

Accelerating the Method of Reflections with Domain Decomposition techniques for Boundary Integral Equations in Multiple Scattering

Stéphanie Chaillat* Marion Darbas† Martin J. Gander‡ Laurence Halpern§

Abstract

The Method of Reflections was historically introduced to obtain approximate solutions as series expansions for the motion of particles in suspension. It can however equally well be used for solving multiple scattering problems numerically. We show for Helmholtz multiple scattering problems that the Method of Reflections, whether applied in its alternating or parallel version, suffers from convergence problems when scatterers are close. We use boundary integral equations to formulate the methods, and then identify their discretization as algebraic Schwarz methods, thereby interpreting them as boundary domain decomposition techniques. This connection allows us to introduce remedies such as overlap (which can be partial, covering only the illuminating region of the obstacles) and coarse spaces from domain decomposition into the Method of Reflections. This leads to substantially accelerated variants, and also naturally makes them suitable preconditioners for GMRES. These new approaches are particularly efficient for closely obstacles. Moreover, numerical experiments show that the number of iterations increases only very slowly with respect to the wavenumber.

1 Introduction

The Method of Reflections is an elegant technique to obtain approximate solutions for problems involving particles in a solution. It was invented in 1833 by Murphy [23], and works as follows: suppose a single spherical particle in a solution. Then in simple settings, its motion can be given in closed form. If there is a second particle, its motion is influenced by the first particle, so a correction must be added to its motion. This in turn requires a correction again for the first particle, and so on. This algorithm is the Method of Reflections; for a detailed historical introduction, see [8]. In scattering problems, one can imagine a similar procedure: the solution for a single scatterer will be influenced if a second scatterer is present, and this influence thus requires a correction on the first scatterer, which in turn will require again a correction on the second scatterer, and so on.

When multiple scattering problems are formulated using boundary integral equations, the Method of Reflections provides an iterative solver for the resulting dense linear systems. It also defines a preconditioner for Krylov acceleration, see [7] for a general introduction to this equivalence. In this paper, we are interested in understanding the Method of Reflections within the framework of boundary domain decomposition methods. This approach decomposes the multiple scattering problem into a sequence of single scattering problems. In a volume setting, however, establishing a connection to classical domain decomposition methods is challenging due to the very large overlap, which corresponds to the entire exterior domain. Moreover, the single scattering problems provide solutions that are formally defined even inside the other scatterers [9]. By contrast, within the boundary integral framework for multiple scattering problems, and upon discretization by boundary element methods, we demonstrate that the Method of Reflections provides a natural way to apply domain decomposition techniques to the resulting dense linear systems.

We are not the first to exploit single-scattering information to tackle multibody scattering problems and to precondition the dense matrices. Under the name ‘single scattering preconditioner’, the parallel Method of Reflections was considered as a preconditioner for boundary integral equations in [31] [30], and a condition number estimate for the case of a multiple scattering problem by disks of the same radius was derived at low-frequency. It was shown that the condition number depends on the mesh size, the wavenumber, and the distance between the obstacles. The Method of Reflections was also studied as a boundary decomposition technique for Helmholtz and Maxwell scattering problems in the PhD thesis [32]. An abstract convergence analysis for low-frequency problems can be found in [2], and further results are presented in [1]. Projection techniques were also used to obtain convergence proofs for the Method of Reflections in [19], for Stokes flow in a suspension; for

*Laboratoire POEMS, CNRS-INRIA-ENSTA, Institut Polytechnique de Paris, Palaiseau, France. stephanie.chaillat@ensta.fr

†LAGA, CNRS UMR 7539, Université Sorbonne Paris Nord, Villetaneuse, France. darbas@math.univ-paris13.fr

‡Section de Mathématiques, Université de Genève, Genève, Switzerland. Martin.Gander@unige.ch

§LAGA, CNRS UMR 7539, Université Sorbonne Paris Nord, Villetaneuse, France. halpern@math.univ-paris13.fr

Laplace type problems, see the more recent results in [18]. A different approach based on the phase information for preconditioning Krylov subspace methods has been proposed to handle high frequencies in [3].

More generally, domain decomposition methods for boundary integral equations have historically received less attention than those developed for partial differential equations. A parallel Schwarz algorithm was introduced for strongly elliptic integral equations in [15], and the Additive Schwarz preconditioner has been numerically tested for boundary integral equations of the first kind in [22]; see also [29] for h-p versions, and [26] for an example in electromagnetics. In contrast to these approaches, the Boundary Element Tearing and Interconnect (BETI) methods [17] (see also the earlier development in [16]) use boundary integral equation techniques for the subdomain solutions in classical volume domain decomposition methods, which can provide advantages if some subdomains are unbounded [25]. This approach was also followed in the first part of the PhD thesis [20], while the second part is more close to our approach here, albeit in an abstract Schwarz setting.

We investigate the numerical behavior of the classical Method of Reflections in a boundary element setting, with particular emphasis on increasing wavenumbers and decreasing inter-scatterer distances, adopting a fresh perspective inspired by domain decomposition methods. To address the degradation observed in these regimes, we develop robust variants capable of handling closely spaced scatterers. The relation to domain decomposition permits the application of techniques such as overlap and coarse spaces to accelerate the Method of Reflections. This viewpoint leads to a new interpretation of overlap in the context of dense-matrix domain decomposition. We further show that coarse spaces not only ensure scalability, but also improve convergence for a fixed number of subdomains (here, the scatterers).

This paper is organized as follows. In Section 2, we introduce the multiple scattering problem for the Helmholtz equation and its integral representation, which will serve as the model problem throughout this paper. In Section 3, we present the original (alternating¹) and parallel Method of Reflections for scattering problems written in integral equation form, both at the continuous and discrete levels. We then demonstrate their performance through numerical experiments, which clearly indicate that convergence may be significantly hampered, or even completely destroyed when obstacles are positioned too close to one another. Even though Krylov acceleration can help alleviate this, as we demonstrate as well in Section 3, we prove in Section 4 that the Method of Reflections, in both its alternating and parallel versions, is very much related to classical algebraic Schwarz domain decomposition methods. This allows us to view the Method of Reflections as a boundary domain decomposition method, and to introduce overlap (even partially, for the illuminating region only) of obstacles, as well as coarse space corrections, which can greatly enhance the performance when obstacles are close to each other. In Section 5, we present a large-scale illustration of diffraction by a school of fish using our new, enhanced methods. We conclude in Section 6.

2 Acoustic multiple scattering problems and boundary integral equations

This section establishes the theoretical and numerical framework of the study, presenting the exterior Helmholtz problem, its integral formulation in the form of the EFIE, and its discretization using the boundary element method.

2.1 Integral representation

Let $O \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain representing an impenetrable body with a smooth boundary $\Gamma := \partial O$. We denote by $\mathbf{n}$ the unit outer normal vector to Γ , *i.e.* exiting O , and by $\Omega := \mathbb{R}^d \setminus \overline{O}$ the homogeneous domain of propagation. For a time-harmonic incident wave u^{inc} and a wavenumber k , the sound-soft scattering problem consists in computing the scattered wave u that solves the exterior problem

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } \Omega, \\ u = g := -u^{\text{inc}} \text{ on } \Gamma, \\ \lim_{|\mathbf{x}| \rightarrow +\infty} |\mathbf{x}|^{(d-1)/2} \left(\nabla u \cdot \frac{\mathbf{x}}{|\mathbf{x}|} - iku \right) = 0 \quad (\text{Radiation Condition}). \end{cases} \quad (1)$$

The analysis of Problem (1) can be found in [10, 24]. It establishes the existence and uniqueness of the scattered wave $u \in H_{\text{loc}}^1(\Omega)$, and the single-layer potential representation

$$u(\mathbf{x}) = \mathcal{S}(q)(\mathbf{x}) := \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) d\Gamma(\mathbf{y}), \quad \mathbf{x} \in \Omega, \quad (2)$$

¹We will not always mention in what follows 'alternating', since the original Method of Reflections was alternating, and distinguish it by systematically saying 'parallel method of reflections' for the parallel variant.

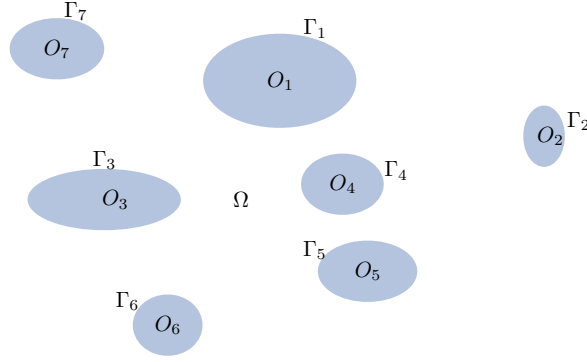


Figure 1: Illustration of a multiple scattering configuration for $I = 7$ obstacles.

where G is the fundamental solution of the Helmholtz equation² in $\mathbb{R}^d$ with wavenumber k , and the single-layer density $q = -\partial_{\mathbf{n}}(u + u^{\text{inc}})$ is the normal derivative of the total wave, with $\partial_{\mathbf{n}}u = \nabla u \cdot \mathbf{n}$. If $g \in H^{\frac{1}{2}}(\Gamma)$, then $q \in H^{-\frac{1}{2}}(\Gamma)$. The density q becomes the unknown of the problem and can be determined by setting the trace of u from (2) onto Γ equal to g , and solving the resulting boundary integral equation (namely the Electric Field Integral Equation or, EFIE) defined on Γ for q . Once q is computed, the solution u of (1) is given by (2).

We consider here a collection of I disjoint bounded domains $\{O_i\}_{1 \leq i \leq I}$ of $\mathbb{R}^d$ with smooth boundaries $\{\Gamma_i\}_{1 \leq i \leq I}$. Let $\Omega := \mathbb{R}^d \setminus \cup_{i=1}^I \overline{O_i}$ be the associated domain of propagation, see Figure 1. The corresponding multiple scattering Helmholtz problem [21] is then given by

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = g_i := -u^{\text{inc}} & \text{on } \Gamma_i, \text{ for } i = 1, \dots, I \quad + \text{ Radiation Condition,} \end{cases} \quad (3)$$

where $g_i \in H^{\frac{1}{2}}(\Gamma_i)$, $i = 1, \dots, I$. Since the obstacles O_i are disjoint, the single-layer representation of the solution u of problem (3) can be written as a sum of I single-layer potentials,

$$u(\mathbf{x}) = \sum_{i=1}^I \mathcal{S}_i(q_i)(\mathbf{x}) := \sum_{i=1}^I \int_{\Gamma_i} G(\mathbf{x}, \mathbf{y}) q_i(\mathbf{y}) d\Gamma_i(\mathbf{y}), \quad \mathbf{x} \in \Omega, \quad (4)$$

where $q_i := -\partial_{\mathbf{n}_i}(u + u^{\text{inc}})$ is the normal derivative of the total wave $u + u^{\text{inc}}$ on Γ_i . We denote by $\mathbf{n}_i$ the unit outer normal vector to Γ_i in Ω , *i.e.* exiting O_i . The layer densities q_i , $i = 1, \dots, I$, serve as the new unknowns of the problem (3).

