A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model

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Abstract

We provide a new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. The proof applies to infinite-range models on arbitrary locally finite transitive infinite graphs.

For Bernoulli percolation, we prove finiteness of the susceptibility in the subcritical regime $\beta < \beta_c$, and the mean-field lower bound $\mathbb{P}_\beta[0 \longleftrightarrow \infty] \geq (\beta - \beta_c)/\beta$ for $\beta > \beta_c$. For finite-range models, we also prove that for any $\beta < \beta_c$, the probability of an open path from the origin to distance n decays exponentially fast in n.

For the Ising model, we prove finiteness of the susceptibility for $\beta < \beta_c$, and the mean-field lower bound $\langle \sigma_0 \rangle_\beta^+ \geq \sqrt{(\beta^2 - \beta_c^2)/\beta^2}$ for $\beta > \beta_c$. For finite-range models, we also prove that the two-point correlation functions decay exponentially fast in the distance for $\beta < \beta_c$.

The paper is organized in two sections, one devoted to Bernoulli percolation, and one to the Ising model. While both proofs are completely independent, we wish to emphasize the strong analogy between the two strategies.

General notation. Let $G = (V, E)$ be a locally finite (vertex-)transitive infinite graph, together with a fixed origin $0 \in V$. For $n \geq 0$, let

$$\Lambda_n := \{x \in V : \mathrm{d}(x, 0) \leq n\},$$

where $\mathrm{d}(\cdot, \cdot)$ is the graph distance. Consider a set of coupling constants $(J_{x,y})_{x,y \in V}$ with $J_{x,y} = J_{y,x} \geq 0$ for every x and y in V. We assume that the coupling constants are *invariant* with respect to some transitively acting group. More precisely, there exists a group Γ of automorphisms acting transitively on V such that $J_{\gamma(x),\gamma(y)} = J_{x,y}$ for all $\gamma \in \Gamma$. We say that $(J_{x,y})_{x,y \in V}$ is *finite-range* if there exists $R > 0$ such that $J_{x,y} = 0$ whenever $\mathrm{d}(x, y) > R$.

1 Bernoulli percolation

1.1 The main result

Let $\mathbb{P}_\beta$ be the bond percolation measure on G defined as follows: for $x, y \in V$, $\{x, y\}$ is *open* with probability $1 - e^{-\beta J_{x,y}}$, and *closed* with probability $e^{-\beta J_{x,y}}$.

We say that x and y are *connected in $S \subset V$* if there exists a sequence of vertices $(v_k)_{0\le k\le K}$ in S such that $v_0 = x$, $v_K = y$, and $\{v_k, v_{k+1}\}$ is open for every $0 \le k < K$. We denote this event by $x \overset{S}{\longleftrightarrow} y$. For $A \subset V$, we write $x \overset{S}{\longleftrightarrow} A$ for the event that x is connected in S to a vertex in A. If $S = V$, we drop it from the notation. Finally, we set $0 \longleftrightarrow \infty$ if 0 is connected to Λ_n^c for all $n \ge 1$. The critical parameter is defined by

$$\beta_c := \inf\{\beta \ge 0 : \mathbb{P}_\beta[0 \longleftrightarrow \infty] > 0\}.$$

Theorem 1.1. *1. For $\beta > \beta_c$, $\mathbb{P}_\beta[0 \longleftrightarrow \infty] \ge \frac{\beta-\beta_c}{\beta}$.*

2. For $\beta < \beta_c$, the susceptibility is finite, i.e.

$$\sum_{x\in V} \mathbb{P}_\beta[0 \longleftrightarrow x] < \infty.$$

3. If $(J_{x,y})_{x,y\in V}$ is finite-range, then for any $\beta < \beta_c$, there exists $c = c(\beta) > 0$ such that

$$\mathbb{P}_\beta[0 \longleftrightarrow \Lambda_n^c] \le e^{-cn} \quad \textit{for all } n \ge 0.$$

Let us describe the proof quickly. For $\beta > 0$ and a finite subset S of V, define

$$\varphi_\beta(S) := \sum_{x\in S}\sum_{y\notin S} (1 - e^{-\beta J_{x,y}})\mathbb{P}_\beta\big(0 \overset{S}{\longleftrightarrow} x\big). \tag{1.1}$$

This quantity can be interpreted as the expected number of open edges on the "external boundary" of S that are connected to 0 by an open path of vertices in S. Also introduce

$$\tilde{\beta}_c := \sup\{\beta \ge 0 : \varphi_\beta(S) < 1 \text{ for some finite } S \subset V \text{ containing } 0\}. \tag{1.2}$$

In order to prove Theorem 1.1, we show that Items 1, 2 and 3 hold with $\tilde{\beta}_c$ in place of β_c. This directly implies that $\tilde{\beta}_c = \beta_c$, and thus Theorem 1.1.

The quantity $\varphi_\beta(S)$ appears naturally when differentiating the probability $\mathbb{P}_\beta[0 \longleftrightarrow \Lambda_n^c]$ with respect to β. Indeed, a simple computation presented in Lemma 1.4 provides the following differential inequality

$$\frac{d}{d\beta}\mathbb{P}_\beta[0 \longleftrightarrow \Lambda_n^c] \ge \tfrac{1}{\beta} \inf_{\substack{S\subset\Lambda_n \\ 0\in S}} \varphi_\beta(S)\cdot(1 - \mathbb{P}_\beta[0 \longleftrightarrow \Lambda_n^c]). \tag{1.3}$$

By integrating (1.3) between $\tilde{\beta}_c$ and $\beta > \tilde{\beta}_c$ and then letting n tend to infinity, we obtain $\mathbb{P}_\beta[0 \longleftrightarrow \infty] \ge \frac{\beta-\tilde{\beta}_c}{\beta}$.

Now consider $\beta < \tilde{\beta}_c$. The existence of a finite set S containing the origin such that $\varphi_\beta(S) < 1$, together with the BK-inequality, imply that the expected size of the cluster the origin is finite.

1.2 Comments and consequences

Bibliographical comments. Theorem 1.1 was first proved in [AB87] and [Men86] for Bernoulli percolation on the d-dimensional hypercubic lattice. The proof was extended to general quasi-transitive graphs in [AV08]. The first item was proved in [CC87].

Nearest-neighbor percolation. We recover easily the standard results for nearest-neighbor model by setting $J_{x,y} = 0$ if $\{x,y\} \notin E$, $J_{x,y} = 1$ if $\{x,y\} \in E$, and $p = 1 - e^{-\beta}$. In this context, one can obtain the inequality $\mathbb{P}_p[0 \longleftrightarrow \infty] \geq \frac{p-p_c}{p(1-p_c)}$ for $p \geq p_c$ by introducing

$$\varphi_p(S) = p \sum_{x \in S} \sum_{\substack{y \notin S \\ \{x,y\} \in E}} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x].$$

This lower bound is slightly better than Item 1 of Theorem 1.1 and is provided by little modifications in our proof (see [DT15] for a presentation of the proof in this context).

Site percolation. As in [AB87], the proof may be adapted to site percolation on transitive graphs. In this context, one can obtain the inequality $\mathbb{P}_p[0 \longleftrightarrow \infty] \geq \frac{1}{d-1}\frac{p-p_c}{1-p_c}$ (d is the degree of G) for $p \geq p_c$ by introducing

$$\varphi_p(S) = \sum_{x \in S} \sum_{\substack{y \notin S \\ \{x,y\} \in E}} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x].$$

Finite susceptibility against exponential decay. Finite susceptibility does not always imply exponential decay of correlations for infinite-range models. Conversely, on graphs with exponential growth, exponential decay does not imply finite susceptibility. Hence, in general, the second condition of Theorem 1.1 is neither weaker nor stronger than the third one.

Percolation on the square lattice. On the square lattice, the inequality $p_c \geq 1/2$ was first obtained by Harris in [Har60] (see also the short proof of Zhang presented in [Gri99]). The other inequality $p_c \leq 1/2$ was first proved by Kesten in [Kes80] using a delicate geometric construction involving crossing events. Since then, many other proofs invoking exponential decay in the subcritical phase (see [Gri99]) or sharp threshold arguments (see e.g. [BR06]) have been found. Here, Theorem 1.1 provides a short proof of exponential decay and therefore a short alternative to these proofs. For completeness, let us sketch how exponential decay implies that $p_c \leq 1/2$: item 3 implies that for $p < p_c$ the probability of an open path from left to right in a n by n square tends to 0 as n goes to infinity. But self-duality implies that this does not happen when $p = 1/2$, thus implying that $p_c \leq 1/2$.

Lower bound on β_c. Since $\varphi_\beta(\{0\}) = \sum_{y \in V} 1 - e^{-\beta J_{0,y}}$, we obtain a lower bound on β_c by taking the solution of the equation $\sum_{y \in V} 1 - e^{-\beta J_{0,y}} = 1$.

Behaviour at β_c. Under the hypothesis that $\sum_{y\in V} J_{0,y} < \infty$, the set

$$\{\beta \geq 0 : \varphi_\beta(S) < 1 \text{ for some finite } S \subset V \text{ containing } 0\ \}$$

defining $\tilde{\beta}_c$ in Equation (1.2) is open. In particular, we have that at $\beta = \beta_c = \tilde{\beta}_c$, $\varphi_\beta(S) \geq 1$ for every finite $S \ni 0$. This implies the following classical result.

Proposition 1.2 ([AN84]). *We have* $\sum_{x\in V} \mathbb{P}_{\beta_c}[0 \longleftrightarrow x] = \infty$.

