigma_J^{-d})$ by (6.185). For the third term, we again use Riemann sum approximation to see that, for $d > 6 + 3\varepsilon$,

$$(\bar{G}_t * \bar{G}_t * F_t)(0) \lesssim \frac{1}{\sigma_J^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{1}{1 \vee |u|} \frac{1}{1 \vee |v|} \frac{1}{1 \vee |u-v|} \right)^{d-2-\varepsilon} \mathrm{d}u \, \mathrm{d}v \lesssim \frac{1}{\sigma_J^d}. \tag{6.193}$$

This shows that $E_0(t) = O(\sigma_J^{-d})$.

We did not use the exponential decay for $E_0(t)$, but for $E_2(t)$ we will, as follows. Let

$$W_t(x) = \frac{|x|^2}{\xi(t)^2} G_t(x) = \frac{|x|^2}{\xi(t)^2} \bar{G}_t(x). \tag{6.194}$$

Then

$$E_2(t) = \sum_{x \in \mathbb{Z}^d} W_t(x)(G_t * F_t)(x) = (W_t * G_t * F_t)(0). \tag{6.195}$$

Since $s^2 e^{-\mathbf{c}s}$ is bounded for $s \in [0,1]$, it follows from (6.180) that

$$W_t(x) \lesssim \frac{1}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|} \right)^{d-2-\varepsilon}. \tag{6.196}$$

The factors in $(W_t * G_t * F_t)(0)$ therefore obey the same estimates as the factors in (6.193), so we also have $E_2(t) = O(\sigma^{-d})$.

6.8.4 Lattice trees

By definition of H_t in (6.179), and with $\bar{G}_t = G_t - \delta_0$,

$$\begin{aligned} E_0(t) &= (G_t^{*4}(0) - 1) + 2(F_t * G_t^{*3}(0)) \\ &= 3\bar{G}_t(0) + 6\bar{G}_t^{*2}(0) + 3\bar{G}_t^{*3}(0) + \bar{G}_t^{*4}(0) \\ &\quad + 2F_t(0) + 6(F_t * \bar{G}_t)(0) + 6(F_t * \bar{G}_t^{*2})(0) + 2(F_t * \bar{G}_t^{*3})(0). \end{aligned} \tag{6.197}$$

A bound $O(\sigma_J^{-d})$ was obtained above for the first three terms in each of second and third lines of (6.197), assuming $d > 6 + 3\varepsilon$. The last terms on each of those lines obey the same upper bound, which is a Riemann sum approximation to

$$\frac{1}{\sigma_J^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{1}{1 \vee |u|} \frac{1}{1 \vee |v-u|} \frac{1}{1 \vee |w-v|} \frac{1}{1 \vee |w|} \right)^{d-2-\varepsilon} \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w. \tag{6.198}$$

The integral converges for $d > 8 + 4\varepsilon$, so $E_0(t) = O(\sigma_J^{-d})$.

The estimate for $E_2(t)$ follows similarly, again using (6.196). We present the details only for the contribution from the first term in (6.179). For this term, we seek a bound $O(\sigma_J^{-d})$ on

$$\frac{1}{\xi(t)^2} \sum_{x,y \in \mathbb{Z}^d} |x|_2^2 G_\beta(y)[G_\beta * G_\beta * G_\beta](y) J(x-y). \tag{6.199}$$

We can replace $|x|_2^2$ by $|y|_2^2 + |x-y|_2^2$, since the cross term gives zero contribution to the sum by the $\mathbb{Z}^d$-symmetry. For $|y|_2^2$, we use $\sum_{x \in \mathbb{Z}^d} J(x-y) = 1$ and bound $|y|_2^2 G_t(y)/\xi(t)$ using (6.196). This brings us to an expression that we have treated already for $E_0(t)$. For $|x-y|_2^2$, we use

$$\frac{1}{\xi(t)^2} \sum_{x \in \mathbb{Z}^d} |x-y|_2^2 J(x-y) = \frac{\sigma_J^2}{\xi(t)^2} = \frac{\xi(0)^2}{\xi(t)^2} \le 1, \tag{6.200}$$

due to the monotonicity of ξ proved in Proposition 4.1. The remaining sum over y has been shown in the bound on $E_0(t)$ to be $O(\sigma_J^{-d})$. This completes the proof.

A Random walk theorems

In this appendix, we prove the averaged and pointwise anti-concentration and Green function estimates stated in Theorem 2.10 and Proposition 2.1, respectively. We prove Theorem 2.10 in Section A.2, and use it to prove Proposition 2.1 in Section A.3. An important preliminary ingredient is the anti-concentration estimate proved in Section A.1.

A.1 Esseen's anti-concentration estimate

A fundamental ingredient in our proof of Theorem 2.10 is the following result from Esseen [29]. For its statement, for any $M \ge 0$ and $z \in \mathbb{R}^d$, we introduce the set

$$D(z; M) := \{x \in \mathbb{R}^d : \min_{1 \le i \le d} |x_i - z_i| \le M\}. \tag{A.1}$$

Recall from (2.27) that for $y \in \mathbb{R}^d$ we set $B_M(y) = \{z \in \mathbb{R}^d : |z-y| \le M\}$, and we also write $B_M = B_M(0)$. A random variable $Y = (Y^{(1)}, \dots, Y^{(d)})$ on $\mathbb{R}^d$ is called *sign invariant* if the 2^d random vectors $(\pm Y^{(1)}, \dots, \pm Y^{(d)})$ have the same distribution.