By taking the trace on Γ_i , $i = 1, \dots, I$, in the integral representation (4) and imposing the Dirichlet boundary condition $u|_{\Gamma_i} = g_i$, we get a system of I integral equations. The boundary integral formulation for problem (3) is to find $(q_1, q_2, \dots, q_I) \in H^{-\frac{1}{2}}(\Gamma_1) \times \dots \times H^{-\frac{1}{2}}(\Gamma_I)$ such that

$$\sum_{j=1}^I S_{ij}(q_j) = g_i, \text{ on } \Gamma_i, \quad i = 1, \dots, I, \quad (5)$$

where $S_{ij} : H^{-\frac{1}{2}}(\Gamma_j) \rightarrow H^{\frac{1}{2}}(\Gamma_i)$ is the boundary integral operator defined by

$$S_{ij}(\varphi)(\mathbf{x}) := \int_{\Gamma_j} G(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) d\Gamma_j(\mathbf{y}), \quad \mathbf{x} \in \Gamma_i. \quad (6)$$

We consider here a single-layer potential representation, although other representations using the double-layer potential can also be considered, see e.g. [3]. System (5) is well posed except at interior Dirichlet eigenvalues (e.g., [30]). Even at these frequencies, the radiating exterior solution remains uniquely determined by the boundary value problem (3) [10].

2.2 Discretization by Boundary Element Methods

We discretize Problem (5) using a Galerkin Boundary Element Method (BEM) (see e.g. [27]). We consider $\mathbb{P}_1$ boundary finite elements on polygonal meshes $\Gamma_{i,h}$ approximating Γ_i . The number of vertices on $\Gamma_{i,h}$ is $N_{i,h}$

² G is a radial distribution, equal to $\frac{1}{4\pi} e^{ikr}/r$ for $d = 3$ and $H_0^{(1)}(kr)$ for $d = 2$, $r = |\mathbf{x} - \mathbf{y}|$, $\mathbf{x} \neq \mathbf{y}$, where $H_0^{(1)}(kr)$ is the Hankel function of the first kind and order zero.

and the total number of vertices is $N_h := \sum_{i=1}^I N_{i,h}$. We then get the linear system

$$\mathbb{S}\mathbf{q} = \mathbf{g} \Leftrightarrow \begin{pmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} & \dots & \mathbb{S}_{1I} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{S}_{I1} & \mathbb{S}_{I2} & \dots & \mathbb{S}_{II} \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_I \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_I \end{pmatrix}, \quad (7)$$

where $\mathbb{S}_{ij} \in \mathcal{M}_{N_{i,h} \times N_{j,h}}(\mathbb{C})$ is the dense matrix associated with the discretization of the single-layer boundary integral operator S_{ij} in (6), $\mathbf{q}_i \in \mathbb{C}^{N_{i,h}}$ is a $\mathbb{P}_1$ approximation of the unknown density q_i , and $\mathbf{g}_i \in \mathbb{C}^{N_{i,h}}$ is the interpolation on the boundary mesh of the data g_i . The matrix $\mathbb{S}$ is thus partitioned into dense blocks, each representing the interaction between a given pair of obstacles. From $\mathbf{q}_i$, $i = 1, \dots, I$, an approximation of the scattered wave u is computed by discretizing the integral representation (4).

We are interested in interpreting the Method of Reflections as a solver and preconditioner for the system (7), and improving its performance using techniques from Domain Decomposition.

3 Block Iterative Methods and the Method of Reflections

We introduce block iterative methods of Gauss-Seidel and Jacobi type for the solution of the linear system (7), and explain how they are related to the Method of Reflections.

3.1 Block Gauss-Seidel and the alternating Method of Reflections

To apply a block Gauss-Seidel method for solving (7), we consider a block decomposition of the matrix $\mathbb{S}$ into its lower triangular, diagonal, and upper triangular parts, $\mathbb{S} = \mathbb{L} + \mathbb{D} + \mathbb{U}$ and, at iteration index n , solve

$$(\mathbb{D} + \mathbb{L})\mathbf{q}^n = -\mathbb{U}\mathbf{q}^{n-1} + \mathbf{g}. \quad (8)$$

With the I vectors $\mathbf{q}_i^n \in \mathbb{C}^{N_{i,h}}$, $i = 1, \dots, I$, and $\mathbf{q}^n := (\mathbf{q}_1^n, \dots, \mathbf{q}_I^n) \in \mathbb{C}^{N_h}$, we obtain Algorithm 1.

Algorithm 1 Block Gauss-Seidel, to be identified as Method of Reflections in (11)

```

Initialize  $\mathbf{q}_i^0$  for  $i = 1, \dots, I$ ;
for  $n = 1, 2, \dots$  do
  for  $i = 1, \dots, I$  do
    Solve  $\mathbb{S}_{ii}\mathbf{q}_i^n = \mathbf{g}_i - \sum_{j=1}^{i-1} \mathbb{S}_{ij}\mathbf{q}_j^n - \sum_{j=i+1}^I \mathbb{S}_{ij}\mathbf{q}_j^{n-1}$ ;
  end for
end for
```

The vector solution $\mathbf{q}_i^n$ corresponds to the boundary element discretization of a density q_i^n on $\Gamma_{i,h}$ (approximation of q_i at iteration n). This approximation provides a way to compute, for each scatterer O_i , a numerical approximation $\mathbf{u}_i^n$ of a scattered wave u_i^n (see (11)). Given the discrete densities $\mathbf{q}_i^n$, the corresponding approximate exterior fields at points in $\Omega_i = \mathbb{R}^d \setminus \overline{O_i}$ (with $\Omega \subset \Omega_i$) are obtained by discretizing the single-layer representation

$$u_i^n(\mathbf{x}) = \int_{\Gamma_i} G(\mathbf{x}, \mathbf{y}) q_i^n(\mathbf{y}) d\Gamma_i(\mathbf{y}), \quad \mathbf{x} \in \Omega_i. \quad (9)$$

The full approximate solution at iteration n is then formed by summation over all indices $i = 1, \dots, I$, based on the representation (4)

$$u^{(n)} := \sum_{i=1}^I u_i^n|_{\Omega}. \quad (10)$$

The iterates $u^{(n)}$ satisfy for all $\mathbf{x}$ in Ω that $\lim_{n \rightarrow +\infty} u^{(n)}(\mathbf{x}) = u(\mathbf{x})$ where u is the solution of (3).

Algorithm 1 is illustrated in Figure 2. We present the steps for a model problem with three well-separated obstacles. The problem involves an incoming plane wave from the top, with wavenumber $k = 8\pi$, and a zero initial guess. The reference solution is at the top left, and the results are read from left to right and top to bottom. The first block solve, corresponding to the lower-left obstacle O_1 , yields a solution that involves only that obstacle; the other two obstacles have not yet influenced the solution. The second block solve, which incorporates the right obstacle O_2 , adds a correction, while the top obstacle O_3 still has no effect at this stage. The first plot in the second row shows the result after completing the first Gauss-Seidel iteration over all three

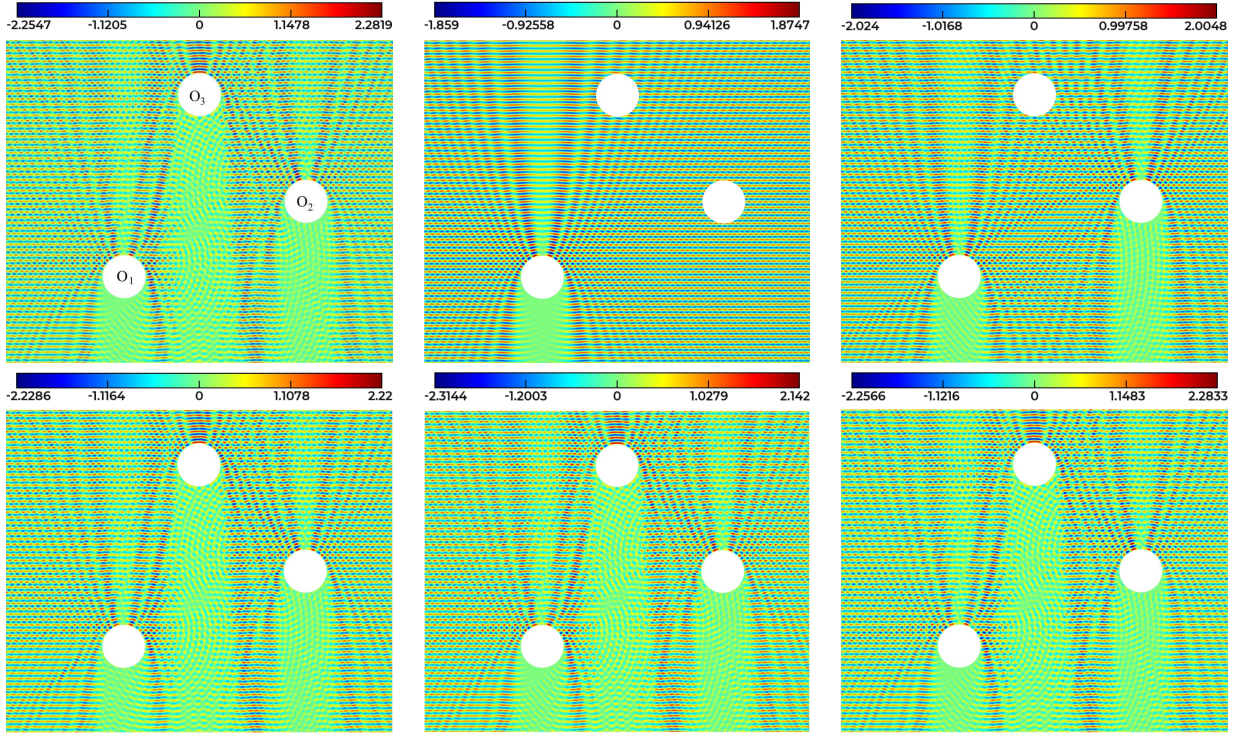


Figure 2: Top left: solution of a model problem with three well-separated obstacles for an incoming plane wave from the top and $k = 8\pi$, and after the three steps of the first block Gauss-Seidel iteration, followed by the two steps of the second block Gauss-Seidel iteration, using a zero initial guess.

obstacles, which already provides an accurate approximation of the reference solution. The first and second block solves at the next iteration of Algorithm 1, shown to the right, further improve the solution.

Algorithm 1 could also be written at the continuous level using the boundary integral operators S_{ij} from (6) instead of their discretizations $\mathbb{S}_{ij}$. This demonstrates that it provides an iterative BEM solver for the multiple scattering problem, based on the successive solution of I exterior problems defined as follows: for given initial data $u^0 = \{u_i^0\}_{i=1,\dots,I} \in H_{\text{loc}}^1(\Omega_1) \times \dots \times H_{\text{loc}}^1(\Omega_I)$, compute at the iteration n the sequence $u^n = \{u_i^n\}_{i=1,\dots,I}$ such that u_i^n solves the problem

$$\begin{cases} \Delta u_i^n + k^2 u_i^n = 0 & \text{in } \Omega_i, \\ u_i^n = g_i - \sum_{j=1}^{i-1} u_j^n - \sum_{j=i+1}^I u_j^{n-1} & \text{on } \Gamma_i \quad + \text{Radiation Condition.} \end{cases} \quad (11)$$

This is known as the alternating Method of Reflections, as each step involves solving the problem (11) for a single obstacle while incorporating the influence of the others through updated boundary data, effectively modeling the reflections between obstacles, see e.g. [8, 9] and references therein. These iterative reflections progressively capture the complex multiple scattering behavior.

3.2 Block Jacobi and the Parallel Method of Reflections

Using the same block decomposition $\mathbb{S} = \mathbb{L} + \mathbb{D} + \mathbb{U}$ from Section 3.1, we can also apply the block Jacobi iteration

$$\mathbb{D}\mathbf{q}^n = (-\mathbb{L} - \mathbb{U})\mathbf{q}^{n-1} + \mathbf{g}, \quad (12)$$

which leads to Algorithm 2.