Proof. Simply write

$$\Big(\sum_{y\in V} 1 - e^{-\beta_c J_{0,y}}\Big) \cdot \sum_{x\in V} \mathbb{P}_{\beta_c}[0 \longleftrightarrow x] \geq \sum_{n\geq 1} \varphi_\beta(\Lambda_n) = \infty.$$

□

Semi-continuity of β_c. Consider the nearest-neighbor model. Since $\tilde{\beta}_c$ is defined in terms of finite sets, one can see that $\tilde{\beta}_c$ is lower semi-continuous when seen as a function of the graph in the following sense. Let G be an infinite locally finite transitive graph. Let (G_n) be a sequence of infinite locally finite transitive graphs such that the balls of radius n around the origin in G_n and G are the same. Then,

$$\liminf \tilde{\beta}_c(G_n) \geq \tilde{\beta}_c(G). \tag{1.4}$$

The equality $\beta_c = \tilde{\beta}_c$ implies that the semi-continuity (1.4) also holds for β_c (this also followed from [Ham57] and the exponential decay in subcritical, but the definition of $\tilde{\beta}_c$ illustrates this property readily). The locality conjecture, due to Schramm and presented in [BNP11], states that for any $\varepsilon > 0$, the map $G \mapsto \beta_c(G)$ should be continuous on the set of graphs with $\beta_c < 1 - \varepsilon$. The discussion above shows that the hard part in the locality conjecture is the upper semi-continuity.

Dependent models. For dependent percolation models, the proof does not extend in a trivial way, mostly due to the fact that the BK inequality is not available in general. Nevertheless, this new strategy may be of some use. For instance, for random-cluster models on the square lattice, a proof (see [DST15]) based on the strategy of this paper and the parafermionic observable offers an alternative to the standard proof of [BD12a] based on sharp threshold theorems.

Oriented percolation. The proof applies mutatis mutandis to oriented percolation.

Percolation with a magnetic field. In [AB87], the authors consider a percolation model with magnetic field defined as follows. Add a *ghost vertex* $g \notin V$ and consider that $\{x, g\}$ is open with probability $1 - e^{-h}$, independently for any $x \in V$. Let $\mathbb{P}_{\beta,h}$ be the measure obtained from $\mathbb{P}_\beta$ by adding the edges $\{x, g\}$. An important results in [AB87] is the following mean-field lower bound which is instrumental in the study of percolation in high dimensions (see e.g. [AN84]).

Proposition 1.3 ([AB87]). *There exists a constant $c > 0$ such that for any $h > 0$,*

$$\mathbb{P}_{\beta_c,h}[0 \longleftrightarrow g] \geq c\sqrt{h}.$$

In Section 1.5, we provide a short proof of this proposition, using the same strategy as in our proof of Theorem 1.1.

1.3 Proof of Item 1

In this section, we prove that for every $\beta \geq \tilde{\beta}_c$,

$$\mathbb{P}_\beta[0 \longleftrightarrow \infty] \geq \frac{\beta - \tilde{\beta}_c}{\beta}. \tag{1.5}$$

Let us start by the following lemma.

Lemma 1.4. *Let $\beta > 0$ and $\Lambda \subset V$ finite,*

$$\frac{d}{d\beta}\mathbb{P}_\beta[0 \longleftrightarrow \Lambda^c] \geq \tfrac{1}{\beta} \inf_{\substack{S\subset\Lambda \\ 0\in S}} \varphi_\beta(S) \cdot \big(1 - \mathbb{P}_\beta[0 \longleftrightarrow \Lambda^c]\big). \tag{1.6}$$

Before proving this lemma, let us see how it implies (1.5). By setting $f(\beta) = \mathbb{P}_\beta[0 \longleftrightarrow \Lambda^c]$ in (1.6), and observing that $\varphi_\beta(S) \geq 1$ for any $\beta > \tilde{\beta}_c$, we obtain the following differential inequality:

$$\frac{f'(\beta)}{1 - f(\beta)} \geq \frac{1}{\beta}, \quad \text{for } \beta \in (\tilde{\beta}_c, \infty). \tag{1.7}$$

Integrating (1.7) between $\tilde{\beta}_c$ and β implies that $\mathbb{P}_\beta[0 \longleftrightarrow \Lambda^c] = f(\beta) \geq \frac{\beta - \tilde{\beta}_c}{\beta}$ for every $\Lambda \subset V$. By letting Λ tend to V, we obtain (1.5).

Proof of Lemma 1.4. Let $\beta > 0$ and Λ. Define the following random subset of Λ:

$$\mathscr{S} := \{x \in \Lambda \text{ such that } x \not\longleftrightarrow \Lambda^c\}.$$

Recall that $\{x, y\}$ is pivotal for the configuration ω and the event $\{0 \longleftrightarrow \Lambda^c\}$ if $\omega_{\{x,y\}} \notin \{0 \longleftrightarrow \Lambda^c\}$ and $\omega^{\{x,y\}} \in \{0 \longleftrightarrow \Lambda^c\}$. (The configuration $\omega_{\{x,y\}}$, resp. $\omega^{\{x,y\}}$, coincides with ω except that the edge $\{x, y\}$ is closed, resp. open.)

Russo's formula ([Rus78] or [Gri99, Section 2.4]) implies that

$$\begin{aligned} \frac{d}{d\beta}\mathbb{P}_\beta[0 \longleftrightarrow \Lambda^c] &= \sum_{\{x,y\}} J_{x,y}\mathbb{P}_\beta[\{x, y\} \text{ pivotal}] \qquad (1.8)\\ &\geq \tfrac{1}{\beta} \sum_{\{x,y\}} (1 - e^{-\beta J_{x,y}})\mathbb{P}_\beta[\{x, y\} \text{ pivotal and } 0 \not\longleftrightarrow \Lambda^c] \\ &\geq \tfrac{1}{\beta} \sum_{S\ni 0} \sum_{\{x,y\}} (1 - e^{-\beta J_{x,y}})\mathbb{P}_\beta[\{x, y\} \text{ pivotal and } \mathscr{S} = S]. \end{aligned}$$

In the second line, we used the inequality $t \geq 1 - e^{-t}$ for $t \geq 0$. Observe that the event that $\{x,y\}$ is pivotal and $\mathscr{S} = S$ is nonempty only if $x \in S$ and $y \notin S$, or $y \in S$ and $x \notin S$. Furthermore, the vertex in S must be connected to 0 in S. We can assume without loss of generality that $x \in S$ and $y \notin S$. Rewrite the event that $\{x,y\}$ is pivotal and $\mathscr{S} = S$ as $\{0 \overset{S}{\longleftrightarrow} x\} \cap \{\mathscr{S} = S\}$. Since the event $\{\mathscr{S} = S\}$ and $\{0 \overset{S}{\longleftrightarrow} x\}$ are measurable with respect to the state of edges having one endpoint in $V \setminus S$, and edges having both endpoints in S respectively. Therefore, the two events above are independent. Thus,

$$\mathbb{P}_\beta[\{x,y\} \text{ pivotal and } \mathscr{S} = S] = \mathbb{P}_\beta[0 \overset{S}{\longleftrightarrow} x]\mathbb{P}_\beta[\mathscr{S} = S].$$

Plugging this equality in the computation above, we obtain

$$\begin{aligned}\frac{d}{d\beta}\mathbb{P}_\beta[0 \longleftrightarrow \Lambda^c] &\geq \tfrac{1}{\beta}\sum_{S \ni 0} \varphi_\beta(S)\mathbb{P}_\beta[\mathscr{S} = S] \\ &\geq \tfrac{1}{\beta}\Big(\inf_{S \ni 0} \varphi_\beta(S)\Big) \cdot \sum_{S \ni 0} \mathbb{P}_\beta[\mathscr{S} = S].\end{aligned}$$

The proof follows readily since

$$\sum_{S \ni 0} \mathbb{P}_\beta[\mathscr{S} = S] = \mathbb{P}_\beta[0 \in \mathscr{S}] = \mathbb{P}_\beta[0 \not\longleftrightarrow \Lambda^c] = 1 - \mathbb{P}_\beta[0 \longleftrightarrow \Lambda^c].$$

□

Remark 1.1. In the proof above, Russo's formula is possibly used in infinite volume, since the model can be infinite-range. There is no difficulty resolving this technical issue (which does not occur for finite-range) by finite volume approximation. The same remark applies below when we use the BK inequality.

1.4 Proof of Items 2 and 3

In this section, we show that Items 2 and 3 in Theorem 1.1 hold with $\tilde{\beta}_c$ in place of β_c.

Lemma 1.5. *Let $\beta > 0$, and $u \in S \subset A$ and $B \cap S = \varnothing$. We have*

$$\mathbb{P}_\beta[u \overset{A}{\longleftrightarrow} B] \leq \sum_{x \in S}\sum_{y \notin S}(1 - e^{-\beta J_{x,y}})\mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} x]\mathbb{P}_\beta[y \overset{A}{\longleftrightarrow} B].$$

Proof of Lemma 1.5. Let $u \in S$ and assume that the event $u \overset{A}{\longleftrightarrow} B$ holds. Consider an open path $(v_k)_{0 \leq k \leq K}$ from u to B. Since $B \cap S = \varnothing$, one can define the first k such that $v_{k+1} \notin S$. We obtain that the following events occur disjointly (see [Gri99, Section 2.3] for a definition of disjoint occurrence):

- u is connected to v_k in S,
- $\{v_k, v_{k+1}\}$ is open,
- v_{k+1} is connected to B in A.

