Asymptotic Analysis for Different Partitionings of RLC Transmission Lines

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1 Introduction

Among many applications of parallel computing, solving large systems of ordinary differential equations (ODEs) which arise from large scale electronic circuits, or discretizations of partial differential equations (PDEs), form an important part. A systematic approach to their parallel solution are Waveform Relaxation (WR) techniques, which were introduced in 1982 as a tool for circuit solvers (see [5]). These techniques are based on partitioning large circuits into smaller sub-circuits, which are then solved separately over multiple time steps, and the overall solution is obtained by an iteration between the sub-circuits. However, these techniques can lead to non-uniform and potentially slow convergence over large time windows. To overcome this issue, optimized waveform relaxation techniques (OWR) were introduced, which are based on optimizing parameters. The application of OWR to RC circuits and its asymptotic analysis can be found in [2]. We introduce overlap and analyze these methods for an RLCG transmission line type circuits with $G = 0$, which corresponds to no current loss in the dielectric medium. For the one node overlapping case, see [1]. We show that these circuit equations represent Yee scheme discretizations of the well known Maxwell equations in 1D, and give some asymptotic results.

2 Circuit Equations

We consider an infinitely long RLC transmission line where the constants R, L, C represent resistance, inductance, and capacitance per unit length of the line. The circuit equations are specified using Modified Nodal Analysis [4], whose principal

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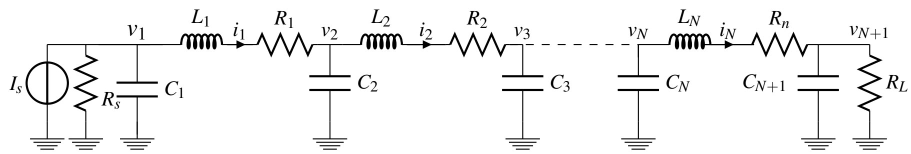

Fig. 1: RLC Transmission Line of length N.

element is Kirchhoff's circuit law. These circuit equations for an RLC transmission line (see Fig 1), with the length of the circuit, N, going to infinity, lead to a system of differential equations in time,

$$\dot{\mathbf{x}} = \begin{bmatrix} \ddots & \ddots & \ddots & & & & & \\ & a & b & -a & & & & \\ & & -c & 0 & c & & & \\ & & & a & b & -a & & \\ & & & & -c & 0 & c & \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix} \mathbf{x} + \mathbf{f}, \tag{1}$$

where the entries in the above tridiagonal matrix are

$$a = \frac{1}{L}, \qquad b = -\frac{R}{L}, \qquad c = -\frac{1}{C},$$

with an unknown vector $\mathbf{x}(t) := (..,x_{-1}(t),x_0(t),x_1(t),..)^T$ and $\mathbf{f}$(t)=$(I_s(t)/C,0,..,0)^T$. The unknowns in $\mathbf{x}(t)$ are the voltages $v(t)$ and currents $i(t)$ at the nodes aligned in a systematic way, $x_{2j}(t) = i_j(t)$ and $x_{2j-1} = v_j(t)$ for $j \in \mathbb{Z}$. Thus the even index rows, which have a and b elements correspond to current unknowns while the odd index rows correspond to voltage unknowns. We assume that all the constants R,L,C are bounded to have a well posed problem.

Before analyzing the WR algorithm, we link these circuit equations to the well known Maxwell's equations in 1D. The coupled differential equations of system (1) can be explicitly written as

$$\frac{\partial x_{2m}}{\partial t} = a x_{2m-1} + b x_{2m} - a x_{2m+1} \qquad \text{and} \qquad \frac{\partial x_{2m+1}}{\partial t} = -c x_{2m} + c x_{2m+2},$$

for $m \in \mathbb{Z}$. The parameters R,C,L are defined per unit length, $R = R_T \Delta x$, $C = C_T \Delta x$ and $L = L_T \Delta x$. Hence, substituting the values of the constants a,b,c and interpreting the differences as derivatives, we arrive at

$$\frac{\partial i}{\partial t} + \frac{1}{L_T}\frac{\partial v}{\partial x} = -\frac{R_T}{L_T} i \qquad \text{and} \qquad \frac{\partial v}{\partial t} + \frac{1}{C_T}\frac{\partial i}{\partial x} = 0.$$