The approximate diffracted field for Problem (3) is recovered as in the block Gauss-Seidel method from vectors $\mathbf{q}_i^n$, using a BEM discretization of the single-layer integral representations (see (9)). We could also have expressed Algorithm 2 at the continuous level in terms of the integral operators in (6). It thus corresponds to starting with an initial guess $u^0 = \{u_i^0\}_{i=1,\dots,I} \in H_{\text{loc}}^1(\Omega_1) \times \dots \times H_{\text{loc}}^1(\Omega_I)$, and then, at iteration n , solving the exterior problems

$$\begin{cases} \Delta u_i^n + k^2 u_i^n = 0 & \text{in } \Omega_i, \\ u_i^n = g_i - \sum_{j=1, j \neq i}^I u_j^{n-1} & \text{on } \Gamma_i \quad + \text{Radiation Condition,} \end{cases} \quad (13)$$

Algorithm 2 Block Jacobi, to be identified as Parallel Method of Reflections (13)

```
Initialize  $\mathbf{q}_i^0$  for  $i = 1, \dots, I$ ;  
for  $n = 1, 2, \dots$  do  
  for  $i = 1, \dots, I$  do  
    Solve  $\mathbb{S}_{ii}\mathbf{q}_i^n = \mathbf{g}_i - \sum_{\substack{j=1 \\ j \neq i}}^I \mathbb{S}_{ij}\mathbf{q}_j^{n-1}$ ;  
  end for  
end for
```

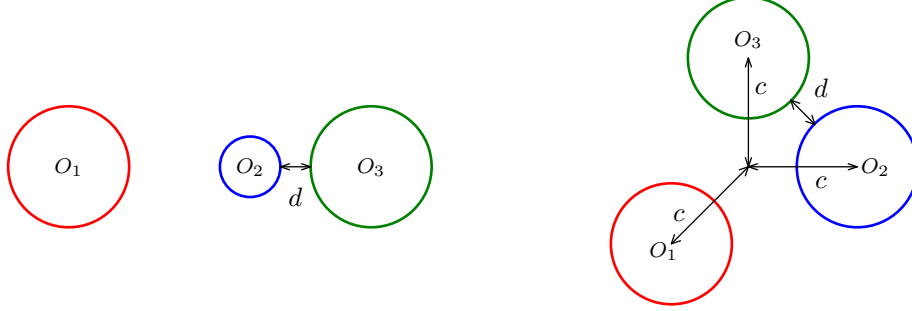


Figure 3: Obstacle configurations for the numerical study. Left: aligned obstacles O_1 centered at $(-1, 0)$ (red) with radius $r_1 = 1$, O_2 centered at $(2, 0)$ (blue) with radius $r_2 = 0.5$, and obstacle O_3 (green) at distance d of O_2 with radius $r_3 = 1$. Right: non-aligned obstacles with radius $r_i = 1$, O_1 centered at $(-c/\sqrt{2}, -c/\sqrt{2})$, O_2 at $(c, 0)$ and O_3 at $(0, c)$, $c > 0$ a parameter. The distance parameter is the distance d between O_2 and O_3 , and $c = (2 + d)/\sqrt{2}$.

by applying boundary integral equations. This is known as the parallel Method of Reflections, since all obstacles can be treated now in parallel, see e.g. [8, 9].

3.3 Numerical Study

The following numerical experiments investigate the impact of the inter-obstacle distance and the wavenumber on the convergence behavior of the alternating and parallel versions of the Method of Reflections, considered both as standalone solvers and as preconditioners for GMRES.

3.3.1 Method of Reflections as solvers

We study the performance of the alternating and parallel Method of Reflections represented by the block Gauss-Seidel and block Jacobi Algorithms 1 and 2 for solving Helmholtz scattering problems involving three obstacles in two configurations (see Figure 3).

We use $\mathbb{P}_1$ boundary finite elements for the discretization and 20 points per wavelength $\lambda = 2\pi/k$. In Figure 4, we show the spectral radius of the block iteration matrices $-\mathbb{D}^{-1}(\mathbb{L} + \mathbb{U})$ and $-(\mathbb{L} + \mathbb{D})^{-1}\mathbb{U}$ as a function of the distance d between the obstacles and for various values of the wavenumber.

From these results, we observe that the spectral radius decreases as the distance d between the obstacles increases, and therefore both algorithms are expected to converge faster. A linear regression indicates that, in these experiments, the spectral radius scales like $1/\sqrt{d}$ with respect to the distance for both the alternating and parallel Methods of Reflections. In absolute terms, however, the parallel Method of Reflections converges significantly more slowly than the alternating method. Furthermore, as the distance parameter d becomes small, both methods deteriorate, and the parallel version may even diverge when the spectral radius exceeds 1.

Concerning the dependence on the wavenumber k , a closer inspection of the spectral radii (for example, for the parallel variant at $d = 5$) reveals only a mild deterioration. A linear regression suggests a dependence of order $\log(k)$. This weak deterioration is due to the fact that we are dealing with exterior problems, in contrast to problems posed in bounded domains such as cavities or waveguides. Free-space Helmholtz problems can, in some cases, even be solved in a manner fully robust with respect to k using domain decomposition methods [13].

A final important remark concerns the peaks (local maxima and minima) observed in the spectral radii, which we believe are related to a local resonance phenomenon between the obstacles. In the aligned configuration for the parallel variant, these peaks appear to occur at distances $d = \ell\lambda/2 = \ell\pi/k$, with $\ell \in \mathbb{N}^*$. In Figure 5, we

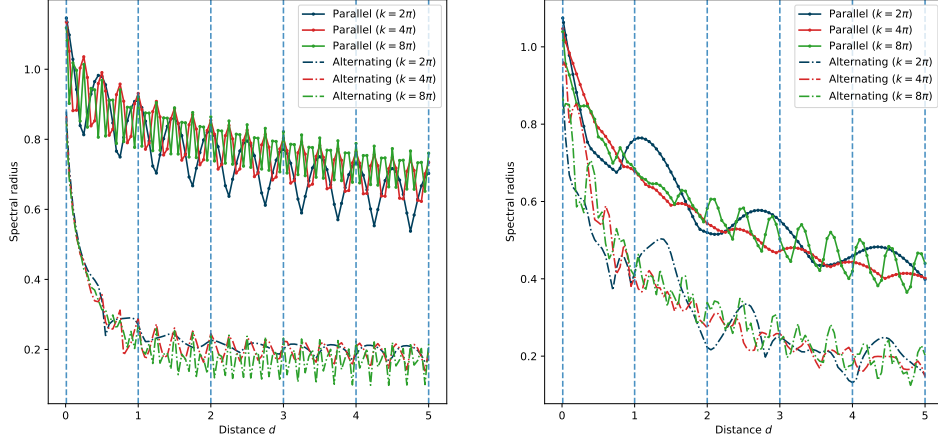


Figure 4: Spectral radius of the block iteration matrices for the alternating and parallel Method of Reflections as a function of the distance d for various wavenumbers (Left: aligned obstacles. Right: non-aligned obstacles; note that c varies with d here, since $d = \sqrt{2}c - 2$).

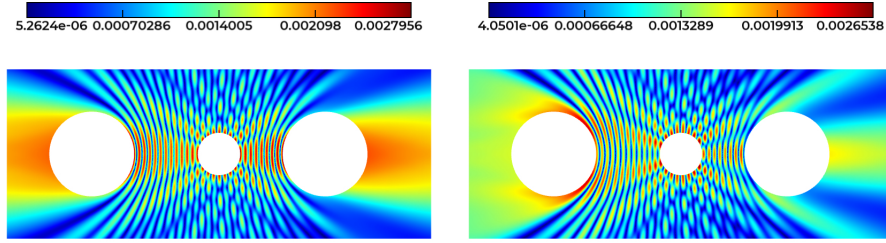


Figure 5: Parallel Method of Reflections. Modulus of the mode corresponding to the largest eigenvalue (in modulus), for the iteration block matrix. $k = 8\pi$. (Left: $d = 1$. Right: $d = 1.0625$).

show the modulus of the mode corresponding to the largest eigenvalue (in modulus) of the iteration matrix of the parallel variant for $k = 8\pi$ and for the two distances $d = 1 = 4\lambda$ (left, corresponding to a maximum) and $d = 1.0625$ (right, corresponding to a minimum). We observe that when $d = 1$ (maximum of the spectral radius), the signal between O_2 and O_3 at distance $d = 1$ is as strong as that between O_1 and O_2 , which are separated by a distance of 1.5. In contrast, for $d = 1.0625$ (minimum of the spectral radius), the signal between O_2 and O_3 remains strong, while it is significantly weaker between O_1 and O_2 and its phase is inverted. At present, we do not have a more precise understanding of these resonance-like phenomena.

To see how well the spectral radius predicts convergence, we run the algorithms for several values of the distance parameter d and wavenumber k . We directly simulate the error equations, i.e., we use zero data and a random initial guess. For the importance of this setting in thoroughly testing iterative methods, see [11, end of Section 5.1]. We use the residual $\|\mathbf{g} - \mathbb{S}\mathbf{q}\|_2$ as the stopping criterion with a tolerance of 10^{-8} . The number of iterations in Tables 1 and 2 clearly reflects the value of the corresponding spectral radius, as expected in such tests using a random initial guess.

3.3.2 Method of Reflections as preconditioners for GMRES

Having identified the two versions of the Method of Reflections with block iterations, it is natural to employ them as preconditioners³ for Krylov subspace methods [34]. Rewriting the alternating Method of Reflections or block Gauss-Seidel method (8) in residual form, we get

$$\mathbf{q}^n = (\mathbb{D} + \mathbb{L})^{-1}(\mathbb{D} + \mathbb{L} - \mathbb{S})\mathbf{q}^{n-1} + (\mathbb{D} + \mathbb{L})^{-1}\mathbf{g} = \mathbf{q}^{n-1} + (\mathbb{D} + \mathbb{L})^{-1}(\mathbf{g} - \mathbb{S}\mathbf{q}^{n-1}).$$

Similarly for the parallel Method of Reflections (12), we obtain

$$\mathbf{q}^n = \mathbf{q}^{n-1} + \mathbb{D}^{-1}(\mathbf{g} - \mathbb{S}\mathbf{q}^{n-1}).$$

³For more details, see [7, Chapter 4.1], where it is shown that in practice, one should always use Krylov acceleration.

Method	d	Wavenumber k					
		2π		4π		8π	
		ρ	#iter	ρ	#iter	ρ	#iter
Alternating	0.02	0.82	82	0.81	74	0.79	72
	1	0.27	15	0.29	14	0.27	14
	3	0.20	13	0.23	13	0.23	12
	5	0.17	12	0.20	12	0.22	12
Parallel	0.02	1.13	—	1.12	—	1.08	—
	1	0.91	174	0.93	244	0.92	213
	3	0.77	63	0.80	80	0.82	87
	5	0.70	46	0.73	57	0.76	62

Table 1: Aligned obstacles (Figure 3 left): number of iterations (#iter) for the alternating and parallel Method of Reflections and corresponding spectral radius (ρ) for various distance parameters d and wavenumbers k . Zero right hand side and random initial guess. Tolerance 10^{-8} .

Method	d	Wavenumber k					
		2π		4π		8π	
		ρ	#iter	ρ	#iter	ρ	#iter
Alternating	0.02	0.85	96	0.96	423	0.84	85
	1	0.42	21	0.40	21	0.41	20
	3	0.25	13	0.26	14	0.24	12
	5	0.15	10	0.15	10	0.20	11
Parallel	0.02	1.06	—	1.04	—	1.02	—
	1	0.76	64	0.68	46	0.68	44
	3	0.55	30	0.47	24	0.50	24
	5	0.40	19	0.40	20	0.44	20

Table 2: Corresponding results as in Table 1 but for non-aligned obstacles (see Figure 3 right). Zero right hand side and random initial guess. Tolerance 10^{-8} .

