Minimizing the convergence factor of a heterogeneous space-time domain decomposition method

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1 Introduction

Heterogeneous domain decomposition methods are a very active field of research, for a brief introduction, see [11]. Since Optimized Schwarz Methods (OSMs) can be used with non-overlapping subdomains, see [5], they are ideal candidates for heterogeneous domain decomposition where the physics in different subdomains is different and requires different numerical treatment with possibly different codes. In addition, in OSMs one optimizes transmission conditions between subdomains for fast convergence, and OSMs can even take advantage of the different physics to converge faster than if the physical properties were the same, see [6] for a typical example of diffusion with jumping coefficients.

Recently, also time dependent heterogeneous domain decomposition methods were proposed and analyzed, see, e.g., [7, 12, 8, 13, 9]. In [2], an Optimized Schwarz Waveform Relaxation algorithm (OSWR) was studied for a heat-wave coupling in 1D on unbounded domains, which is a minimal example of relevance for fluid-structure interaction. On unbounded domains in 1D, optimal transmission conditions for OSWR turn out to be particularly simple for the wave equation domain, since the best transmission condition choice is still local. In [1], the authors study the heat-wave coupling problem and associated OSWR algorithms for the case of bounded domains. This is closer to practical applications in which fluid and solid domains are bounded and displacements, velocities and tractions are imposed on the external

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boundaries, see, *e.g.*, [3]. In [1], there are two new such approximations, a first one where both the heat and the wave domain use the same optimized parameter, and a second one where the wave domain uses a local optimal parameter, and the heat parameter is then optimized for this setting. We investigate here the optimization with two independent parameters, for each subdomain. Surprisingly, we find that the best choice corresponds to the same value for both parameters.

Let $l_w > 0$ be the domain length of the wave domain, $\Omega_w := (-l_w, 0)$ and $\Omega_h := (0, +\infty)$ be the heat domain. Let $c > 0$ be the wave speed and $\kappa > 0$ be the heat diffusion coefficient. Let $\alpha_w \in \mathbb{R}$. The source terms are denoted by f and g, and u_0, v_0 and $\dot{v}_0$ are the initial conditions. We are interested in designing and studying a heterogeneous OSWR algorithm for the heat and wave coupled problem

$$\begin{cases} \partial_t^2 v - c^2\, \partial_x^2 v = f & \text{in } (0,T)\times\Omega_w, \\ -\partial_x v + \alpha_w\, v = 0 & \text{on } (0,T)\times\{x=-l_w\}, \\ v(0,\cdot) = v_0,\ \partial_t v(0,\cdot) = \dot{v}_0 & \text{in } \Omega_w, \end{cases}$$
$$\begin{cases} \partial_t u - \kappa\, \partial_x^2 u = g & \text{in } (0,T)\times\Omega_h, \\ u(0,\cdot) = u_0 & \text{in } \Omega_h. \end{cases} \tag{1}$$

Equations (1) are simplified versions of (9.3)–(9.4) in [4, Chapter 9, p. 308]. The heat equation retains the parabolic structure of time-dependent fluid models, while the wave equation retains the hyperbolic structure of elastodynamics. In this analogy, v represents a displacement and u a velocity (time derivative of a displacement). To model fluid–structure interaction, equations (1) must be supplemented by coupling conditions at $x = 0$. Simplified from (9.5) in [4, Chapter 9, p. 308], these enforce continuity of velocities $\partial_t v$ and u (essential), and fluxes $c^2\partial_x v$ and $\kappa\partial_x u$ (natural) across the interface, crucial for the energy balance of the coupled system (cf. Proposition 9.1, p. 309), and are thus given by

$$\begin{cases} \partial_t v = u & \text{on } (0,T)\times\{x=0\}, \\ c^2\partial_x v = \kappa\,\partial_x u & \text{on } (0,T)\times\{x=0\}. \end{cases} \tag{2}$$