Theorem A.1 ([29, Theorem 3])**.** *Let $d \ge 1$. There exists $C = C(d) > 0$ such that the following holds. Let $n \ge 1$, and let $Y_1, \dots, Y_n$ be mutually independent and sign-invariant random variables on $\mathbb{R}^d$. Set $S_n := Y_1 + \ldots + Y_n$. Then, for every $M \ge 1$,*

$$\sup_{y \in \mathbb{R}^d} \mathbb{P}[S_n \in B_M(y)] \le \frac{C}{\Big(\sum_{k=1}^{n} \big(1 - \sup_{z \in \mathbb{R}^d} \mathbb{P}[Y_k \in D(z; M)]\big)\Big)^{d/2}}. \tag{A.2}$$

As stated, Theorem A.1 is not well-suited for regular random walks. Indeed, to use it directly would require showing that for a generic $(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}})$-regular random walk X there exists $M > 0$ such that $\mathbb{P}[X_1 \notin D(z; M)] \ge c_0(d, \mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}) > 0$, uniformly in z. This property is false in general: it does not hold for the simple random walk on $\mathbb{Z}^d$ with $z = 0$ and $M = 1$. Nevertheless, as we will see below in (A.27), $(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}})$-regular random walks satisfy a closely related bound of the form $\mathbb{P}[X_1 \notin B_M] \ge c_1(d, \mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}) > 0$, for an M depending on the variance σ^2 of X_1. The purpose of the next result is to adapt Theorem A.1 to apply under this alternative (weaker) assumption.

We say that a random variable Y on $\mathbb{R}^d$ is *symmetric* if the $2^d d!$ random vectors $(\pm Y^{(\pi(1))}, \dots, \pm Y^{(\pi(d))})$, where π ranges over all permutations of $\{1, \dots, d\}$, have the same distribution. This is a stronger condition than being sign invariant. A random walk $X = (X_k)_{k \geq 0}$ on $\mathbb{R}^d$ is called *symmetric* if X_1 (and therefore $X_{i+1} - X_i$ for every $i \geq 2$) is symmetric.

Corollary A.2. *Let $d \geq 1$ and $a_0 > 0$. Let $X = (X_k)_{k \geq 0}$ be a symmetric random walk started at 0. Let $M \geq 1$ and assume that*

$$\mathbb{P}[X_1 \notin B_M] \geq a_0. \tag{A.3}$$

Then there exists $C = C(a_0, d) > 0$ such that, for every $n \geq 1$,

$$\sup_{y \in \mathbb{R}^d} \mathbb{P}[X_n \in B_M(y)] \leq \frac{C}{n^{d/2}}. \tag{A.4}$$

Proof. By Theorem A.1, it would be sufficient to show that there exists $c > 0$ such that

$$\sup_{z \in \mathbb{R}^d} \mathbb{P}[X_1 \in D(z; M)] \leq 1 - c. \tag{A.5}$$

However, with our hypotheses, this is not necessarily true. Instead, we prove that there exists $c_1 = c_1(a_0, d) > 0$ such that

$$\sup_{z \in \mathbb{R}^d} \mathbb{P}[X_d \in D(z; M)] \leq 1 - c_1, \tag{A.6}$$

as follows.

For $1 \leq i \leq d$, let $\mathcal{E}_i := \{x \in \mathbb{R}^d : x_i > M,\ x_j \geq 0 \text{ if } j \neq i\}$. Since X_1 is symmetric, we have

$$\mathbb{P}[X_1 \in \mathcal{E}_i] \geq \frac{a_0}{2d \cdot 2^{d-1}} =: \varepsilon. \tag{A.7}$$

Again by symmetry,

$$\max_{z \in \mathbb{R}^d} \mathbb{P}[X_d \in D(z; M)] = \max_{z \in \mathbb{R}^d \colon \max_{1 \leq i \leq d} z_i \leq 0} \mathbb{P}[X_d \in D(z; M)]. \tag{A.8}$$

Finally, if $z \in \mathbb{R}^d$ with $z_i \leq 0$ for all i, then

$$\mathbb{P}[X_d \notin D(z; M)] \geq \mathbb{P}\Big[\bigcap_{i=1}^{d} \{X_i - X_{i-1} \in \mathcal{E}_i\}\Big] \geq \varepsilon^d =: c_1. \tag{A.9}$$

This proves (A.6).

We now prove (A.4). If $n \leq d - 1$, we just bound the probability by 1. Assume that $n \geq d$. In this case, we set $\ell := \lfloor n/d \rfloor$, and for $1 \leq k \leq \ell$ define

$$Y_k := \sum_{j=(k-1)d+1}^{kd} (X_j - X_{j-1}), \qquad Y_{\ell+1} := X_n - (Y_1 + \ldots + Y_\ell), \tag{A.10}$$

so that

$$X_n = Y_1 + \ldots + Y_{\ell+1}. \tag{A.11}$$

The random variables $Y_1, \dots, Y_{\ell+1}$ are mutually independent and sign invariant. Also, for every $1 \le k \le \ell$, Y_k has the same law as X_d. We apply Theorem A.1 to the random variables $Y_1, \dots, Y_{\ell+1}$ and find $C_1 = C_1(d) > 0$ such that for every $n \ge d$,

$$\begin{aligned}\sup_{y\in\mathbb{R}^d} \mathbb{P}[X_n \in B_M(y)] &\le \frac{C_1}{\Big((1-\mathbb{P}[Y_{\ell+1}\in D(z;M)]) + \sum_{k=1}^{\ell}(1-\sup_{z\in\mathbb{R}^d}\mathbb{P}[Y_k \in D(z;M)])\Big)^{d/2}} \\ &\le \frac{C_1}{\Big(\ell(1-\sup_{z\in\mathbb{R}^d}\mathbb{P}[X_d\in D(z;M)])\Big)^{d/2}} \le \frac{C_2}{n^{d/2}}, \end{aligned} \tag{A.12}$$

