# A sweeping domain decomposition method for elliptic problems

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## 1 Introduction

Our goal is to introduce and study at the continuous level a domain decomposition method (DDM) that computes the exact solution in a finite number of iterations. Such DD-based solvers are often referred to as truly *optimal*. In early work [7] on the decomposition into non-overlapping strips, it has been proved that transmission conditions based on the Dirichlet-to-Neumann map led to an optimal DDM. Here, for decompositions into strips of 1D and 2D domains, we study how to choose *local* transmission conditions to obtain an optimal DDM at the continuous level.

We consider the Laplace problem in $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2\}$, with a source term $f \in L^2(\Omega)$ and a Dirichlet boundary condition $g$ on $\partial\Omega$:

$$-\Delta u = f \;\text{ in } \Omega, \quad u = g \;\text{ on } \partial\Omega. \tag{1}$$

When $\Omega$ is a square in 2D, we already have a few examples of such optimal DDMs to solve (1). If $\Omega$ is divided into two identical rectangles, then both the Dirichlet-Neumann method [1] and the Neumann-Neumann method [2] converge to the exact solution in two iterations for specific values of the relaxation parameter, regardless of the initial, as detailed in [6, Sections 4.7 and 4.8]. However, those DDMs fail to converge in a finite number of iterations as soon as the relaxation parameter is modified or the interface is slightly moved, suggesting that this ”direct” solver property relies on the underlying symmetry of the problem considered, see [3]. This observation motivates the search for transmission conditions that are adapted to the symmetry of the problem.

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## 2 The one-dimensional case

Let $L > 0$ and $a, b \in \mathbb{R}$. Taking $\Omega = (-L, L)$, problem (1) reads: find $u$ solution to

$$-u'' = f \ \text{ in } \Omega, \quad u(-L) = a, \ u(L) = b. \tag{2}$$

**Two subdomain analysis.** Let us divide $\Omega$ into $\Omega_1 = (-L, 0)$ and $\Omega_2 = (0, L)$, separated by the interface $\Gamma = \{x = 0\}$. In the same spirit as in [5], we introduce the even and odd parts of the solution $u$ denoted by $u_e$ and $u_o$, defined as: for all $x \in (-L, L)$, $u_{e,o}(x) := \frac{1}{2}\big(u(x) \pm u(-x)\big)$. This decomposition is unique and such that $u = u_e + u_o$ in $\Omega$. Moreover, since $u \in H^2(\Omega)$, we get $u_e, u_o \in H^2(\Omega)$. Given the symmetry of the problem, we are able to characterize the even/odd parts as the solutions to two boundary value problems posed on one subdomain only.

**Lemma 1** *The functions $u_e$ and $u_o$ are the unique solutions to the following boundary value problems in $\Omega_1$, where $a_e = \frac{1}{2}(a + b)$ and $a_o = \frac{1}{2}(a - b)$,*

$$\begin{cases} -u_e'' = f_e \ \text{ in } \Omega_1, \\ u_e(-L) = a_e, \ u_e'(0) = 0, \end{cases} \qquad \begin{cases} -u_o'' = f_o \ \text{ in } \Omega_1, \\ u_o(-L) = a_o, \ u_o(0) = 0. \end{cases} \tag{3}$$

***Proof*** Since the Laplace operator preserves evenness/oddness, taking the even and odd parts in (2) and restricting to $\Omega_1$ gives the two state equations and the boundary conditions at $-L$ in (3). As for the condition at the interface $\Gamma$, it follows from the Sobolev embedding theorem in 1D that $u_e, u_o \in C^1(\overline{\Omega})$, thus the standard properties of smooth even and odd functions yield at the interface $\Gamma$: $u_e'(0) = 0$ and $u_o(0) = 0$. Finally, since $f \in L^2(\Omega)$, we get that $f_e, f_o \in L^2(\Omega_1)$, which ensures that these subproblems are well-posed in $H^2(\Omega_1)$. □

Therefore, instead of solving (2), it suffices to solve the two subproblems in (3) in order to get both $u_e$ and $u_o$ in $\Omega_1$. Then we extend these functions to the whole domain $\Omega$ by symmetry, which yields $u = u_e + u_o$. In other words, we are able to obtain the solution in $\Omega$ by solving only **once** in each subdomain.