The lemma is then a direct consequence of the BK inequality applied twice (v_k plays the role of x, and v_{k+1} of y). □

Let us now prove the second item of Theorem 1.1. Fix $\beta < \tilde{\beta}_c$ and S such that $\varphi_\beta(S) < 1$. For $\Lambda \subset V$ finite, introduce

$$\chi(\Lambda,\beta) := \max\Big\{ \sum_{v\in\Lambda} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} v] \ ; \ u \in \Lambda \Big\}.$$

For every u, let S_u be the image of S by a fixed automorphism sending 0 to u. Lemma 1.5 implies that for every $v \in \Lambda \setminus S_u$,

$$\mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} v] \le \sum_{x\in S_u} \sum_{y\notin S_u} \mathbb{P}_\beta[u \overset{S_u}{\longleftrightarrow} x](1 - e^{-\beta J_{x,y}})\mathbb{P}_\beta[y \overset{\Lambda}{\longleftrightarrow} v].$$

Summing over all $v \in \Lambda \setminus S_u$, we find

$$\sum_{v\in\Lambda\setminus S_u} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} v] \le \varphi_\beta(S)\chi(\Lambda,\beta).$$

Using the trivial bound $\mathbb{P}_\beta[u \longleftrightarrow v] \le 1$ for $v \in \Lambda \cap S_u$, we obtain

$$\sum_{v\in\Lambda} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} v] \le |S| + \varphi_\beta(S)\chi(\Lambda,\beta).$$

Optimizing over u, we deduce that

$$\chi(\Lambda,\beta) \le \frac{|S|}{1 - \varphi_\beta(S)}$$

which implies in particular that

$$\sum_{x\in\Lambda} \mathbb{P}_\beta[0 \overset{\Lambda}{\longleftrightarrow} x] \le \frac{|S|}{1 - \varphi_\beta(S)}.$$

The result follows by taking the limit as Λ tends to V.

We now turn to the proof of the third item of Theorem 1.1. A similar proof was used in [Ham57]. Let R be the range of the $(J_{x,y})_{x,y\in V}$, and let L be such that $S \subset \Lambda_{L-R}$. Lemma 1.5 implies that for $n \ge L$,

$$\begin{aligned}\mathbb{P}_\beta[0 \longleftrightarrow \Lambda_n^c] &\le \sum_{x\in S} \sum_{y\notin S} (1 - e^{-\beta J_{x,y}})\mathbb{P}_\beta[0 \overset{S}{\longleftrightarrow} x]\mathbb{P}_\beta[y \longleftrightarrow \Lambda_n^c] \\ &\le \varphi_\beta(S)\mathbb{P}_\beta[0 \longleftrightarrow \Lambda_{n-L}^c].\end{aligned}$$

In the last line, we used that y is connected to distance larger than or equal to $n - L$ since $1 - e^{-\beta J_{x,y}} = 0$ if $x \in S$ and y is not in Λ_L. By iterating, this immediately implies that

$$\mathbb{P}_\beta[0 \longleftrightarrow \Lambda_n^c] \le \varphi_\beta(S)^{\lfloor n/L \rfloor}.$$

1.5 Proof of Proposition 1.3

Let us introduce $M(\beta,h) = \mathbb{P}_{\beta,h}[0 \longleftrightarrow g]$.

Lemma 1.6 ([AB87])**.**

$$\frac{\partial M}{\partial \beta} \le \Big(\sum_{x\in V} J_{0,x}\Big) M \frac{\partial M}{\partial h}. \tag{1.9}$$

Proof. Consider a finite subset Λ of V. Russo's formula leads to the following version of (1.8):

$$\frac{\partial \mathbb{P}_{\beta,h}[0 \longleftrightarrow \Lambda^c \cup \{g\}]}{\partial \beta} = \sum_{\{x,y\}} J_{x,y}\, \mathbb{P}_{\beta,h}[\{x,y\} \text{ pivotal}].$$

The edge $\{x,y\}$ is pivotal if, without using $\{x,y\}$, one of the two vertices is connected to 0 but not to $\Lambda^c \cup \{g\}$, and the other one to $\Lambda^c \cup \{g\}$. Without loss of generality, let us assume that x is connected to 0, and y is not. Conditioning on the set

$$\mathscr{S} = \big\{z \in \Lambda : z \longleftrightarrow \Lambda^c \cup \{g\} \text{ without using } \{x,y\}\big\},$$

we obtain

$$\mathbb{P}_{\beta,h}[\{x,y\} \text{ pivotal}] \le \mathbb{P}_{\beta,h}[y \leftrightarrow \Lambda^c \cup \{g\}] \cdot \mathbb{P}_{\beta,h}[0 \longleftrightarrow x, 0 \not\longleftrightarrow \Lambda^c \cup \{g\}].$$

Plugging this inequality in (1.8) and letting Λ tend to V, we find

$$\frac{\partial M}{\partial \beta} \le \Big(\sum_{\{0,y\}} J_{0,y}\Big) M \Big(\sum_{x\in V} \mathbb{P}_{\beta,h}[0 \longleftrightarrow x, 0 \not\longleftrightarrow g]\Big).$$

We conclude by observing that if C denotes the cluster of 0 in V, we find

$$\begin{aligned}\frac{\partial M}{\partial h} &= \frac{\partial}{\partial h}\Big(1 - \sum_{n=0}^{\infty} \mathbb{P}_{\beta,h}[|\mathsf{C}| = n] e^{-nh}\Big) = \sum_{n=0}^{\infty} n\mathbb{P}_{\beta,h}[|\mathsf{C}| = n] e^{-nh} \\ &= \sum_{n=0}^{\infty} \sum_{x\in V} \mathbb{P}_{\beta,h}[0 \longleftrightarrow x, |\mathsf{C}| = n] e^{-nh} = \sum_{x\in V} \mathbb{P}_{\beta,h}[0 \longleftrightarrow x, 0 \not\longleftrightarrow g].\end{aligned}$$

□

Another differential inequality, which is harder to obtain, usually complements (1.9):

$$M \le h\frac{\partial M}{\partial h} + M^2 + \beta M \frac{\partial M}{\partial \beta}. \tag{1.10}$$

This other inequality may be avoided using the following observation. The differential inequality (1.6) is satisfied with $\mathbb{P}_\beta[0 \longleftrightarrow \Lambda^c]$ replaced by $\mathbb{P}_{\beta,h}[0 \longleftrightarrow \{g\} \cup \Lambda^c]$, thus giving us for $\beta \ge \beta_c$ and $h \ge 0$,

$$\frac{\partial M}{\partial \beta} \ge \tfrac{1}{\beta}(1 - M)$$

(at $\beta = \beta_c$, we use the fact that $\varphi_{\beta_c}(S) \geq 1$ for every finite $S \ni 0$, see the comment before Proposition 1.2). When $\beta \geq \beta_c$ this implies that

$$1 - M \;\leq\; \beta \frac{\partial M}{\partial \beta} \;\leq\; \beta \Big(\sum_{x \in V} J_{0,x} \Big) M \frac{\partial M}{\partial h}, \tag{1.11}$$

which immediately implies the following mean-field lower bound: there exists a constant $c > 0$ such that for any $h > 0$,

$$\mathbb{P}_{\beta_c,h}[0 \longleftrightarrow g] = M(\beta_c, h) \;\geq\; c\sqrt{h}.$$

Remark 1.2. While (1.11) is slightly shorter to obtain that (1.10), the later is very useful when trying to obtain an upper bound on $M(\beta_c, h)$.

2 The Ising model

2.1 The main result

For a finite subset Λ of V, consider a spin configuration $\sigma = (\sigma_x : x \in \Lambda) \in \{-1,1\}^\Lambda$. For $\beta > 0$ and $h \in \mathbb{R}$, introduce the Hamiltonian

$$H_{\Lambda,\beta,h}(\sigma) \;:=\; -\,\beta \sum_{x,y \in \Lambda} J_{x,y} \sigma_x \sigma_y \;-\; h \sum_{x \in \Lambda} \sigma_x.$$

Define the Gibbs measures on Λ with free boundary conditions, inverse-temperature β and external field $h \in \mathbb{R}$ by the formula

$$\langle f \rangle_{\Lambda,\beta,h} = \frac{\displaystyle\sum_{\sigma \in \{-1,1\}^\Lambda} f(\sigma) \exp[-H_{\Lambda,\beta,h}(\sigma)]}{\displaystyle\sum_{\sigma \in \{-1,1\}^\Lambda} \exp[-\beta H_{\Lambda,\beta,h}(\sigma)]}$$

for $f : \{-1,1\}^\Lambda \longrightarrow \mathbb{R}$. Let the infinite-volume Gibbs measure $\langle \cdot \rangle_{\beta,h}$ be the weak limit of $\langle \cdot \rangle_{\Lambda,\beta,h}$ as $\Lambda \nearrow V$. Also write $\langle \cdot \rangle_\beta^+$ for the weak limit of $\langle \cdot \rangle_{\beta,h}$ as $h \searrow 0$.

Introduce

$$\beta_c := \inf\{\beta > 0 : \langle \sigma_0 \rangle_\beta^+ > 0\}.$$

Theorem 2.1. *1. For $\beta > \beta_c$, $\langle \sigma_0 \rangle_\beta^+ \geq \sqrt{\frac{\beta^2 - \beta_c^2}{\beta^2}}$.*

2. For $\beta < \beta_c$, the susceptibility is finite, i.e.

$$\sum_{x \in V} \langle \sigma_0 \sigma_x \rangle_\beta^+ < \infty.$$

3. If $(J_{x,y})_{x,y \in V}$ is finite-range, then for any $\beta < \beta_c$, there exists $c = c(\beta) > 0$ such that

$$\langle \sigma_0 \sigma_x \rangle_\beta^+ \leq e^{-c\mathrm{d}(0,x)} \quad \textit{for all } x \in V.$$

This theorem was first proved in [ABF87] for the Ising model on the d-dimensional hypercubic lattice. The proof presented here improves the constant in the mean-field lower bound, and extends to general transitive graphs.

