Optimised Compact High Order Schemes for the Helmholtz Equation by Interpolation with Polynomials and Plane Waves

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1 Introduction

Babuska and Sauter in [1] derived a dispersion free scheme for the 1D Helmholtz equation by requiring the symmetric three-point stencil equation to be exactly satisfied by the plane wave solutions (for zero source) and simultaneously exactly satisfied by the solutions for element wise constant sources. A similar idea was used earlier by Babuska, Ihlenburg, Paik and Sauter in [2] to define a quasi-stabilised nine-point stencil equation for the 2D Helmholtz equation, which is exactly satisfied by the plane waves in the following angles (from x-axis) $\frac{\pi}{16} + j\frac{\pi}{8}$ for $j = 0, 1, .., 15$, and constant solutions. We will see by dispersion analysis that the selected angles give quasi-minimised dispersion error. In this work, we are interested in the generalisation of the idea to compact high order schemes and 3D.

Fine tuning the stencil coefficients for the Helmholtz equation can significantly enhance the accuracy of numerical solutions, as shown by Stolk in [7], where both the phase error and asymptotic amplitude errors are minimised numerically. Closed-form asymptotic formulas for minimising dispersion errors of any given scheme are derived in [4, 3]. In this work, we follow the idea of [1, 2] to let the stencil equation be satisfied by some plane waves and polynomial solutions, and choose the angles of the plane waves to minimise the dispersion error.

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To go high order, we first use generalised Birkhoff interpolation of the solution u at the stencil points to define a smooth interpolant as linear combination of selected plane waves and polynomials. Then we use the Laplacian of the interpolant to approximate the Laplacian of the solution at the stencil center. The use of multidimensional interpolation for discretisation originated from the reconstruction or recovery based methods [9]. The interpolation of Δu and $\Delta^2 u$ is inspired by the Equation-Based methods of Singer and Turkel [6] which substitute Δu and $\Delta^2 u$, that appear in the Taylor expansion of the local truncation error of some low order scheme, with the Helmholtz equation $\Delta u + k^2 u = f$.

2 Interpolative High Order Schemes on a 9-Point Stencil in 2D

To start, we consider the 5-point stencil centered at the origin. We can find an interpolant $p_h \in \operatorname{span}\{1, x, y, x^2, y^2\}$ such that

$$p_h(0,0) = u(0,0), \quad p_h(\pm h, 0) = u(\pm h, 0), \quad p_h(0, \pm h) = u(0, \pm h).$$

Then $(\Delta u)(0,0) \approx (\Delta p_h)(0,0)$ gives us the classical second order scheme. We have the following lemmas for interpolative high order schemes.

Lemma 1 (9-point Fourth Order Discretizations in 2d) *Let u and p_h be six times continuously differentiable in $(-h,h)^2$, and all the sixth partial derivatives of p_h be h-uniformly bounded in $(-h,h)^2$. If $p_h(x,y) = u(x,y)$ for $x, y \in \{0, -h, h\}$ and* **either** *of the following groups of conditions hold:*

(i) $(\Delta p_h)(\pm h, 0) = (\Delta u)(\pm h, 0)$, $(\Delta p_h)(0, \pm h) = (\Delta u)(0, \pm h)$,
(ii)$(\Delta^2 p_h)(0,0) = (\Delta^2 u)(0,0)$,

then $(\Delta u)(0,0) = (\Delta p_h)(0,0) + O(h^4)$.

Note that the lemma is independent of what shape functions will be used for the interpolation. The proof can be done by Taylor expansion.

Lemma 2 (9-point Sixth Order Discretizations in 2d) *Let u and p_h be eight times continuously differentiable in $(-h,h)^2$, and all the eighth-order partial derivatives of p_h be h-uniformly bounded in $(-h,h)^2$. If $p_h(x,y) = u(x,y)$ for $x, y \in \{0, -h, h\}$, $(\Delta^2 p_h)(0,0) = (\Delta^2 u)(0,0)$ and* **one** *of the following groups of conditions hold:*

(i) $(\Delta p_h)(x,y) = (\Delta u)(x,y)$, *for* $x, y \in \{0, \pm h\}$ *but* $(x,y) \neq (0,0)$,
(ii) $(\Delta^m p_h)(x,y) = (\Delta^m u)(x,y)$, *for* $(x,y) \in \{(0, \pm h), (\pm h, 0)\}$ *and* $m = 1, 2$,
(iii) $\frac{\partial^4 \Delta p_h}{\partial x^2 \partial y^2}(0,0) = \frac{\partial^4 \Delta u}{\partial x^2 \partial y^2}(0,0) + O(h^2)$, $(\Delta^3 p_h)(0,0) = (\Delta^3 u)(0,0)$,

then $(\Delta u)(0,0) = (\Delta p_h)(0,0) + O(h^6)$.