We propose to solve (1)-(2) using an alternating heterogeneous Schwarz Waveform Relaxation algorithm (see [1]) that computes for iteration index $k = 1, 2, \ldots$

$$\begin{cases} \partial_t^2 v^k - c^2\, \partial_x^2 v^k = f, & \text{in } (0,T)\times\Omega_w, \\ v^k(0,\cdot) = v_0, & \text{in } \Omega_w, \\ \partial_t v^k(0,\cdot) = \dot{v}_0, & \text{in } \Omega_w, \\ -\partial_x v^k + \alpha_w\, v^k = 0, & \text{on } (0,T)\times\{x=l_w\}, \\ \left(s_1\partial_t + c^2\partial_x\right) v^k = (s_1 + \kappa\partial_x)\, u^{k-1} & \text{on } (0,T)\times\{x=0\}, \end{cases}$$
$$\begin{cases} \partial_t u^k - \kappa\, \partial_x^2 u^k = g, & \text{in } (0,T)\times\Omega_h, \\ u^k(0,\cdot) = u_0, & \text{in } \Omega_h, \\ (s_2 + \kappa\partial_x)\, u^k = \left(s_2\partial_t + c^2\partial_x\right) v^k & \text{on } (0,T)\times\{x=0\}, \end{cases} \tag{3}$$

where $s_1 > 0$ and $s_2 < 0$ are two real numbers to be chosen. In [1] the convergence factor was optimized under the simplifying assumption that $s_1 = |s_2|$. We prove here that $s_1 = |s_2|$ is indeed the optimal choice.

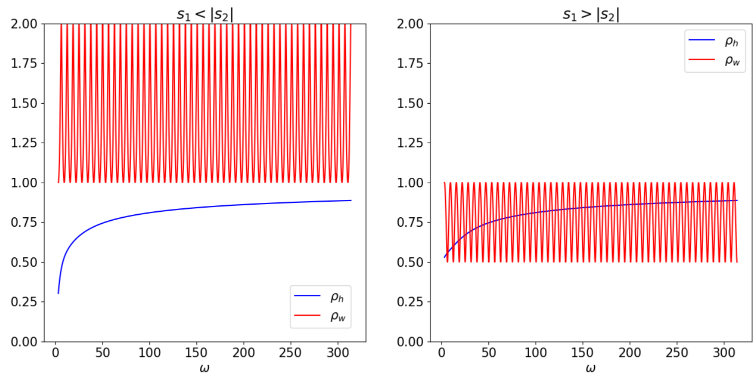

Fig. 1 Heat and wave convergence factors ρ_h and ρ_w as functions of the frequency ω.

2 The Optimization Problem

Let $\hat{v}(i\omega, x)$, $\omega \in \mathbb{R}$ be the Laplace transform of the function v. Solving the Laplace transform of (3), we obtain with a short computation (see [1]) the convergence factor

$$\rho(i\omega; s_1, s_2) := \frac{\hat{v}^k(i\omega, 0)}{\hat{v}^{k-1}(i\omega, 0)} = \rho_h(i\omega; s_1, s_2)\, \rho_w(i\omega; s_1, s_2), \tag{4}$$

where ρ_h and ρ_w are two factors coming from the heat and wave equation,

$$\rho_h(i\omega; s_1, s_2) := \frac{s_1 - \sqrt{\kappa i\omega}}{s_2 - \sqrt{\kappa i\omega}} \text{ and } \rho_w(i\omega; s_1, s_2) := \frac{s_2 i\omega + c^2\phi_w(i\omega)}{s_1 i\omega + c^2\phi_w(i\omega)}, \tag{5}$$

with $\phi_w(i\omega) := \frac{i\omega}{c}\left(1 + \frac{c\alpha_w - i\omega}{c\alpha_w + i\omega} e^{\frac{-2i\omega l_w}{c}}\right)\left(1 - \frac{c\alpha_w - i\omega}{c\alpha_w + i\omega} e^{\frac{-2i\omega l_w}{c}}\right)^{-1}$. We will study separately the behavior of ρ_h and ρ_w, see Figure 1 for an illustration of the two behaviors.