The proof of Theorem 2.1 follows closely the proof for percolation. For $\beta > 0$ and a finite subset S of V, define

$$\varphi_S(\beta) := \sum_{x\in S}\sum_{y\in V\setminus S} \tanh(\beta J_{x,y})\langle\sigma_0\sigma_x\rangle_{S,\beta,0},$$

which bears a resemblance to (1.1). Similarly to (1.2), set

$$\tilde{\beta}_c := \sup\{\beta \geq 0 : \varphi_\beta(S) < 1 \text{ for some finite } S \subset V \text{ containing } 0\}.$$

In order to prove Theorem 2.1, we show that Items 1, 2 and 3 hold with $\tilde{\beta}_c$ in place of β_c. This directly implies that $\tilde{\beta}_c = \beta_c$, and thus Theorem 2.1. The proof of Theorem 2.1 proceeds in two steps.

As for percolation, the quantity $\varphi_\beta(S)$ appears naturally in the derivative of a "finite-volume approximation" of $\langle\sigma_0\rangle_{\beta,h}$. Roughly speaking (see Lemma 2.6 for a precise statement), one obtains a finite-volume version of the following inequality:

$$\frac{d}{d\beta}\langle\sigma_0\rangle_{\beta,h}^2 \geq \tfrac{2}{\beta}\inf_{S\ni 0}\varphi_\beta(S)\cdot\big(1-\langle\sigma_0\rangle_{\beta,h}^2\big).$$

This inequality implies, for every $\beta > \tilde{\beta}_c$,

$$\langle\sigma_0\rangle_{\beta,h} \geq \sqrt{\tfrac{\beta^2-\tilde{\beta}_c^2}{\beta^2}}$$

and therefore Item 1 by letting h tend to 0.

The remaining items follow from an improved Simon's inequality, proved below.

Remark 2.1. The proof uses the random-current representation. In this context, the derivative of $\langle\sigma_0\rangle_{\beta,h}^2$ has an interpretation which is very close to the differential inequality (1.6). In some sense, percolation is replaced by the trace of the sum of two independent random sourceless currents. Furthermore, the strong Simon's inequality plays the role of the BK inequality for percolation.

2.2 Comments and consequences

1. The random-cluster model (also called Fortuin-Kasteleyn percolation) with cluster weight $q = 2$ is naturally coupled to the Ising model (see [Gri06] for details). The previous theorem implies exponential decay in the subcritical phase for this model.

2. Exactly like in the case of Bernoulli percolation, the critical parameter of the random-cluster model on the square lattice with $q = 2$ can be proved to be equal to $\sqrt{2}/(1+\sqrt{2})$ using the exponential decay in the subcritical phase together with the self-duality.

3. The previous item together with the coupling with the Ising model implies that $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ on the square lattice (see [Ons44, BD12b] for alternative proofs).

4. Exactly as for Bernoulli percolation, we get that $\varphi_{\beta_c}(S) \geq 1$ for any finite set $S \ni 0$, which implies the following classical proposition.

   Proposition 2.2 ([Sim80]). *We have* $\sum_{x \in V} \langle \sigma_0 \sigma_x \rangle^+_{\beta_c} = \infty$.

   Proof. Use Griffiths' inequality (2.1) below to show that for $x \in \Lambda_n \setminus \Lambda_{n-1}$,

$$\langle \sigma_0 \sigma_x \rangle^+_{\beta_c} \geq \langle \sigma_0 \sigma_x \rangle_{\beta_c,0} \geq \langle \sigma_0 \sigma_x \rangle_{\Lambda_n,\beta_c,0}$$

   so that
$$\Big(\sum_{y \in V} \tanh(\beta J_{0,y}) \Big) \cdot \sum_{x \in V} \langle \sigma_0 \sigma_x \rangle^+_{\beta_c} \geq \sum_{n \geq 1} \varphi_{\beta_c}(\Lambda_n) = \infty.$$

   □

5. The equality $\beta_c = \tilde{\beta}_c$ implies that β_c is lower semi-continuous with respect to the graph (see the discussion for Bernoulli percolation).

6. In [ABF87], the authors also prove the following result.

   Proposition 2.3 ([ABF87]). *There exists a constant $c > 0$ such that for any $h > 0$,*
$$\langle \sigma_0 \rangle_{\beta_c,h} \geq ch^{1/3}.$$

   We present in Section 2.6 a short proof of this proposition, using the same strategy as in our proof of Proposition 1.3.

2.3 Preliminaries

Griffiths' inequality. The following is a standard consequence of the second Griffiths' inequality [Gri67]: for $\beta > 0$, $h \geq 0$ and $S \subset \Lambda$ two finite subsets of V,

$$\langle \sigma_0 \rangle_{S,\beta,h} \leq \langle \sigma_0 \rangle_{\Lambda,\beta,h}. \tag{2.1}$$

Random-current representation. This section presents a few basic facts on the random-current representation. We refer to [Aiz82, AF86, ABF87] for details on this representation.

Let Λ be a finite subset of V and $S \subset \Lambda$. We consider an additional vertex g not in V, called the *ghost vertex*, and write $\mathscr{P}_2(S \cup \{g\})$ for the set of pairs $\{x, y\}$, $x, y \in S \cup \{g\}$. We also define $J_{x,g} = h/\beta$ for every $x \in \Lambda$.

Definition 2.4. *A current* $\mathbf{n}$ *on S (also called a current configuration) is a function from $\mathscr{P}_2(S \cup \{g\})$ to $\{0, 1, 2, ...\}$. A source of* $\mathbf{n} = (\mathbf{n}_{x,y} : \{x, y\} \in \mathscr{P}_2(S \cup \{g\}))$ *is a vertex $x \in S \cup \{g\}$ for which $\sum_{y \in S} \mathbf{n}_{x,y}$ is odd. The set of sources of* $\mathbf{n}$ *is denoted by* $\partial \mathbf{n}$. *We say that x and y are connected in* $\mathbf{n}$ *(denoted by $x \overset{\mathbf{n}}{\longleftrightarrow} y$) if there exists a sequence of vertices $v_0, v_1, \ldots, v_K$ in $S \cup \{g\}$ such that $v_0 = x$, $v_K = y$ and $\mathbf{n}_{v_k,v_{k+1}} > 0$ for every $0 \leq k < K$.*

For a finite subset Λ of V and a current $\mathbf{n}$ on Λ, define

$$w(\mathbf{n}) = w(\mathbf{n},\beta,h) := \prod_{\{x,y\}\in\mathscr{P}_2(\Lambda\cup\{g\})} \frac{(\beta J_{x,y})^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!}.$$

From now on, we will write $\sum_{\partial\mathbf{n}=A}$ for the sum running on currents on S with sources A. Sometimes, the current $\mathbf{n}$ will be on $S' \subset S$ (and therefore the sum will run on such currents), but this will be clear from context.

An important property of random currents is the following: for every subset A of Λ, we have

$$\Big\langle \prod_{a\in A} \sigma_a \Big\rangle_{\Lambda,\beta,h} = \begin{cases} \dfrac{\sum_{\partial\mathbf{n}=A} w(\mathbf{n})}{\sum_{\partial\mathbf{n}=\varnothing} w(\mathbf{n})} & \text{if } A \text{ is even,} \\[2ex] \dfrac{\sum_{\partial\mathbf{n}=A\cup\{g\}} w(\mathbf{n})}{\sum_{\partial\mathbf{n}=\varnothing} w(\mathbf{n})} & \text{if } A \text{ is odd.} \end{cases} \tag{2.2}$$

We will use the following standard lemma on random currents.

Lemma 2.5 (Switching Lemma, [Aiz82, Lemma 3.2])**.** *Let $A \subset \Lambda$ and $u, v \in \Lambda \cup \{g\}$. Let F be a function from the set of currents on Λ to $\mathbb{R}$. We have*

$$\sum_{\substack{\partial\mathbf{n}_1=A\Delta\{u,v\}\\ \partial\mathbf{n}_2=\{u,v\}}} F(\mathbf{n}_1+\mathbf{n}_2)w(\mathbf{n}_1)w(\mathbf{n}_2) = \sum_{\substack{\partial\mathbf{n}_1=A\\ \partial\mathbf{n}_2=\varnothing}} F(\mathbf{n}_1+\mathbf{n}_2)w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[u \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} v], \tag{2.3}$$

where Δ is the symmetric difference between sets.

Backbone representation for random currents. Fix a finite subset Λ of V. Choose an arbitrary order of the oriented edges of the lattice. Consider a current $\mathbf{n}$ on Λ with $\partial\mathbf{n} = \{x,y\}$. Let $\omega(\mathbf{n})$ be the edge self-avoiding path from x to y passing only through edges e with $\mathbf{n}_e$ odd which is minimal for the lexicographical order on paths induced by the previous ordering on oriented edges. Such an object is called the *backbone* of the current configuration. For the backbone ω with endpoints $\partial\omega = \{x,y\}$, set

$$\rho_\Lambda(\omega) = \rho_\Lambda(\beta,h,\omega) := \frac{\sum_{\partial\mathbf{n}=\{x,y\}} w(\mathbf{n})\mathbf{I}[\omega(\mathbf{n})=\omega]}{\sum_{\partial\mathbf{n}=\varnothing} w(\mathbf{n})}.$$

The backbone representation has the following properties (see (4.2), (4.7) and (4.11) of [AF86] for **P1**, **P2** and **P3** respectively):

- **P1** $\langle\sigma_x\sigma_y\rangle_{\Lambda,\beta,h} = \sum_{\partial\omega=\{x,y\}} \rho_\Lambda(\omega)$.
- **P2** If the backbone ω is the concatenation of two backbones ω_1 and ω_2 (this is denoted by $\omega = \omega_1 \circ \omega_2$), then
$$\rho_\Lambda(\omega) = \rho_\Lambda(\omega_1)\rho_{\Lambda\setminus\overline{\omega}_1}(\omega_2),$$
where $\overline{\omega}_1$ is the set of bonds whose state is determined by the fact that ω_1 is an admissible backbone (this includes bonds of ω_1 together with some neighboring bonds).
- **P3** For the backbone ω not using any edge outside $T \subset \Lambda$, we have
$$\rho_\Lambda(\omega) \le \rho_T(\omega).$$

2.4 Proof of Item 1

In this section, we prove that for every $\beta \geq \tilde{\beta}_c$,

$$\langle\sigma_0\rangle^+_\beta \geq \sqrt{\tfrac{\beta^2-\tilde{\beta}_c^2}{\beta^2}}. \tag{2.4}$$

In order to do so, we will based our analysis on the following lemma.