Comparing with the Maxwell's equations in 1D,

$$\frac{\partial E}{\partial t}+\frac{1}{\varepsilon}\frac{\partial H}{\partial x}=-\frac{\sigma}{\varepsilon}E \qquad \text{and} \qquad \frac{\partial H}{\partial t}+\frac{1}{\mu}\frac{\partial E}{\partial x}=0,$$

we see that $i \sim E$, $v \sim H$, $L_T \sim \varepsilon$, $C_T \sim \mu$ and $R_T \sim \sigma$.

3 The Classical WR Algorithm

In this section, we apply the classical waveform relaxation algorithm to an RLC transmission line of infinite length and analyze its convergence. To start with this algorithm, we divide system (1) into two subsystems with unknowns $\mathbf{x}(s_1)$ and $\mathbf{x}(s_2)$, where both unknowns depend on time t but for simplicity, we have removed t from the notation. Since the system (1) consists of two different equations, one for current and the other for voltage, the type of partitioning is interesting. We first partition the system at an odd row, say at $x_{-1}(t)$, and overlap n nodes of the circuit (which corresponds to an overlap of $2n$ nodes of the two subsystems in (2) below). Thus, initially, both the subsystems have equal length and then we increase the size of $\mathbf{x}(s_1)$ by $2n-1$ to include the overlap while the size of $\mathbf{x}(s_2)$ remains unchanged. This leads to two new subsystems of differential equations

$$\begin{aligned}
\dot{\mathbf{x}}^{k+1}(s_1) &= \begin{bmatrix} \ddots & \ddots & \ddots & \\ & a & b & -a \\ & & -c & 0 \end{bmatrix} \mathbf{x}^{k+1}(s_1) + \begin{bmatrix} \vdots \\ f_{2n-4} \\ f_{2n-3} \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \\ c\,x^{k+1}_{2n-2}(s_1) \end{bmatrix}, \\
\dot{\mathbf{x}}^{k+1}(s_2) &= \begin{bmatrix} 0 & c & & \\ a & b & -a & \\ & \ddots & \ddots & \ddots \end{bmatrix} \mathbf{x}^{k+1}(s_2) + \begin{bmatrix} f_{-1} \\ f_0 \\ \vdots \end{bmatrix} + \begin{bmatrix} -c\,x^{k+1}_{-2}(s_2) \\ 0 \\ \vdots \end{bmatrix},
\end{aligned} \tag{2}$$

where k is the iteration index and the unknowns $x^{k+1}_{2n-2}(s_1)$ and $x^{k+1}_{-2}(s_2)$ are given by transmission conditions,

$$x^{k+1}_{2n-2}(s_1)=x^{k}_{2n-2}(s_2), \qquad x^{k+1}_{-2}(s_2)=x^{k}_{-2}(s_1), \tag{3}$$

which exchange only current at the interfaces. For the convergence study, we consider the homogeneous problem $\mathbf{f}=0$ and zero initial conditions $\mathbf{x}(0)=0$. The Laplace transform with $s \in \mathbb{C}$ for the subsystems (2) yields

$$\begin{aligned}
s\,\hat{\mathbf{x}}^{k+1}(s_1) &= \begin{bmatrix} \ddots & \ddots & \ddots & \\ & a & b & -a \\ & & -c & 0 \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{x}^{k+1}_{2n-4}(s_1) \\ \hat{x}^{k+1}_{2n-3}(s_1) \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \\ c\,\hat{x}^{k}_{2n-2}(s_2) \end{bmatrix}, \\
s\,\hat{\mathbf{x}}^{k+1}(s_2) &= \begin{bmatrix} 0 & c & & \\ a & b & -a & \\ & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \hat{x}^{k+1}_{-1}(s_2) \\ \hat{x}^{k+1}_{0}(s_2) \\ \vdots \end{bmatrix} + \begin{bmatrix} -c\,\hat{x}^{k}_{-2}(s_1) \\ 0 \\ \vdots \end{bmatrix}.
\end{aligned} \tag{4}$$

