Hierarchical Coarse Basis by Randomised SVD: the Helmholtz Problem

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1 Introduction

The Helmholtz problem is hard to solve by established iterative methods for positive definite problems. Special techniques have been developed to make domain decomposition methods (DDM) work for the Helmholtz problem. However, the efficiency of DDMs is still restricted by the oscillatory nature of propagating waves. One big concern is the large size of coarse problems for coupling many small subdomains. This motivated us to rethink the traditional framework of coarse correction which aims at small subdomains and treats large coarse problems recursively by DDM. An alternative way is to start with big subdomains and thus a small coarse problem, then treat big subdomains recursively by DDM. This leads to a hierarchy of subdomains and coarse problems.

The hierarchical coarse basis is generated very naturally by this approach. To see this, we consider the Helmholtz problem in 1D with impedance boundary conditions:

$$-u'' - k^2 u = f \text{ in } \Omega = (0,1), \quad -(u' + \mathrm{i}ku)(0) = g_0, \ (u' - \mathrm{i}ku)(1) = g_1. \quad (1)$$

We first decompose Ω into $\Omega_1 = (0, b_1)$ and $\Omega_2 = (a_2, 1)$ with $a_2 < b_1$. Let $a_1 = 0$ and $b_2 = 1$. Suppose (1) after discretisation becomes $A\mathbf{u} = \mathbf{f}$. Then we can solve $A\mathbf{u} = \mathbf{f}$ by the optimised Schwarz iteration with coarse correction

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$$\tilde{\mathbf{u}}^{(n)} = (P_1\tilde{A}_1^{-1}R_1 + P_2\tilde{A}_2^{-1}R_2)(\mathbf{f} - A\mathbf{u}^{(n-1)}), \tag{2}$$

$$\mathbf{u}^{(n)} = \tilde{\mathbf{u}}^{(n)} + CA_c^{-1}C^H(\mathbf{f} - A\tilde{\mathbf{u}}^{(n)}), \tag{3}$$

where R_i is the identity restriction from $\overline{\Omega}$ to $\overline{\Omega_i}$, P_i is a prolongation from $\overline{\Omega_i}$ to $\overline{\Omega}$ such that $P_1R_1 + P_2R_2$ is the identity, $\tilde{A}_i$ is from discretisation of the Helmholtz equation in Ω_i with impedance boundary conditions $-(u' + \mathrm{i}ku)(a_i) = 0$, $(u' - \mathrm{i}ku)(b_i) = 0$, $A_c = C^HAC$ is the Galerkin approximation of A in the range of C (the column space of C), and $C = [C_1^0, C_2^0]$ with $C_i^0 = P_iU_i$ with $U_i \approx u_i$ obtained from solving the discretised version of

$$-u_1'' - k^2u_1 = 0 \text{ in } \Omega_1, \quad -(u_1' + \mathrm{i}ku_1)(0) = 0, \ (u_1' - \mathrm{i}ku_1)(b_1) = 1, \tag{4}$$

$$-u_2'' - k^2u_2 = 0 \text{ in } \Omega_2, \quad -(u_2' + \mathrm{i}ku_2)(a_2) = 1, \ (u_2' - \mathrm{i}ku_2)(1) = 0, \tag{5}$$

which is well-known as the a-harmonic[1] extension of the interface data. The motivation of using a-harmonic extensions is that the error of the optimised general additive Schwarz iteration (2) for $n \geq 1$ is essentially coming from incorrect interface values at b_1 and a_2; see [12]. In other words, the error space is exactly the a-harmonic extension of the interface space. In particular for (2), the error space is of dimension two, and the coarse correction (3) would result in the exact solution of (1). To get the hierarchical domain decomposition method, we now perturb (2) and (4)-(5) by solving the problems on Ω_1 and Ω_2, again with the method (2)-(3). For example, we decompose Ω_1 into $\Omega_3 = (0, b_3)$ and $\Omega_4 = (a_4, b_1)$, and construct $C_1 = [C_3^1, C_4^1]$. If the problems on Ω_3, Ω_4 are solved exactly, then the method of (2)-(3) applied to the Ω_1 problem would converge in one step. Or, we can further decompose Ω_3, Ω_4, and so on. Note that the set of coarse basis $\{C, C_1, C_2\}$ or $\{C, C_1, C_2, C_3, C_4, C_5, C_6\}$ (with Ω_i decomposed into Ω_{2i+1} and Ω_{2i+2}, and C_i the basis on Ω_i) after prolongation forms a hierarchical coarse basis on Ω. An example of the hierarchical Helmholtz coarse basis is shown in Fig. 1, where we used the restricted prolongation P_i (see [2, 12]) from optimised restriced additive Schwarz (ORAS).

