A New Hierarchical Domain Decomposition Method with Coarse Correction

Martin J. Gander[0000−0001−8450−9223] and
Hui Zhang[0000−0001−7245−0674]

1 Introduction

The philosophy of traditional multilevel domain decomposition methods [9, 8] consists in (i) decomposing the domain into sufficiently small subdomains, (ii) constructing a coarse problem to couple all the subdomains and accelerate subdomain iterations, and (iii) if the coarse problem is too large, solving it again by (i)-(ii). To think differently, we can (i') decompose the domain into a few large subdomains, (ii) (as before), and (iii') if a subdomain is too large, then solve it again by (i')-(ii). We call this method hierarchical domain decomposition method with coarse correction. While a traditional multilevel method recalls itself recursively in the coarse problem on the original domain, our new approach recalls itself recursively in the subdomain problems. Especially when the geometry or physics are simpler in subdomains than in the original domain, our new approach can thus have direct advantages.

The hierarchical coarse basis is generated very naturally by this method. To see this, we consider the Laplace problem in 1D,

$$-u'' = f \text{ in } \Omega = (0,1), \quad u(0) = g_0,\ u(1) = g_1. \tag{1}$$

We first decompose Ω into $\Omega_1 = (0, b_1)$ and $\Omega_2 = (a_2, 1)$ with $a_2 < b_1$. Let $a_1 = 0$ and $b_2 = 1$. Suppose (1) after discretisation becomes $A\mathbf{u} = \mathbf{f}$. Then we can solve $A\mathbf{u} = \mathbf{f}$ by the Restricted Additive Schwarz (RAS) iteration with partition of unity and coarse correction,

$$\tilde{u}^{(n)} = (P_1 A_1^{-1} R_1 + P_2 A_2^{-1} R_2)(f - A u^{(n-1)}), \tag{2}$$

$$u^{(n)} = \tilde{u}^{(n)} + C A_c^{-1} C^T (f - A\tilde{u}^{(n)}), \tag{3}$$

Martin J. Gander
Department of Mathematics, University of Geneva, Rue du Conseil-Général 9, CP 64, 1211 Genève 4, Switzerland, e-mail: martin.gander@unige.ch

Hui Zhang (corresponding)
School of Mathematics & Physics, Xi'an Jiaotong-Liverpool University, Ren'ai Road 111, 215123 Suzhou, China, e-mail: hui.zhang@xjtlu.edu.cn

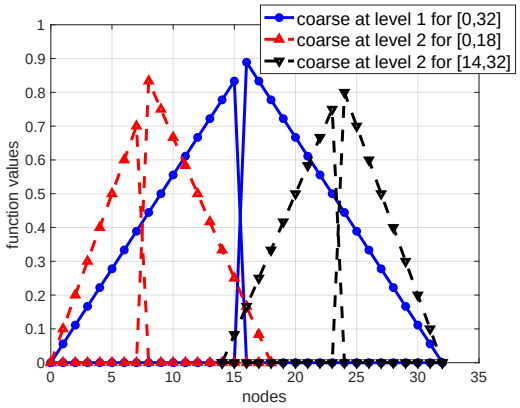

Fig. 1 Hierarchical coarse basis $\{C, C_1^0, C_2^0\}$ from two-level bisections and restricted P_i.

where R_i is the identity restriction from Ω to Ω_i, P_i is a prolongation from Ω_i to Ω such that $P_1R_1 + P_2R_2$ is the identity, $A_i = R_iAR_i^T$, $A_c = C^TAC$ is the Galerkin approximation of A in the range of C (the column space of C), and $C = [C_1^0, C_2^0]$ with $C_i^0 = P_iU_i$ obtained from solving the discretised version of