**Non-symmetric two subdomain analysis.** Let us now divide $\Omega$ into $\Omega_1 = (-L, \alpha)$ and $\Omega_2 = (\alpha, L)$ with $\alpha \neq 0$, separated by $\Gamma = \{x = \alpha\}$. In this case, the standard even/odd parts of $u$ are no longer useful since we know nothing about $u_e$ and $u_o$ at $\Gamma$. Therefore, we need to work with different definitions for the even/odd parts.

First, we define a continuous and piecewise linear map $\varphi : \Omega \to \Omega$, together with its inverse $\varphi^{-1}$, such that for all $x, \tilde{x} \in \Omega$,

$$\tilde{x} = \varphi(x) = \begin{cases} \frac{L}{L+\alpha}(x - \alpha) & \text{if } x \leq \alpha, \\ \frac{L}{L-\alpha}(x - \alpha) & \text{if } x > \alpha, \end{cases} \quad \text{and} \quad x = \varphi^{-1}(\tilde{x}) = \begin{cases} \frac{L+\alpha}{L}\,\tilde{x} + \alpha & \text{if } \tilde{x} \leq 0, \\ \frac{L-\alpha}{L}\,\tilde{x} + \alpha & \text{if } \tilde{x} > 0. \end{cases}$$

Note that $\varphi$ is not a classical change of coordinates as it is not a $C^1$-diffeomorphism. However, we can compute its weak derivative, denoted by $\sigma$ and defined in $\Omega_1 \cup \Omega_2$ by : $\sigma(x) = \varphi'(x) = \begin{cases} \sigma_- & \text{if } x < \alpha, \\ \sigma_+ & \text{if } x > \alpha. \end{cases}$ where $\sigma_\pm = \frac{L}{L\mp\alpha}$.

With this new variable $\tilde{x}$, the interface $\tilde{\Gamma}$ is located at $\{\tilde{x} = 0\}$, and the two subdomains are $\tilde{\Omega}_1 = (-L, 0)$ and $\tilde{\Omega}_2 = (0, L)$. This will allow us to reuse the arguments of the previous subsection for the even/odd parts with respect to $\tilde{x}$.

*Notation.* For any function $h$, we will denote by $\tilde{h}$ the function such that $h = \tilde{h} \circ \varphi$, and by $\tilde{h}_e$, $\tilde{h}_o$ its even/odd parts with respect to $\tilde{x}$.

Before using the even/odd decomposition, we need to analyze how this change of variable affects the original model problem (2).

**Lemma 2** *The function $\tilde{u}$ is the unique solution in $H^1(\Omega)$ to:*

$$-\tilde{\sigma}(\tilde{\sigma}\tilde{u}')' = \tilde{f} \ \text{ in } \Omega, \quad \tilde{u}(-L) = a, \ \tilde{u}(L) = b. \tag{4}$$

*In addition, $\tilde{u}$ satisfies at the interface $\tilde{\Gamma}$ the conditions:*

$$\tilde{u}(0^-) = \tilde{u}(0^+) \quad \text{and} \quad \sigma_- \tilde{u}'(0^-) = \sigma_+ \tilde{u}'(0^+). \tag{5}$$

***Proof*** First, let us show that $\tilde{u}$ solves (4). The state equation can be obtained using the standard chain rule for smooth functions in $\Omega_1 \cup \Omega_2$ and the boundary conditions using $\varphi(\pm L) = \pm L$.

Now, let us prove that the problem (4) is well-posed in $H^1(\Omega)$. Since $\tilde{\sigma} > 0$ a.e. in $\Omega$, the differential equation in (4) can be rewritten as $-(\tilde{\sigma}\tilde{u}')' = \frac{1}{\tilde{\sigma}}\tilde{f}$ a.e. in $\Omega$. This corresponds to the usual 1D diffusion equation with non constant diffusion coefficient. Given that $\tilde{\sigma}$ is piecewise constant, we get $\frac{1}{\tilde{\sigma}} \in L^\infty(\Omega)$, and so $\frac{1}{\tilde{\sigma}}\tilde{f} \in L^2(\Omega)$. It follows that the solution $\tilde{u}$ to (4) exists and is unique in $H^1(\Omega)$.