2 Hierarchical Coarse Basis in 2D

We now extend the idea introduced in Sec. 1 to 2D. Since there are many dofs on an interface of a 2D domain, it is no longer efficient to lift the whole interface space to the coarse space. The key question is what kind of interface modes to pick. We follow the general idea of using the error modes that are difficult to contract by the Schwarz iteration; see [3, 1].

To this end, for each subdomain Ω_i (or each interface $\overline{\partial\Omega_i \cap \Omega_j}$), we consider the interface map T_i (or T_{ij}) as composition of: 1) the a-harmonic extension of the interface data g_i on $\overline{\partial\Omega_i \cap \Omega}$ (or g_{ij} on $\overline{\partial\Omega_i \cap \Omega_j}$) to the subdomain solution u_i, followed by 2) the evaluation of interface data $\{g_{ji}\}$ on $\{\overline{\partial\Omega_j \cap \Omega_i}\}$ for the neigh-

[1] Here, a-harmonic means the weak form zero source problem with the bilinear form $a(u, v)$.

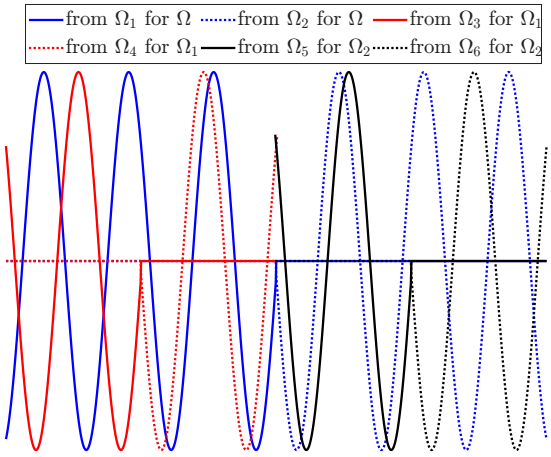

Fig. 1 Hierarchical coarse basis $\{C, C_1, C_2\}$ (real parts) from 2-level bisections & restricted P_i.

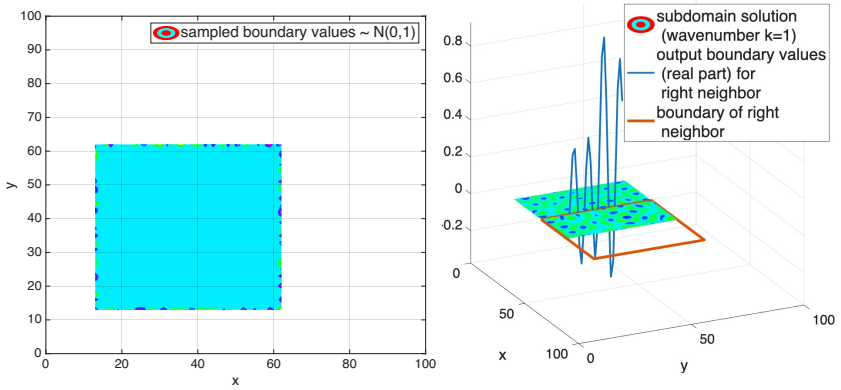

Fig. 2 Interface map from one subdomain to its right neighbor with boundary operator $\partial_{\mathbf{n}} - \mathrm{i}k$.

boring subdomains $\{\Omega_j\}$ using u_i. The interface map T_i (or T_{ij}) is a "rectangular" operator in the sense that its domain and codomain are different spaces. An example of the interface map from one subdomain to one of its neighbors is shown in Fig. 2.