Theorem 1 *The convergence factor $\omega \to |\rho_w(i\omega; s_1, s_2)|$ coming from the wave domain is oscillating in ω between 1 and $|s_2|/s_1$.*

Proof. In [1] it was shown that ϕ_w is a real number, which implies that

$$|\rho_w(i\omega; s_1, s_2)|^2 \quad = \frac{s_2^2\omega^2 + c^4\phi_w(i\omega)^2}{s_1^2\omega^2 + c^4\phi_w(i\omega)^2} = 1 - \frac{1 - (\frac{s_2}{s_1})^2}{1 + \frac{c^4}{s_1^2}\frac{\phi_w(i\omega)^2}{\omega^2}}. \tag{6}$$

Letting $e^{i\theta} := \frac{c\alpha_w - i\omega}{c\alpha_w + i\omega} = \frac{c^2\alpha_w^2 - \omega^2}{c^2\alpha_w^2 + \omega^2} - i\frac{2c\alpha_w\omega}{c^2\alpha_w^2 + \omega^2}$ for the coefficient in ϕ_w, we obtain

$$\frac{c}{\omega}\phi_w(i\omega) = i\frac{1 + e^{i(\theta - \frac{2\omega}{c}l_w)}}{1 - e^{i(\theta - \frac{2\omega}{c}l_w))}} = -\frac{\cos(\psi/2)}{\sin(\psi/2)},$$

where $\psi := \theta - \frac{2\omega}{c} l_w$. This leads for the wave convergence factor in (6) to

$$|\rho_w(i\omega; s_1, s_2)|^2 = 1 - \frac{1 - (\frac{s_2}{s_1})^2}{1 + \frac{c^2}{s_1^2} \frac{\cos^2(\psi/2)}{\sin^2(\psi/2)}}.$$

Let $\beta := \frac{c^2}{s_1^2}$ and $g_\beta : t \longrightarrow (1 + \beta \frac{\cos^2(t)}{\sin^2(t)})^{-1}$. With the derivative $g'_\beta(t) = \frac{\beta \sin(2t)}{(1+(\beta-1)\cos^2(t))^2}$, the variations of g_β are given by the sign of $\sin(2t)$, and g_β is oscillating between 0 and 1 (including these values). Since

$$|\rho_w(i\omega; s_1, s_2)|^2 = 1 - \left(1 - \left(\frac{s_2}{s_1}\right)^2\right) g_\beta(t),$$

$|\rho_w(i\omega; s_1, s_2)|$ is oscillating between 1 (when $g_\beta = 0$) and $|s_2|/s_1$ (when $g_\beta = 1$). □

Corollary 1 *If $|s_2| < s_1$, then $|\rho(i\omega; s_1, s_2)| \le |\rho_h(i\omega; s_1, s_2)|$. And if $|s_2| = s_1$, then $|\rho(i\omega; s_1, s_2)| = |\rho_h(i\omega; s_1, s_2)|$.*

Based on Corollary 1, we propose to solve the min-max problem

$$\min_{(p,q)\in K} \mathcal{G}(p,q), \quad \mathcal{G}(p,q) := \max_{\omega \in [\omega_{min}, \omega_{max}]} |\rho_h(i\omega; p, -q)|^2 \tag{7}$$

where $K := \{(p,q) \in \mathbb{R}^2, q \ge 0, p - q \ge 0\}$ and where $\omega_{min} > 0$ and $\omega_{max} > 0$ are the smallest and the largest frequency occurring in the simulation, see [10, Figure 3.17] for an illustration.

3 Solving the Optimization Problem

We first show that the min-max problem is solved when $p = q$ and then provide the optimal value of this parameter.