$$-U_1'' = 0 \text{ in } \Omega_1, \quad U_1(0) = 0,\ U_1(b_1) = 1, \tag{4}$$

$$-U_2'' = 0 \text{ in } \Omega_2, \quad U_2(a_2) = 1,\ U_2(1) = 0, \tag{5}$$

which is well-known as harmonic extension of interface data. The motivation of using harmonic extension is that the error of RAS in (2) for $n \geq 1$ is essentially coming from incorrect interface values at b_1 and a_2; see [2]. In other words, the error space is exactly the harmonic extension of the interface space. In particular, for RAS in (2), the error space is of dimension two, and the coarse correction (3) would result in the exact solution of (1). To get the hierarchical domain decomposition method, we now perturb (2) and (4)-(5) by solving the problems on Ω_1 and Ω_2 again with the method (2)-(3). For example, we decompose Ω_1 into $\Omega_3 = (0, b_3)$ and $\Omega_4 = (a_4, b_1)$, and construct $C_1 = [C_3^1, C_4^1]$. If the problems on Ω_3, Ω_4 are solved exactly, then the method of (2)-(3) applied to the Ω_1 problem would converge in one step. Or, we can further decompose Ω_3, Ω_4, and so on. Note that the set of coarse basis $\{C, C_1, C_2\}$ or $\{C, C_1, C_2, C_3, C_4, C_5, C_6\}$ (with Ω_i decomposed into Ω_{2i+1} and Ω_{2i+2}, and C_i the basis on Ω_i) after prolongation forms a hierarchical coarse basis on Ω. An example of the hierarchical coarse basis is shown in Fig. 1, where we used the restricted prolongation P_i (see [1]).

Remark 1 By appropriate (e.g. restricted) prolongation, the hierarchical coarse basis may also be used for a traditional two-level Schwarz method with the bottom level subdomains e.g. $\{C, C_1, C_2\}$ for the decomposition $\Omega = \cup\{\Omega_3, \Omega_4, \Omega_5, \Omega_6\}$. The resulting coarse problem could be easily parallelised because of weak or no coupling between e.g. Ω_4, Ω_5. But this is still to be investigated.

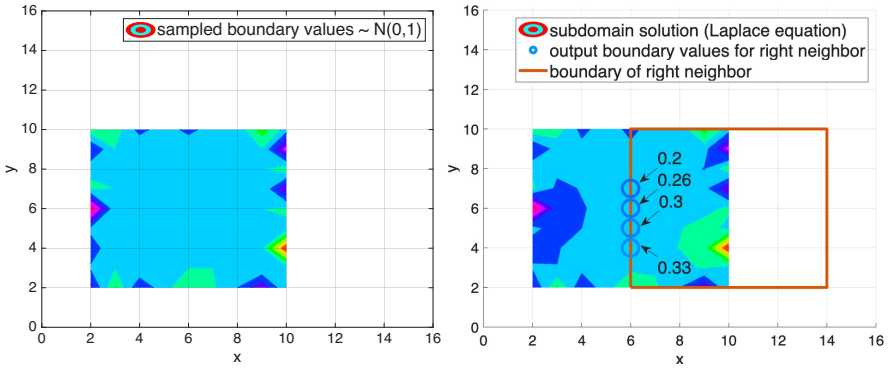

Fig. 2 Interface map from one subdomain to its right neighbor, which due to the partition of unity used needs only the 4 indicated values, the other not indicated ones coming from other subdomains.

2 Hierarchical Coarse Basis in 2D

Now we extend the idea introduced in Sec. 1 to 2D. Since there are many dofs on an interface of a 2D domain, it is no longer efficient to lift the whole interface space to the coarse space. The key question is what kind of interface modes to pick. We follow the general idea of using the error modes that are difficult to contract by the Schwarz iteration; see [3] and references therein.

To this end, for each subdomain Ω_i, we consider the interface map T_i as composition of: 1) the harmonic extension of the interface data g_i on $\partial\Omega_i \cap \Omega$ to the subdomain solution u_i, followed by 2) the evaluation of interface data $\{g_{ji}\}$ on $\{\partial\Omega_j \cap \Omega_i\}$ for the neighboring subdomains $\{\Omega_j\}$ using u_i as needed by the partition of unity used; see Fig. 2 for an example for which the partition of unity needs 4 values as indicated in the figure to be passed on. The interface map T_i is a "rectangular" operator in the sense that its domain (interface data g_i to solve with) and codomain (interface data g_{ji} to give to neighbors via partition of unity) are different spaces.

By the maximum principle, $\max|g_{ji}|$ should be a fraction of $\max|g_i|$, which results in contraction. The error modes g_i that lead to slow convergence are those not well contracted. A good idea is thus to work with the Singular Value Decomposition[1] (SVD), e.g. $T_i V = U\Sigma$, which diagonalises T_i to Σ by choosing orthonormal bases (columns of V and U). Then the columns of V corresponding to large singular values (diagonal entries of Σ) are the coarse modes on the interface of Ω_i. Since the SVD is very expensive and we need only a few dominant singular vectors, we will use the randomised SVD (rsvd) instead; see [6, 4], which needs only to apply T_i on a few random samples (a bit more than the required number of modes), orthogonalise, and apply the adjoint T_i^* before the SVD.