By [4], $\sum_{j\neq i} \|g_{ji}\|^2_{L^2}$ should be a fraction of $\|g_i\|^2_{L^2}$ (or $\|g_{ij}\|^2_{L^2}$), which gives contraction. To find out the slowly convergent error modes, we use the Singular Value Decomposition[2] e.g. $T_i V = U\Sigma$. Then the columns of V corresponding to large singular values (diagonal entries of Σ) are the coarse modes on the interface of Ω_i. Since the SVD is very expensive and we need only a few dominant singular vectors, we shall use the randomised SVD (rsvd) instead; see [10, 5], which needs

[2] Ma-Alber-Scheichl [7] used the SVD of a restriction operator to build a coarse basis. Similar works are [6, 8, 9].

only to apply T_i on a few random samples (a bit more than the required number of modes), orthogonalise, and apply the adjoint T_i^* before the SVD.

In this work, we adopt T_i in a purly algebraic form without considering function spaces, and simply use the Euclidean vector inner product. The Q1 finite element method is used for discretisation. Let B_i be the boolean matrix that restricts the nodal values $\mathbf{u}_i$ on $\overline{\Omega_i}$ to the boundary $\partial\Omega_i$. Then the discretised Robin boundary value problem with zero source on Ω_i is in the form $\tilde{A}_i\mathbf{u}_i = B_i^T\mathbf{g}_i$ where $\mathbf{g}_i$ comes from the Robin boundary value. Denote the solution by $\mathbf{u}_i^*$. Let B_{ij} be the matrix that maps $\mathbf{u}_i^*$ to the Robin boundary data $\mathbf{g}_j$ for Ω_j. A convenient definition is $B_{ij} = B_j\tilde{A}_jR_jP_i$ with sufficient overlap and P_i so that the Robin trace (residual) on $\partial\Omega_j \cap \overline{\Omega_i}$ could be evaluated from the side of $\overline{\Omega_j} \cap \overline{\Omega_i}$ rather than $\overline{\Omega_i} - \Omega_j$. We note that B_{ij} has many zero rows corresponding to the boundary values on $\partial\Omega_j - (\partial\Omega_j \cap \overline{\Omega_i})$, and the definition of B_{ij} could be refined to not take the zero rows. Let $B_i^o = [B_{ij}]$ be the vertical concatenation for all the neighbors of Ω_i. The matrix T_i is defined as $T_i = B_i^o\tilde{A}_i^{-1}B_i^T$. After finding the dominant right singular vectors V_i of T_i, we extend them to the coarse vectors $C_i = P_i\tilde{A}_i^{-1}B_i^TV_i$. The coarse basis matrix is defined as the horizontal concatenation of C_i's for all the subdomains, namely $C = (C_i)$.

Remark 1 It is well known that the propagating modes need to travel through the outgoing boundaries of the domain to disappear. A consequence is that if there are m subdomains in one direction across the domain to the outgoing boundaries, then it takes m parallel subdomain iterations to see a significant decrease of the error. So the interface map defined through the input and output of *multiple* Schwarz iterations could be useful for extracting coarse modes, which is still to be investigated.

Remark 2 In the literature [7, 6, 8, 9] for the Helmholtz problem, some weighted H^1-like norms are used for the a-harmonic extension of the boundary values. That could be useful in our setting too. We observed for the Laplace problem that some singular corner modes can "pollute" the desired smoothing modes (e.g. the piecewise constant modes), and that can be removed by using $H^{1/2}$-norm instead of the L^2-norm.

It is worthwhile to mention that in a hierarchical decomposition, e.g. dividing each of the 2-by-2 subdomains into 2-by-2 smaller subdomains, the coarse modes are built with a few steps of the sublevel iteration like (2)-(3) for the a-harmonic extension, and thus contain iterative errors.