Finally, the conditions at the interface $\tilde{\Gamma}$ can be found using the relation $u = \tilde{u} \circ \varphi$ together with the regularities of $u$ and $\varphi$. Indeed, since $u$ and $\varphi^{-1}$ are both continuous in $\Omega$, $\tilde{u}$ is also continuous, and in particular at $\tilde{x} = 0$ we obtain $\tilde{u}(0^-) = \tilde{u}(0^+)$. In addition, differentiating the relation between $u$ and $\tilde{u}$ yields $u' = \varphi' \cdot (\tilde{u}' \circ \varphi)$. We know that $u'$ and $\varphi'$ admit left and right limits at $x = \alpha$. Computing these limits and using the continuity of $u'$ at $\alpha$, we obtain $\sigma_- \tilde{u}'(0^-) = \sigma_+ \tilde{u}'(0^+)$. □

As in the previous subsection, we decompose $\tilde{u}$ into its even/odd parts and characterize these functions as solutions to a problem in one subdomain only.

**Lemma 3** *The couple $(\tilde{u}_e, \tilde{u}_o)$ is the unique solution to a coupled boundary value problem in $\tilde{\Omega}_1$,*

$$\begin{aligned}
&-(\sigma_-^2 + \sigma_+^2)\tilde{u}_e'' - (\sigma_-^2 - \sigma_+^2)\tilde{u}_o'' = 2\tilde{f}_e \ \text{ in } \tilde{\Omega}_1, \\
&-(\sigma_-^2 - \sigma_+^2)\tilde{u}_e'' - (\sigma_-^2 + \sigma_+^2)\tilde{u}_o'' = 2\tilde{f}_o \ \text{ in } \tilde{\Omega}_1, \\
&\tilde{u}_e(-L) = a_e, \ \tilde{u}_o(-L) = a_o, \ \tilde{u}_e'(0) = \frac{\alpha}{L}\tilde{u}_o'(0), \ \tilde{u}_o(0) = 0.
\end{aligned} \tag{6}$$

***Proof*** In order to obtain the desired state equations, we first decompose (4) into its even and odd symmetric parts, which leads to a set of two equations in $\Omega$, involving the even and odd parts of $\tilde{u}$, $\tilde{\sigma}$ and $\tilde{f}$. Restricting these equations to $\tilde{\Omega}_1$ and using that $\tilde{\sigma}_{e,o=}\frac{1}{2}(\sigma_- \pm \sigma_+)$ are constant in this subdomain, we end up with (6). Then, taking the even/odd parts in the boundary conditions, we get $\tilde{u}_e(-L) = a_e$, $\tilde{u}_o(-L) = a_o$.

Now, to derive the interface conditions at $\tilde{\Gamma}$, we compute at $\tilde{x} = 0$ the left and right limits of $\tilde{u}_e$, $\tilde{u}_o$ and their derivatives,

$$\tilde{u}_e(0^-) = \tilde{u}_e(0^+),\ \ \tilde{u}_e'(0^-) = -\tilde{u}_e'(0^+),\ \ \tilde{u}_o(0^-) = -\tilde{u}_o(0^+),\ \ \tilde{u}_o'(0^-) = \tilde{u}_o'(0^+).$$

Then, writing $\tilde{u}$ as $\tilde{u}_e + \tilde{u}_o$ in (5) and using the relations above enables us to obtain two relations in terms of left limits only,

$$\tilde{u}_e(0^-)+\tilde{u}_o(0^-) = \tilde{u}_e(0^-)-\tilde{u}_o(0^-), \sigma_-\big(\tilde{u}_e'(0^-)+\tilde{u}_o'(0^-)\big) = \sigma_+\big(-\tilde{u}_e'(0^-)+\tilde{u}_o'(0^-)\big).$$

From the point of view of $\tilde{\Omega}_1$, the left limit $0^-$ can be replaced by 0, and after some algebraic manipulations, we obtain the interface conditions in (6).