So these methods are fixed point algorithms, or simple stationary Richardson iterations⁴ for the preconditioned systems

$$(\mathbb{D} + \mathbb{L})^{-1}\mathbb{S}\mathbf{q} = (\mathbb{D} + \mathbb{L})^{-1}\mathbf{g} \quad \text{and} \quad \mathbb{D}^{-1}\mathbb{S}\mathbf{q} = \mathbb{D}^{-1}\mathbf{g}. \quad (14)$$

It is much more effective to use a Krylov method to solve the systems in (14). Furthermore, in practice, neither the preconditioners $(\mathbb{D} + \mathbb{L})^{-1}$ and $\mathbb{D}^{-1}$ nor the system matrix $\mathbb{S}$ are ever assembled. Instead, the Method of Reflections is applied to evaluate in a fast way (only by solving local systems) the preconditioned right hand side and the preconditioned iteration. Note that preconditioning GMRES with $\mathbb{D}^{-1}$ corresponds to the so-called “single scattering preconditioner” commonly used in the BEM community (e.g., [30]).

We show in Table 3 the number of GMRES iterations required to solve the linear systems associated with (7), both without preconditioning, and with preconditioning using either the alternating (Alt) or the parallel (Par) Method of Reflections. We fix the tolerance to 10^{-8} . Without preconditioning, GMRES converges slowly in all cases, while preconditioning significantly accelerates convergence. Comparing with Tables 1 and 2, both versions of the Method of Reflections are significantly more efficient as preconditioners than as stationary iterative solvers. Although the parallel Method of Reflections is less effective as a preconditioner and can sometimes even diverge, Krylov acceleration ensures that convergence is always achieved.

⁴Richardson’s original iteration was much more sophisticated, it contained a variable parameter at each iteration step to be optimized, which today Krylov methods do automatically, see [7, Chapter 4.2].

Obstacles	d	Wavenumber k								
		2π			4π			8π		
		No Prec	Alt	Par	No Prec	Alt	Par	No Prec	Alt	Par
Aligned	0.02	488	11	22	888	11	26	697	11	24
	1	405	7	12	898	7	14	657	9	14
	3	397	6	12	819	7	11	643	7	14
	5	395	6	10	848	6	11	645	8	13
Non-aligned	0.02	812	15	35	667	15	40	1030	18	53
	1	551	8	14	405	9	17	742	11	19
	3	515	7	13	340	8	13	634	8	15
	5	446	6	11	322	7	11	642	7	13

Table 3: Number of GMRES iterations (without restart, tolerance 10^{-8}) for the two configurations in Figure 3 and various distance parameters d and wavenumbers k : without preconditioning (No Prec), preconditioned with the alternating (Alt) or parallel (Par) Method of Reflections. Zero right-hand side and random initial guess.

4 Domain Decomposition and the Method of Reflections

We have seen that the Method of Reflections provides powerful preconditioners to solve multiple scattering problems with boundary integral equations. They converge faster when obstacles are far away, and show only a mild deterioration as the wavenumber increases. However, when obstacles are close, convergence becomes slower. In the following, we relate the Method of Reflections to classical algebraic domain decomposition methods, and we introduce techniques from domain decomposition to further enhance performance, namely overlap and coarse spaces, see [33].

4.1 Multiplicative Schwarz and the Method of Reflections

In domain decomposition, Schwarz methods use overlap to improve convergence, and in many cases, increasing the overlap leads to faster convergence. To address the slow convergence caused by closely spaced obstacles, nearby obstacles can be grouped into larger overlapping subdomains and solved together within the Method of Reflections. For clarity, we explain the approach for three obstacles as in Figure 3. The generalization to more obstacles does then not present any difficulties. The linear system (5) for three obstacles becomes

$$\begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} & \mathbb{S}_{13} \\ \mathbb{S}_{21} & \mathbb{S}_{22} & \mathbb{S}_{23} \\ \mathbb{S}_{31} & \mathbb{S}_{32} & \mathbb{S}_{33} \end{bmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix}. \quad (15)$$

For the aligned example from Figure 3 (left), the middle obstacle O_2 is close to one of its neighbors, depending on the value of d . It is therefore natural to alternate between grouping it with O_1 to form the cluster $O_1 \cup O_2$, and with O_3 to form the cluster $O_2 \cup O_3$, solving each cluster in turn within the Method of Reflections. This leads to the overlapping Method of Reflections

$$\begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} \\ \mathbb{S}_{21} & \mathbb{S}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{q}_1^{n-\frac{1}{2}} \\ \mathbf{q}_2^{n-\frac{1}{2}} \end{pmatrix} = - \begin{bmatrix} \mathbb{S}_{13} \\ \mathbb{S}_{23} \end{bmatrix} \mathbf{q}_3^{n-1} + \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix}, \quad (16)$$

$$\begin{bmatrix} \mathbb{S}_{22} & \mathbb{S}_{23} \\ \mathbb{S}_{32} & \mathbb{S}_{33} \end{bmatrix} \begin{pmatrix} \mathbf{q}_2^n \\ \mathbf{q}_3^n \end{pmatrix} = - \begin{bmatrix} \mathbb{S}_{21} \\ \mathbb{S}_{31} \end{bmatrix} \mathbf{q}_1^{n-\frac{1}{2}} + \begin{pmatrix} \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix}. \quad (17)$$

A disadvantage of this approach is that larger subproblems must be solved, and the unknowns on the overlapping obstacle must now be stored twice, once as $\mathbf{q}_2^{n-\frac{1}{2}}$ and then as $\mathbf{q}_2^n$.

In domain decomposition, there exist algebraic methods which avoid operating directly on the global system and still make overlaps possible. This is achieved by using restriction matrices, which are simple to handle. In particular, the Multiplicative Schwarz (MS) method employs this approach: it can be applied without detailed knowledge of the obstacle configuration and even allows overlapping partial obstacles, as we will see. The restriction matrices are constructed using the identity matrix⁵ I . For the example with the three obstacle dimensions $N_{i,h}$, $i = 1, 2, 3$ with $N_h = N_{1,h} + N_{2,h} + N_{3,h}$, the rectangular restriction matrices for overlap with

⁵Not to be confused with the number of obstacles.

the middle obstacle would be

$$\begin{aligned} R_1 &:= I(1 : (N_{1,h} + N_{2,h}), :) = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \\ R_2 &:= I((N_{1,h} + 1) : N_h, :) = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \end{aligned} \quad (18)$$

MS is then defined for the three obstacle problem (15) and the two cluster example in a purely algebraic fashion as the two step residual correction iteration

$$\tilde{\mathbf{q}}^{n-\frac{1}{2}} = \tilde{\mathbf{q}}^{n-1} + R_1^T (R_1 \mathbb{S} R_1^T)^{-1} R_1 (\mathbf{g} - \mathbb{S} \tilde{\mathbf{q}}^{n-1}), \quad (19)$$

$$\tilde{\mathbf{q}}^n = \tilde{\mathbf{q}}^{n-\frac{1}{2}} + R_2^T (R_2 \mathbb{S} R_2^T)^{-1} R_2 (\mathbf{g} - \mathbb{S} \tilde{\mathbf{q}}^{n-\frac{1}{2}}). \quad (20)$$

This is very easy to implement and generalizes immediately to many restriction operators R_i . We prove in Theorem 1 that MS in (19)-(20) and the overlapping Method of Reflections (16)-(17) produce identical iterates. Thus MS applied to the boundary integral equations (5) defines a general, algebraic, overlapping Method of Reflections, which also allows for partially overlapping obstacles.

Theorem 1. *Starting the MS method in (19)-(20) with an initial guess $\tilde{\mathbf{q}}^0$ satisfying $\tilde{\mathbf{q}}_3^0 = \mathbf{q}_3^0$, where $\mathbf{q}_3^0$ is the initial guess used in the overlapping Method of Reflections (16)-(17), the two methods yield equivalent iterates, i.e., for $n \geq 1$,*

$$\tilde{\mathbf{q}}^{n-\frac{1}{2}} = \begin{pmatrix} \mathbf{q}_1^{n-\frac{1}{2}} \\ \mathbf{q}_2^{n-\frac{1}{2}} \\ \mathbf{q}_3^{n-1} \end{pmatrix}, \quad \tilde{\mathbf{q}}^n = \begin{pmatrix} \mathbf{q}_1^{n-\frac{1}{2}} \\ \mathbf{q}_2^n \\ \mathbf{q}_3^n \end{pmatrix}. \quad (21)$$

Proof. First, using the form of the restriction matrix R_1 , we obtain that

$$R_1 (\mathbf{g} - \mathbb{S} \tilde{\mathbf{q}}^{n-1}) = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} \\ \mathbb{S}_{21} & \mathbb{S}_{22} \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{q}}_1^{n-1} \\ \tilde{\mathbf{q}}_2^{n-1} \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{13} \\ \mathbb{S}_{23} \end{bmatrix} \tilde{\mathbf{q}}_3^{n-1}. \quad (22)$$

Applying next the local solve from (19), namely

$$(R_1 \mathbb{S} R_1^T)^{-1} = \begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} \\ \mathbb{S}_{21} & \mathbb{S}_{22} \end{bmatrix}^{-1}, \quad (23)$$

on both sides of (22) and reordering terms, we get

$$(R_1 \mathbb{S} R_1^T)^{-1} R_1 (\mathbf{g} - \mathbb{S} \tilde{\mathbf{q}}^{n-1}) = - \begin{pmatrix} \tilde{\mathbf{q}}_1^{n-1} \\ \tilde{\mathbf{q}}_2^{n-1} \end{pmatrix} + \begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} \\ \mathbb{S}_{21} & \mathbb{S}_{22} \end{bmatrix}^{-1} \left(\begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{13} \\ \mathbb{S}_{23} \end{bmatrix} \tilde{\mathbf{q}}_3^{n-1} \right). \quad (24)$$

Extending this result according to the first half step in MS in (19) by multiplying it with R_1^T and adding it to the previous iterate $\tilde{\mathbf{q}}^{n-1}$ yields

$$\tilde{\mathbf{q}}^{n-\frac{1}{2}} = \tilde{\mathbf{q}}^{n-1} - R_1^T \begin{pmatrix} \tilde{\mathbf{q}}_1^{n-1} \\ \tilde{\mathbf{q}}_2^{n-1} \end{pmatrix} + R_1^T \begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} \\ \mathbb{S}_{21} & \mathbb{S}_{22} \end{bmatrix}^{-1} \left(\begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{13} \\ \mathbb{S}_{23} \end{bmatrix} \tilde{\mathbf{q}}_3^{n-1} \right). \quad (25)$$

Now two of the old subdomain approximation values in $\tilde{\mathbf{q}}^{n-1}$ cancel, because

$$\tilde{\mathbf{q}}^{n-1} - R_1^T \begin{pmatrix} \tilde{\mathbf{q}}_1^{n-1} \\ \tilde{\mathbf{q}}_2^{n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \tilde{\mathbf{q}}_3^{n-1} \end{pmatrix}, \quad (26)$$

and we therefore obtain

$$\tilde{\mathbf{q}}^{n-\frac{1}{2}} = \begin{pmatrix} \begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} \\ \mathbb{S}_{21} & \mathbb{S}_{22} \end{bmatrix}^{-1} \left(\begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{13} \\ \mathbb{S}_{23} \end{bmatrix} \tilde{\mathbf{q}}_3^{n-1} \right) \\ \tilde{\mathbf{q}}_3^{n-1} \end{pmatrix}. \quad (27)$$