The singular values of the interface map from the subdomain to its neighbors are plotted in Fig. 3. We see the initial singular values decay very slowly, which can be attributed to propagating modes. Their number grows linearly with the wavenumber k. This is a big difference from the diffusion problem. There are also two regimes the decaying is slowing down in the middle. Otherwise, the singular values decay exponentially, just as for the diffusion problem, which can be attributed to the evanescent modes. The number of singular values above a fixed value grows about linearly with wavenumber k as $hk = 1$, and hence linearly with subdomain size H by a scaling argument. This number is also inversely proportional to h (essentially the overlap tending to zero). In this sense, it is useful to keep the overlap large and independent of h. However, the price is solving larger subdomain problems.

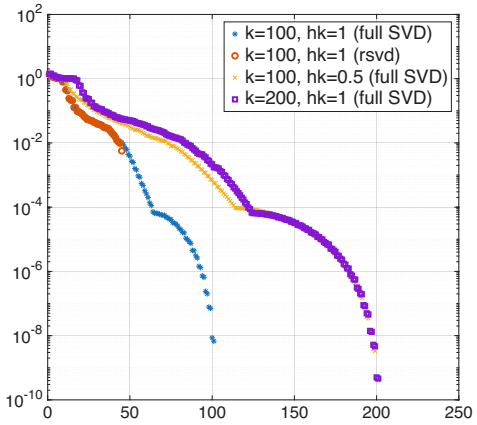

Fig. 3 Singular values of the interface map T_i on the meshes of element size h from one of the 2×2 subdomains to its neighbors with $4h$ overlap, for $\Delta + k^2$ in $\Omega = (0,1)^2$ with impedance conditions. The horizontal axis is the index of the descending singular values. The actual values of the singular values are along the vertical axis.

3 Numerical Experiments

We test the hierarchical restricted additive Schwarz method with the coarse basis generated by the randomised SVD, as described in Sec. 2. We denote the hierarchical *decomposition* by $[m_{x1}\times m_{y1}, m_{x2}\times m_{y2}, \ldots, m_{x\ell}\times m_{y\ell}]$ meaning the domain is first partitioned into $m_{x1}\times m_{y1}$ subdomains, then each subdomain is partitioned into $m_{x2}\times m_{y2}$ subdomains, and so on. For convenience, we use m without subscript when $m_{x1} = m_{y1} = \ldots$. The *number of coarse modes* from a subdomain at each level of partition is collected as $[n_{c1}, n_{c2}, \ldots, n_{c\ell}]$, which means e.g. n_{c1} coarse modes are generated for each subdomain of the partition $m_{x1}\times m_{y1}$. We take $n_{ci}+5$ samples in rsvd. The *number of Schwarz iterations* like (2)-(3) used at level l for both "the coarse basis at the upper level" and "the preconditioner for the original problem" is denoted by n_{il}. For convenience, we use n_c and n_i if the quantities are level independent. A bilinear finite element method on rectangular meshes is used for the discretisation. The partition is element based and first without overlap, then extended outward by two-element width from the non-overlapping interfaces, i.e. *overlap* $\delta = 4h$ with h the element size. Each non-overlapping subdomain at the last level ℓ is discretised with $n\times n$ elements, i.e. $H/h = n$ with H the subdomain size. We use GMRES [11] to solve the non-symmetric indefinite Schwarz preconditioned system. In all the experiments, $k_{\max}h = 1$ is kept, where $k_{\max}$ is the maximum wavenumber in the domain.

3.1 Free Space Problem

We solve $-\Delta u - k^2 u = f$ in $\Omega = (0,1)^2$, $\partial_{\mathbf{n}} u - \mathrm{i}ku = 0$ on $\partial\Omega$ with a random solution (by using a random $\mathbf{u}$ in the discretised problem $A\mathbf{u} = \mathbf{f}$ to get $\mathbf{f}$). The initial guess is zero. The stopping criterion is the relative residual $\|\mathbf{f} - A\mathbf{u}^{(n)}\|_2/\|\mathbf{f}\|_2 < 10^{-5}$.