The stencil coefficients given in [2] can also be obtained by the interpolant $p_h \in \operatorname{span}\{1, \mathrm{e}^{\mathrm{i}k(x\cos\theta + y\sin\theta)}$ for $\theta = \frac{\pi}{16}, \frac{3\pi}{16}\}$ such that $p_h(x,y) = u(x,y)$ for $x, y \in$

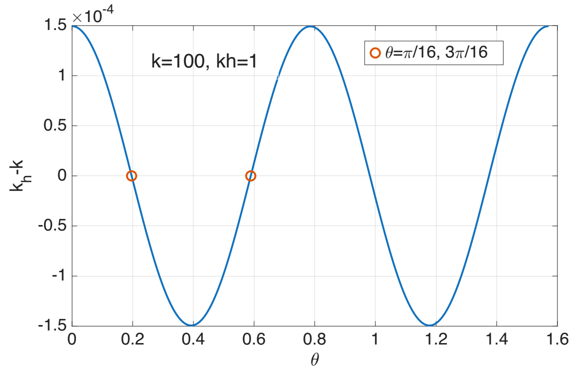

Fig. 1 Dispersion error if $\mathrm{e}^{\mathrm{i}k(x\cos\theta+y\sin\theta)}$ for $\theta=\frac{\pi}{16},\frac{3\pi}{16}$ satisfy homogeneous stencil equation.

$\{0,-h,h\}$. The dispersion error k_h-k is shown in Fig. 1, where $k_h=k_h(\theta)$ is the discrete wavenumber such that $\mathrm{e}^{\mathrm{i}k_h(\theta)(x\cos\theta+y\sin\theta)}$ satisfies homogeneous stencil equations for all $\theta\in\mathbb{R}$.

To enhance the second-order scheme in [2] to order four, we use

$$p_h\in\operatorname{span}\left\{1,x^2,\mathrm{e}^{\mathrm{i}k(x\cos\theta+y\sin\theta)}\text{ for }\theta=\frac{\pi}{16},\frac{3\pi}{16}\right\}$$

to interpolate

$$u(0,0),\ u_s:=u(0,h)+u(0,-h)+u(h,0)+u(-h,0),$$
$$u_c:=u(h,h)+u(h,-h)+u(-h,h)+u(-h,-h),\text{ and }(\Delta u)_s.$$

Then we get a fourth order scheme with the dispersion error that look identical[1] to that of the second order scheme in [2].

To further enhance the order, we use

$$p_h\in\operatorname{span}\left\{1,x^2,x^4,x^2y^2,\mathrm{e}^{\mathrm{i}k(x\cos\theta+y\sin\theta)}\text{ for }\theta=\frac{\pi}{16},\frac{3\pi}{16}\right\}$$

to interpolate

$$u(0,0),\ u_s,\ u_c,\ (\Delta u)_s,(\Delta u)_c\text{ and }(\Delta^2u)(0,0).$$

Then we get a sixth order scheme with the dispersion error that look identical to that of the second order scheme in [2].

We note that the stencil coefficients are obtained numerically for a specific value of kh because analytic formulas are too complicated to get or use. This computation is local and can be parallelised for different stencil centers, or approximated by some interpolants as done in [7] for the parameterised stencil coefficients.

[1] This observation has not been proved yet, so are the similar observations below.

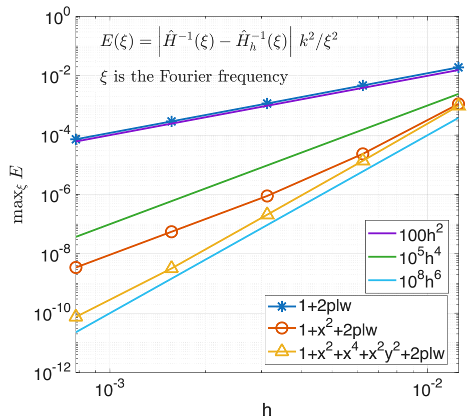

Fig. 2 h-Convergence of the maximum symbol errors for the 2nd, 4th and 6th order schemes in 2D with $k = 80$. The 2nd order scheme uses two plane waves (plw) and the constant one for interpolation. The 4th order scheme uses 1, x^2 and two plane waves. The 6th order scheme uses $1, x^2, x^4, x^2y^2$ and two plane waves.