Regarding well-posedness of problem (6), we have already proved the existence of $\tilde{u}$ in Lemma 2. Thus, defining the couple $(\tilde{u}_e, \tilde{u}_o)$ as the even/odd parts of $\tilde{u}$ provides us with a solution to (6), as shown previously. As for the uniqueness of the solution, it can be proved by showing that the only possible solution to the homogeneous problem is $(0, 0)$, which is rather straightforward. □

Returning to the variable $x$, this lemma gives us a characterization of the functions $\tilde{u}_e \circ \varphi$ and $\tilde{u}_o \circ \varphi$, which motivates the introduction of the notion of *pseudo* even/odd decomposition associated with a piecewise linear change of variables $\varphi$.

**Definition 1** Let $h \in L^p(\Omega)$, with $p \in [1, +\infty]$, we define the pseudo even and odd parts of $h$ as, for almost all $x \in \Omega$, $h_{\bar{e},\bar{o}}(x) := \big[(h \circ \varphi^{-1})_{e,o} \circ \varphi\big](x)$.

These functions provide us with a new splitting of the original function $h = h_{\bar{e}} + h_{\bar{o}}$, which will be referred to as the pseudo even/odd decomposition of $h$.

**Theorem 1** *The couple $(u_{\bar{e}}, u_{\bar{o}})$ is the unique solution to a system of differential equations in $\Omega_1$, with $\rho = \sigma_+/\sigma_-$,*

$$\begin{aligned} &-\frac{1}{2}\begin{bmatrix}(1+\rho^2)\partial_{xx} & (1-\rho^2)\partial_{xx}\\ (1-\rho^2)\partial_{xx} & (1+\rho^2)\partial_{xx}\end{bmatrix}\begin{bmatrix}u_{\bar{e}}\\ u_{\bar{o}}\end{bmatrix} = \begin{bmatrix}f_{\bar{e}}\\ f_{\bar{o}}\end{bmatrix} \ \ \text{in } \Omega_1,\\ &u_{\bar{e}}(-L) = a_e,\ \ u_{\bar{o}}(-L) = a_o,\ \ u_{\bar{e}}'(\alpha) = \frac{\alpha}{L}\,u_{\bar{o}}'(\alpha),\ \ u_{\bar{o}}(\alpha) = 0. \end{aligned} \tag{7}$$

***Proof*** Given the definition of $u_{\bar{e}}$, we have for any $x \in \Omega_1$ that $\frac{1}{\sigma_-^2}\frac{d^2u_{\bar{e}}}{dx^2}(x) = \frac{1}{\sigma_-^2}\frac{d^2}{dx^2}\big[\tilde{u}_e \circ \varphi\big](x) = \frac{d^2\tilde{u}_e}{d\tilde{x}^2}(\tilde{x})$, where $\tilde{x} = \varphi(x)$. And the same relation holds for $\tilde{u}_o$ and $u_{\bar{o}}$. Therefore, dividing each equation in (6) by 2 on both sides and composing to the right with $\varphi$ yields (7). As for the boundary conditions, we can again deduce them directly from those in (6) using $u_{\bar{e}}(-L) = \tilde{u}_e(-L)$ and $\frac{du_{\bar{e}}}{dx}(\alpha) = \sigma_-\frac{d\tilde{u}_e}{d\tilde{x}}(0)$, which also hold for $\tilde{u}_o$ and $u_{\bar{o}}$.

Regarding the existence and uniqueness of the solution to (7), they can be proved in the same way as the existence and uniqueness of $(\tilde{u}_e, \tilde{u}_o)$ in Lemma 3. □

Therefore, the approach followed in the symmetric case can be extended to the non-symmetric case by means of the pseudo even/odd decomposition introduced above. Indeed, solving the coupled problem (7) once in $\Omega_1$ yields $(u_{\tilde{e}}, u_{\tilde{o}})$, from which we deduce $u = u_{\tilde{e}} + u_{\tilde{o}}$ in $\Omega_1$. Now, in order to obtain $u$ in $\Omega_2$, we use the fact that $\varphi^{-1} \circ (-\varphi) : \Omega_1 \longrightarrow \Omega_2$ is one-to-one, together with the relation

$$[u \circ \varphi^{-1} \circ (-\varphi)](x) = u_{\tilde{e}}(x) - u_{\tilde{o}}(x), \quad \forall x \in \Omega_1.$$