Table 1 GMRES #iter with n_c (below #iter) coarse modes per subdomain, $n_i = 1$ iteration for the preconditioner, m^2 subdomains, $(mn)^2 = k^2$ elements and $4h$ overlap for the **free space** problem.

n	$m=2$				$m=4$				$m=8$				$m=16$			
4	6	5	4	3	15	8	4	3	32	6	4	3	61	8	4	3
	0	2	3	4	0	4	5	6	0	6	7	8	0	6	7	8
8	7	9	5	3	17	8	5	3	34	6	4	3	63	7	4	3
	0	1	5	7	0	10	11	13	0	11	13	14	0	11	13	14
16	9	7	6	3	20	6	4	3	39	6	4	3	78	6	4	3
	0	11	12	13	0	19	23	25	0	21	23	25	0	22	24	25
32	10	5	4	3	22	6	4	3	48	6	4	3	92	6	4	3
	0	27	28	29	0	39	43	47	0	42	45	48	0	43	45	49

The number of coarse modes per subdomain n_c listed is the least to reach the corresponding iteration number.

We first test with only one level of partition, which amounts to an optimised two-level Schwarz method with our rsvd coarse basis. The results are shown in Tab. 1. Note that double m or n leads to half element size and in turn half overlap and double wavenumber (because kh is fixed). To keep roughly equal iteration numbers, as we increase the wavenumber k and refine the mesh such that $kh = 1$ on fixed subdomains (double n and fixed m), the number of coarse modes per subdomain n_c needs to grow linearly; but if additionally we increase proportionally also the number of subdomains m in each direction (double m and fixed n), then n_c seems to be stable for $m \geq 4$. To check finer subdomains for fixed wavenumber and meshes, we look along the skew diagonals (half n and double m), and we find the required n_c roughly proportional to the subdomain size $H = 1/m$ for $m \geq 4$. This is consistent with the observation made from Fig. 3 in Sec. 2.

Some differences from the diffusion problem are: 1) insufficient number of coarse modes for the Helmholtz problem causes slower convergence than with no coarse modes, and 2) the sufficient number of coarse modes per subdomain, to be better than without coarse modes, or to keep the same iteration number, grows linearly with k for fixed kh and subdomains. Actually, to make an improvement over without coarse modes, n_c needs to exceed the number of propagating modes per subdomain and cover some evanescent modes. This is much like the rule of thumb for grid based discretisation e.g. ten points per wavelength, even though the coarse problem here resembles a spectral element method.

Now we test the hierarchical decomposition with ℓ levels. The benefit of hierarchical coarse correction consists in the hierarchical structure and the parallelism between the coarse problems at the same level. We fix $m_x = m_y = 2$. The results are shown in Tab. 2. For $\ell = 1$, it is the same as $m = 2$ in Tab. 1. For $\ell = 2, 3, 4$, the subdomain number at the finest level is the same as $m = 4, 8, 16$ in Tab. 1. Compared to Tab. 1, here the coarse problems are hierarchical, and the required $n_{c\ell}$ per subdomain at the finest level follows similar scalings but is generally smaller. The dimension of the coarse problem for solving a subdomain problem at level $j-1$ is half of that at level j. *Without coarse correction and with one DD iteration as*

Table 2 GMRES #iter & $n_{c\ell} = 2^{j-\ell} n_{cj}$ coarse modes per subdomain, $n_i = 1$ iteration for the coarse basis & preconditioner, ℓ levels of 2×2 subdomains, $n^2 4^\ell = k^2$ elements, $4h$ overlap, **free space** problem.