Lemma 2.6. *Let $\beta > 0$, $h > 0$ and Λ a finite subset of V. Then,*

$$\frac{d}{d\beta}\langle\sigma_0\rangle^2_{\Lambda,\beta,h} \geq 2c(\Lambda,\beta,h)\Big[\tfrac{1}{\beta}\inf_{S\ni 0}\varphi_\beta(S)(1-\langle\sigma_0\rangle^2_{\Lambda,\beta,h}) - \epsilon(\Lambda,\beta,h)\Big], \tag{2.5}$$

where

$$c(\Lambda,\beta,h) := \inf_{y\in\Lambda}\frac{\langle\sigma_0\rangle_{\Lambda,\beta,h}}{\langle\sigma_y\rangle_{\Lambda,\beta,h}}$$

and

$$\epsilon(\Lambda,\beta,h) := \sum_{x\in\Lambda}\sum_{y\notin\Lambda} J_{x,y}(\langle\sigma_0\sigma_x\rangle_{\Lambda,\beta,h} - \langle\sigma_0\rangle_{\Lambda,\beta,h}\langle\sigma_x\rangle_{\Lambda,\beta,h}).$$

To conclude the proof, fix $\beta_1,\beta_2 > \tilde{\beta}_c$. Integrating (2.5) between β_1 and β_2 for Λ equal to the box Λ_n of size n, and then letting Λ_n go to infinity, implies that

$$\langle\sigma_0\rangle^2_{\beta_2,h} - \langle\sigma_0\rangle^2_{\beta_1,h} \geq \int_{\beta_1}^{\beta_2}\tfrac{2}{\beta}(1-\langle\sigma_0\rangle^2_{\beta,h})d\beta,$$

where the inequality above follows from Fatou's lemma together with

$$\begin{aligned}
&\lim_{n\to\infty}\langle\sigma_0\rangle_{\Lambda_n,\beta_i,h} = \langle\sigma_0\rangle_{\beta_i,h} && \text{(by weak convergence)},\\
&\lim_{n\to\infty} c(\Lambda_n,\beta,h) = 1 && \text{(see Remark 2.2 below)},\\
&\lim_{n\to\infty} \epsilon(\Lambda_n,\beta,h) = 0 && \text{(see Remark 2.3 below)}.
\end{aligned}$$

The proof of (2.4) follows easily by letting h tend to 0.

Remark 2.2. To see that $c(\Lambda_n,\beta,h)$ tends to 1, observe that Griffiths' inequality (2.1) implies that $\langle\sigma_y\rangle_{\Lambda_n,\beta,h} \leq \langle\sigma_0\rangle_{\Lambda_{2n},\beta,h}$ (we use the invariance under translation and the fact that the translate of Λ_{2n} centered at y contains Λ_n). Therefore, for every $n \geq 1$, we have

$$\frac{\langle\sigma_0\rangle_{\Lambda_n,\beta,h}}{\langle\sigma_0\rangle_{\Lambda_{2n},\beta,h}} \leq c(\Lambda_n,\beta,h) \leq 1. \tag{2.6}$$

Together with the fact that $\langle\sigma_0\rangle_{\Lambda_n,\beta,h}$ tends to $\langle\sigma_0\rangle_{\beta,h}$ as n tends to infinity, (2.6) implies that $c(\Lambda_n,\beta,h)$ tends to 1.

Remark 2.3. To see that $\epsilon(\Lambda_n,\beta,h)$ tends to 0, first observe that the GHS inequality [GHS70] implies that $\langle\sigma_0\rangle_{\Lambda,\beta,h}$ is a concave function of h. We deduce that

$$\sum_{x\in\Lambda}\langle\sigma_0\sigma_x\rangle_{\Lambda,\beta,h} - \langle\sigma_0\rangle_{\Lambda,\beta,h}\langle\sigma_x\rangle_{\Lambda,\beta,h} = \frac{\partial}{\partial h}\langle\sigma_0\rangle_{\Lambda,\beta,h} \leq \frac{\langle\sigma_0\rangle_{\Lambda,\beta,h}}{h} \leq \frac{1}{h}.$$

Applied to $\Lambda = \Lambda_n$, this gives in particular that for each k,

$$\sum_{x\in\Lambda_{n-k}}\sum_{y\in V\setminus\Lambda_n} J_{x,y}\langle\sigma_0\sigma_x\rangle_{\Lambda_n,\beta,h} - \langle\sigma_0\rangle_{\Lambda_n,\beta,h}\langle\sigma_x\rangle_{\Lambda_n,\beta,h} \le \frac{1}{h}\Big(\sum_{y\in V\setminus\Lambda_k} J_{0,y}\Big),$$

which can be made arbitrarily small (uniformly in n) by setting k large enough. Now, a second use of the GHS inequality [GHS70] implies that

$$\begin{aligned}\sum_{x\in\Lambda_n\setminus\Lambda_{n-k}}\langle\sigma_0\sigma_x\rangle_{\Lambda_n,\beta,h} - \langle\sigma_0\rangle_{\Lambda_n,\beta,h}\langle\sigma_x\rangle_{\Lambda_n,\beta,h} &\le \frac{\langle\sigma_0\rangle_{\Lambda_n,\beta,h} - \langle\sigma_0\rangle_{\Lambda_n,\beta,\mathbf{h}}}{h}\\ &\le \frac{\langle\sigma_0\rangle_{\Lambda_n,\beta,h} - \langle\sigma_0\rangle_{\Lambda_{n-k},\beta,h}}{h},\end{aligned}$$

where $\langle\cdot\rangle_{\Lambda_n\beta,\mathbf{h}}$ is the measure with inverse-temperature β, and magnetic field h_x depending on x which is equal to h for $x\in\Lambda_{n-k}$ and 0 in $\Lambda_n\setminus\Lambda_{n-k}$. In the second line, we used Griffiths inequality to show that $\langle\sigma_0\rangle_{\Lambda_{n-k},\beta,h}\le\langle\sigma_0\rangle_{\Lambda_n,\beta,\mathbf{h}}$. For each fixed k, the term on the right converges to 0 as n tends to infinity by weak convergence.

In order to prove Lemma 2.6, we use a computation similar to one provided in [ABF87].

Proof of Lemma 2.6. Let $\beta>0$, $h>0$ and a finite subset Λ of V. Set

$$Z := \sum_{\partial\mathbf{n}=\varnothing} w(\mathbf{n}).$$

The derivative of $\langle\sigma_0\rangle_{\Lambda,\beta,h}$ is given by the following formula

$$\frac{d}{d\beta}\langle\sigma_0\rangle_{\Lambda,\beta,h} = \sum_{\{x,y\}\subset\Lambda} J_{x,y}\big(\langle\sigma_0\sigma_x\sigma_y\rangle_{\Lambda,\beta,h} - \langle\sigma_0\rangle_{\Lambda,\beta,h}\langle\sigma_x\sigma_y\rangle_{\Lambda,\beta,h}\big).$$

Using (2.2) and the switching lemma, we obtain

$$\frac{d}{d\beta}\langle\sigma_0\rangle_{\Lambda,\beta,h} = \frac{1}{Z^2}\sum_{\{x,y\}\subset\Lambda} J_{x,y}\sum_{\substack{\partial\mathbf{n}_1=\{0,g\}\Delta\{x,y\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\not\longleftrightarrow} g].$$

If $\mathbf{n}_1$ and $\mathbf{n}_2$ are two currents such that $\partial\mathbf{n}_1=\{0,g\}\Delta\{x,y\}$, $\partial\mathbf{n}_2=\varnothing$ and 0 and g are not connected in $\mathbf{n}_1+\mathbf{n}_2$, then exactly one of these two cases holds: $0\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} x$ and $y\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g$, or $0\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} y$ and $x\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g$. Since the second case is the same as the first one with x and y permuted, we obtain the following expression,

$$\frac{d}{d\beta}\langle\sigma_0\rangle_{\Lambda,\beta,h} = \frac{1}{Z^2}\sum_{\substack{x\in\Lambda\\ y\in\Lambda}} J_{x,y}\delta_{x,y}, \tag{2.7}$$

where

$$\delta_{x,y} = \sum_{\substack{\partial\mathbf{n}_1=\{0,g\}\Delta\{x,y\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[0\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} x, y\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g, 0\overset{\mathbf{n}_1+\mathbf{n}_2}{\not\longleftrightarrow} g]$$

(see Fig. 1 and notice the analogy with the event involved in Russo's formula, namely that the edge $\{x,y\}$ is pivotal, in Bernoulli percolation).

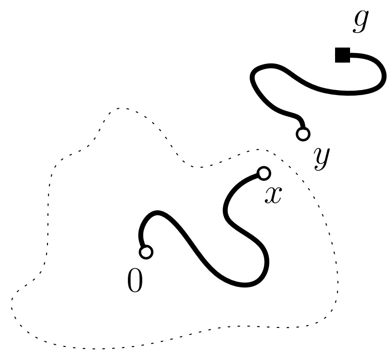

Figure 1: A diagrammatic representation of $\delta_{x,y}$: the solid lines represent the backbones, and the dotted line the boundary of the cluster of 0 in $\mathbf{n}_1 + \mathbf{n}_2$.