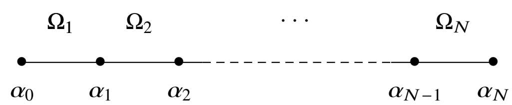


**Fig. 1** Decomposition of $\Omega$ into $N$ subdomains.

**Many subdomain analysis.** Let us consider a chain of $N \geq 3$ non-overlapping subdomains $\Omega_1, \ldots, \Omega_N$ based on a uniform subdivision $(\alpha_0, \alpha_1, \ldots, \alpha_N)$ of $[-L, L]$, see Figure 1. For each $i \in \{1, \ldots, N\}$, we have $\Omega_i = (\alpha_{i-1}, \alpha_i)$, and the interface between $\Omega_i$ and $\Omega_{i+1}$ is denoted by $\Gamma_i = \{x = \alpha_i\}$.

Using the result from the non-symmetric two subdomain case with $\alpha = \alpha_1$, we compute $u$ in $\Omega_1$. In parallel, taking $\alpha = \alpha_{N-1}$, we compute $u$ in $\Omega_N$. Once we know $u$ in $\Omega_1 \cup \Omega_N$, we propagate the information at the interfaces $\Gamma_1$ and $\Gamma_{N-1}$, and reapply the same process in $\Omega_2 \cup \cdots \cup \Omega_{N-1}$. Such a procedure may be understood as a sweeping algorithm, where we start solving with the boundary subdomains and then propagate the information towards the center of the domain, see the details in Algorithm 1.

Using the preliminary results from the previous sections, we are able to prove that this algorithm is well-posed, and that it converges in a finite number of steps.

**Theorem 2** *For a partition of* $\Omega$ *into* $N$ *identical subdomains, Algorithm 1 converges in* $N/2$ *steps if* $N$ *is even, and* $(N+1)/2$ *steps if* $N$ *is odd.*

## 3 The two-dimensional case

For $L > 0$, we consider the rectangle $\Omega = (-L, L) \times (-1, 1)$, and we assume that $f \in H^1(\Omega)$ and $g \in H^{\frac{5}{2}}(\partial\Omega)$, ensuring that the solution $u$ to (1) belongs to $H^2(\Omega)$ and has $H^3$ regularity away from the corners of $\Omega$.

**Two subdomain analysis.** We divide $\Omega$ into two identical subdomains $\Omega_1$ and $\Omega_2$ using the vertical interface $\Gamma = \{0\} \times (-L, L)$. For $i \in \{1, 2\}$, we define $\partial\Omega_i^0 = \partial\Omega_i \setminus \overline{\Gamma}$. Since the geometric configuration is symmetric with respect to the $y$-axis, we decompose $u$ into its even and odd parts with respect to $x$, that is, $u_{e,o}(x, y) := \frac{1}{2}(u(x, y) \pm u(-x, y))$, for all $(x, y) \in \Omega$. Then, the property $u = u_e + u_o$ and the

**Algorithm 1** Cross Sweep 1D

1. Initialize the indices for the left and right subdomains $\ell = 1$, $r = N$, and set $a_\ell = a$, $b_r = b$.
2. Follow one of the following sets of instructions, depending on the value of $r - \ell + 1$.

- If $r - \ell + 1 = 1$ (one subdomain), solve directly (2) in $\Omega_\ell = \Omega_r$ with $a_\ell$ and $b_r$ as left and right Dirichlet boundary conditions.
- If $r - \ell + 1 = 2$ (two subdomains), solve (2) in $\Omega_\ell \cup \Omega_r$ with $a_\ell$ and $b_r$ as left and right Dirichlet boundary conditions, using the classical even/odd decomposition.
- If $r - \ell + 1 > 2$ (more than two subdomains), follow these steps.
  a. Compute: $L_{\ell,r} = \alpha_{r+1} - \alpha_{\ell-1}, \rho_k = \frac{\sigma_{k,+}}{\sigma_{k,-}}$, where $\sigma_{k,\pm} = \frac{L_{\ell,r}}{L_{\ell,r} \mp \alpha_k}$ for $k \in \{\ell, r\}$.
  b. Compute: $a_{e_\ell} = \frac{1}{2}(a_\ell + b_r),\ a_{o_\ell} = \frac{1}{2}(a_\ell - b_r),\ b_{e_r} = a_{e_\ell},\ b_{o_r} = -a_{o_\ell}$.
  c. Compute the pseudo even/odd parts of $u$: $(u_{\tilde{e}_\ell}, u_{\tilde{o}_\ell})$ in $\Omega_\ell$ and $(u_{\tilde{e}_r}, u_{\tilde{o}_r})$ in $\Omega_r$ s.t.