Theorem 1. *The convergence factor of the classical algorithm for an RLC transmission line of infinite length with n nodes overlap is*

$$\rho_{cla}(s,a,b,c) = \begin{cases} (\lambda_1)^{2n} & , \quad |\lambda_1| < 1, \\ (\lambda_2)^{2n} & , \quad |\lambda_1| > 1, \end{cases} \tag{5}$$

where $\lambda_{1,2} := \frac{2ac - s(s-b) \pm \sqrt{(2ac - s(s-b))^2 - 4a^2c^2}}{2ac}$ *with the property* $\lambda_1\lambda_2 = 1$.

Proof. Solving the first subsystem of (4) corresponds to solving coupled recurrence equations, for $j = n-2, ..., 0, -1, -2, ...$

$$\begin{aligned} -a\hat{x}_{2j-1}^{k+1}(s_1) + (s-b)\hat{x}_{2j}^{k+1}(s_1) + a\hat{x}_{2j+1}^{k+1}(s_1) = 0, \\ c\hat{x}_{2j}^{k+1}(s_1) + s\hat{x}_{2j+1}^{k+1}(s_1) - c\hat{x}_{2j+2}^{k+1}(s_1) = 0. \end{aligned}$$

To simplify, we introduce the new notations $\hat{p}_j^{k+1} := \hat{x}_{2j}^{k+1}(s_1)$ and $\hat{q}_j^{k+1} := \hat{x}_{2j+1}^{k+1}(s_1)$ for $j = n-2, ...0, -1, ..,$ to get

$$-a\,\hat{q}_{j-1}^{k+1} + (s-b)\,\hat{p}_j^{k+1} + a\,\hat{q}_j^{k+1} = 0 \quad \text{and} \quad c\,\hat{p}_j^{k+1} + s\,\hat{q}_j^{k+1} - c\,\hat{p}_{j+1}^{k+1} = 0. \tag{6}$$

Solving the first equation for $\hat{p}_j^{k+1}$ and substituting it into the second equation yields $ac\,\hat{q}_{j-1}^{k+1} + [s\,(s-b) - 2\,ac]\,\hat{q}_j^{k+1} + ac\,\hat{q}_{j+1}^{k+1} = 0$. The general solution of this recurrence equation is

$$\hat{q}_j^{k+1} = A^{k+1}\lambda_1^j + B^{k+1}\lambda_2^j,$$

where $\lambda_{1,2} := \frac{2ac - s(s-b) \pm \sqrt{(2ac - s(s-b))^2 - 4a^2c^2}}{2ac}$ are the roots of the characteristic equation and A^{k+1}, B^{k+1} are constants to be determined. We first consider the case $|\lambda_1| < 1$. Since $|\lambda_1^{2j-1}| \to \infty$ as $j \to -\infty$ and $\hat{q}_j^{k+1}$ are bounded, we have $A^{k+1} = 0$. The coupled equations (6) gives $\hat{q}_j^{k+1} = B^{k+1}\,\lambda_2^j$ and $\hat{p}_j^{k+1} = \frac{aB^{k+1}}{s-b}[\lambda_2^{j-1} - \lambda_2^j]$.