where in the second line we used (A.6), and where $C_2 = C_2(a_0, d) > 0$. As a result, for every $n \ge 1$,

$$\sup_{y\in\mathbb{R}^d} \mathbb{P}[X_n \in B_M(y)] \le \frac{C}{n^{d/2}}, \tag{A.13}$$

where $C := d^{d/2} \vee C_2$. This concludes the proof. □

A.2 Regular random walk: proof of Theorem 2.10

We now prove Theorem 2.10. Recall from Definition 2.6 that a $(\mathsf{c}_{\rm reg}, \mathsf{C}_{\rm reg})$-*regular* random walk on $\mathbb{R}^d$ has $X_0 = 0$, variance $\sigma^2 = \mathbb{E}[|X_1|_2^2] < \infty$, is invariant under permutation of coordinates and/or replacement of a coordinate by its negative, and its moment generating function $M(t) = \mathbb{E}[e^{t(\mathsf{e}_1\cdot X_1)/\sigma}]$ obeys

$$M(\mathsf{c}_{\rm reg}) \le \mathsf{C}_{\rm reg}. \tag{A.14}$$

It follows from the elementary inequality (4.56) that, for $0 \le t \le \mathsf{c}_{\rm reg}$,

$$M(t) \le 1 + \frac{t^2}{2d} + \frac{t^4}{\mathsf{c}_{\rm reg}^4} M(\mathsf{c}_{\rm reg}) \le 1 + \frac{t^2}{4}\left(\frac{2}{d} + \frac{4t^2\mathsf{C}_{\rm reg}}{\mathsf{c}_{\rm reg}^4}\right). \tag{A.15}$$

Therefore, for $d > 2$, we can choose

$$t_0 = \frac{\mathsf{c}_{\rm reg}^2}{2}\sqrt{\frac{d-2}{d\mathsf{C}_{\rm reg}}} \wedge \mathsf{c}_{\rm reg} \wedge 1 \in (0,1] \tag{A.16}$$

to achieve

$$M(t) \le 1 + \frac{t^2}{4} \quad \text{for all} \quad t \in [0, t_0]. \tag{A.17}$$

We prove the anti-concentration part of Theorem 2.10, in the next proposition.

Proposition A.3. (Anti-concentration inequality). *Let $d > 2$. There is a constant $\mathsf{C}_{\rm ac}$ (depending on $d, \mathsf{c}_{\rm reg}, \mathsf{C}_{\rm reg}$) such that for every $m \ge 1$, every $y \in \mathbb{R}^d$, and every $\tau \in [0, t_0]$,*

$$\mathbb{P}[X_m \in B_\sigma(y)] \le \frac{\mathsf{C}_{\rm ac}}{m^{d/2}} e^{m\tau^2/8} e^{-\tau|y|/2\sigma}. \tag{A.18}$$

We will apply two special cases of (A.18), obtained from two choices of τ. The first is

$$\mathbb{P}[X_m \in B_\sigma(y)] \le \frac{\mathsf{C}_{\rm ac} e^{t_0^2/8}}{m^{d/2}} \exp\left(-t_0 \frac{|y|}{2\sigma\sqrt{m}}\right) \qquad (\tau = t_0/\sqrt{m}). \tag{A.19}$$

The anti-concentration bound (2.29) of Theorem 2.10 follows immediately from (A.19), once we require that

$$\mathsf{c}_{\mathrm{RW}} \le \frac{t_0}{2}, \qquad \mathsf{C}_{\mathrm{RW}} \ge \mathsf{C}_{\mathrm{ac}} e^{t_0^2/8}. \tag{A.20}$$

For the second choice, given $s \in [0,1]$, let $\tau = s \wedge t_0$. Then $m^2\tau/8 \le m^2 s/8$ and $-\tau \le -t_0 s$ (since $0 \le s \le 1$ and $0 \le t_0 \le 1$), so (A.18) implies that

$$\mathbb{P}[X_m \in B_\sigma(y)] \le \frac{\mathsf{C}_{\mathrm{ac}}}{m^{d/2}} e^{ms^2/8} \exp\left(-\frac{t_0 s |y|}{2\sigma}\right) \qquad (\tau = s \wedge t_0). \tag{A.21}$$

The constants C_i below depend only on $\mathsf{c}_{\mathrm{reg}}$, $\mathsf{C}_{\mathrm{reg}}$, and d. Before proving Proposition A.3, we start with a weaker estimate.

Lemma A.4. *Let $d \ge 1$. There exists $C = C(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}, d) > 0$ such that, for every $m \ge 1$ and every $y \in \mathbb{R}^d$,*

$$\mathbb{P}[X_m \in B_\sigma(y)] \le \frac{C}{m^{d/2}}. \tag{A.22}$$

Proof. Let $\tilde{X}_1 = X_1/\sigma$ and $\delta > 0$. By the Cauchy–Schwarz inequality,

$$1 = \mathbb{E}[|\tilde{X}_1|^2] \le \delta^2 + \mathbb{E}[\mathbb{1}_{|\tilde{X}_1|>\delta}|\tilde{X}_1|^2] \le \delta^2 + \sqrt{\mathbb{P}[|\tilde{X}_1| > \delta]\mathbb{E}[|\tilde{X}_1|^4]}. \tag{A.23}$$

This gives

$$\frac{(1-\delta^2)^2}{\mathbb{E}[|\tilde{X}_1|^4]} \le \mathbb{P}[|X_1| > \delta\sigma]. \tag{A.24}$$

By symmetry, for any $t \ge 0$ we have

$$\mathbb{E}[e^{t|\tilde{X}_1|}] \le 2d\, \mathbb{E}[e^{t(\mathbf{e}_1 \cdot \tilde{X}_1)} \mathbb{1}_{|\tilde{X}_1| = (\mathbf{e}_1 \cdot \tilde{X}_1)}] \le 2d\, M(t), \tag{A.25}$$

so

$$\frac{\mathsf{c}_{\mathrm{reg}}^4}{4!} \mathbb{E}[|\tilde{X}_1|^4] \le \mathbb{E}[e^{\mathsf{c}_{\mathrm{reg}}|\tilde{X}_1|}] \le 2d\mathsf{C}_{\mathrm{reg}}. \tag{A.26}$$

Therefore, by choosing $\delta = \delta(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}, d)$ small enough, we obtain

$$\mathbb{P}[X_1 \notin B_{\delta\sigma}] \ge \delta. \tag{A.27}$$

The inequality (A.27) is a form of "non-lazyness" of the random walk.