Given two currents $\mathbf{n}_1$ and $\mathbf{n}_2$, and $z \in \{0, g\}$, define $\mathscr{S}_z$ to be the set of vertices in $\Lambda \cup \{g\}$ that are *not* connected to z in $\mathbf{n}_1 + \mathbf{n}_2$. Let us compute $\delta_{x,y}$ by summing over the different possible values for $\mathscr{S}_0$:

$$\delta_{x,y} = \sum_{S \subset \Lambda \cup \{g\}} \sum_{\substack{\partial \mathbf{n}_1 = \{0,g\}\Delta\{x,y\} \\ \partial \mathbf{n}_2 = \varnothing}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S, 0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} x, y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g, 0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\not\longleftrightarrow} g]$$

$$= \sum_{\substack{S \subset \Lambda \cup \{g\} \\ \text{s.t. } y,g \in S \\ \text{and } 0,x \in \Lambda \setminus S}} \sum_{\substack{\partial \mathbf{n}_1 = \{0,g\}\Delta\{x,y\} \\ \partial \mathbf{n}_2 = \varnothing}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S, y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g].$$

Since 0 and x are not connected to y in $\mathbf{n}_1 + \mathbf{n}_2$ (recall that $y \in S$), we deduce that y must be connected to g in $\mathbf{n}_1$ because of the constraints on sources. Thus, the indicator $\mathbf{I}[y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g]$ equals 1 for any currents $\mathbf{n}_1$ and $\mathbf{n}_2$ satisfying $\mathscr{S}_0 = S$. Therefore,

$$\delta_{x,y} = \sum_{\substack{S \subset \Lambda \cup \{g\} \\ \text{s.t. } y,g \in S \\ \text{and } 0,x \in \Lambda \setminus S}} \sum_{\substack{\partial \mathbf{n}_1 = \{0,g\}\Delta\{x,y\} \\ \partial \mathbf{n}_2 = \varnothing}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S]. \tag{2.8}$$

Let us now focus on the following claim, which enables us to remove the sources y and g.

Claim 1: Let $S \subset \Lambda$ containing y and g but neither x nor 0. We have

$$\sum_{\substack{\partial \mathbf{n}_1 = \{0,g\}\Delta\{x,y\} \\ \partial \mathbf{n}_2 = \varnothing}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S] \tag{2.9}$$

$$\geq \frac{1}{\langle \sigma_y \rangle_{\Lambda,\beta,h}} \sum_{\substack{\partial \mathbf{n}_1 = \{0\}\Delta\{x\} \\ \partial \mathbf{n}_2 = \varnothing}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S, y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g].$$

Proof of Claim 1. Let

$$\Theta = \sum_{\substack{\partial \mathbf{n}_1 = \{0,g\}\Delta\{x,y\} \\ \partial \mathbf{n}_2 = \varnothing}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S].$$

When $\mathscr{S}_0 = S$, the two currents $\mathbf{n}_1$ and $\mathbf{n}_2$ vanish on every $\{u,v\}$ with $u \in S$ and $v \notin S$. Thus, for $i = 1,2$, we can decompose $\mathbf{n}_i$ as

$$\mathbf{n}_i = \mathbf{n}_i^S + \mathbf{n}_i^{\Lambda \setminus S},$$

where $\mathbf{n}_i^A$ denotes the current

$$\mathbf{n}_i^A(\{u,v\}) = \begin{cases} \mathbf{n}_i(\{u,v\}) & \text{if } u,v \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\partial \mathbf{n}_i^A = A \cap \partial \mathbf{n}_i$ and $w(\mathbf{n}_i) = w(\mathbf{n}_i^{\Lambda \setminus S}) w(\mathbf{n}_i^S)$.

Since $\mathbf{I}[\mathscr{S}_0 = S]$ does not depend on $\mathbf{n}_1^S$, the decomposition $\mathbf{n}_1 = \mathbf{n}_1^S + \mathbf{n}_1^{\Lambda \setminus S}$ gives

$$\Theta = \sum_{\substack{\partial \mathbf{n}_1^{\Lambda \setminus S} = \{0\}\Delta\{x\} \\ \partial \mathbf{n}_2 = \varnothing}} w(\mathbf{n}_1^{\Lambda \setminus S}) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S] \Big(\sum_{\partial \mathbf{n}_1^S = \{y,g\}} w(\mathbf{n}_1^S) \Big).$$

Using (2.2), we find

$$\Theta = \sum_{\substack{\partial \mathbf{n}_1^{\Lambda \setminus S} = \{0\}\Delta\{x\} \\ \partial \mathbf{n}_2 = \varnothing}} w(\mathbf{n}_1^{\Lambda \setminus S}) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S] \langle \sigma_y \rangle_{S,\beta,h} \Big(\sum_{\partial \mathbf{n}_1^S = \varnothing} w(\mathbf{n}_1^S) \Big).$$

Multiply the expression above by $\langle \sigma_y \rangle_{\Lambda,\beta,h} \geq \langle \sigma_y \rangle_{S,\beta,h}$ (which follows from (2.1)), and then decompose $\mathbf{n}_2$ into $\mathbf{n}_2^S$ and $\mathbf{n}_2^{\Lambda \setminus S}$ to find

$$\begin{aligned}
\langle \sigma_y \rangle_{\Lambda,\beta,h} \Theta &\geq \sum_{\substack{\partial \mathbf{n}_1^{\Lambda \setminus S} = \{0\}\Delta\{x\} \\ \partial \mathbf{n}_2 = \varnothing}} w(\mathbf{n}_1^{\Lambda \setminus S}) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S] \langle \sigma_y \rangle_{S,\beta,h}^2 \Big(\sum_{\partial \mathbf{n}_1^S = \varnothing} w(\mathbf{n}_1^S) \Big) \\
&= \sum_{\substack{\partial \mathbf{n}_1^{\Lambda \setminus S} = \{0\}\Delta\{x\} \\ \partial \mathbf{n}_2^{\Lambda \setminus S} = \varnothing}} w(\mathbf{n}_1^{\Lambda \setminus S}) w(\mathbf{n}_2^{\Lambda \setminus S}) \mathbf{I}[\mathscr{S}_0 = S] \langle \sigma_y \rangle_{S,\beta,h}^2 \\
&\qquad \Big(\sum_{\substack{\partial \mathbf{n}_1^S = \varnothing \\ \partial \mathbf{n}_2^S = \varnothing}} w(\mathbf{n}_1^S) w(\mathbf{n}_2^S) \Big) \\
&= \sum_{\substack{\partial \mathbf{n}_1^{\Lambda \setminus S} = \{0\}\Delta\{x\} \\ \partial \mathbf{n}_2^{\Lambda \setminus S} = \varnothing}} w(\mathbf{n}_1^{\Lambda \setminus S}) w(\mathbf{n}_2^{\Lambda \setminus S}) \mathbf{I}[\mathscr{S}_0 = S] \\
&\qquad \Big(\sum_{\substack{\partial \mathbf{n}_1^S = \{y,g\} \\ \partial \mathbf{n}_2^S = \{y,g\}}} w(\mathbf{n}_1^S) w(\mathbf{n}_2^S) \Big) \\
&= \sum_{\substack{\partial \mathbf{n}_1 = \{0\}\Delta\{x\}\Delta\{y,g\} \\ \partial \mathbf{n}_2 = \{y,g\}}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{I}[\mathscr{S}_0 = S].
\end{aligned}$$

The switching lemma (2.3) applied to $F = \mathbf{I}[\mathscr{S}_0 = S]$ implies

$$\langle\sigma_y\rangle_{\Lambda,\beta,h}\,\Theta \geq \sum_{\substack{\partial\mathbf{n}_1=\{0\}\Delta\{x\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_0 = S, y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g].$$

□

Inserting (2.9) into (2.8) gives us

$$\delta_{x,y} \geq \frac{1}{\langle\sigma_y\rangle_{\Lambda,\beta,h}} \sum_{\substack{\partial\mathbf{n}_1=\{0\}\Delta\{x\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g, 0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\not\longleftrightarrow} g].$$

We now decompose over the possible values of $\mathscr{S}_g$ (recall that $\mathscr{S}_g$ is the set of vertices *not* connected to g):

$$\begin{aligned}\delta_{x,y} &\geq \frac{1}{\langle\sigma_y\rangle_{\Lambda,\beta,h}} \sum_{S\subset\Lambda} \sum_{\substack{\partial\mathbf{n}_1=\{0\}\Delta\{x\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S, y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g, 0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\not\longleftrightarrow} g] \\ &= \frac{1}{\langle\sigma_y\rangle_{\Lambda,\beta,h}} \sum_{\substack{S\subset\Lambda\\ \text{s.t. } 0,x\in S\\ \text{and } y\in\Lambda\setminus S}} \sum_{\substack{\partial\mathbf{n}_1=\{0\}\Delta\{x\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S]. \end{aligned} \tag{2.10}$$

In the second line, we used the constraint on the sources, which implies that x is connected to 0, and therefore, belong to $\mathscr{S}_g$.

We now focus on a second claim, which enables us to remove the sources 0 and x.