Similarly for the second half step, we obtain going through each step in (20) that

$$\tilde{\mathbf{q}}^n = \begin{pmatrix} \tilde{\mathbf{q}}_1^{n-\frac{1}{2}} \\ \begin{bmatrix} \mathbb{S}_{22} & \mathbb{S}_{23} \\ \mathbb{S}_{32} & \mathbb{S}_{33} \end{bmatrix}^{-1} \left(\begin{pmatrix} \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{21} \\ \mathbb{S}_{31} \end{bmatrix} \tilde{\mathbf{q}}_1^{n-\frac{1}{2}} \right) \end{pmatrix}. \quad (28)$$

The proof is now by induction. Comparing (27,28) with (16,17), we see that starting with $\mathbf{q}_3^0 = \tilde{\mathbf{q}}_3^0$, we compute $\mathbf{q}_j^{\frac{1}{2}} = \tilde{\mathbf{q}}_j^{\frac{1}{2}}$ for $j = 1, 2$, and then from $\mathbf{q}_1^{\frac{1}{2}} = \tilde{\mathbf{q}}_1^{\frac{1}{2}}$, we compute $\tilde{\mathbf{q}}_j^1 = \mathbf{q}_j^1$ for $j = 2, 3$, i.e.

$$\tilde{\mathbf{q}}^{\frac{1}{2}} = \begin{pmatrix} \mathbf{q}_1^{\frac{1}{2}} \\ \mathbf{q}_2^{\frac{1}{2}} \\ \mathbf{q}_3^0 \end{pmatrix}, \quad \tilde{\mathbf{q}}^1 = \begin{pmatrix} \mathbf{q}_1^{\frac{1}{2}} \\ \mathbf{q}_2^1 \\ \mathbf{q}_3^1 \end{pmatrix}.$$

So the property holds at the first step. We now suppose that $\tilde{\mathbf{q}}^n = \begin{pmatrix} \mathbf{q}_1^{n-\frac{1}{2}} \\ \mathbf{q}_2^n \\ \mathbf{q}_3^n \end{pmatrix}$. Then by (27), we compute

$$\tilde{\mathbf{q}}^{n-\frac{1}{2}} = \begin{pmatrix} \begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} \\ \mathbb{S}_{21} & \mathbb{S}_{22} \end{bmatrix}^{-1} \left(\begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{13} \\ \mathbb{S}_{23} \end{bmatrix} \mathbf{q}_3^{n-1} \right) \\ \mathbf{q}_3^{n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{q}_1^{n-\frac{1}{2}} \\ \mathbf{q}_2^{n-\frac{1}{2}} \\ \mathbf{q}_3^{n-1} \end{pmatrix},$$

and

$$\tilde{\mathbf{q}}^n = \begin{pmatrix} \mathbf{q}_1^{n-\frac{1}{2}} \\ \begin{bmatrix} \mathbb{S}_{22} & \mathbb{S}_{23} \\ \mathbb{S}_{32} & \mathbb{S}_{33} \end{bmatrix}^{-1} \left(\begin{pmatrix} \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{21} \\ \mathbb{S}_{31} \end{bmatrix} \mathbf{q}_1^{n-\frac{1}{2}} \right) \\ \mathbf{q}_3^n \end{pmatrix} = \begin{pmatrix} \mathbf{q}_1^{n-\frac{1}{2}} \\ \mathbf{q}_2^n \\ \mathbf{q}_3^n \end{pmatrix},$$

which concludes the proof. $\square$

The proof of this result can be easily generalized to the case of multiple obstacles, as is commonly done when MS is applied directly to the discretization of partial differential equations, see e.g. [11]. We have thus shown that the Method of Reflections formulated with boundary integral equations can be identified with MS to obtain equivalent iterates, both with and without overlapping obstacle boundaries, for an arbitrary large number of obstacles.

The key insight of the present work is that the Method of Reflections becomes a boundary domain decomposition method regardless of the number of obstacles [1]. This is not the case for the Method of Reflections seen as a volume domain decomposition method, which differs from MS as soon as there are more than two obstacles [9, Section 3.4].

4.2 Restricted Additive Schwarz and the parallel Method of Reflections

Corrections in the overlapping Method of Reflections (16)-(17) can also be performed in parallel yielding a parallel overlapping variant, namely

$$\begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} \\ \mathbb{S}_{21} & \mathbb{S}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{q}_1^n \\ \mathbf{q}_{2,1}^n \end{pmatrix} = - \begin{bmatrix} \mathbb{S}_{13} \\ \mathbb{S}_{23} \end{bmatrix} \mathbf{q}_3^{n-1} + \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix}, \quad (29)$$

$$\begin{bmatrix} \mathbb{S}_{22} & \mathbb{S}_{23} \\ \mathbb{S}_{32} & \mathbb{S}_{33} \end{bmatrix} \begin{pmatrix} \mathbf{q}_{2,2}^n \\ \mathbf{q}_3^n \end{pmatrix} = - \begin{bmatrix} \mathbb{S}_{21} \\ \mathbb{S}_{31} \end{bmatrix} \mathbf{q}_1^{n-1} + \begin{pmatrix} \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix}. \quad (30)$$

This corresponds to a block Jacobi method applied to the augmented system (see [28, Section 3.4])

$$\left[\begin{array}{cc|cc} \mathbb{S}_{11} & \mathbb{S}_{12} & \mathbb{O} & \mathbb{S}_{13} \\ \mathbb{S}_{21} & \mathbb{S}_{22} & \mathbb{O} & \mathbb{S}_{23} \\ \hline \mathbb{S}_{21} & \mathbb{O} & \mathbb{S}_{22} & \mathbb{S}_{23} \\ \mathbb{S}_{31} & \mathbb{O} & \mathbb{S}_{32} & \mathbb{S}_{33} \end{array} \right] \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_{2,1} \\ \mathbf{q}_{2,2} \\ \mathbf{q}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix}, \quad (31)$$

where the middle corresponding to the middle obstacle O_2 and its unknowns are written twice.

The parallel Method of Reflections (29)-(30) shares the same drawbacks as the original alternating methods (16)-(17), namely multiple storage of overlapping domains (there could be many and multiple overlaps) and the need to solve larger subproblems. There is again a clever alternative in the Schwarz domain decomposition community, called the Restricted Additive Schwarz (RAS) method [4]⁶. In addition to the restriction matrices (18) of MS, RAS needs modified restriction matrices $\tilde{R}_i$, $i = 1, 2$ of the same size as the R_i that contain a partition of unity,

$$\tilde{R}_1 := \begin{bmatrix} I & 0 & 0 \\ 0 & \tilde{I}_1 & 0 \end{bmatrix}, \quad \tilde{R}_2 := \begin{bmatrix} 0 & \tilde{I}_2 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \tilde{I}_1 + \tilde{I}_2 = I.$$

⁶RAS reduces to the Additive Schwarz method in the absence of algebraic overlap but differs when overlap is present [11], and we consider both overlapping and non-overlapping cases here.

From a partition of unity expressed as diagonal matrices χ_i with $\sum_i \chi_i = I$, we define $\tilde{R}_i := R_i \chi_i$, such that $\sum_i \tilde{R}_i^T R_i = I$. An iteration of RAS is then defined purely algebraically as the residual correction

$$\tilde{\mathbf{q}}^n = \tilde{\mathbf{q}}^{n-1} + \sum_{i=1}^2 \tilde{R}_i^T (R_i \mathbb{S} R_i^T)^{-1} R_i (\mathbf{g} - \mathbb{S} \tilde{\mathbf{q}}^{n-1}), \quad (32)$$

and the extension to multiple obstacle clusters is straightforward. Theorem 2 establishes that the overlapping parallel Method of Reflections (29)-(30) is equivalent to RAS applied to the original dense system (15). It provides a direct relation between the parallel Method of Reflections and RAS for an overlapping boundary domain decomposition.

Theorem 2. *Starting the first iteration of RAS defined in (32) with an initial guess $\tilde{\mathbf{q}}^0$ satisfying $\tilde{\mathbf{q}}_1^0 = \mathbf{q}_1^0$ and $\tilde{\mathbf{q}}_3^0 = \mathbf{q}_3^0$, where $\mathbf{q}^0$ is the initial guess used in the overlapping parallel Method of Reflections (29)-(30) for the augmented system (31), the two methods yield equivalent iterates, i.e., for $n \geq 1$,*

$$\tilde{\mathbf{q}}^n = \tilde{R}_1^T \begin{pmatrix} \mathbf{q}_1^n \\ \mathbf{q}_{2,1}^n \end{pmatrix} + \tilde{R}_2^T \begin{pmatrix} \mathbf{q}_{2,2}^n \\ \mathbf{q}_3^n \end{pmatrix}. \quad (33)$$

Proof. Similarly to the proof of Theorem 1, we obtain from RAS, for the term corresponding to $i = 1$ in (32), exactly the same sequence of results as in (22)-(24); only the application of $\tilde{R}_1^T$ differs. A similar modification is used for the term corresponding to $i = 2$ in (32). Combining these two expressions, we get

$$\begin{aligned} \tilde{\mathbf{q}}^n = \tilde{\mathbf{q}}^{n-1} + \tilde{R}_1^T & \left(\begin{bmatrix} \mathbb{S}_{11} & \mathbb{S}_{12} \\ \mathbb{S}_{21} & \mathbb{S}_{22} \end{bmatrix}^{-1} \left(\begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{13} \\ \mathbb{S}_{23} \end{bmatrix} \tilde{\mathbf{q}}_3^{n-1} \right) - \begin{pmatrix} \tilde{\mathbf{q}}_1^{n-1} \\ \tilde{\mathbf{q}}_2^{n-1} \end{pmatrix} \right) \\ + \tilde{R}_2^T & \left(\begin{bmatrix} \mathbb{S}_{22} & \mathbb{S}_{23} \\ \mathbb{S}_{32} & \mathbb{S}_{33} \end{bmatrix}^{-1} \left(\begin{pmatrix} \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix} - \begin{bmatrix} \mathbb{S}_{21} \\ \mathbb{S}_{31} \end{bmatrix} \tilde{\mathbf{q}}_1^{n-1} \right) - \begin{pmatrix} \tilde{\mathbf{q}}_2^{n-1} \\ \tilde{\mathbf{q}}_3^{n-1} \end{pmatrix} \right). \end{aligned} \quad (34)$$

The proof is again by induction. For $n = 1$, using the initial assumption $\tilde{\mathbf{q}}_1^0 = \mathbf{q}_1^0$ and $\tilde{\mathbf{q}}_3^0 = \mathbf{q}_3^0$ in (34) and (29)-(30), we obtain

$$\tilde{\mathbf{q}}^1 = \tilde{\mathbf{q}}^0 + \tilde{R}_1^T \left(\begin{pmatrix} \mathbf{q}_1^1 \\ \mathbf{q}_{2,1}^1 \end{pmatrix} - \begin{pmatrix} \tilde{\mathbf{q}}_1^0 \\ \tilde{\mathbf{q}}_2^0 \end{pmatrix} \right) + \tilde{R}_2^T \left(\begin{pmatrix} \mathbf{q}_{2,2}^1 \\ \mathbf{q}_3^1 \end{pmatrix} - \begin{pmatrix} \tilde{\mathbf{q}}_2^0 \\ \tilde{\mathbf{q}}_3^0 \end{pmatrix} \right). \quad (35)$$

Applying the partition of unity, the three old terms cancel,

$$\tilde{\mathbf{q}}^0 - \tilde{R}_1^T \begin{pmatrix} \tilde{\mathbf{q}}_1^0 \\ \tilde{\mathbf{q}}_2^0 \end{pmatrix} - \tilde{R}_2^T \begin{pmatrix} \tilde{\mathbf{q}}_2^0 \\ \tilde{\mathbf{q}}_3^0 \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{q}}_1^0 - \tilde{\mathbf{q}}_1^0 \\ \tilde{\mathbf{q}}_2^0 - \tilde{I}_1 \tilde{\mathbf{q}}_2^0 - \tilde{I}_2 \tilde{\mathbf{q}}_2^0 \\ \tilde{\mathbf{q}}_3^0 - \tilde{\mathbf{q}}_3^0 \end{pmatrix} = \mathbf{0},$$

which shows (33) for $n = 1$. Now suppose that (33) holds for $n - 1$. Then, using (34) and again (29)-(30) gives

$$\tilde{\mathbf{q}}^n = \tilde{\mathbf{q}}^{n-1} + \tilde{R}_1^T \left(\begin{pmatrix} \mathbf{q}_1^n \\ \mathbf{q}_{2,1}^n \end{pmatrix} - \begin{pmatrix} \tilde{\mathbf{q}}_1^{n-1} \\ \tilde{\mathbf{q}}_2^{n-1} \end{pmatrix} \right) + \tilde{R}_2^T \left(\begin{pmatrix} \mathbf{q}_{2,2}^n \\ \mathbf{q}_3^n \end{pmatrix} - \begin{pmatrix} \tilde{\mathbf{q}}_2^{n-1} \\ \tilde{\mathbf{q}}_3^{n-1} \end{pmatrix} \right),$$

and as for $n = 1$, the three terms from step $n - 1$ cancel, which concludes the proof of (33) for $n \geq 1$. $\square$

4.3 Effect of Overlap on the Method of Reflections

The effect of overlap can be studied when the methods are used as standalone solvers or as preconditioners for GMRES, and both are considered in this section.