Now, thanks to Corollary A.2 (applied to X, $a_0 = \delta$, and $M = \delta\sigma$), there exists $C_1 > 0$ which depends on $\mathsf{c}_{\mathrm{reg}}$, $\mathsf{C}_{\mathrm{reg}}$, and d, such that for every $m \ge 1$ and every $y \in \mathbb{R}^d$,

$$\mathbb{P}[X_m \in B_{\delta\sigma}(y)] \le \frac{C_1}{m^{d/2}}. \tag{A.28}$$

Then, after adding the contributions to the big box from small boxes, (A.28) gives the existence of $C_2 = C_2(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}, d) > 0$ such that, for every $m \ge 1$,

$$\sup_{y \in \mathbb{R}^d} \mathbb{P}[X_m \in B_\sigma(y)] \le \frac{C_2}{m^{d/2}}. \tag{A.29}$$

This concludes the proof. ☐

We are now in a position to prove Proposition A.3.

Proof of Proposition A.3. By Lemma A.4, there exists $C > 0$ such that for every $m \geq 1$ and every $y \in \mathbb{R}^d$,

$$\mathbb{P}[X_m \in B_\sigma(y)] \leq \frac{C}{m^{d/2}}. \tag{A.30}$$

To improve (A.30) with an exponential factor, we proceed as follows. Suppose that $X_m \in B_\sigma(y)$. If $|y| \leq \sigma$ then we can insert the exponential factor in the upper bound (A.30) at the cost of increasing C, so we assume now that $|y| > \sigma$.

Let $k = \lceil m/2 \rceil$. Either $|X_k| \leq (|y| - \sigma)/2$ or $|X_k| \geq (|y| - \sigma)/2$. In the former case, $|X_m - X_k| \geq (|y| - \sigma)/2$. Therefore,

$$\begin{aligned}\mathbb{P}[X_m \in B_\sigma(y)] \leq \mathbb{P}\Big[X_m \in B_\sigma(y) \,\Big|\, |X_m - X_k| \geq (|y| - \sigma)/2\Big]\mathbb{P}[|X_m - X_k| \geq (|y| - \sigma)/2] \\ + \mathbb{P}\Big[X_m \in B_\sigma(y) \,\Big|\, |X_k| \geq (|y| - \sigma)/2\Big]\mathbb{P}[|X_k| \geq (|y| - \sigma)/2]. \end{aligned} \tag{A.31}$$

For the conditional probabilities, we first observe that $(X_0, X_1, \ldots, X_m)$ has the same law as $(X_m - X_m, \ldots, X_m - X_0)$. With $k' = \lfloor m/2 \rfloor$, this gives

$$\mathbb{P}\Big[X_m \in B_\sigma(y) \,\Big|\, |X_m - X_k| \geq (|y| - \sigma)/2\Big] = \mathbb{P}\Big[X_m \in B_\sigma(y) \,\Big|\, |X_{k'}| \geq (|y| - \sigma)/2\Big]. \tag{A.32}$$

This allows the two terms in (A.31) to be bounded in the same way. By the Markov property and (A.30),

$$\mathbb{P}\Big[X_m \in B_\sigma(y) \,\Big|\, |X_{k'}| \geq (|y| - \sigma)/2\Big] \leq \frac{C}{(m - k')^{d/2}}. \tag{A.33}$$

With (A.30), and with $\ell = k = \lceil m/2 \rceil$ or $\ell = k' = m - k = \lfloor m/2 \rfloor$ giving the maximum of the two options on the right-hand side, this gives the existence of C_1 such that

$$\mathbb{P}[X_m \in B_\sigma(y)] \leq \frac{C_1}{m^{d/2}} \mathbb{P}[|X_\ell| \geq (|y| - \sigma)/2]. \tag{A.34}$$

Let Z_j denote the first coordinate of X_j. By symmetry, $\mathbb{E}[Z_1^2] = d^{-1}\sigma^2$ and

$$\mathbb{P}[|X_\ell| \geq (|y| - \sigma)/2] \leq (2d)\mathbb{P}[Z_\ell \geq (|y| - \sigma)/2]. \tag{A.35}$$

The moment generating function $M(t) = \mathbb{E}[\exp(tZ_1/\sigma)]$ is well defined for $|t| \leq \mathsf{c}_{\text{reg}}$. Let t_0 be chosen in (A.17) so that $M(t) \leq 1 + \frac{t^2}{4}$ for all $|t| \leq t_0$. By Markov's inequality, if $\tau \in [0, t_0]$ then

$$\mathbb{P}[Z_\ell \geq (|y| - \sigma)/2] \leq [M(\tau)]^{\lceil m/2 \rceil} \exp\left(-\frac{\tau}{2} \frac{|y| - \sigma}{\sigma}\right) \leq e^{t_0/2} \exp\left(\frac{\tau^2 m}{8} - \frac{\tau}{2} \frac{|y|}{\sigma}\right). \tag{A.36}$$

The combination of (A.34) and (A.36) gives the existence of C_{ac} and completes the proof. □