Remark 2 We can also use the interface map of more Schwarz iterations to see further propagation of the error and faster decay of the singular values.

[1] Ma-Alber-Scheichl [5] used the SVD of a restriction operator to build a coarse basis.

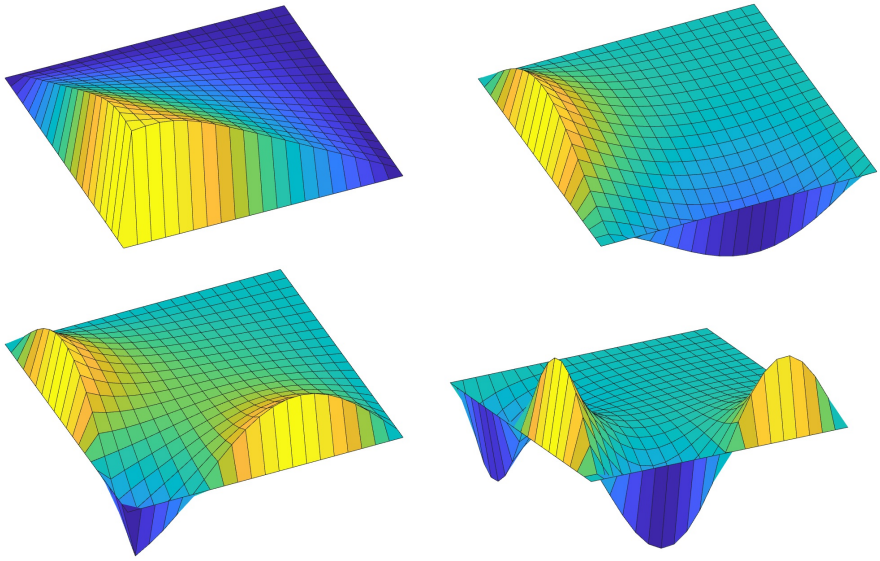

Fig. 3 The first four coarse modes from one of the 2×2 subdomains for Δu in $\Omega = (0,1)^2$, $u|_{\partial\Omega} = 0$.

For the Laplace problem in the unit square with homogeneous Dirichlet boundary conditions and RAS on 2-by-2 subdomains, the first four coarse modes from a subdomain are shown in Fig. 3. The singular values of the interface map from the subdomain to its neighbors are plotted in Fig. 4; they decay exponentially. It is worthwhile to mention that in a hierarchical decomposition, e.g. dividing each of the 2-by-2 subdomains into 2-by-2 smaller subdomains, the coarse modes are built with a few steps of the sublevel iteration like (2)-(3) for the harmonic extension, and thus contain iterative errors; but they look like in Fig. 3.

Fig. 4 Singular values of the interface map T_i on the meshes of step size h from one of the 2×2 subdomains to its neighbors with $4h$ overlap (except for the triangle markers where $8h$ overlap is used), for Δu in $\Omega = (0,1)^2$, $u|_{\partial\Omega} = 0$. The horizontal axis is the index of the descending singular values. The actual values of the singular values are along the vertical axis.

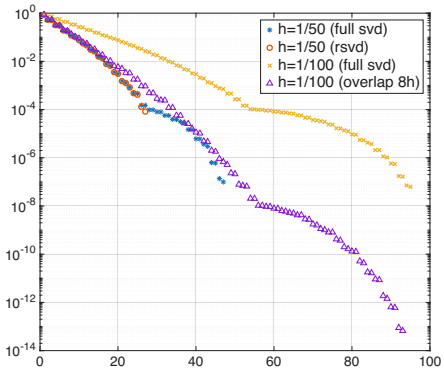

Table 1 GMRES #iter with $n_c = 1, 3, 5, 7$ (four columns each m) coarse modes per subdomain, $n_i = 1$ iter. for precond., m^2 subdomains, $(mn)^2$ elements and $4h$ overlap for the **Laplace** problem.