4.3.1 As solvers

We now compare numerically the performance of the Method of Reflections for various overlapping configurations. We show in Figure 6 the real part of the error at iteration $n = 1$ of the alternating Method of Reflections, both without overlap and with overlap involving obstacle O_2 , for a fixed distance parameter $d = 0.2$, wavenumber $k = 8\pi$, and an incoming plane wave from above; see Figure 3 (left) and Figure 7 (top left) for illustrations of the geometry and overlap. The introduction of an overlap reduces the error already at iteration $n = 1$, particularly in the zones between the obstacles, where the non-overlapping variant exhibits quasi-trapping modes. The version with overlap converges to the given tolerance 10^{-8} in 4 iterations, compared to 29 iterations required without overlap.

We next study the error behavior for partially overlapping configurations shown in Figure 7, for a fixed distance parameter $d = 0.02$. For these tests we use a random initial guess and solve the homogeneous problems, and study convergence to zero. For all cases with overlap, convergence is achieved within a few iterations.

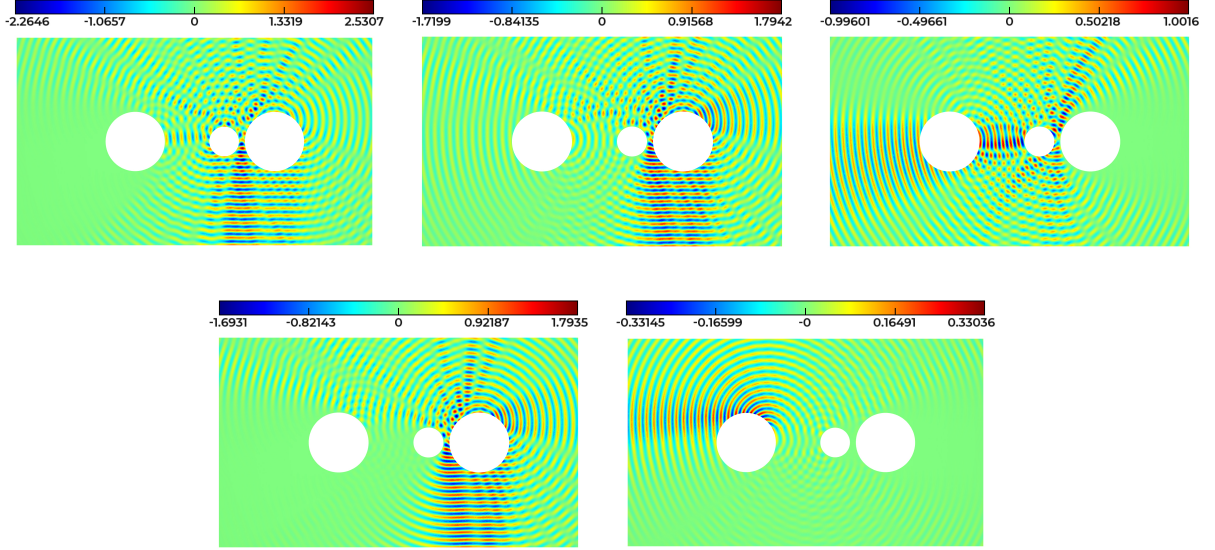


Figure 6: Difference between the real parts of the reference solution and the numerical solution with the Method of Reflections. Incoming plane wave from the top and wavenumber $k = 8\pi$. Distance $d = 0.2$. Top: iteration $n = 1$ without overlap, solve on O_1 then O_2 then O_3 . Bottom: with overlap O_2 , solve on $O_1 \cup O_2$, then solve on $O_2 \cup O_3$.

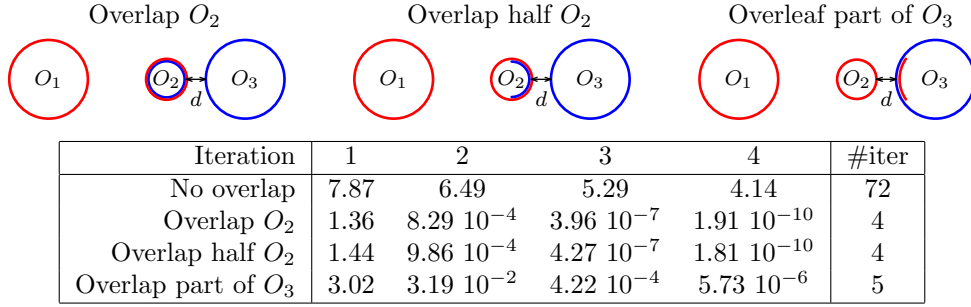


Figure 7: Euclidean norm of the the first four iterations (zero reference solution) for the three aligned obstacles without and with various overlaps using the alternating Method of Reflections. Zero right hand side and a random initial guess, $k = 8\pi$ and $d = 0.02$.

4.3.2 As preconditioners for GMRES

In Table 4, we examine the effect of overlapping obstacle O_2 on the performance of GMRES preconditioned with the alternating Method of Reflections. The test setup is the same as in Table 3, with the addition of higher wavenumbers k . Results show that overlap considerably accelerates convergence, often reducing the number of iterations by a factor of two, or more especially for small d , and significantly decreases the sensitivity to the obstacle distance, for both aligned and non-aligned configurations.

The good preconditioning capabilities of the overlapping Method of Reflections for GMRES can be seen in the improved eigenvalue distribution (see Figure 8). The eigenvalues of the original system matrix $\mathbb{S}$ are clustered around the origin, which leads to poor GMRES convergence. The Method of Reflections used as a preconditioner shifts them away from the origin, and the overlap leads to a further, much tighter clustering around the point $(1, 0)$; a desired behavior for GMRES.

4.4 Impact of coarse space correction on the Method of Reflections

The identification of the Method of Reflections (1) and (2) as algebraic Schwarz methods MS (19)-(20) and RAS (32) also allows one to easily add a coarse space correction, see e.g. [12]. Assume that an eigenvalue of the iteration matrices $G := \prod_j (I - \hat{R}_j^T (R_j \mathbb{S} R_j^T)^{-1} R_j \mathbb{S})$ for the alternating and $G := I - \sum_j \hat{R}_j^T (R_j \mathbb{S} R_j^T)^{-1} R_j \mathbb{S}$ for the parallel Method of Reflections, has been noted as a source of slow convergence, and that an associated eigenvector can be computed. This can be achieved for example by running the method like a power iteration, or for multiple vectors by using the orthogonal iteration, see e.g. [14, Section 7.4.3]. Denoting $G \mathbf{v}_j = \lambda_j \mathbf{v}_j$ for the largest eigenvalues in modulus $|\lambda_j|$, $j = 1, 2, \dots, J$, we put these vectors row wise into a coarse restriction matrix

Obstacles	d	Wavenumber k									
		2π		4π		8π		16π		24π	
		Alt	Over	Alt	Over	Alt	Over	Alt	Over	Alt	Over
Aligned	0.02	11	4	11	3	11	3	11	3	11	3
	1	7	4	7	4	9	3	8	3	8	3
	3	6	3	7	4	7	3	7	3	7	3
	5	6	3	6	3	8	3	7	3	7	3
Non-aligned	0.02	15	6	15	7	18	8	26	10	35	12
	1	8	5	9	6	11	6	12	7	13	7
	3	7	5	8	5	8	5	9	6	9	6
	5	6	4	7	4	7	5	8	5	9	5

Table 4: Number of GMRES iterations (without restart, tolerance 10^{-8}) as a function of the distance parameter d for various wavenumbers, preconditioned with the alternating Method of Reflections without overlap (Alt) and with overlap of O_2 (Over), zero right-hand side and a random initial guess.

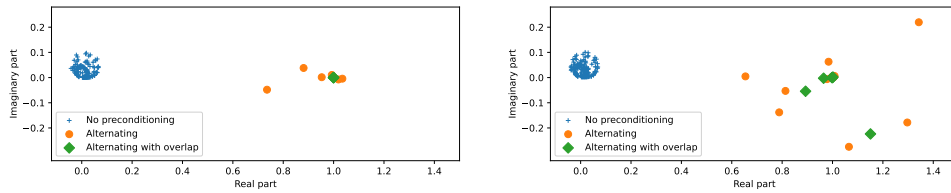


Figure 8: Eigenvalue distribution of the matrix of the global linear system (7) before and after preconditioning for $k = 8\pi$ and $d = 1$ (Left: aligned obstacles. Right: non-aligned obstacles, see Fig. 3).

$R_c^* := [\mathbf{v}_1, \dots, \mathbf{v}_J]$. Then after each iteration of the underlying Method of Reflections we perform algebraically the coarse correction

$$\tilde{\mathbf{q}}^{n+1} = \tilde{\mathbf{q}}^{n+1} + R_c^*(R_c \mathbb{S} R_c^*)^{-1} R_c(\mathbf{g} - \mathbb{S} \tilde{\mathbf{q}}^{n+1}) \quad (36)$$

in order to remove these modes. The new convergence factor is given by the modulus of the first eigenvalue $|\lambda_{J+1}|$ that has not been removed with the coarse space.

As an illustration, we consider an incident plane wave from the top and $k = 4\pi$ on the non-aligned obstacles in Figure 3 (right) with distance parameter $d = 0.2$. Using a zero initial guess and a stopping criterion of $\varepsilon = 10^{-8}$, the alternating Method of Reflections converges in 121 iterations. We show in Figure 9 (left) the modulus of the final error for the scattered wave and, on the right, the modulus of the quantity radiated by the first mode of the associated iteration matrix. We see that it is precisely this mode that hinders convergence. When the contribution of the first mode is removed, the second dominant mode becomes the main factor limiting convergence, and so on. In Figure 10, we show in log10 scale the relative residual $\|\mathbf{g} - \mathbb{S} \mathbf{q}^n\|_2 / \|\mathbf{g}\|_2$ as a function of the iteration n for the alternating Method of Reflections without and with $J = 1$ and $J = 2$ coarse functions in the coarse space correction. The expected convergence rates are observed.