The following theorem is a restatement of the Green function estimate (2.30) of Theorem 2.10. In order to achieve the constant c_{RW} in (2.30), we now strengthen the demand in (A.20) to require that

$$\mathsf{c}_{\text{RW}} \leq \frac{t_0}{4\sqrt{d}}. \tag{A.37}$$

The proof of Theorem A.5 uses the following elementary fact. Given $\lambda > 0$, for $x \geq 0$ let $f(x) = x^{-d/2}e^{-\lambda/\sqrt{x}}$. Since $f'(x) = f(x)[-\frac{d}{2x} + \frac{\lambda}{2x^{3/2}}]$, the function f has a unique maximum at $x = (\lambda/d)^2$, and

$$f(x_0) \leq f(x) \leq f((\lambda/d)^2) = (d/\lambda)^d e^{-d} \quad \text{if } x_0 \leq x \leq (\lambda/d)^2. \tag{A.38}$$

Theorem A.5 (Green function estimate for regular random walks)**.** *Let $d > 2$. For every $\mathsf{c}_{\text{reg}}, \mathsf{C}_{\text{reg}} > 0$, there exist $\mathsf{C}_{\text{RW}} = \mathsf{C}_{\text{RW}}(\mathsf{c}_{\text{reg}}, \mathsf{C}_{\text{reg}}, d)$ and $\mathsf{c}_{\text{RW}} = \mathsf{c}_{\text{RW}}(\mathsf{c}_{\text{reg}}, \mathsf{C}_{\text{reg}}, d) > 0$ such that, for every $(\mathsf{c}_{\text{reg}}, \mathsf{C}_{\text{reg}})$-regular random walk X (started at 0) on $\mathbb{R}^d$ of law $\mathbb{P}$, Green function $\mathbb{G}$, and variance σ^2, every $\mu \in [0,1]$, and every $y \in \mathbb{Z}^d$,*

$$\mathbb{G}_\mu(B_\sigma(y)) \le \mathsf{C}_{\text{RW}} \left(\frac{\sigma}{\sigma \vee |y|}\right)^{d-2} \exp\left(-\sqrt{1-\mu}\frac{t_0}{4\sqrt{d}}\frac{|y|}{\sigma}\right). \tag{A.39}$$

Proof. We first consider $\mu = 1$, for which there is no exponential decay. For $|y| < 2\sigma$, the desired bound follows from (A.19) together with the fact that $\sum_{m\ge1}\frac{1}{m^{d/2}}$ is finite when $d > 2$. For the case $|y| \ge 2\sigma$, we consider large and small m separately. By Lemma A.4,

$$\sum_{m \ge (|y|/\sigma)^2} \mathbb{P}[X_m \in B_\sigma(y)] \le C_1 \sum_{m \ge (|y|/\sigma)^2} m^{-d/2} \le C_2 \left(\frac{\sigma}{|y|}\right)^{d-2}. \tag{A.40}$$

For small m, we use (A.19) and apply (A.38) to see that

$$\mathbb{P}[X_m \in B_\sigma(y)] \le \frac{C_3}{m^{d/2}} \exp\left(-c\frac{|y|}{\sigma\sqrt{m}}\right) \le C_4 \left(\frac{\sigma}{|y|}\right)^{d}. \tag{A.41}$$

Therefore,

$$\sum_{m \le (|y|/\sigma)^2} \mathbb{P}[X_m \in B_\sigma(y)] \le C_5 \left(\frac{\sigma}{|y|}\right)^{d-2}. \tag{A.42}$$

This proves (A.39) for the case $\mu = 1$.

Let

$$a = \frac{t_0}{2}\frac{|y|}{\sigma}, \qquad s = \sqrt{1-\mu}. \tag{A.43}$$

For $as \le K$ (any fixed $K > 0$) the exponential factor in (A.39) plays no role, so (A.39) holds in this case for all μ by using monotonicity in μ with the result for $\mu = 1$. It therefore suffices to assume in the following that as is bounded below by whatever value is convenient. Fix $s \in (0,1]$. We again consider large and small m separately, but now with a different division between large and small.

Consider the contribution to the Green function due to $m \le d/s^2$. In this case, we simply use $\mu \le 1$, and then apply (A.19). This gives

$$\sum_{m \le d/s^2} \mu^m \mathbb{P}[X_m \in B_\sigma(y)] \le \mathsf{C}_{\text{ac}} e^{t_0^2/8} \sum_{m \le d/s^2} \frac{1}{m^{d/2}} e^{-a/\sqrt{m}}. \tag{A.44}$$

We apply (A.38) a second time to see that the terms in the sum on the above right-hand side are bounded above by their value at $m = (a/d)^2$. We can assume that $d/s^2 \le (a/d)^2$, because this is a statement that as is bounded below and we have already dealt with the case when as is bounded above. In this case, the terms in the sum on the right-hand side of (A.44) are bounded above by their value for $m = d/s^2$, and we obtain the desired estimate via

$$\begin{aligned} \sum_{m \le d/s^2} \frac{1}{m^{d/2}} e^{-a/\sqrt{m}} &\lesssim s^{-2} s^d e^{-as/\sqrt{d}} \\ &= s^{d-2} e^{-as/2\sqrt{d}} (as)^{-(d-2)} [(as)^{d-2} e^{-as/2\sqrt{d}}] \\ &\lesssim a^{-(d-2)} e^{-as/2\sqrt{d}}. \end{aligned} \tag{A.45}$$

We now turn to the remaining case, which is $m \geq d/s^2$. We use $\mu^m = (1-s^2)^m \leq e^{-s^2 m}$ and apply (A.21). Therefore $\mu^m e^{s^2 m/8} \leq 1$, and we find that

$$\sum_{m \geq d/s^2} \mu^m \mathbb{P}[X_m \in B_\sigma(y)] \leq \mathsf{C}_{\mathrm{ac}} e^{-as/2} \sum_{m \geq d/s^2} \frac{1}{m^{d/2}} \tag{A.46}$$
$$\lesssim e^{-as/2} s^{d-2} \lesssim a^{-(d-2)} e^{-as/4},$$

where the last inequality is as in (A.46).