n	$m=2$				$m=4$				$m=8$				$m=16$			
4	4	2	2	1	6	4	3	3	9	4	3	3	12	4	3	3
8	5	3	3	2	8	6	4	4	13	7	5	5	18	8	7	6
16	7	5	4	3	13	8	6	5	18	10	7	6	24	13	9	7
32	10	8	6	5	18	12	9	8	28	14	10	9	42	18	12	10

3 Numerical Experiments

We now test hierarchical RAS with a coarse basis generated by randomised SVD, as described in Sec. 2. We denote the hierarchical *decomposition* by $[m_{x1} \times m_{y1}, m_{x2} \times m_{y2}, \ldots, m_{x\ell} \times m_{y\ell}]$ meaning the domain is first partitioned into $m_{x1} \times m_{y1}$ subdomains, then each subdomain is partitioned into $m_{x2} \times m_{y2}$ subdomains, and so on. For convenience, we use m without subscript when $m_{x1} = m_{y1} = \ldots$. The *number of coarse modes* from a subdomain at each level of partition is collected as $[n_{c1}, n_{c2}, \ldots, n_{c\ell}]$, which means e.g. n_{c1} coarse modes are generated for each subdomain of the partition $m_{x1} \times m_{y1}$. The *number of Schwarz iterations* like (2)-(3) used at level l for both "the coarse basis at the upper level" and "the preconditioner for the original problem" is denoted by n_{il}. For convenience, we use n_c and n_i if the quantities are level independent. A bilinear finite element method on rectangular meshes is used for the discretisation. The partition is element based and first without overlap, then extended outward by two-element width from the non-overlapping interfaces, i.e. *overlap* $\delta = 4h$ with h the element size. Each non-overlapping subdomain at the last level ℓ is discretised with $n \times n$ elements, i.e. $H/h = n$ with H the subdomain size. We take here $n_c + \frac{n}{2}$ samples in rsvd, and use GMRES [7] to solve the non-symmetric Schwarz preconditioned system.

Laplace Problem: We solve $-\Delta u = f$ in $\Omega = (0,1)^2$, $u = 0$ on $\partial\Omega$ with a random solution $\mathbf{u}$ in the discretised problem $A\mathbf{u} = \mathbf{f}$ and the corresponding source $\mathbf{f}$. The initial guess is zero. The stopping criterion is the relative residual $\|\mathbf{f} - A\mathbf{u}^{(n)}\|_2 / \|\mathbf{f}\|_2 < 10^{-5}$.

We first test with only one level of partition, which amounts to a two-level Schwarz method with our rsvd coarse basis. The results are shown in Tab. 1. To keep a roughly equal iteration number, as we halve the mesh (and thus overlap) size on fixed subdomains (double n, fixed m) or halve both mesh and subdomain sizes (fixed n, double m), the coarse mode number per subdomain n_c needs to be increased. The latter scenario corresponds to fixed H/δ for which the classical two-level Schwarz method would converge at a speed independent of h and H, but here we see a logarithmic growth of the iteration numbers with $m = 1/H$. It would be interesting to find out the cause of the logarithmic growth and get rid of the growth. At the time of writing this manuscript, we do not have an understanding yet.

Next we test the hierarchical decomposition with ℓ levels. The benefit of a hierarchical coarse correction consists in the hierarchical structure and the parallelism

Table 2 GMRES #iter with n_c coarse modes per subdomain, $n_i = 1$ iteration for coarse & preconditioner, ℓ levels of 2×2 subdomains, $n^2 4^\ell$ elements, $4h$ overlap for the **Laplace** problem.

$n_c = 1, 3, 5, 7$:																
n	$\ell = 1$				$\ell = 2$				$\ell = 3$				$\ell = 4$			
4	4	2	2	1	6	4	3	2	9	6	4	3	15	9	6	5
8	5	3	3	2	8	5	4	3	14	8	6	5	23	17	12	9
16	7	5	4	3	12	8	6	5	21	14	10	8	36	26	20	16
32	10	8	6	5	18	12	9	7	31	23	16	14	54	38	30	28
$n_c = 9, 11, 13, 15$:																
16	3	3	2	2	4	4	4	3	7	6	5	5	13	11	9	9
32	4	4	4	3	6	6	5	5	11	9	8	7	23	21	18	17