Finally, in Table 5 we compare the alternating Method of Reflections without and with coarse space corrections ($J = 2$ and $J = 6$). We consider a zero right-hand side, a random initial guess, and a tolerance of 10^{-8} , while fixing $k = 4\pi$ and $d = 0.02$. In this configuration, the three non-aligned obstacles are very close to each other and the spectral radius is $\rho = 0.96$. In this challenging setting, the coarse space correction yields a marked improvement in convergence over the standard method.

5 Diffraction by a school of fish

To conclude, we consider the diffraction by multiple NACA-shaped scatterers, chosen here to simulate the diffraction by a school of fish (Fig. 11). We use the NACA 0035 profile, which is a symmetric airfoil characterized by a substantial relative thickness of 35% of the chord, and can be parameterized by its chord length, with the maximum half-thickness defining its width. We consider an incident plane wave coming from the top. Our goal is to study numerically the influence of the number and size of the fish and the frequency on the number of iterations required for convergence.

Table 6 shows the effect of the number and size of the fish on the number of iterations required by the Method of Reflections, both with and without overlap, and used as iterative solver itself or as preconditioner for GMRES. The overlap is chosen in the simplest way by taking the fish immediately to the right, as in the

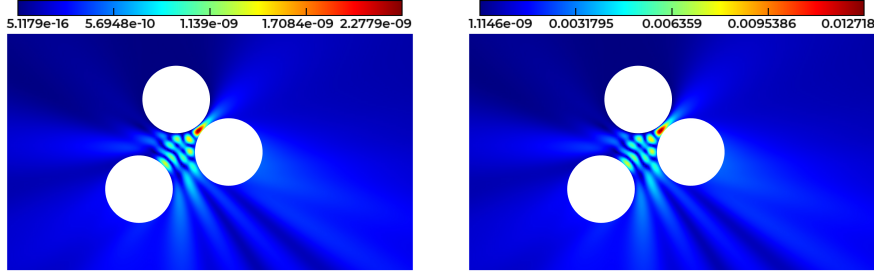


Figure 9: Left: Modulus of the error after the alternating Method of Reflections has converged to the tolerance $1e-8$ with an incident plane wave from the top, $k = 4\pi$ and $d = 0.2$. Right: Modulus of the quantity radiated by the first eigenmode of the associated iteration matrix G .

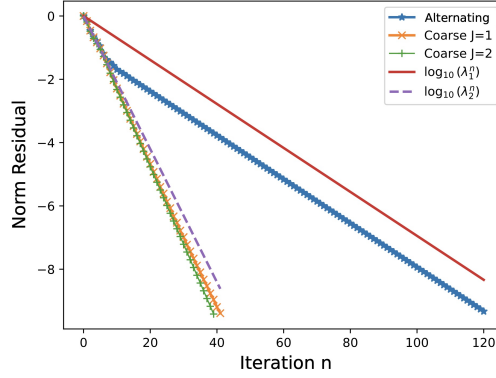


Figure 10: Alternating Method of Reflections without and with coarse space correction ($\lambda_1 \approx 0.85$, $\lambda_2 \approx 0.62$). Incident plane wave from the top. $k = 4\pi$. $d = 0.2$.

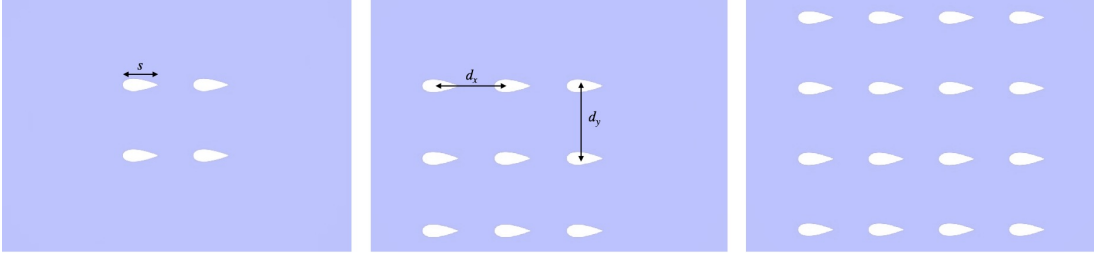


Figure 11: Definition of parameters for the school of fish.

previously considered aligned case. We emphasize that unpreconditioned GMRES for the original system always fails to converge for these examples. As expected, the convergence rate deteriorates as either the size or the number of fish increases. Indeed, a larger size results in shorter distances d_x and d_y . GMRES preconditioned with the overlapping Method of Reflections consistently performs best, with very low iteration counts, and the dependence is approximately proportional to the number of fish. In future work, we will study strategies to automatically cluster; and overlap configurations based on the distances between the obstacles.

Table 7 shows the number of preconditioned GMRES iterations for the same configurations and size $s = 1$ for three different frequencies. We observe that iteration numbers increase slowly as the frequency increases, and the small variations are due to the oscillations we also observed in Figure 4 depending on how the wavelength is aligned with the distance. Figure 12 gives the real part of the total field for arrays of 2×2 (left) and 4×7 fish (right) for the wavenumber $k = 24\pi$.

As a final illustration, we present additional simulations with randomly distributed fish positions and orientations (see Fig. 13 for the configurations). Table 8 reports the number of preconditioned GMRES iterations for configurations involving 4, 9, 16, 20, and 28 fish, for $k = 2\pi$ and $k = 24\pi$. For comparison, we note that GMRES without preconditioning requires an impractically large number of iterations. Even for the most favorable configuration considered, namely 4 fish at $k = 2\pi$, convergence is achieved only after 1390 iterations. These results further confirm the effectiveness of the overlap-based preconditioner, which consistently reduces

Iteration	1	2	3	4	10	#iter
Alternating	8.28	6.28	3.74	2.13	0.86	423
Coarse $J = 2$	8.23	6.26	3.35	1.48	0.16	46
Coarse $J = 6$	6.43	3.58	1.54	0.66	$4.88 \cdot 10^{-3}$	23

Table 5: Three non-aligned obstacles. Euclidean norm of the numerical solution (zero reference solution) for the alternating Method of Reflections without and with coarse space correction. Zero right hand side and a random initial guess. Tolerance 10^{-8} . $k = 4\pi$. $d = 0.02$.

Fish	Fish size s											
	0.5					1						
	Alt	Over	Prec	Alt	Prec	Over	Alt	Over	Prec	Alt	Prec	Over
4	17	11	7		5		47	22	8		6	
9	35	25	12		10		86	70	15		13	
16	71	36	18		15		—	—	31		23	

Table 6: Influence of the number of fish and their size on the number of iterations for the alternating Method of Reflections, both with and without overlap, used iteratively and as preconditioners for GMRES (without restart, tolerance 10^{-8} , $k = 2\pi$). Incident plane wave coming from the top.

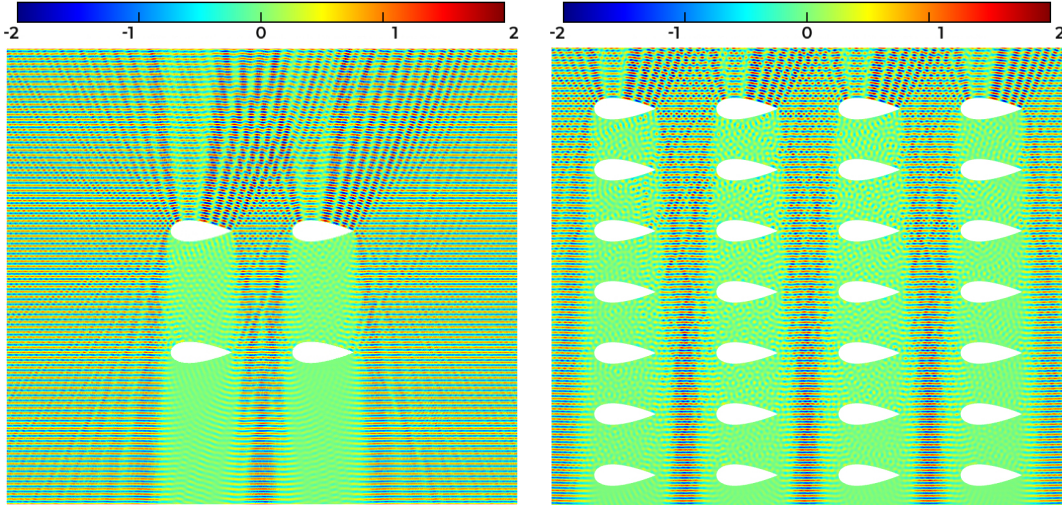


Figure 12: Diffraction by a school of fish. The obstacles are NACA 0035 profiles, with centers regularly spaced by 2 units in both horizontal and vertical directions. Two configurations are shown: 2×2 (left) and 4×7 obstacles (right). The real part of the total field for a wavenumber $k = 24\pi$ is plotted. Incident plane wave coming from the top.

the number of GMRES iterations for all configurations considered. Figure 14 provides a visualization of the influence of the distance between fish and the number of scatterers on the total field at $k = 24\pi$. Furthermore, tests with different incident wave directions showed no significant influence of the direction of incidence on the convergence of the preconditioned GMRES solver.

For the case of 28 scatterers, Table 9 reports the number of preconditioned GMRES iterations as a function of the wavenumber k . A dependence on the wavenumber is observed, as in the aligned configurations. Nevertheless, the overlap-based preconditioning strategy improves the convergence for these randomly distributed test cases.

Fish	Wavenumber					
	$k = 2\pi$		$k = 4\pi$		$k = 6\pi$	
	Prec Alt	Prec Over	Prec Alt	Prec Over	Prec Alt	Prec Over
4	8	6	9	6	9	8
9	15	13	18	16	23	19
16	31	23	40	34	39	37

Table 7: Influence of the frequency and number of fish ($s = 1$) on the number of preconditioned GMRES iterations, alternating Method of Reflections, both with and without overlap (without restart, tolerance 10^{-8}). Incident plane wave coming from the top.

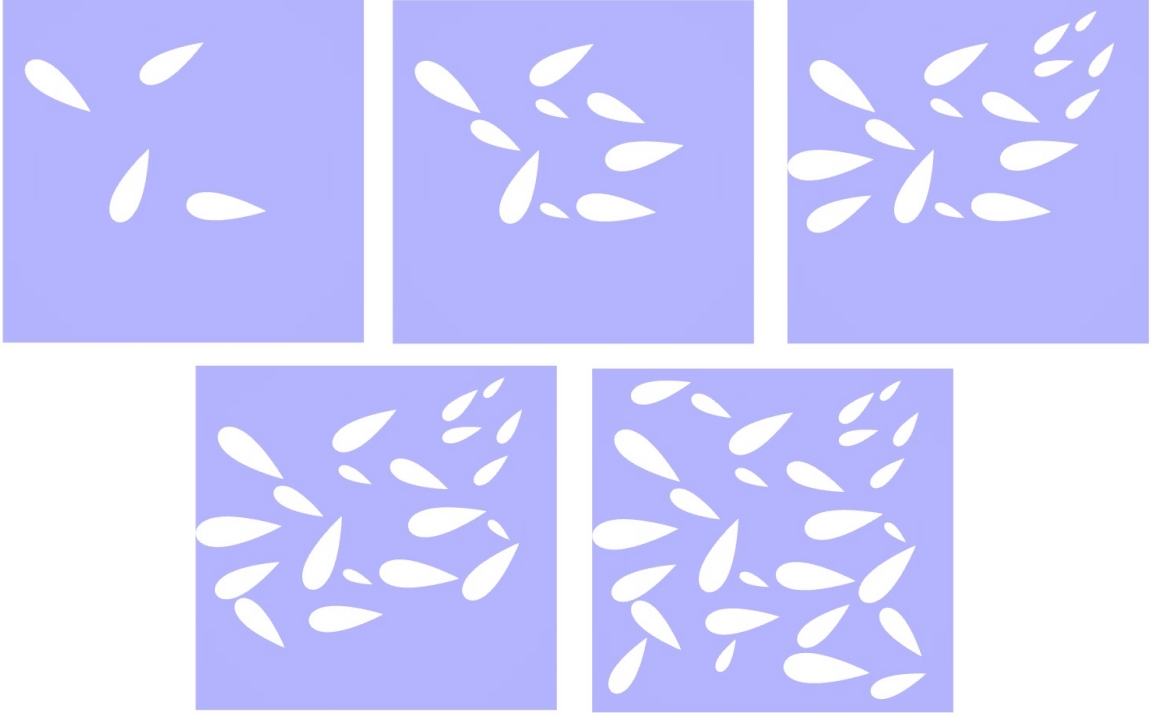


Figure 13: Definition of parameters for the school of fish with randomly distributed positions and orientations.