This completes the proof. □

The following is an example of a regular random walk which does not obey a pointwise version of (A.39). The averaging in Theorem A.5 therefore plays an important role.

Example A.6. Let $d > 2$ and $N > 1$ be integers. Consider the random walk on $\mathbb{Z}^d$ whose transition probabilities are given, for $1 \leq i \leq d$, by $\mathbb{P}[X_1 = \pm N \mathbf{e}_i] = \frac{1}{2d}$. Then $\sigma = N$ and

$$M(s) = \mathbb{E}[e^{s(\mathbf{e}_1 \cdot X_1)/\sigma}] = \frac{d-1}{d} + \frac{\cosh(s)}{d}. \tag{A.47}$$

Therefore $M(1) \leq 1 + e$ for every N, so the walk is uniformly regular in N. At criticality, a non-averaged estimate of (A.39) would have to state that

$$\mathbb{G}_1(0, N\mathbf{e}_1) \leq \frac{\mathsf{C}_{\mathrm{RW}}}{\sigma^d} \left(\frac{\sigma}{\sigma \vee |N\mathbf{e}_1|} \right)^{d-2} = \frac{\mathsf{C}_{\mathrm{RW}}}{N^d}. \tag{A.48}$$

This cannot hold for every N because $\mathbb{G}_1(0, N\mathbf{e}_1) \geq \frac{1}{2d}$.

A.3 Pointwise estimates: proof of Proposition 2.1

We now apply Theorem A.5 to prove its pointwise counterpart for any random walk on $\mathbb{Z}^d$ whose transition function is given by an admissible kernel, as defined in Definition 1.1.

Let J be an admissible kernel. In particular, there is a constant $c_0 > 0$ such that

$$c_0 R_J \leq \sigma_J, \qquad J_x \leq c_0^{-1} R_J^{-d} \quad (x \in \mathbb{Z}^d). \tag{A.49}$$

Let $\mathbb{P}_J$ denote the law of the random walk $(X_k)_{k \geq 0}$ started at 0 and of step distribution given by J, and let $\sigma_J^2 := \mathbb{E}_J[|X_1|_2^2]$. For $\mu \in [0,1]$ and $x \in \mathbb{Z}^d$, the Green function is $\mathbb{C}_\mu(x) = \sum_{m \geq 0} \mu^m \mathbb{P}_J[X_m = x]$, and we define the moment generating function $M_J(s) = \mathbb{E}_J[\exp(s(\mathbf{e}_1) \cdot X_1)]$.

We restate Proposition 2.1 here as Proposition A.7. For the nearest-neighbour J, it provides a different proof of the Green function estimate of [84, Proposition 2.1]. For the specific spread-out J of (1.12), Proposition A.7 improves the Green function estimate of [65, Proposition B.1], which has less precise control in terms of σ_J.

Proposition A.7. *Let $d > 2$. There exist $\mathsf{c}, \mathsf{C} > 0$, which depend on d and c_0 but not on J, such that for every $\mu \leq 1$, every $m \geq 1$, and every $x \in \mathbb{Z}^d$,*

$$\mathbb{P}_J[X_m = x] \leq \frac{\mathsf{C}}{\sigma_J^d} \frac{1}{m^{d/2}} \exp\left(-\mathsf{c} \frac{|x|}{\sigma_J \sqrt{m}} \right), \tag{A.50}$$

$$\mathbb{C}_\mu(x) \leq \delta_0(x) + \frac{\mathsf{C}}{\sigma_J^d} \left(\frac{\sigma_J}{\sigma_J \vee |x|} \right)^{d-2} \exp\left(-\mathsf{c} \sqrt{1-\mu} \frac{|x|}{\sigma_J} \right). \tag{A.51}$$

Proof. To begin, we notice that

$$M_J(2/\sigma_J) = \sum_{x\in\Lambda_R} \exp(2(\mathbf{e}_1\cdot x)/\sigma_J)J_x \le \exp(2R_J/\sigma_J) \le \exp(2/c_0), \tag{A.52}$$

so the random walk is $(2, \exp(2/c_0))$-regular, uniformly in $R_J \ge 1$. We can therefore conclude from Theorem A.5 that, with the d- and c_0-dependent constants $(\mathsf{c}_{\mathrm{RW}}, \mathsf{C}_{\mathrm{RW}})$ determined by $(\mathsf{c}_{\mathrm{reg}}, \mathsf{C}_{\mathrm{reg}}) = (2, \exp(2/c_0))$, and for every $y \in \mathbb{Z}^d$,

$$\mathbb{C}_\mu(\Lambda_{\sigma_J}(y)) \le \mathsf{C}_{\mathrm{RW}} \left(\frac{\sigma_J}{\sigma_J \vee |y|}\right)^{d-2} \exp\left(-\mathsf{c}_{\mathrm{RW}}\sqrt{1-\mu}\frac{|y|}{\sigma_J}\right). \tag{A.53}$$

We first prove (A.51), by "unaveraging" (A.53). For this, we start by using $J_x \le c_0^{-1}R_J^{-d}$ (by (A.49)) and the definition of R_J to see that, for $m \ge 1$,

$$\mathbb{P}_J[X_m = x] \le \sum_{u\in\Lambda_{R_J}(x)\setminus\{x\}} \mathbb{P}_J[X_{m-1} = u]J_{u-x} \le \frac{1}{c_0R_J^d}\mathbb{P}_J[X_{m-1} \in \Lambda_{R_J}(x)]. \tag{A.54}$$