Similarly, from the second subsystem of (4), for $j = 0, 1, 2, ..,$ we define $\hat{u}_j^{k+1} := \hat{x}_{2j}^{k+1}(s_2)$ and $\hat{w}_j^{k+1} := \hat{x}_{2j-1}^{k+1}(s_2)$ to arrive at $\hat{w}_j^{k+1} = D^{k+1}\lambda_1^j$ and $\hat{u}_j^{k+1} = \frac{aD^{k+1}}{s-b}[\lambda_1^j - \lambda_1^{j+1}]$. To determine the constants B^{k+1} and D^{k+1}, we use transmission conditions in (3). The last equation of the first subsystem of (4) gives $c\,\hat{p}_{n-2}^{k+1} + s\,\hat{q}_{n-2}^{k+1} = c\,\hat{u}_{n-1}^k$. Using the properties $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 = 2 - \frac{s\,(s-b)}{ac}$, we have $B^{k+1} = -D^k(\lambda_1^2)^{n-1}$. Similarly, the first equation of the second subsystem of (4) gives $D^{k+1} = -B^k\lambda_1^2$. Therefore, we have $B^{k+1} = (\lambda_1)^{2n}B^{k-1}$ and $D^{k+1} = (\lambda_1)^{2n}D^{k-1}$ which implies $\hat{x}_j^{k+1}(s_1) = (\lambda_1)^{2n}\,\hat{x}_j^{k-1}(s_1)$, and $\hat{x}_j^{k+1}(s_2) = (\lambda_1)^{2n}\,\hat{x}_j^{k-1}(s_2)$. Similarly, we have the same convergence factor when $|\lambda_1| > 1$.

We observe that the convergence factor is the same for all the nodes irrespective of which subsystem they belong to. Also, the convergence becomes faster by increasing the number of nodes in the overlap. Note also that, if we partition the system at an even row corresponding to a current equation, we still obtain the same convergence factor.

4 Optimized WR Algorithm

It has been observed that increasing the number of nodes in the overlap does not increase the convergence speed very much, especially for large time windows. This forces us to look for better transmission conditions to make the exchange of information between the subsystems more effective. Thus, we propose general transmission conditions for splitting the circuit at a voltage node,

$$\begin{aligned} x_{2n-2}^{k+1}(s_1) + \alpha x_{2n-3}^{k+1}(s_1) &= x_{2n-2}^{k}(s_2) + \alpha x_{2n-3}^{k}(s_2), \\ x_{-1}^{k+1}(s_2) + \beta x_{-2}^{k+1}(s_2) &= x_{-1}^{k}(s_1) + \beta x_{-2}^{k}(s_1), \end{aligned} \tag{7}$$

where α and β are weighting factors. We can have similar transmission conditions for splitting at a current node. These transmission conditions can be viewed as Robin transmission conditions which transfer both current and voltage at the boundary. Under the condition, $\alpha = 0,\ \beta = \infty$, we recover the classical transmission conditions (3).

Theorem 2. *The convergence factor of the OWR algorithm for an RLC transmission line of infinite length with n nodes overlap and with splitting at a voltage node is given by*

$$\rho_n^v(s,a,b,c,\alpha,\beta) = \begin{cases} \left(\frac{s-\alpha c(\lambda_2-1)}{s+\alpha c(1-\lambda_1)}\right)\cdot\left(\frac{\beta s+c(\lambda_2-1)}{\beta s-c(1-\lambda_1)}\right)\cdot\left(\lambda_1\right)^{2n}, & |\lambda_1|<1, \\ \left(\frac{s-\alpha c(\lambda_1-1)}{s+\alpha c(1-\lambda_2)}\right)\cdot\left(\frac{\beta s+c(\lambda_1-1)}{\beta s-c(1-\lambda_2)}\right)\cdot\left(\lambda_2\right)^{2n}, & |\lambda_1|>1. \end{cases} \tag{8}$$

Similarly, for the splitting at a current node, the convergence factor $\rho_n^c(s,a,b,c,\alpha,\beta)$ *is*

$$\rho_n^c(s,a,b,c,\alpha,\beta) = \begin{cases} \left(\frac{s-b+a\alpha(\lambda_2-1)}{s-b-a\alpha(1-\lambda_1)}\right)\cdot\left(\frac{\beta(s-b)-a(\lambda_2-1)}{\beta(s-b)+a(1-\lambda_1)}\right)\cdot\left(\lambda_1\right)^{2n}, & |\lambda_1|<1, \\ \left(\frac{s-b+a\alpha(\lambda_1-1)}{s-b-a\alpha(1-\lambda_2)}\right)\cdot\left(\frac{\beta(s-b)-a(\lambda_1-1)}{\beta(s-b)+a(1-\lambda_2)}\right)\cdot\left(\lambda_2\right)^{2n}, & |\lambda_1|>1. \end{cases} \tag{9}$$