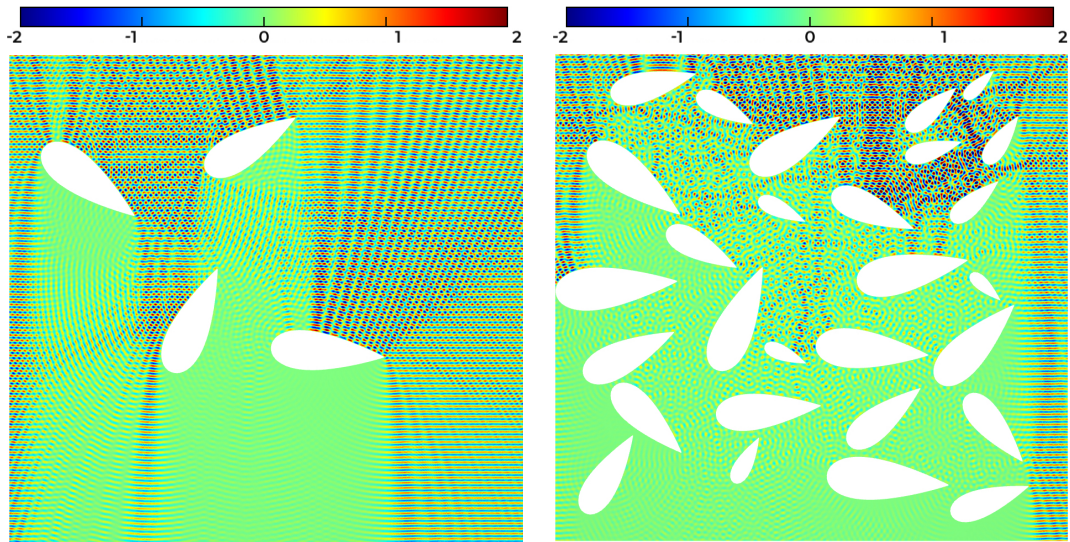


Figure 14: Diffraction by a school of fish. The real part of the total field is shown for a school of 4 fish (left) and 28 fish (right), with $k = 24\pi$. Incident plane wave coming from the top.

Fish	Wavenumber			
	$k = 2\pi$		$k = 24\pi$	
	Prec Alt	Prec Over	Prec Alt	Prec Over
4	7	6	8	7
9	20	12	57	32
16	70	19	91	42
20	116	37	179	70
28	214	101	227	142

Table 8: Configuration of randomly distributed fish. Influence of the frequency and number of fish on the number of preconditioned GMRES iterations, alternating Method of Reflections, both with and without overlap (without restart, tolerance 10^{-8}). Incident plane wave coming from the top.

Wavenumber	Prec Alt	Prec Over
2π	214	101
4π	312	175
8π	278	148
16π	255	132
24π	227	142
32π	209	121

Table 9: Configuration of 28 randomly distributed fish. Influence of the wavenumber on the number of preconditioned GMRES iterations using the alternating Method of Reflections, with and without overlap (no restart, tolerance 10^{-8}). The incident plane wave propagates from the top.

6 Conclusion

We have combined boundary integral equation methods with domain decomposition techniques to address Helmholtz multiple scattering problems. For each obstacle, the solution is expressed through a single-layer boundary integral representation, although the method is not limited to this choice. Various formulations of the Method of Reflections are analyzed, with a focus on frequency, inter-scatterer distance, and geometry. The convergence of both the alternating and parallel forms exhibits only a mild dependence on the wavenumber under Dirichlet boundary conditions. When the scatterers are closely spaced, convergence may slow down or, in the parallel formulation, the method might even fail. By connecting the Method of Reflections to algebraic Schwarz methods, we introduce overlap and coarse space corrections that significantly improve convergence. The main strength of the approach lies in its effectiveness as an algebraic preconditioner for Krylov methods, effectively reducing iteration counts as scatterers become closer and depending only slightly on the wavenumber. Combining these ideas with concepts from analytical preconditioners [5, 6] could define a promising avenue for future developments.

Although validated in two dimensions, the proposed framework naturally extends to three dimensions and to other PDE models. The main challenge lies in the treatment of large-scale three-dimensional high-frequency problems, where an appropriate balance must be achieved between the improved convergence provided by overlap or coarse-space techniques and the additional computational cost associated with these strategies. Furthermore, although the proposed methods remain effective as the number of obstacles increases, we plan to further investigate their scalability for very large configurations in future work.

References

- [1] M. Balabane. Boundary decomposition for Helmholtz and Maxwell equations. I. Disjoint sub-scatterers. *Asymptot. Anal.*, 38(1):1–10, 2004.
- [2] M. Balabane and V. Tirel. Décomposition de domaine pour un calcul hybride de l'équation de Helmholtz. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(3):281–286, 1997.
- [3] Yassine Boubendir, Fatih Ecevit, and Fernando Reitich. Acceleration of an iterative method for the evaluation of high-frequency multiple scattering effects. *SIAM J. Sci. Comput.*, 39(6):B1130–B1155, 2017.
- [4] Xiao-Chuan Cai and Marcus Sarkis. A restricted additive Schwarz preconditioner for general sparse linear systems. *SIAM J. Sci. Comput.*, 21(2):792–797, 1999.
- [5] Stéphanie Chaillat, Marion Darbas, and Frédérique Le Louër. Fast iterative boundary element methods for high-frequency scattering problems in 3D elastodynamics. *Journal of Computational Physics*, 341:429–446, 2017.
- [6] Stéphanie Chaillat, Marion Darbas, and Frédérique Le Louër. Analytical preconditioners for Neumann elastodynamic boundary element methods. *SN Partial Differential Equations and Applications*, 2(22), 2021.
- [7] Gabriele Ciaramella and Martin J. Gander. *Iterative methods and preconditioners for systems of linear equations*. SIAM, Philadelphia, 2022.
- [8] Gabriele Ciaramella, Martin J. Gander, Laurence Halpern, and Julien Salomon. Review of the methods of reflections. *Mathematisches Forschungsinstitut Oberwolfach*, 2017.
- [9] Gabriele Ciaramella, Martin J. Gander, Laurence Halpern, and Julien Salomon. Methods of reflections: relations with Schwarz methods and classical stationary iterations, scalability and preconditioning. *SMAI J. Comput. Math.*, 5:161–193, 2019.
- [10] D. Colton and R. Kress. *Inverse Acoustic and Electromagnetic Scattering Theory*. Springer-Verlag, Berlin, 1992.
- [11] Martin J Gander et al. Schwarz methods over the course of time. *Electron. Trans. Numer. Anal.*, 31(5):228–255, 2008.
- [12] Martin J. Gander and Laurence Halpern. Piece-wise constant, linear and oscillatory: a historical introduction to spectral coarse spaces with focus on Schwarz methods. In *Int. Conf. Domain Decomposition Methods*, pages 189–198. Springer, 2022.
- [13] Martin J. Gander and Hui Zhang. Schwarz methods by domain truncation. *Acta Numerica*, 31:1–134, 2022.
- [14] Walter Gander, Martin J. Gander, and Felix Kwok. *Scientific computing - An introduction using Maple and MATLAB*, volume 11. Springer Science & Business, Cham, 2014.
- [15] F.K. Hebekker. On a parallel Schwarz algorithm for symmetric strongly elliptic integral equations. In *Proc. 4th Int. Symp. Domain Decomposition Methods PDEs*, pages 382–393, Philadelphia, 1991. SIAM.
- [16] G.C. Hsiao and W.L. Wendland. Domain decomposition in boundary element methods. In *Proc. 4th Int. Symp. Domain Decomposition Methods PDEs*, pages 41–49. SIAM Philadelphia, 1991.
- [17] Ulrich Langer and Olaf Steinbach. Boundary element tearing and interconnecting methods. *Computing*, 71(3):205–228, 2003.
- [18] P. Laurent, G. Legendre, and J. Salomon. On the method of reflections. *Numerische Mathematik*, 148:449–493, 2021.
- [19] J. H. C. Luke. Convergence of a multiple reflection method for calculating Stokes flow in a suspension. *SIAM J. Appl. Math.*, 49(6):1635–1651, 1989.
- [20] Pierre Marchand. *Schwarz methods and boundary integral equations*. PhD thesis, Sorbonne Université, 2020.
- [21] Paul A. Martin. *Multiple scattering: interaction of time-harmonic waves with N obstacles*, volume 107. Cambridge University Press, Cambridge, 2006.
- [22] Maischak Matthias, Ernst P. Stephen, and Tran Thanh. Domain decomposition methods for boundary integral equations of the first kind: numerical results. *Applicable Analysis*, 63(1-2):111–132, 1996.
- [23] Robert Murphy. *Elementary principles of the theories of electricity, heat, and molecular actions*. Printed at the Pitt Press by J. Smith for J. & J. Deighton, Cambridge, 1833.
- [24] J.-C. Nédélec. *Acoustic and Electromagnetic Equations*, volume 144 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001.
- [25] Clemens Pechstein. Boundary element tearing and interconnecting methods in unbounded domains. *Applied numerical mathematics*, 59(11):2824–2842, 2009.
- [26] Zhen Peng, Xiao-chuan Wang, and Jin-Fa Lee. Integral equation based domain decomposition method for solving electromagnetic wave scattering from non-penetrable objects. *IEEE Transactions on Antennas and Propagation*, 59(9):3328–3338, 2011.
- [27] S.A. Sauter and C.. Schwab. *Boundary element methods*. Springer series in computational mathematics. Springer, Heidelberg, 2011.
- [28] Amik St-Cyr, Martin J Gander, and Stephen J Thomas. Optimized multiplicative, additive, and restricted additive Schwarz preconditioning. *SIAM J. Sci. Comput.*, 29(6):2402–2425, 2007.
- [29] Ernst P. Stephan, Matthias Maischak, and Florian Leydecker. Some Schwarz methods for integral equations on surfaces-h and p versions. *Comput. Vis. Sci.*, 8(3):211–216, 2005.
- [30] B. Thierry. A remark on the single scattering preconditioner applied to boundary integral equations. *J. Math. Anal. Appl.*, 413(1):212–228, 2014.
- [31] B. Thierry and X. Antoine. Spectral and condition number estimates of the acoustic single-layer operator for low-frequency multiple scattering in dense media. *Journal of Computational and Applied Mathematics*, 239(0):380–395, 2013.
- [32] V. Tirel. *Hybridation par méthode de décomposition de bord appliquée à l'étude des ondes acoustiques et électromagnétiques: résultats théoriques et numériques*. PhD thesis, Paris 13 university, 1998.
- [33] A. Toselli and O. Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer, Berlin, 2005.
- [34] Y.Saad. *Iterative methods for sparse linear systems*. PWS Publishing Company, Boston, 1996.