Lemma 1 *For $\omega_{min} \le \omega_{max}/2$, the solution of the min-max problem* (7) *satisfies $p = q$.*

Proof. We let $r(\omega) := \sqrt{\kappa\omega}$ and study the function $\omega \mapsto |\rho_h(i\omega; p, -q)|^2$ with

$$|\rho_h(i\omega; p, -q)|^2 = \left|\frac{\sqrt{2}p - (1+i)r(\omega)}{\sqrt{2}q + (1+i)r(\omega)}\right|^2 = \frac{p^2 - \sqrt{2}pr(\omega) + r(\omega)^2}{q^2 + \sqrt{2}qr(\omega) + r(\omega)^2}.$$

Taking a derivative with respect to ω gives

$$\frac{\partial}{\partial\omega}|\rho_h(i\omega; p, -q)|^2 = (p+q)\frac{\sqrt{\kappa}}{2\sqrt{\omega}} \frac{\sqrt{2}r(\omega)^2 + 2(q-p)r(\omega) - \sqrt{2}pq}{(q^2 + \sqrt{2}qr(\omega) + r(\omega)^2)^2}.$$

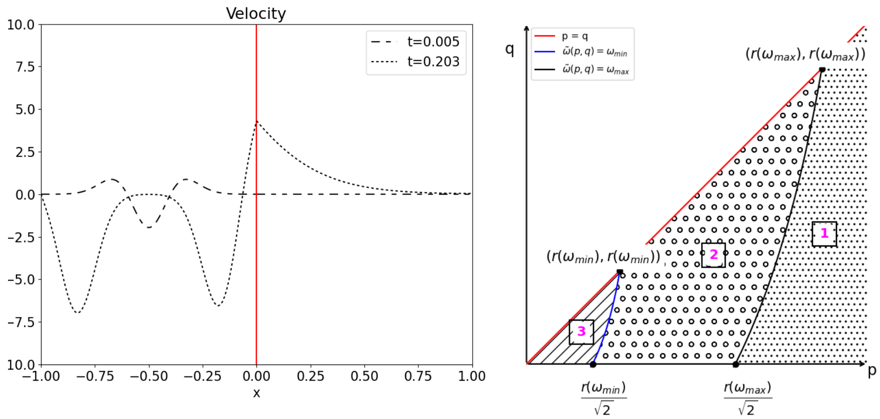

Fig. 2 Left : reference velocity $(\partial_t v, u)$ when $f = g = \dot{v}_0 = 0$ and $v_0(x) = e^{-50(x+0.5)^2}$. Right : the domain of definition of $\mathcal{G}(p, q)$ is partitioned into three parts, separated by curves given in implicit form by $\tilde{\omega}(p, q) = \omega_l$ ($l = \omega_{min}$ or $l = \omega_{max}$), which equivalently can be expressed explicitly as $q = r(\omega_l)\frac{\sqrt{2}p - r(\omega_l)}{\sqrt{2}r(\omega_l) - p}$ or $p = r(\omega_l)\frac{\sqrt{2}q + r(\omega_l)}{\sqrt{2}r(\omega_l) + q}$.

For the numerator to vanish, we must study the roots of the quadratic polynomial in $r(\omega)$, and the term $\sqrt{\tilde{\omega}} := \frac{1}{\sqrt{2\kappa}}(p - q + \sqrt{p^2 + q^2})$ depending on p and q appears. This shows that $\omega \to |\rho_h(i\omega; p, -q)|^2$ is decreasing for $\omega < \tilde{\omega}$ and increasing for $\omega > \tilde{\omega}$. We thus have to distinguish for the maximum $\mathcal{G}(p, q)$ three cases,

$$\mathcal{G}(p, q) = \begin{cases} |\rho_h(i\omega_{min}; p, -q)|^2 & \text{if } \omega_{max} \le \tilde{\omega}(p, q), \\ |\rho_h(i\omega_{max}; p, -q)|^2 & \text{if } \tilde{\omega}(p, q) \le \omega_{min}, \\ \max(|\rho_h(i\omega_{min}; p, -q)|^2, \rho_h(i\omega_{max}; p, -q)|^2) & \text{otherwise,} \end{cases}$$

and hence need to study three minimization problems, see Figure 2 (right):