Claim 2: Let $S \subset \Lambda$ containing 0 and x but neither y nor g. We have

$$\sum_{\substack{\partial\mathbf{n}_1=\{0\}\Delta\{x\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S] = \sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\langle\sigma_0\sigma_x\rangle_{S,0}\,\mathbf{I}[\mathscr{S}_g = S].$$

Proof of Claim 2. For currents $\mathbf{n}_1$ and $\mathbf{n}_2$ such that $\mathscr{S}_g = S$, $\mathbf{n}_1$ can be decomposed as $\mathbf{n}_1 = \mathbf{n}_1^S + \mathbf{n}_1^{\Lambda\cup\{g\}\setminus S}$ as we did for $\mathscr{S}_0 = S$ in the previous claim. Using that $w(\mathbf{n}_1) = w(\mathbf{n}_1^S)w(\mathbf{n}_1^{\Lambda\cup\{g\}\setminus S})$ together with the fact that $\mathbf{I}[\mathscr{S}_g = S]$ does not depend on $\mathbf{n}_1^S$, we find that

$$\begin{aligned}&\sum_{\substack{\partial\mathbf{n}_1=\{0\}\Delta\{x\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S] \\ &\quad= \sum_{\substack{\partial\mathbf{n}_1^{\Lambda\cup\{g\}\setminus S}=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1^{\Lambda\cup\{g\}\setminus S})w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S]\Big(\sum_{\partial\mathbf{n}_1^S=\{0\}\Delta\{x\}} w(\mathbf{n}_1^S)\Big) \\ &\quad= \sum_{\substack{\partial\mathbf{n}_1^{\Lambda\cup\{g\}\setminus S}=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1^{\Lambda\cup\{g\}\setminus S})w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S]\Big(\sum_{\partial\mathbf{n}_1^S=\varnothing} w(\mathbf{n}_1^S)\Big)\langle\sigma_0\sigma_x\rangle_{S,\beta,0} \\ &\quad= \sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\langle\sigma_0\sigma_x\rangle_{S,\beta,0}\,\mathbf{I}[\mathscr{S}_g = S].\end{aligned}$$

In the third line we used (2.2) and in the fourth line, we recombined $\mathbf{n}_1^S$ with $\mathbf{n}_1^{\Lambda\cup\{g\}\setminus S}$. □

Inequality (2.10) and Claim 2 imply that for any $x, y \in \Lambda$,

$$\delta_{x,y} \geq \frac{1}{\langle\sigma_y\rangle_{\Lambda,\beta,h}} \sum_{\substack{S\subset\Lambda \\ \text{s.t. } 0,x\in S \\ \text{and } y\in\Lambda\setminus S}} \sum_{\substack{\partial\mathbf{n}_1=\varnothing \\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\langle\sigma_0\sigma_x\rangle_{S,\beta,0}\,\mathbf{I}[\mathscr{S}_g = S].$$

By plugging the inequality above in (2.7), we find

$$\begin{aligned}
\frac{d}{d\beta}\langle\sigma_0\rangle^2_{\Lambda,\beta,h} &= 2\langle\sigma_0\rangle_{\Lambda,\beta,h}\frac{d}{d\beta}\langle\sigma_0\rangle_{\Lambda,\beta,h} \\
&\geq \frac{2}{Z^2}\sum_{\substack{S\subset\Lambda\\ S\ni 0}}\sum_{\substack{x\in S\\ y\in\Lambda\setminus S}}\frac{\langle\sigma_0\rangle_{\Lambda,\beta,h}}{\langle\sigma_y\rangle_{\Lambda,\beta,h}}\sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)J_{x,y}\langle\sigma_0\sigma_x\rangle_{S,\beta,0}\mathbf{I}[\mathscr{S}_g = S] \\
&\geq \frac{2}{Z^2}\sum_{\substack{S\subset\Lambda\\ S\ni 0}}\Big(\sum_{\substack{x\in S\\ y\in\Lambda\setminus S}}\frac{\langle\sigma_0\rangle_{\Lambda,\beta,h}}{\langle\sigma_y\rangle_{\Lambda,\beta,h}}J_{x,y}\langle\sigma_0\sigma_x\rangle_{S,\beta,0}\Big)\Big(\sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S]\Big) \\
&\geq \frac{2c(\Lambda,\beta,h)}{Z^2}\sum_{\substack{S\subset\Lambda\\ S\ni 0}}\Big(\sum_{\substack{x\in S\\ y\in\Lambda\setminus S}} J_{x,y}\langle\sigma_0\sigma_x\rangle_{S,\beta,0}\Big)\Big(\sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S]\Big).
\end{aligned}$$

Using that $J_{x,y} \geq \frac{1}{\beta}\tanh(\beta J_{x,y})$ gives that

$$\sum_{\substack{x\in S\\ y\in\Lambda\setminus S}} J_{x,y}\langle\sigma_0\sigma_x\rangle_{S,\beta,0} \geq \tfrac{1}{\beta}\varphi_\beta(S) - \sum_{\substack{x\in S\\ y\in V\setminus\Lambda}} J_{x,y}\langle\sigma_0\sigma_x\rangle_{S,\beta,0}.$$

We deduce that

$$\begin{aligned}
\frac{d}{d\beta}\langle\sigma_0\rangle^2_{\Lambda,\beta,h} \geq 2c(\Lambda,\beta,h)\Big(\frac{1}{\beta}\cdot\underbrace{\sum_{\substack{S\subset\Lambda\\ S\ni 0}}\varphi_\beta(S)\cdot\frac{1}{Z^2}\sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S]}_{(A)} \\
-\underbrace{\sum_{\substack{S\subset\Lambda\\ S\ni 0}}\sum_{\substack{x\in S\\ y\in V\setminus\Lambda}} J_{x,y}\langle\sigma_0\sigma_x\rangle_{S,\beta,0}\frac{1}{Z^2}\sum_{\substack{\partial\mathbf{n}_1=\varnothing\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{I}[\mathscr{S}_g = S]}_{(B)}\Big)
\end{aligned} \tag{2.11}$$

Taking the infimum over all the $\varphi_\beta(S)$ and then using (2.3) and (2.2) one more time, we obtain that

$$(A) \;\geq\; \inf_{S\ni 0}\varphi_\beta(S)\,(1-\langle\sigma_0\rangle^2_{\Lambda,\beta,h}).$$

Now, summing on S after applying Claim 2 (backward compared to the last use of Claim 2) gives that

$$\begin{aligned}
(B) &= \sum_{\substack{x\in\Lambda\\ y\in V\setminus\Lambda}} J_{x,y}\frac{1}{Z^2}\sum_{\substack{\partial\mathbf{n}_1=\{0\}\Delta\{x\}\\ \partial\mathbf{n}_2=\varnothing}} w(\mathbf{n}_1)w(\mathbf{n}_2)(1-\mathbf{I}[0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} g]) \\
&= \sum_{\substack{x\in\Lambda\\ y\in V\setminus\Lambda}} J_{x,y}(\langle\sigma_0\sigma_x\rangle_{\Lambda,\beta,h} - \langle\sigma_0\rangle_{\Lambda,\beta,h}\langle\sigma_x\rangle_{\Lambda,\beta,h}),
\end{aligned}$$

where, in the second line, we used (2.3) and (2.2) one last time. Plugging the expressions for (A) and (B) obtained above in (2.11) implies the claim. □

2.5 Proof of Items 2 and 3

In this section, we show that Items 2 and 3 in Theorem 2.1 hold with $\tilde{\beta}_c$ in place of β_c.

We need a replacement for the BK inequality used in the case of Bernoulli percolation. The relevant tool for the Ising model will be a modified version of Simon's inequality. The original inequality can be found in [Sim80], see also [Lie80] for an improvement. (Those previous versions do not suffice for our application).

Lemma 2.7 (Modified Simon's inequality). *Let S be a finite subset of V containing 0. For every $z \in V \setminus S$,*

$$\langle \sigma_0 \sigma_z \rangle_\beta^+ \le \sum_{x \in S} \sum_{y \notin S} \tanh(\beta J_{x,y}) \langle \sigma_0 \sigma_x \rangle_{S,\beta,0} \langle \sigma_y \sigma_z \rangle_\beta^+.$$

Proof. Fix $h \ge 0$ and Λ a finite subset of V containing S. We introduce the ghost vertex g as before.

We consider the backbone representation of the Ising model on $\Lambda \cup \{g\}$ defined in the previous section. Let $\omega = (v_k)_{0 \le k \le K}$ be a backbone from 0 to z (it may go through g). Since $z \notin S$, one can define the first k such that $v_k \in \Lambda \setminus S$ and set $y = v_k$. Also set x to be the vertex of S visited last by the backbone before reaching y. The following occurs:

- ω goes from 0 to x staying in $S \cup \{g\}$,
- then ω goes from x to y either in one step by using the edge $\{x, y\}$ or in two steps by going through $\{x, g\}$ and then $\{g, y\}$,
- finally ω goes from y to z in $\Lambda \cup \{g\}$.

Call ω_1 the part of the walk ω from 0 to x, ω_2 the walk from x to y, and ω_3 the reminder of the walk ω.

Using Property **P1** of the backbone representation, we can write

$$\langle \sigma_0 \sigma_z \rangle_{\Lambda,\beta,h} = \sum_{\partial\omega = \{0,z\}} \rho_\Lambda(\omega).$$

Then, **P2** applied with ω_1 and $\omega_2 \circ \omega_3$ and then with ω_2 and ω_3 implies that $\langle \sigma_0 \sigma_z \rangle_{\Lambda,\beta,h}$ is bounded from above by

$$\sum_{x \in S} \sum_{y \in \Lambda \setminus S} \sum_{\partial\omega_1 = \{0,x\}} \rho_\Lambda(\omega_1) \Big(\sum_{\partial\omega_2 = \{x,y\}} \rho_{\Lambda \setminus \overline{\omega_1}}(\omega_2) \Big(\sum_{\partial\omega_3 = \{y,z\}} \rho_{\Lambda \setminus \overline{\omega_1 \circ \omega_2}}(\omega_3) \Big) \Big).$$

P1 and then Griffiths' inequality (2.1) imply that

$$\sum_{\partial\omega_3 = \{y,z\}} \rho_{\Lambda \setminus \overline{\omega_1 \circ \omega_2}}(\omega_3) = \langle \sigma_y \sigma_z \rangle_{\Lambda \setminus \overline{\omega_1 \circ \omega_2},\beta,h} \le \langle \sigma_y \sigma_z \rangle_{\Lambda,\beta,h}.$$