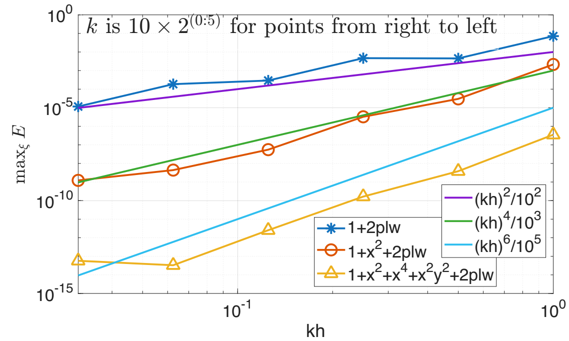

Fig. 3 kh-Convergence of maximum symbol errors for the 2nd, 4th and 6th order schemes in 2D.

To evaluate the accuracy of the above schemes for general sources, we use the tool developed in [8, 5]. The convergence is predicted in the Fourier domain by the symbol errors. The symbol of the Helmholtz operator is $\hat{H} = k^2 - |\xi|^2$ with $\xi = (\xi_x, \xi_y)$ the spatial Fourier frequency in x and y. The symbol of the solution operator for the discretised Helmholtz problem is denoted by $\hat{H}_h^{-1}$. More precisely, for the 9-point stencil equation

$$a_0 u(0,0) + a_s u_s + a_c u_c = b_0 f(0,0) + b_s f_s + b_c f_c + c_0 (\Delta f)_0$$

with the mesh step size h, the symbol of the solution operator is

$$\hat{H}_h^{-1} = \frac{b_0 + b_s S(\xi) + b_c C(\xi) - c_0 |\xi|^2}{a_0 + a_s S(\xi) + a_c C(\xi)},$$

where $S(\xi) = 2(\cos(\xi_x h) + \cos(\xi_y h))$, $C(\xi) = 4\cos(\xi_x h)\cos(\xi_y h)$. Then we define the error symbol for the 9-point stencil as $E(\xi) = \left|\hat{H}^{-1}(\xi) - \hat{H}_h^{-1}(\xi)\right| k^2/|\xi|^2$. First, we check the order in h only; see Fig. 2. Then we measure the performance when kh decreases in Fig. 3 where little dispersion error is seen in the range of k.

Table 1 Numerical values of the unit vectors **l** for the selected plane waves $e^{ik(l_x x+l_y y+l_z z)}$

l_x	l_y	l_z
0.164949174608596	0.537613423217881	0.826899980028960
0.199565210241023	0.096899997954037	0.975081595179585
0.481948496020002	0.401141513744529	0.778980829760788

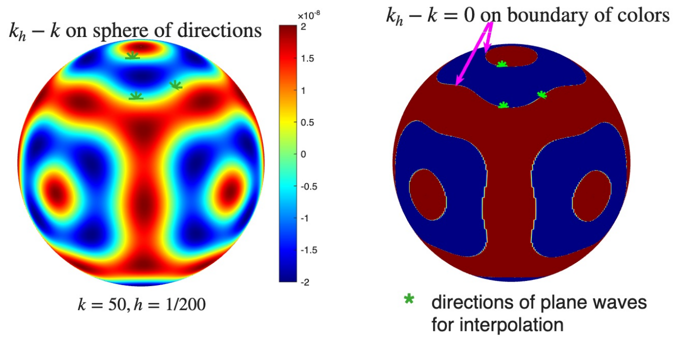

Fig. 4 Left: dispersion error if $e^{ik(l_x x+l_y y+l_z z)}$ for (l_x, l_y, l_z) in Tab. 1 are satisfied by the homogeneous stencil equation. Right: zero level set of the dispersion error.