The Green function therefore obeys

$$\mathbb{C}_\mu(x) = \delta_0(x) + \sum_{m\ge1} \mu^m\mathbb{P}_J[X_m = x] \le \delta_0(x) + \frac{1}{c_0R_J^d}\mathbb{C}_\mu(\Lambda_{R_J}(x)). \tag{A.55}$$

With (A.53), this is almost what we need, but there is a mismatch between the boxes of size R_J and σ_J. To deal with this, we let $b = \lceil c_0^{-1}\rceil$ and $B = (2b+1)^d$, so that, by (A.53)

$$\begin{aligned}\mathbb{C}_\mu(\Lambda_{R_J}(x)) &\le \sum_{j\in\Lambda_b} \mathbb{C}_\mu(\Lambda_{\sigma_J}(x + j\sigma)) \qquad\qquad \text{(A.56)}\\ &\le B\mathsf{C}_{\mathrm{RW}} \sup_{j\in\Lambda_b} \left(\frac{\sigma_J}{\sigma_J \vee |x + j\sigma_J|}\right)^{d-2} \exp\left(-\mathsf{c}_{\mathrm{RW}}\sqrt{1-\mu}\frac{|x + j\sigma_J|}{\sigma_J}\right).\end{aligned}$$

Suppose first that $|x| \le 2b\sigma_J$. Then,

$$\frac{\sigma_J}{\sigma_J \vee |x + j\sigma_J|} \le 1 \le 2b\frac{\sigma_J}{\sigma_J \vee |x|}, \tag{A.57}$$

$$\exp\left(-\mathsf{c}_{\mathrm{RW}}\sqrt{1-\mu}\frac{|x + j\sigma_J|}{\sigma_J}\right) \le 1 \le e^{\mathsf{c}_{\mathrm{RW}}2b} \exp\left(-\mathsf{c}_{\mathrm{RW}}\sqrt{1-\mu}\frac{|x|}{\sigma_J}\right). \tag{A.58}$$

These give the desired bound for the case $|x| \le 2b\sigma_J$, with suitable constants c, C. Suppose instead that $|x| \ge 2b\sigma_J$. In this case, for every $j \in \Lambda_b$,

$$|x + j\sigma_J| \ge |x| - b\sigma_J \ge \frac{1}{2}|x|, \tag{A.59}$$

which implies the desired bound (A.51) with suitable constants c, C.

We prove (A.50) similarly. For $m = 1$ it is straightforward, so we assume $m \ge 2$. We start with (A.54) and apply the averaged anti-concentration bound of (A.19), and $\sigma_J \le R_J$, to obtain

$$\begin{aligned}\mathbb{P}_J[X_m = x] &\le \frac{B}{c_0\sigma_J^d}\frac{\mathsf{C}_{\mathrm{ac}}e^{t_0^2/8}}{(m-1)^{d/2}} \sup_{j\in\Lambda_b} \exp\left(-t_0\frac{|x + j\sigma_J|}{2\sigma_J\sqrt{m-1}}\right)\\ &\le \frac{B}{c_0\sigma_J^d}\frac{2^{d/2}\mathsf{C}_{\mathrm{ac}}e^{t_0^2/8}}{m^{d/2}}e^{2t_0b} \sup_{j\in\Lambda_b} \exp\left(-\frac{t_0}{4}\frac{|x|}{\sigma_J\sqrt{m}}\right). \qquad \text{(A.60)}\end{aligned}$$

This gives (A.50) and, after relaxing the values of c, C if necessary, completes the proof. □

B Convolution estimates

In this appendix, we collect two convolution estimates. First, we prove Lemma 5.2, which we restate as the following lemma.

Lemma B.1. *Let $a, b, c_1, c_2, \sigma, \xi > 0$, $\mu > 0$ and $\varepsilon \in [0, 1]$. Suppose that the functions $f, g : \mathbb{Z}^d \to [0, \infty)$ satisfy*

$$f(x) \le c_1 \frac{1}{\sigma^d} \left(\frac{\sigma}{\sigma \vee |x|} \right)^{d-2-\varepsilon} e^{-a|x|/\xi}, \tag{B.1}$$

$$g(\Lambda_\xi(x)) \le c_2 \left(\frac{\xi}{\xi \vee |x|} \right)^{d-2} e^{-b|x|/\xi}. \tag{B.2}$$

Then, there is a constant $C_{a,\mu}$ such that, for every $|x| \ge 2(\sigma \vee \xi)$,

$$\sum_{y \notin \Lambda_{\mu\xi}(0)} f(y) g(x-y) \le \frac{c_1}{\sigma^2 |x|^{d-2}} \Big(\frac{|x|}{\sigma} \Big)^{\varepsilon} \left(2^d \|g\|_1 e^{-a|x|/2\xi} + c_2 C_{a,\mu} \Big(\frac{\xi}{|x|} \Big)^{\varepsilon} e^{-b|x|/2\xi} \right). \tag{B.3}$$

Proof. We first observe that we may assume that $\mu \in (0, 1)$. Indeed, if $\mu \ge 1$ then we can bound the left-hand side of (B.3) by the left-hand side with $\mu = \frac{1}{2}$. That new left-hand side is bounded by the corresponding right-hand side, which suffices.