Inserting this in the last displayed equation gives

$$\langle \sigma_0 \sigma_z \rangle_{\Lambda,\beta,h} \le \sum_{x \in S} \sum_{y \in \Lambda \setminus S} \Big(\sum_{\partial\omega_1 = \{0,x\}} \rho_\Lambda(\omega_1) \Big(\sum_{\partial\omega_2 = \{x,y\}} \rho_{\Lambda \setminus \overline{\omega_1}}(\omega_2) \Big) \Big) \langle \sigma_y \sigma_z \rangle_{\Lambda,\beta,h}.$$

Since ω_2 uses only vertices x, y and g, **P3** and then **P1** lead to

$$\sum_{\partial\omega_2=\{x,y\}} \rho_{\Lambda\setminus\overline{\omega_1}}(\omega_2) \le \sum_{\partial\omega_2=\{x,y\}} \rho_{\{x,y\}}(\omega_2) = \langle\sigma_x\sigma_y\rangle_{\{x,y\},\beta,h}$$

which gives

$$\langle\sigma_0\sigma_z\rangle_{\Lambda,\beta,h} \le \sum_{x\in S}\sum_{y\in\Lambda\setminus S}\Big(\sum_{\partial\omega_1=\{0,x\}} \rho_\Lambda(\omega_1)\Big)\langle\sigma_x\sigma_y\rangle_{\{x,y\},\beta,h}\langle\sigma_y\sigma_z\rangle_{\Lambda,\beta,h}.$$

Finally, **P3** can be used with the fact that $\omega_1 \subset S\cup\{g\}$ to show that

$$\begin{aligned}\langle\sigma_0\sigma_z\rangle_{\Lambda,\beta,h} &\le \sum_{x\in S}\sum_{y\in\Lambda\setminus S}\Big(\sum_{\partial\omega_1=\{0,x\}} \rho_S(\omega_1)\Big)\langle\sigma_x\sigma_y\rangle_{\{x,y\},\beta,h}\langle\sigma_y\sigma_z\rangle_{\Lambda,\beta,h}\\ &\le \sum_{x\in S}\sum_{y\in\Lambda\setminus S}\langle\sigma_0\sigma_x\rangle_{S,\beta,h}\,\langle\sigma_x\sigma_y\rangle_{\{x,y\},\beta,h}\,\langle\sigma_y\sigma_z\rangle_{\Lambda,\beta,h}\end{aligned}$$

(we used **P1** in the second line). Let Λ tend to V to obtain

$$\langle\sigma_0\sigma_z\rangle_{\beta,h} \le \sum_{x\in S}\sum_{y\in V\setminus S}\langle\sigma_0\sigma_x\rangle_{S,\beta,h}\,\langle\sigma_x\sigma_y\rangle_{\{x,y\},\beta,h}\,\langle\sigma_y\sigma_z\rangle_{\beta,h}.$$

Let now h tend to 0 to find

- $\langle\sigma_0\sigma_z\rangle_{\beta,h}$ and $\langle\sigma_y\sigma_z\rangle_{\beta,h}$ tend to $\langle\sigma_0\sigma_z\rangle^+_\beta$ and $\langle\sigma_y\sigma_z\rangle^+_\beta$ respectively.
- $\langle\sigma_0\sigma_x\rangle_{S,\beta,h}$ tends to $\langle\sigma_0\sigma_x\rangle_{S,\beta,0}$ (since S is finite).
- $\langle\sigma_x\sigma_y\rangle_{\{x,y\},\beta,h}$ tends to $\langle\sigma_x\sigma_y\rangle_{\{x,y\},\beta,0} = \tanh(\beta J_{x,y})$.

Using one last time that S is finite, we deduce that

$$\langle\sigma_0\sigma_z\rangle^+_\beta \le \sum_{x\in S}\sum_{y\in V\setminus S}\tanh(\beta J_{x,y})\langle\sigma_0\sigma_x\rangle_{S,\beta,0}\langle\sigma_y\sigma_z\rangle^+_\beta.$$

□

We are now in a position to conclude the proof. Let $\beta < \tilde\beta_c$. Fix a finite set S such that $\varphi_\beta(S) < 1$. Define,

$$\chi_n(\beta) := \sup\Big\{\sum_{z\in\Lambda}\langle\sigma_0\sigma_z\rangle^+_\beta : \Lambda\subset V \text{ with } |\Lambda|\le n\Big\}.$$

By the same reasoning as for percolation, Lemma 2.7 shows that

$$\sum_{z\in\Lambda}\langle\sigma_0\sigma_z\rangle^+_\beta \le |S| + \sum_{x\in S}\sum_{y\in V\setminus S}\tanh(\beta J_{x,y})\langle\sigma_0\sigma_x\rangle_{S,\beta,0}\Big(\sum_{z\in\Lambda\setminus S}\langle\sigma_y\sigma_z\rangle^+_\beta\Big).$$

Using the invariance under translations and taking the supremum over sets Λ of volume n, we immediately get that $\chi_n(\beta) < |S|/[1-\varphi_\beta(S)]$ uniformly in n. Letting n tend to infinity ∞ gives the second item.

We finish by the proof of the third item. Let R be the range of the $(J_{x,y})_{x,y\in V}$, and let L be such that $S\subset\Lambda_{L-R}$. Lemma 2.7 implies that for any z with $\mathrm{d}(0,z)\ge n > L$,

$$\langle\sigma_0\sigma_z\rangle^+_\beta \le \sum_{x\in S}\sum_{y\in V\setminus S}\tanh(\beta J_{x,y})\langle\sigma_0\sigma_x\rangle_{S,\beta,0}\langle\sigma_y\sigma_z\rangle^+_\beta \le \varphi_\beta(S)\max_{y\in\Lambda_L}\langle\sigma_y\sigma_z\rangle^+_\beta.$$

Note that $d(y,z)\ge n-L$. If $d(y,z)\le L$, we bound $\langle\sigma_y\sigma_z\rangle^+_\beta$ by 1, while if $d(y,z) > L$, we apply the previous inequality to y and z instead of 0 and z. The proof follows by iterating $\lfloor n/L\rfloor$ times this strategy.

2.6 Proof of Proposition 2.3

Let us introduce $M(\beta,h) = \langle\sigma_0\rangle_{\beta,h}$. Recall that $M(\beta,h)$ is differentiable in (β,h) away from the line $h = 0$.

As in the case of percolation, the proof in [ABF87] invokes three inequalities (the pages below refer to the numbering in [ABF87]): the differential inequality (1.12) page 348,

$$\frac{\partial M}{\partial \beta} \le \Big(\sum_{y\in V} J_{0,y}\Big) M \frac{\partial M}{\partial h}, \tag{2.12}$$

the more difficult differential inequality (1.9) page 347, as well as (1.13) page 348. Below, we combine Lemma 2.6 with (2.12) to conclude the proof without using (1.9) or (1.13) of [ABF87].

Since $M(\beta,h)$ is differentiable for $h > 0$, we may pass to the limit $\Lambda \nearrow V$ in Lemma 2.6 to get

$$M\frac{\partial M}{\partial \beta} \ge \frac{2}{\beta}(1-M^2)$$

for $h > 0$ and $\beta \ge \beta_c$ (once again we used that $\varphi_\beta(S) \ge 1$ for any finite $S \ni 0$ and for any $\beta \ge \beta_c$, see the comment before Proposition 2.2). Together with (2.12), we find

$$\frac{2}{\beta}(1-M^2) \le M\frac{\partial M}{\partial \beta} \le \Big(\sum_{y\in V} J_{0,y}\Big) M^2 \frac{\partial M}{\partial h}$$

which immediately implies that there exists a constant $c > 0$ such that for any $h > 0$,

$$\langle\sigma_0\rangle_{\beta_c,h} = M(\beta_c,h) \ \ge\ ch^{1/3}.$$

To conclude this article, let us recall the proof of (2.12) for completeness.

Lemma 2.8 ((1.12) page 348 of [ABF87])**.** *On $(0,1)\times(0,\infty)$, the function M satisfies the following differential inequality:*

$$\frac{\partial M}{\partial \beta} \le \Big(\sum_{y\in V} J_{0,y}\Big) M \frac{\partial M}{\partial h}.$$

Proof. Let $\beta > 0$, $h > 0$. We have

$$\frac{\partial M}{\partial \beta} = \sum_{\{x,y\}} J_{x,y}\big(\langle\sigma_0\sigma_x\sigma_y\rangle_{\beta,h} - \langle\sigma_0\rangle_{\beta,h}\langle\sigma_x\sigma_y\rangle_{\beta,h}\big).$$

The Griffith-Hurst-Sherman inequality [GHS70, (2.8)] gives

$$\begin{aligned}\langle\sigma_0\sigma_x\sigma_y\rangle_{\beta,h} - \langle\sigma_0\rangle_{\beta,h}\langle\sigma_x\sigma_y\rangle_{\beta,h} \le (\langle\sigma_0\sigma_x\rangle_{\beta,h} - \langle\sigma_0\rangle_{\beta,h}\langle\sigma_x\rangle_{\beta,h})\langle\sigma_y\rangle_{\beta,h} \\ + (\langle\sigma_0\sigma_y\rangle_{\beta,h} - \langle\sigma_0\rangle_{\beta,h}\langle\sigma_y\rangle_{\beta,h})\langle\sigma_x\rangle_{\beta,h}.\end{aligned}$$

This implies that

$$\frac{\partial M}{\partial \beta} \le \Big(\sum_{y\in V} J_{0,y}\Big) M \Big(\sum_x \langle\sigma_0\sigma_x\rangle_{\beta,h} - \langle\sigma_0\rangle_{\beta,h}\langle\sigma_x\rangle_{\beta,h}\Big) = \Big(\sum_{y\in V} J_{0,y}\Big) M \frac{\partial M}{\partial h}.$$

□

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