$$\begin{cases} -\dfrac{1}{2}\begin{bmatrix} (1+\rho_\ell^2)\partial_{xx} & (1-\rho_\ell^2)\partial_{xx} \\ (1-\rho_\ell^2)\partial_{xx} & (1+\rho_\ell^2)\partial_{xx} \end{bmatrix}\begin{bmatrix} u_{\tilde{e}_\ell} \\ u_{\tilde{o}_\ell} \end{bmatrix} = \begin{bmatrix} f_{\tilde{e}_\ell} \\ f_{\tilde{o}_\ell} \end{bmatrix} \text{ in } \Omega_\ell, \\ u_{\tilde{e}_\ell}(\alpha_{\ell-1}) = a_{e_\ell},\ u_{\tilde{o}_\ell}(\alpha_{\ell-1}) = a_{o_\ell},\ u'_{\tilde{e}_\ell}(\alpha_\ell) = \dfrac{\alpha_\ell}{L_{\ell,r}} u'_{\tilde{o}_\ell}(\alpha_\ell),\ u_{\tilde{o}_\ell}(\alpha_\ell) = 0. \end{cases}$$

$$\begin{cases} -\dfrac{1}{2}\begin{bmatrix} (1+\rho_r^2)\partial_{xx} & (1-\rho_r^2)\partial_{xx} \\ (1-\rho_r^2)\partial_{xx} & (1+\rho_r^2)\partial_{xx} \end{bmatrix}\begin{bmatrix} u_{\tilde{e}_r} \\ u_{\tilde{o}_r} \end{bmatrix} = \begin{bmatrix} f_{\tilde{e}_r} \\ f_{\tilde{o}_r} \end{bmatrix} \text{ in } \Omega_r, \\ u_{\tilde{e}_r}(\alpha_{r+1}) = b_{e_r},\ u_{\tilde{o}_r}(\alpha_{r+1}) = b_{o_r},\ u'_{\tilde{e}_r}(\alpha_r) = -\dfrac{\alpha_r}{L_{\ell,r}} u'_{\tilde{o}_r}(\alpha_r),\ u_{\tilde{o}_r}(\alpha_r) = 0. \end{cases}$$

  d. Build $u = u_{\tilde{e}_\ell} + u_{\tilde{o}_\ell}$ in $\Omega_\ell$ and $u = u_{\tilde{e}_r} + u_{\tilde{o}_r}$ in $\Omega_r$.
  e. Update the Dirichlet boundary conditions $a_\ell = u(\alpha_\ell)$, $b_r = u(\alpha_r)$, and the indices $\ell = \ell + 1$, $r = r - 1$, then go back to step 2.

uniqueness of the decomposition still hold. In addition, as in the one-dimensional case, we are able to deduce from (1) the two boundary value problems satisfied by $u_e$ and $u_o$.

**Lemma 4** *The functions $u_e$ and $u_o$ are the unique solutions to boundary value problems in $\Omega_1$,*

$$\begin{cases} -\Delta u_e = f_e \text{ in } \Omega_1, \\ u_e = g_e \text{ on } \partial\Omega_1^0,\ \partial_n u_e = 0 \text{ on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta u_o = f_o \text{ in } \Omega_1\ , \\ u_o = g_o \text{ on } \partial\Omega_1^0,\ u_o = 0 \text{ on } \Gamma. \end{cases} \tag{8}$$