Proof. The proof is similar to the proof of Theorem 1, with the change in the transmission conditions which are now given by the new transmission conditions (7). For $\beta \neq 0$,

$$x_{2n-2}^{k+1}(s_1) = x_{2n-2}^{k}(s_2) + \alpha x_{2n-3}^{k}(s_2) - \alpha x_{2n-3}^{k+1}(s_1),$$
$$x_{-2}^{k+1}(s_2) = x_{-2}^{k}(s_1) + x_{-1}^{k}(s_1)/\beta - x_{-1}^{k+1}(s_2)/\beta.$$

Performing similar operations for both cases, $|\lambda_1| < 1$ and $|\lambda_1| > 1$, we obtain the new convergence factor (8). Similarly for splitting at a current node one can obtain the convergence factor (9).

The analysis to find optimized α and β for both convergence factors $\rho_n^v(s,a,b,c,\alpha,\beta)$ and $\rho_n^c(s,a,b,c,\alpha,\beta)$ is similar. Hence, in this article we present the analysis for the convergence factor obtained by splitting at a voltage node, i.e for $\rho_n^v(s,a,b,c,\alpha,\beta)$.

Corollary 1 *The optimized waveform relaxation algorithm for splitting at a voltage node converges in two iterations, independently of the initial waveforms if*

$$\alpha_{opt} := \frac{s}{c\,(\lambda_2 - 1)} \quad \textit{and} \quad \beta_{opt} := \frac{c\,(1-\lambda_2)}{s}.$$

Proof. Equating the convergence factor (8) with zero provides us the expressions for the optimal α and β.

Note that $\alpha_{opt}, \beta_{opt}$ are complicated functions of s, which would lead to non-local transmission conditions in time. Hence one searches for the optimized α, β by approximating them by a constant. For this, we solve the min-max problem

$$\min_{\alpha,\beta}\left(\max_{s} |\rho_n^v(s,a,b,c,\alpha,\beta)|\right). \tag{10}$$

By equating the denominator of $\rho_n^v(s,a,b,c,\alpha,\beta)$ with zero, we can show, provided that $\alpha < 0$ and $\beta > 0$, that $\rho_n^v(s,a,b,c,\alpha,\beta)$ is an analytic function in the right half of the complex plane. We also prove that $\rho_n^v(s,a,b,c,\alpha,\beta) \to 0$ as $s \to \infty$. These proofs are technical and will appear in [3]. The maximum principle states that the maximum of $|\rho_n^v(s,a,b,c,\alpha,\beta)|$ lies on the imaginary axis, i.e. $s = i\omega$. Further, $\rho_n^v(s,a,b,c,\alpha,\beta)$ is an even function of ω. From Corollary 1, we observe that $\beta_{opt} = \frac{-1}{\alpha_{opt}}$. This motivates to choose $\beta = \frac{-1}{\alpha}$, which means that the current in both sub-circuits at the point of partition is equal but opposite in direction. All these results and assumptions reduce our optimization problem (10) to

$$\min_{\alpha<0}\left(\max_{\omega_{min}<\omega<\omega_{max}} |\rho_n^v(\omega,a,b,c,\alpha)|\right), \tag{11}$$

where $\omega_{min} := \frac{2\pi}{T}$ and $\omega_{max} := \frac{2\pi}{\Delta t}$ with T as the total time window we are computing and Δt as the time discretization parameter.