Table 3 GMRES #iter with $n_c = 1, 3, 5, 7$ coarse modes per subdomain, $n_i = 1$ iteration for preconditioner, m^2 subdomains, $(mn)^2$ elements and $4h$ overlap for **random diffusion**.

n	$m = 2$				$m = 4$				$m = 8$				$m = 16$			
4	4	2	2	1	7	4	3	3	10	5	4	3	14	6	4	3
8	5	4	3	2	9	6	5	4	14	7	5	5	21	10	8	7
16	8	5	4	4	13	9	7	5	20	12	8	7	29	16	10	9
32	11	7	6	5	19	13	10	9	30	18	12	10	49	24	17	14

between the coarse problems at the same level. We fix $m_x = m_y = 2$. The results are shown in Tab. 2. For $\ell = 1$, this is the same as $m = 2$ in Tab. 1. For $\ell = 2, 3, 4$, the subdomain number at the finest level is the same as $m = 4, 8, 16$ in Tab. 1. Also, with the partition of $\ell \geq 3$ levels, more coarse modes are needed to reach the same iteration numbers as in Tab. 1. So too many levels of hierarchy can slow down the convergence. We used the same n_c at all levels of decomposition. A probably better idea is to use larger coarse spaces on larger subdomains. For example, if we double $n_{ci} = 2n_{c,i+1}$, then #iter is 48, 21, 13, 10 for $\ell = 4$ and $n_{c\ell} = 1, 3, 5, 7$ in Tab. 2.

Random Diffusion Problem: The setting is almost the same as for the Laplace problem, but instead of Δu we use now $\nabla(a(x, y)\nabla u)$, where $a(x, y) \sim \mathcal{U}(0, 1)$ is a mesh dependent element-wise constant random (fixed once generated) field. The corresponding results with one level partition are shown in Tab. 3. Compared to Tab. 1 for the Laplace problem, the iteration numbers here are a little higher. Again, we see the need of more coarse modes per subdomain when n or m increases.

The results with multiple levels are shown in Tab. 4. Compared to Tab. 3 for the one-level partition, the iteration number here decreases slowlier with n_c when $\ell \geq 3$. Compared to Tab. 2 for the Laplace problem, the iteration numbers here are also a little higher.

Layered Diffusion Problem: The diffusion coefficient in this subsection is $a(x, y) = a(y)$ piecewise constants in $y \in (0, 1)$. We assume $a(y)$ takes the values $a_1 = 1$ and $a_0 \in (0, 1)$ alternatively in equally spaced 8 or 64 layers. The results are given in Tab. 5 and Tab. 6.

Table 4 GMRES #iter with n_c coarse modes per subdomain, $n_i = 1$ iteration for coarse & preconditioner, ℓ levels of 2×2 subdomains, $n^2 4^\ell$ elements, $4h$ overlap for **random diffusion**.

$n_c = 1, 3, 5, 7$																
n	$\ell = 1$				$\ell = 2$				$\ell = 3$				$\ell = 4$			
4	4	2	2	1	6	4	3	2	9	6	4	4	16	10	8	5
8	5	4	3	2	9	5	4	4	16	10	7	5	25	18	13	10
16	8	5	4	4	14	8	6	5	23	15	11	9	39	30	23	19
32	11	7	6	5	20	14	10	8	33	25	17	14	56	37	35	31
$n_c = 9, 11, 13, 15$:																
16	3	2	2	2	5	4	4	3	8	6	6	5	15	12	10	10
32	5	4	4	3	7	6	6	5	12	11	10	9	27	23	21	20

Table 5 GMRES #iter with $n_c = 1, 3, 5, 7$ coarse modes per subdomain, $n_i = 1$ iteration for preconditioner, m^2 subdomains, $(mn)^2$ elements and $4h$ overlap for **layered diffusion**.