3 Interpolative High Order Schemes on a 27-Point Stencil in 3D

Unlike in 2D, no analytical formula is available in 3D for which directions of plane waves one should interpolate. We studied this question numerically by minimizing $\max_{|\mathbf{d}|=1} |k_h(\mathbf{d}) - k|$ and found numerical values that look quasi-optimal. Three unit vectors can be chosen as $\mathbf{l} = (l_x, l_y, l_z)$ in the plane wave $e^{ik(l_x x+l_y y+l_z z)}$ to be satisfied by the homoegeneous stencil equation. They are listed in Tab. 1. Using those plane waves, the dispersion error is plotted on the surface of the unit sphere; see Fig. 4 where the positive maxima and negative minima look equal, indicating an almost equioscillating situation and thus a quasi-optimal choice of the interpolation points on the unit spherical surface. Note that although we choose only three plane waves to interpolate, in effect twelve (essentially two by symmetry) closed curves on the surface are interpolated automatically. The numbers in Tab. 1 and the zero level set in Fig. 4 are quite stable and nearly independent of kh.

Results similar to Lem. 1-2 can be derived in 3D. Based on that, using $p_h \in \mathrm{span}\{1, x^2, e^{ik(l_x x+l_y y+l_z z)}$ for (l_x, l_y, l_z) in Tab. 1$\}$ to interpolate

$$u(0,0,0),\ u_s := \sum_{\substack{x,y,z\in\{0,\pm h\}:\\ x^2+y^2+z^2=h^2}} u(x,y,z),\ u_{sc} := \sum_{\substack{x,y,z\in\{0,\pm h\}:\\ x^2+y^2+z^2=2h^2}} u(x,y,z),$$

$$u_{cc} := \sum_{x,y,z\in\{\pm h\}} u(x,y,z),\ \text{and } (\Delta u)_s,$$

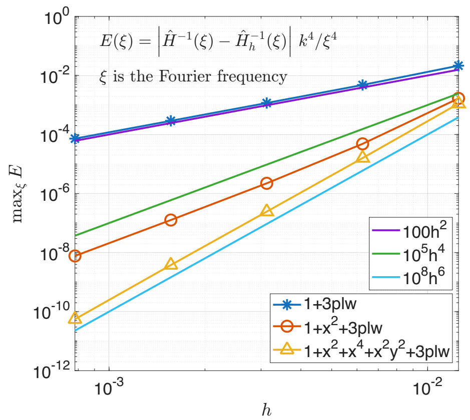

Fig. 5 h-Convergence of the maximum symbol errors for the 2nd, 4th and 6th order schemes in 3D with $k = 80$. The 2nd order scheme uses three plane waves (plw) and the constant one for interpolation. The 4th order scheme uses 1, x^2 and three plane waves. The 6th order scheme uses 1, x^2, x^4, x^2y^2 and three plane waves.

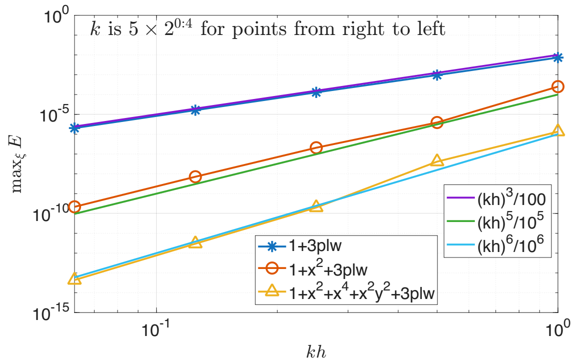

Fig. 6 kh-Convergence of maximum symbol errors for the 2nd, 4th and 6th order schemes in 3D.

we get a fourth order scheme with the dispersion error as in Fig. 4.

To get a sixth order scheme with the dispersion error as in Fig. 4, we use $p_h \in \mathrm{span}\{1, x^2, x^4, x^2y^2, \mathrm{e}^{\mathrm{i}k(l_x x + l_y y + l_z z)}$ for (l_x, l_y, l_z) in Tab. 1$\}$ to interpolate

$$u(0,0,0),\ u_s := \sum_{\substack{x,y,z\in\{0,\pm h\}:\\ x^2+y^2+z^2=h^2}} u(x,y,z),\ u_{sc} := \sum_{\substack{x,y,z\in\{0,\pm h\}:\\ x^2+y^2+z^2=2h^2}} u(x,y,z),$$

$$u_{cc} := \sum_{x,y,z\in\{\pm h\}} u(x,y,z),\ (\Delta u)_s,\ (\Delta u)_{sc},\ \text{and}\ (\Delta^2 u)(0,0,0).$$

The convergence is analysed in the Fourier domain; see Fig. 5-6.