- Minimize $(p, q) \to |\rho_h(i\omega_{min}; p, -q)|^2$ over the subregion 1 defined by

$$q \le r(\omega_{max})\frac{\sqrt{2}p - r(\omega_{max})}{\sqrt{2}r(\omega_{max}) - p}, \quad 0 \le q \le p.$$

- Minimize $(p, q) \to |\rho_h(i\omega_{max}; p, -q)|^2$ over the subregion 3 defined by

$$0 \le p \le r(\omega_{min}), \quad \max\left(0, r(\omega_{min})\frac{\sqrt{2}p - r(\omega_{min})}{\sqrt{2}r(\omega_{min}) - p}\right) \le q \le p.$$

- Minimize $(p, q) \to \max(|\rho_h(i\omega_{min}; p, -q)|^2, \rho_h(i\omega_{max}; p, -q)|^2)$ over the subregion 2 defined by

p	0	$r(\omega_l)/\sqrt{2}$	$+\infty$
$p \to \lvert\rho_h(i\omega_l;p,-q)\rvert^2$	$\searrow$		$\nearrow$

q	0	$+\infty$
$q \to \lvert\rho_h(i\omega_l;p,-q)\rvert^2$	$\searrow$	

Table 1 Variations of the convergence factor $|\rho_h|^2$ as a function of p and q.

$$r(\omega_{max})\frac{\sqrt{2}p - r(\omega_{max})}{\sqrt{2}r(\omega_{max}) - p} \le q \le r(\omega_{min})\frac{\sqrt{2}p - r(\omega_{min})}{\sqrt{2}r(\omega_{min}) - p}, \quad 0 \le q \le p.$$

If the minimum of one of these problems were to be attained in the interior of its domain of definition, then the gradient of $(p,q) \to |\rho_h(i\omega;p,-q)|^2$ would vanish. Computing the gradient,

$$\frac{\partial}{\partial p}|\rho_h(i\omega_l;p,-q)|^2 = \frac{2p - \sqrt{2}r(\omega_l)}{q^2 + \sqrt{2}qr(\omega_l) + r(\omega_l)^2},$$
$$\frac{\partial}{\partial q}|\rho_h(i\omega_l;p,-q)|^2 = -\frac{(p^2 - \sqrt{2}pr(\omega_l) + r(\omega_l)^2)(2q + \sqrt{2}r(\omega_l))}{(q^2 + \sqrt{2}qr(\omega_l) + r(\omega_l)^2)^2},$$

shows that it can not be zero in the quarter of plan $p \ge 0, q \ge 0$, and hence the minimum of each of the three problems is attained on the boundary. Note that from the computation of the gradient, we also obtain the variations of $|\rho_h(i\omega_l;p,-q)|^2$ as function of p and q, see Table 1, which will be useful in what follows.

Let us first study the minimization problem in subregion 3: First note that $(p,q) \in$ 3 is equivalent to (see the Figure 2 (right))

$$0 \le q \le r(\omega_{min}), \quad q \le p \le r(\omega_{min})\frac{\sqrt{2}q + r(\omega_{min})}{\sqrt{2}r(\omega_{min}) + q} =: \varphi(q).$$

Second, we can easily show that $q \mapsto \varphi(q)$ is increasing, and recalling the assumption $\omega_{min} \le \omega_{max}/2$, we find for any $q \in [0, r(\omega_{min})]$ that

$$\varphi(q) \le r(\omega_{min})\frac{\sqrt{2}+1}{\sqrt{2}+1} = r(\omega_{min}) \le \frac{r(\omega_{max})}{\sqrt{2}},$$

and thus $[q,\varphi(q)] \subset \left[0, \frac{r(\omega_{max})}{\sqrt{2}}\right]$. Hence, using Table 1 (left), for any $q \in [0, r(\omega_{min})]$, the function $p \to |\rho_h(i\omega_{max};p,-q)|^2$ is decreasing on $[q,\varphi(q)]$, and therefore, for any (p,q) in subregion 3, we must have

$$|\rho_h(i\omega_{max};p,-q)|^2 \ge |\rho_h\,(i\omega_{max};\varphi(q),-q)|^2.$$