Theorem 3. *For splitting at a voltage node, and for small $\omega_{min} = \varepsilon > 0$, if $\alpha_v^* = K_v \varepsilon^{1/3}$, where $K_v = (a^2/(2nb^2c))^{1/3}$, then the convergence factor ρ_n^v satisfies*

$$|\rho_n^v(\omega,a,b,c,\alpha_v^*)| \le |\rho_n^v(\omega_{min},a,b,c,\alpha_v^*)| \sim 1 + \frac{2\sqrt{2a}\,\omega_{min}^{1/6}}{K_v\sqrt{bc}} + \mathscr{O}(\omega_{min}^{1/2}). \tag{12}$$

Similarly, for a splitting at a current node and with $n > 1$, if $\alpha_c^* = K_c \varepsilon^{-1/3}$, where $K_c = ((2(n-1)b^2c)/a^2)^{1/3}$, then the convergence factor ρ_n^c satisfies

$$|\rho_n^c(\omega,a,b,c,\alpha_c^*)| \le |\rho_n^c(\omega_{min},a,b,c,\alpha_c^*)| \sim 1 + \frac{2\sqrt{2a}\,K_c\omega_{min}^{1/6}}{\sqrt{bc}} + \mathscr{O}(\omega_{min}^{1/2}).$$

Proof. We observe numerically (see right plot of Figure 2) that a solution of our min-max problem (11) is given by equioscillation of $|\rho_n^v(\omega,a,b,c,\alpha)|$ for $\omega = \omega_{min}$ and $\omega = \tilde{\omega}$ and hence can be found by solving the coupled equations $\rho_n^v(\omega_{min} = \varepsilon, a,b,c,\alpha_v^*) = \rho_n^v(\bar{\omega}_v,a,b,c,\alpha_v^*)$ and $\frac{\partial}{\partial\omega}\rho_n^v(\bar{\omega}_v,a,b,c,\alpha_v^*) = 0$, where $\omega_{min} < \bar{\omega}_v \le$

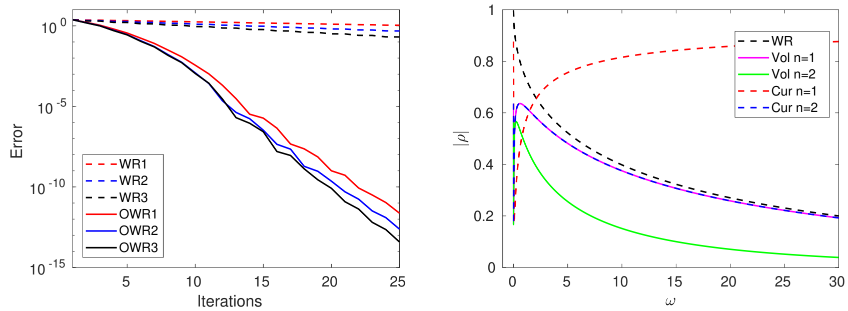

Fig. 2: Convergence for long time $T = 100$ (left) and convergence factor in Laplace space (right).

ω_{max}. Asymptotic calculations for $\varepsilon \to 0$, yield $\alpha_v^* \sim K_v \cdot \varepsilon^{1/3}$ and $\bar{\omega}_v \sim \frac{2K_v c}{n} \cdot \varepsilon^{1/3}$. Similar calculations yield expressions for α_c^* and $\bar{\omega}_c$. The details of this proof are complicated and too long to present in this short paper and will appear in [3].

Theorem 4. *The convergence of OWR is faster for the splitting at a voltage node.*

Proof. We substitute the values of K_c and K_v into the expression of $\rho_n^c(\omega_{min} = \varepsilon, a, b, c, \alpha_c^*)$ and $\rho_n^v(\omega_{min} = \varepsilon, a, b, c, \alpha_v^*)$ respectively to prove $\rho_n^c(\omega_{min} = \varepsilon, a, b, c, \alpha_c^*) > \rho_n^v(\omega_{min} = \varepsilon, a, b, c, \alpha_v^*)$. The details of this proof will also appear in [3].