4 Numerical Experiments in 2D

We solve the Helmholtz equation $\Delta u + k^2 u = f$ in the unit square with Dirichlet boundary conditions. For $k = 50, 100, 200$, we did the h-refinement tests for the 2nd, 4th and 6th order schemes described in Sec. 2. The results are shown in Fig. 7. With the given exact solution, higher wavenumber k usually leads to larger error at the same mesh size h, but some exceptions occur for the 2nd and 4th order schemes, which may be caused by the changing distance of k to the spectra of the discretised Helmholtz operator.

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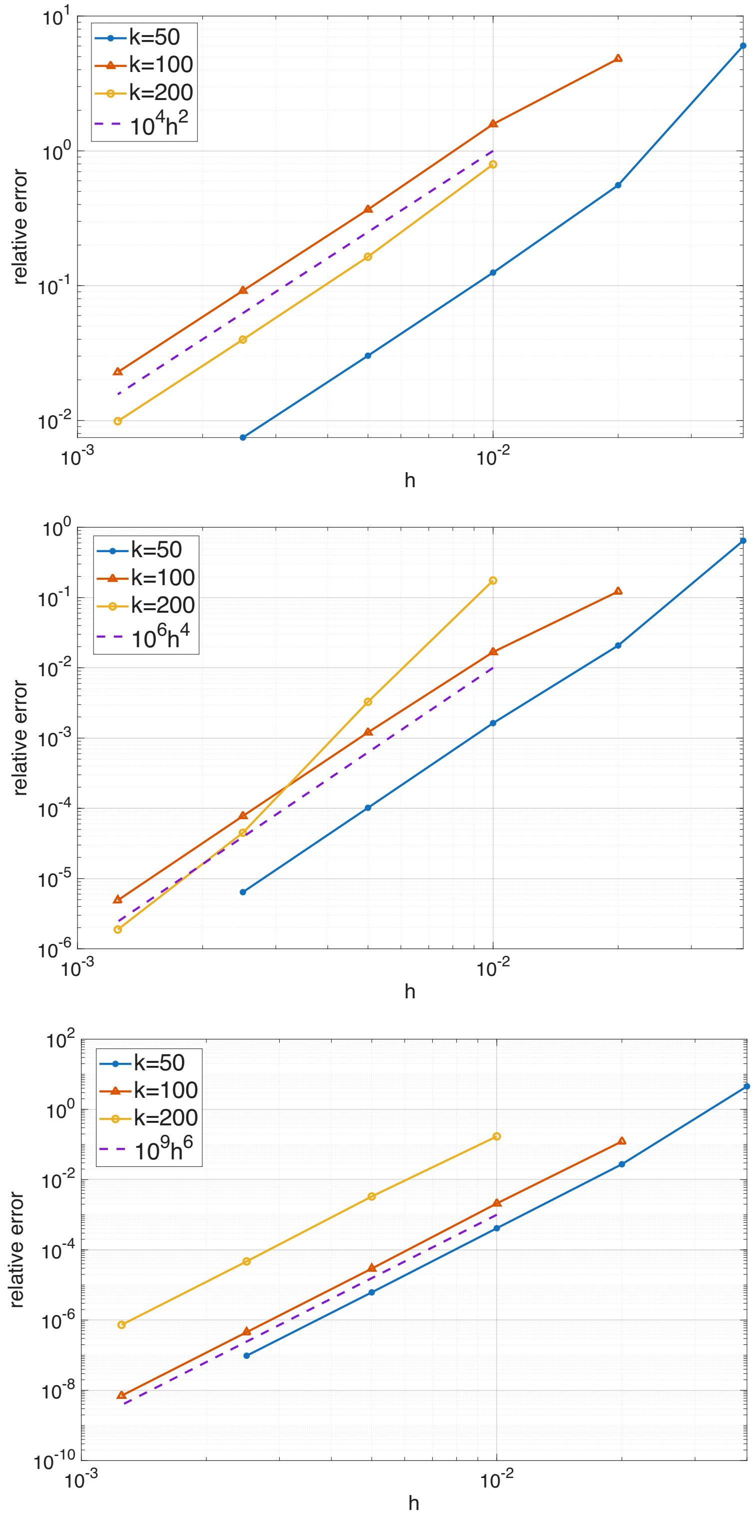

Fig. 7 *h*-Convergence of the 2nd, 4th and 6th order schemes in 2D for the wavenumber k-dependent exact solution $u = 2\exp(\mathrm{i}k(x\cos\frac{\pi}{4} + y\sin\frac{\pi}{4})) + \exp(x+y)\sin(90xy)$.