$a_0 = 0.1$, 8 layers:																
n	$m = 2$				$m = 4$				$m = 8$				$m = 16$			
4	4	2	2	1	6	3	3	3	10	4	3	3	13	6	4	3
8	6	3	3	2	9	5	4	3	15	8	6	5	22	11	9	7
16	8	5	4	3	12	7	6	5	21	11	9	8	32	16	13	11
32	12	7	6	5	18	11	9	8	32	19	13	12	57	30	19	17
$a_0 = 0.01$, 8 layers:																
n	$m = 2$				$m = 4$				$m = 8$				$m = 16$			
4	4	2	2	1	6	3	3	2	9	5	3	3	14	7	4	3
8	6	3	2	2	7	4	4	3	13	8	5	5	23	12	8	5
16	8	5	4	3	10	6	6	5	17	11	9	7	38	18	12	10
32	10	7	5	5	14	10	8	7	33	23	13	12	69	36	19	16
$a_0 = 0.01$, 64 layers:																
32	16	13	12	10	30	23	17	14	49	35	21	13	75	25	18	12

References

1. Cai, X.C., Sarkis, M.: A restricted additive Schwarz preconditioner for general sparse linear systems. SIAM J. Sci. Comput. **21**(2), 792–797 (1999)
2. Efstathiou, E., Gander, M.J.: Why restricted additive Schwarz converges faster than additive Schwarz. BIT Numer. Math. **43**(5), 945–959 (2003)
3. Gander, M.J., Halpern, L.: Piece-wise constant, linear and oscillatory: a historical introduction to spectral coarse spaces with focus on Schwarz methods. In: International Conference on Domain Decomposition Methods, pp. 189–198. Springer (2022)
4. Halko, N., Martinsson, P.G., Tropp, J.A.: Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. SIAM Rev. **53**(2), 217–288 (2011)
5. Ma, C., Alber, C., Scheichl, R.: Wavenumber explicit convergence of a multiscale generalized finite element method for heterogeneous Helmholtz problems. SIAM J. Numer. Anal. **61**(3),

Table 6 GMRES #iter with n_c coarse modes per subdomain, n_i=1 iteration for coarse & preconditioner, ℓ levels of 2×2 subdomains, $n^2 4^\ell$ elements, $4h$ overlap for **layered diffusion**.

$a = 0.1$, 8 layers, $n_c = 1, 3, 5, 7$:																
n	$\ell = 1$				$\ell = 2$				$\ell = 3$				$\ell = 4$			
4	4	2	2	1	6	3	3	2	10	5	4	3	16	9	6	5
8	6	3	3	2	9	5	4	3	16	9	6	5	29	16	11	9
16	8	5	4	3	13	8	6	5	27	14	10	8	41	28	20	15
32	12	7	6	5	19	13	9	8	36	25	17	14	59	44	33	26
$a = 0.1$, 8 layers, $n_c = 9, 11, 13, 15$:																
16	3	3	2	2	5	4	4	3	8	7	6	5	13	11	10	9
32	5	4	4	3	8	7	6	5	12	11	10	8	23	20	18	16
$a = 0.01$, 8 layers, $n_c = 1, 3, 5, 7$:																
n	$\ell = 1$				$\ell = 2$				$\ell = 3$				$\ell = 4$			
4	4	2	2	1	6	3	3	2	9	5	4	3	16	8	6	5
8	6	3	2	2	8	5	4	3	16	8	6	5	29	16	10	8
16	8	5	4	3	10	7	6	5	27	13	9	8	44	28	18	15
32	10	7	5	5	15	10	8	8	37	19	15	13	65	35	30	26
$a = 0.01$, 8 layers, $n_c = 9, 11, 13, 15$:																
16	3	3	2	2	5	5	4	3	8	7	7	6	12	11	11	10
32	5	5	4	4	8	7	6	6	12	10	11	9	23	20	17	16
$a_0 = 0.01$, 64 layers, $n_c = 1, 3, 5, 7$ (row 1) or $n_c = 9, 11, 13, 15$ (row 2):																
32	16	13	12	10	29	20	17	14	49	34	26	20	80	53	43	39
	8	7	7	6	12	10	9	9	18	17	15	12	33	29	25	20

1546–1584 (2023)
6. Rokhlin, V., Szlam, A., Tygert, M.: A randomized algorithm for principal component analysis. SIAM J. Matrix Anal. Appl. **31**(3), 1100–1124 (2010)
7. Saad, Y., Schultz, M.H.: GMRES: A generalized minimal residual algorithm for solving non-symmetric linear systems. SIAM J. Sci. Stat. Comp. **7**(3), 856–869 (1986)
8. Smith, B., Bjørstad, P., Gropp, W.: Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations. Cambridge University Press, United Kingdom (1996)
9. Zhang, X.: Multilevel Schwarz methods. Numer. Math. **63**(1), 521–539 (1992)