***Proof*** Since the Laplace operator preserves evenness/oddness with respect to $x$, decomposing (1) into its even/odd parts and restricting to $\Omega_1$ gives the state equations and the boundary condition on $\partial\Omega_1^0$ in (8). Then, if we introduce a smooth neighborhood of the interface, say $\Gamma \subset \omega_\Gamma \subset \Omega$, we have $u \in H^3(\omega_\Gamma) \hookrightarrow C^1(\omega_\Gamma)$ (since $\omega_\Gamma$ has a smooth boundary). Therefore, given the definitions of $u_e$ and $u_o$, they both belong to $C^1(\omega_\Gamma)$, and we obtain at the interface $\Gamma$: $\partial_n u_e = 0$ and $u_o = 0$. The well-posedness of these two subproblems in $H^1(\Omega_1)$ follows from standard arguments. □

Again, we can solve these two subproblems in $\Omega_1$, then extend $u_e$ and $u_o$ to $\Omega$ by symmetry, and finally recompose $u = u_e + u_o$ in the whole domain $\Omega$.

**Non-symmetric two subdomain analysis.** We divide $\Omega$ into two subdomains $\Omega_1$ and $\Omega_2$ by means of the vertical interface $\Gamma = \{x = \alpha\}$, with $\alpha \neq 0$. This time, we recover a symmetric configuration using the map $\phi(x, y) = \big(\varphi(x), y\big)$, where $\varphi$ has been defined in the previous section. Following the exact same steps as in the 1D case, we end up defining the notion of pseudo even/odd decomposition.

**Definition 2** Let $h \in L^p(\Omega)$, with $p \in [1, +\infty]$, we define the pseudo even and odd parts of $h$ as, for almost all $(x, y) \in \Omega$, $h_{\bar{e},\bar{o}}(x, y) := \big[(h \circ \phi^{-1})_{e,o} \circ \phi\big](x, y)$.

This new pseudo even/odd decomposition of $h$ still gives the splitting $h = h_{\bar{e}} + h_{\bar{o}}$.

**Theorem 3** *The couple* $(u_{\bar{e}}, u_{\bar{o}})$ *is the unique solution to a boundary value problem in* $\Omega_1$*, where* $\rho = \sigma_+/\sigma_-$*,*

$$
\begin{aligned}
&-\begin{bmatrix} \frac{1}{2}(1+\rho^2)\partial_{xx} + \partial_{yy} & \frac{1}{2}(1-\rho^2)\partial_{xx} \\ \frac{1}{2}(1-\rho^2)\partial_{xx} & \frac{1}{2}(1+\rho^2)\partial_{xx} + \partial_{yy} \end{bmatrix} \begin{bmatrix} u_{\bar{e}} \\ u_{\bar{o}} \end{bmatrix} = \begin{bmatrix} f_{\bar{e}} \\ f_{\bar{o}} \end{bmatrix} \quad \text{in } \Omega_1, \\
&u_{\bar{e}} = g_{\bar{e}} \ \text{ and } \ u_{\bar{o}} = g_{\bar{o}} \text{ on } \partial\Omega_1^0, \ \ \partial_n u_{\bar{e}} = \frac{\alpha}{L}\,\partial_n u_{\bar{o}} \ \text{ and } \ u_{\bar{o}} = 0 \text{ on } \Gamma.
\end{aligned} \tag{9}
$$

***Proof*** The proof of this result relies on the same arguments as those in the proof of Theorem 1. The approach and the steps are also the same, except for a few technicalities that cannot be detailed here due to page limitations. □

**Many subdomain analysis.** The algorithm presented in the 1D case can be naturally extended to one-level decompositions into $N \geq 3$ identical and non-overlapping subdomains $\Omega_1, \ldots, \Omega_N$. More specifically, using the subdivision $(\alpha_0, \ldots, \alpha_N)$ from the 1D case, and setting $\Omega_i = (\alpha_{i-1}, \alpha_i) \times (-1, 1)$ for each $i \in \{1, \ldots, N\}$, we are able to prove the following result.