n	$\ell = 1$				$\ell = 2$				$\ell = 3$				$\ell = 4$			
4	6	5	4	3	15	7	4	3	32	7	4	3	61	9	4	3
	0	2	3	4	0	3	4	5	0	4	5	6	0	4	5	6
8	7	9	5	3	17	8	5	3	34	7	4	3	63	14	6	3
	0	1	5	7	0	6	7	8	0	7	8	9	0	7	8	9
16	9	7	6	3	20	7	5	3	39	5	4	3	78	7	5	3
	0	11	12	13	0	13	14	15	0	15	16	17	0	15	16	17
32	10	5	4	3	22	6	4	3	48	5	4	3	92	5	4	3
	0	27	28	29	0	29	30	31	0	31	32	33	0	32	33	34

preconditioner, #iter for ℓ here is the same as for $m = 2^\ell$ in Tab. 1 because one hierarchical DD iteration is exactly one flat DD iteration in this case!

We could fine tune n_{cj} at each level and on each subdomain. For example, subdomains on the boundary of the original domain have less interface dofs and may need less coarse modes. But we have not done so in this paper.

3.2 Layered Media Problem

The wavenumber in this subsection is $k(x, y) = \omega/c(y)$ piecewise constants in $y \in (0, 1)$. We assume $c(y)$ takes the values $c_1 = 1$ and $c_0 > 1$ alternatively in equally spaced layers. We keep $\omega h = 1$. The results are given in Tab. 3 and Tab. 4.

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Table 3 GMRES #iter with n_c (below #iter) coarse modes per subdomain, $n_i = 1$ iteration for the preconditioner, m^2 subdomains, $(mn)^2 = \omega^2$ elements and $4h$ overlap, **layered media** problem.

$c_0 = 5$, 8 layers:

n	$m=2$				$m=4$				$m=8$				$m=16$			
4	7	5	4	3	19	6	4	3	60	10	5	3	103	6	4	3
	0	2	3	4	0	4	5	6	0	4	5	7	0	6	7	8
8	12	5	4	3	34	5	4	3	47	8	4	3	154	6	5	3
	0	4	5	6	0	9	10	11	0	10	11	13	0	11	12	13
16	16	5	4	3	27	5	4	3	81	5	4	3	216	5	4	3
	0	9	10	11	0	17	19	20	0	19	21	22	0	22	23	25
32	17	5	4	3	45	5	4	3	109	5	4	3	321	5	4	3
	0	17	20	21	0	35	37	40	0	39	40	44	0	43	44	47
									$c_0 = 10$, 8 layers:							
16	15	5	4	3	31	5	4	3	44	5	4	3	245	5	4	3
	0	9	10	11	0	17	19	20	0	19	21	22	0	22	23	25
32	16	5	4	3	27	5	4	3	118	5	4	3	358	5	4	3
	0	17	20	21	0	36	37	40	0	39	40	44	0	43	44	48
									$c_0 = 10$, 64 layers:							
32	14	7	5	3	58	5	4	3	235	5	4	3	185	5	4	3
	0	21	22	23	0	33	35	38	0	39	41	43	0	40	42	44

Table 4 GMRES #iter with n_c coarse modes per subdomain, $n_i = 1$ iteration for the coarse basis & preconditioner, ℓ levels of 2×2 subdomains, $n^2 4^\ell = \omega^2$ elements, $4h$ overlap, **layered media** problem.

$c_0 = 5$, 8 layers:

n	$\ell=1$				$\ell=2$				$\ell=3$				$\ell=4$			
4	7	5	4	3	19	6	4	3	60	8	5	3	103	8	4	3
	0	2	3	4	0	3	4	5	0	3	4	5	0	4	5	6
8	12	5	4	3	34	6	4	3	47	6	4	3	154	7	5	3
	0	4	5	6	0	5	6	7	0	6	7	8	0	7	8	9
16	16	5	4	3	27	7	5	3	81	7	4	3	216	10	5	3
	0	9	10	11	0	10	11	12	0	12	13	14	0	14	15	16
32	17	5	4	3	45	5	4	3	109	5	4	3	321	6	5	3
	0	17	20	21	0	23	24	25	0	26	27	28	0	28	29	30

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