The first term in the upper bound on the convolution arises from

$$\begin{aligned} \sum_{y \notin \Lambda_{|x|/2}(0)} f(y) g(x-y) &\le \frac{c_1}{\sigma^d} \left(\frac{\sigma}{|x|/2} \right)^{d-2-\varepsilon} e^{-a|x|/2\xi} \sum_{y \notin \Lambda_{|x|/2}(0)} g(x-y) \\ &\le \frac{c_1}{\sigma^2 |x|^{d-2}} \Big(\frac{|x|}{\sigma} \Big)^{\varepsilon} 2^d e^{-a|x|/2\xi} \|g\|_1. \end{aligned} \tag{B.4}$$

The remaining contribution to (B.3) is due to the sum over $y \in \Lambda_{|x|/2}(0) \setminus \Lambda_{\mu\xi}(0)$. We do not have a pointwise hypothesis on g, so averaging is required. For $k \ge 0$, we define the annulus $A_k := \{z \in \mathbb{Z}^d : 2^k \mu\xi < |z| \le 2^{k+1}\mu\xi\}$. We decompose the annulus $\Lambda_{|x|/2}(0) \setminus \Lambda_{\mu\xi}(0)$ into the annuli A_k, with $0 \le k \le \log_2(|x|/2\mu\xi) - 1$. This gives

$$\sum_{y \in \Lambda_{|x|/2}(0)} f(y) g(x-y) \le \sum_{k=0}^{\log_2(|x|/2\mu\xi)-1} \sum_{y \in A_k} f(y) g(x-y). \tag{B.5}$$

By hypothesis, if $k \ge 0$ and $y \in A_k$, we have the uniform bound

$$f(y) \le \frac{c_1}{\sigma^d} \left(\frac{\sigma}{\sigma \vee 2^k \xi\mu} \right)^{d-2-\varepsilon} e^{-a 2^k \mu\xi/\xi} \le \frac{c_1}{\sigma^{2+\varepsilon}} \left(\frac{1}{2^k \xi\mu} \right)^{d-2-\varepsilon} e^{-a 2^k \mu}. \tag{B.6}$$

Also, since $\Lambda_{\mu\xi}(x-y) \subset \Lambda_\xi(x-y)$,

$$\sum_{y \in A_k} g(x-y) \le 2^{kd} \sup_{y \in A_k} g(\Lambda_{\mu\xi}(x-y)) \le 2^{kd} \sup_{y \in A_k} g(\Lambda_\xi(x-y)). \tag{B.7}$$

It follows from the hypothesis on g, and the fact that $|y| \le |x|/2$, that

$$\sup_{y \in A_k} g(\Lambda_\xi(x-y)) \le c_2 \left(\frac{\xi}{|x|/2} \right)^{d-2} e^{-b|x|/2\xi}. \tag{B.8}$$

We abbreviate the notation for the annulus of interest by defining $A := \{y \in \Lambda_{|x|/2}(0) \setminus \Lambda_{\mu\xi}(0)\}$. Altogether, we find that

$$\sum_{y\in A} f(y)g(x-y) \le \frac{c_1}{\sigma^{2+\varepsilon}} \left(\frac{1}{|x|}\right)^{d-2-\varepsilon} \left(\frac{\xi}{|x|}\right)^{\varepsilon} c_2 e^{-b|x|/2\xi} \\ \times \Big[\left(\frac{2}{\mu}\right)^{d-2} \sum_{k\ge 0} e^{-a2^k\mu} 2^{k(2+\varepsilon)} \Big]. \tag{B.9}$$

The factor in square brackets is bounded by a constant $C_{a,\mu}$ which depends on a, μ, d. We therefore obtain the desired upper bound

$$\frac{c_1}{\sigma^2|x|^{d-2}} \left(\frac{|x|}{\sigma}\right)^{\varepsilon} c_2 C_{a,\mu} \left(\frac{\xi}{|x|}\right)^{\varepsilon} e^{-b|x|/2\xi}. \tag{B.10}$$

This completes the proof. □

Next, we prove Lemma 5.4, which we restate as follows.

Lemma B.2. *Let $p, a > 0$. For $i = 1, 2$, suppose that $f_i \in \ell^1(\mathbb{Z}^d)$ satisfy $0 \le f_i(x) \le a(1 \vee |x|)^{-p}$ for all $x \in \mathbb{Z}^d$. Let $k \ge 1$. Then*

$$(f_1 * f_2)(x) \le \frac{a}{(1\vee|x|)^p} \Big(\frac{1}{k^p} \|f_1\|_1 + 2^p \sum_{y\in\Lambda_{k|x|}(0)} (f_1(y) + f_2(y)) \Big). \tag{B.11}$$

Proof. We divide the sum $\sum_y f_1(x-y) f_2(y)$ into three parts:

$$y \in \Lambda_{|x|/2}(x), \qquad y \in \Lambda_{k|x|}(0) \setminus \Lambda_{|x|/2}(x), \qquad y \notin \Lambda_{k|x|}(0). \tag{B.12}$$

For the first case, we have $|y| \ge |x|/2$, so by hypothesis, and by $\Lambda_{|x|/2}(0) \subset \Lambda_{k|x|}(0)$,

$$\sum_{y\in\Lambda_{|x|/2}(x)} f_1(x-y) f_2(y) \le \frac{2^p a}{(1\vee|x|)^p} \sum_{y\in\Lambda_{k|x|}(0)} f_1(y). \tag{B.13}$$

For the second case, $|x-y| \ge |x|/2$, so

$$\sum_{y\in\Lambda_{k|x|}(0)\setminus\Lambda_{|x|/2}(x)} f_1(x-y) f_2(y) \le \frac{2^p a}{(1\vee|x|)^p} \sum_{y\in\Lambda_{k|x|}(0)} f_2(y). \tag{B.14}$$

For the third case, $|y| \ge k|x|$, so

$$\sum_{y\notin\Lambda_{|x|}(x)} f_1(x-y) f_2(y) \le \frac{a}{k^p(1\vee|x|)^p} \|f_1\|_1. \tag{B.15}$$

This completes the proof. □

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