**Theorem 4** *For a partition of* $\Omega$ *into N identical subdomains in strips, the 2D version of Algorithm 1 converges in* $N/2$ *steps if N is even, and* $(N+1)/2$ *steps if N is odd.*

## 4 Numerical results

We test our sweeping algorithm on two simple benchmarks. The first is one-dimensional : we divide $\Omega = (-1, 1)$ into $N = 6$ subdomains, discretized with mesh size $h = 1.7 \cdot 10^{-3}$. We set $f$, $a$, $b$ so that the exact solution is $u(x) = e^{2x}$. As predicted by Theorem 2, the algorithm converges in 3 iterations, see Figure 2. Moreover, the problem is first solved in the boundary subdomains, then the information is propagated towards the center of $\Omega$. The second example is analogous, but two-dimensional: we divide $\Omega = (-1, 1)^2$ into $N = 8$ subdomains into strips, discretized with mesh size $h = 8.3 \cdot 10^{-3}$. We set $f$ and $g$ so that the exact solution is $u(x, y) = e^x \cos\left(\frac{\pi}{2} y\right)$. As shown in Figure 2, the algorithm behaves as in the one-dimensional case, solving from the boundary subdomains towards the center of $\Omega$, and converges in 4 iterations, illustrating the result of Theorem 4.

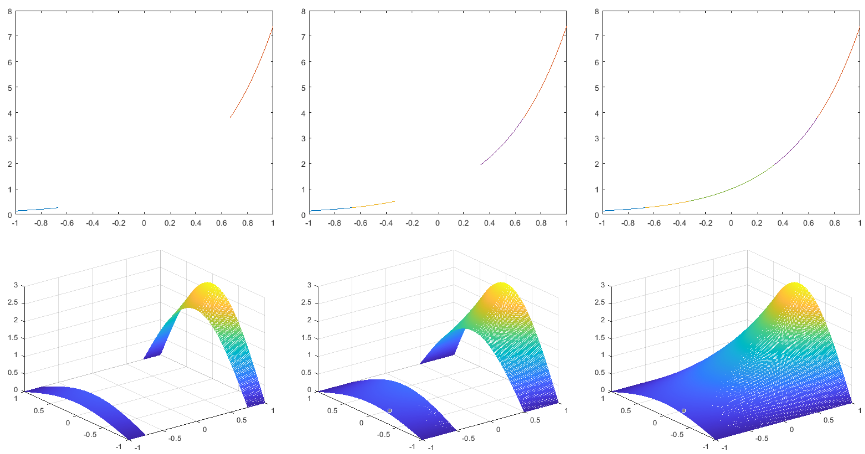

**Fig. 2** Iterations 1, 2 3 for the 1D example (top), and 1, 2, 4 for the 2D example (bottom).

In this short paper, we presented a DDM that converges in a finite number of iterations. In the case of two non-symmetric subdomains, we found an analytical decomposition of the solution into pseudo even/odd components, for which we were able to derive exact interface conditions. This result enabled us to build a sweeping DDM, for 1D domains and 2D domains decomposed into strips, that solves the problem from the boundary subdomains towards the center of the domain. The algorithm has been validated on two simple benchmarks. A natural extension of this work would be the analysis of regular decompositions with cross-points in 2D. The key step would be the generalization of the even/odd decomposition presented in [5, 4] to the case of four non-symmetric subdomains, which could be done using a change of variables $\phi$ acting on both $x$ and $y$ coordinates.

## References


1. P. E. Bjørstad and O. B. Widlund. Solving elliptic problems on regions partitioned into substructures. In *Elliptic Problem Solvers*, pages 245–255. Elsevier, 1984.
2. J.-F. Bourgat, R. Glowinski, P. Le Tallec, and M. Vidrascu. *Variational formulation and algorithm for trace operation in domain decomposition calculations*. PhD thesis, INRIA, 1988.
3. F. Chaouqui, M. J. Gander, and K. Santugini. On nilpotent subdomain iterations. In *Domain Decomposition Methods in Science and Engineering XXIII*, pages 125–133. Springer, 2017.
4. B. Chaudet-Dumas and M. J. Gander. Cross-points in the Neumann-Neumann method. In *International Conference on Domain Decomposition Methods*, pages 117–124. Springer, 2022.
5. B. Chaudet-Dumas and M. J. Gander. Cross-points in the Dirichlet-Neumann method I: well-posedness and convergence issues. *Numerical Algorithms*, 92(1):301–334, 2023.
6. Gabriele Ciaramella and Martin J Gander. *Iterative methods and preconditioners for systems of linear equations*. SIAM, 2022.
7. F. Nataf, F. Rogier, and E. De Sturler. *Optimal interface conditions for domain decomposition methods*. PhD thesis, CMAP Ecole Polytechnique, 1994.