5 Numerical Results

We consider an RLC transmission line of length $N = 149$ with $R = 2K\Omega/cm$, $L = 4.95 \times 10^{-3} \mu H/cm$ and $C = 0.021 pF/cm$. For the time discretization, we use backward Euler with $\Delta t = T/5000$, where T is the total time. We first compare the classical WR and OWR algorithm for large time $T = 100$. The left plot in Figure 2 clearly shows the improvement in the convergence factor when optimized transmission conditions are used. The dashed lines show the results for classical WR while solid lines represent the OWR algorithm. We also see the effect of overlapping nodes (e.g. WR1 denotes the WR algorithm with one node overlap). Increasing the overlap increases the convergence speed. However, the gain is very small. The right plot of Figure 2 compares the convergence factor for OWR in Laplace space for both splittings, at a current node and a voltage node. The dotted black line is for WR with single node overlap while the other lines are for OWR. For OWR, the splitting at a voltage node leads to faster convergence. This is also true in the time domain, see the left plot of Figure 3. But for classical WR, splitting does not matter, see Theorem 1. Finally, the right plot of Figure 3 validates our asymptotic result (12). Both numerically computed and asymptotically derived values of the optimal α for splitting at a voltage node are very close.

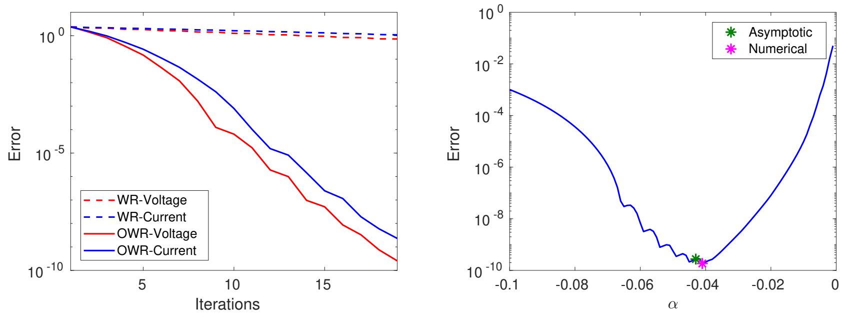

Fig. 3: Comparison of different splittings in time (left) and of values of optimized alpha (right).

6 Conclusion

This is the first analysis of WR and OWR for an RLC transmission line with overlap and with splitting either at a current or voltage node. We show that using optimized transmission conditions, we can achieve a drastic improvement in the convergence rate. Note that our analysis is in the Laplace domain since the analysis is easier and the convergence in the Laplace domain implies convergence in the time domain, see Remark 1 in [2]. We also see that overlapping nodes increase the convergence rate for both WR and OWR algorithms but the improvement (by the factor of $(\lambda_1)^{2n}$) is not large. Further, for OWR, the splitting at a voltage node leads to a little faster convergence than the splitting at a current node, while this splitting does not effect the convergence of WR. We finally compared the values of the optimized α found numerically and by asymptotic analysis and they are very close.

References

1. Gander, M.J., Al-Khaleel, M., Ruchli, A.E.: Optimized waveform relaxation methods for longitudinal partitioning of transmission lines. IEEE Transactions on Circuits and Systems I: Regular Papers **56**(8), 1732–1743 (2009). DOI 10.1109/TCSI.2008.2008286
2. Gander, M.J., Kumbhar, P.M., Ruehli, A.E.: Analysis of overlap in waveform relaxation methods for rc circuits. In: Domain Decomposition Methods in Science and Engineering XXIV, pp. 281–289. Springer International Publishing, Cham (2018)
3. Gander, M.J., Kumbhar, P.M., Ruehli, A.E.: Optimized waveform relaxation methods applied to RLCG transmission line and their asymptotic analysis. In Preparation (2019)
4. Ho, C.W., Ruehli, A., Brennan, P.: The modified nodal approach to network analysis. IEEE Transactions on Circuits and Systems **22**(6), 504–509 (1975)
5. Lelarasmee, E., Ruehli, A.E., Sangiovanni-Vincentelli, A.L.: The waveform relaxation method for time-domain analysis of large scale integrated circuits. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems **1**(3), 131–145 (1982)
6. Menkad, T., Dounavis, A.: Resistive coupling-based waveform relaxation algorithm for analysis of interconnect circuits. IEEE Transactions on Circuits and Systems I: Regular Papers **64**(7), 1877–1890 (2017). DOI 10.1109/TCSI.2017.2665973