Now since $q \to |\rho_h(i\omega_{max};p,-q)|^2$ is decreasing from Table 1 (right), we must have for any (p,q) in subregion 3 that

$$|\rho_h(i\omega_{max};p,-q)|^2 \ge |\rho_h(i\omega_{max};\varphi(q),-q)|^2 \ge |\rho_h(i\omega_{max};\varphi(q),-\varphi(q))|^2,$$

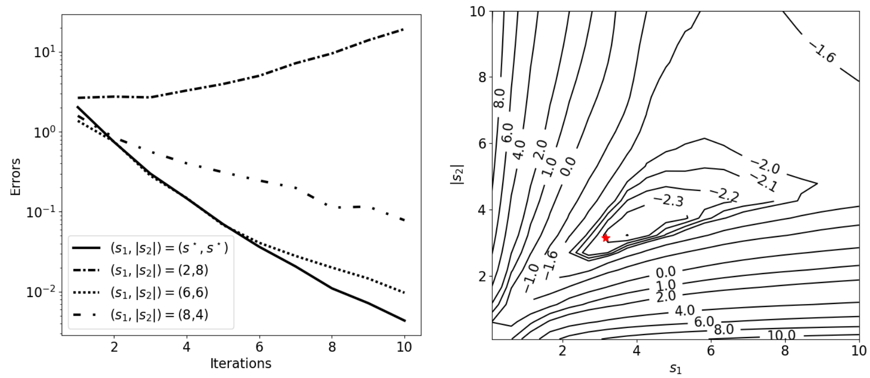

Fig. 3 Left: error in Ω_h for several values of (s_1, s_2). Right: contour lines of the error in Ω_h obtained after 10 iterations. The red star is for $(s_1, |s_2|) = (s^\star, s^\star)$, $s^\star \simeq 3.16$.

and the minimum of the convergence factor is reached on the boundary $p = q$. Similar arguments show that this is also the case in subregions 1 and 2. □

Theorem 2 *The optimal parameter* $s > 0$ *solving the min-max problem* (7) *is given by*

$$s^\star = \sqrt{\kappa}(\omega_{min}\omega_{max})^{\frac{1}{4}}. \tag{8}$$

Proof. Using Lemma 1, we obtain that the optimal parameters are attained for $p = q$, and it was shown in [1] that (8) is this solution. □

4 Numerical Results and Conclusions

To solve problem (1)-(2) with algorithm (3), we use the method of lines, where we combine finite volumes for the discretization in space and the Crank-Nicolson scheme for the discretization in time; see [1, Section 4] for details. We chose for our numerical experiments $T = 20$, split the domain $(-1, 1)$ into the two subdomains $\Omega_w := (-1, 0)$ and $\Omega_h := (0, 1)$, and set $\kappa = 1$ and $c = 2$. We simulate directly the error equations, i.e. set f, g, v_0, $\dot{v}_0$ and u_0 to zero, and start the iteration with an initial guess containing all error frequencies, $(s_1 + \kappa\partial_x)u^0 = (s_1 + \kappa)\frac{\sum_{j=1}^{100} t \sin(j\pi t)}{\max_{t\in[0,T]} |\sum_{j=1}^{100} t \sin(j\pi t)|}$. The mesh size is $\Delta x = 2\Delta t = \frac{1}{101}$. In Figure 3 (left), we show the error in the heat domain as function of the iterations. We see that the theoretically best choice $(s_1, |s_2|) = (s^\star, s^\star)$ performs best, and also choosing the same value $(s_1, |s_1|)$ performs quite well. On the right we show the contour lines of the error obtained after 10 iterations, and we see that the analysis predicts very well the optimal parameter.

To conclude, we have proved that the simplifying assumption from [1] to only use one parameter to make the solution of the min-max problem easier, and to use

$s_1 = -s_2$, is indeed leading to the optimal solution of the min-max problem when using two parameters, and this choice is also performing very well in practice in our numerical experiments.

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