Computational & Applied Mathematics

Understanding our World

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Optimal Intercept Time

What direction does the police boat have to choose to approach the suspect vessel up to a distance $R$ as quickly as possible?
Imagine the police boat going into all directions at the same time $\implies y_0^2 + (vt - x_0)^2 = (ut + R)^2$.

The Donut Problem

What is the maximum number of pieces one can get from a donut when cutting it with three planar cuts?
Starting with an apple, how many pieces can we get?
The most we can get with a single planar cut is two pieces.
We get six pieces in total, each of the first two is cut twice.
Third Cut

How many pieces do we have now?
The Total Number of Pieces is

Upper top octant: 3 pieces
Upper left octant: 2 pieces
Upper right octant: 1 piece
Upper bottom octant: 3 pieces

Lower top octant: 2 pieces
Lower left octant: 1 piece
Lower right octant: 1 piece
Lower bottom octant: 2 pieces

15 Pieces with 3 planar cuts.
Is this Solution Really Correct?

The answer is NO, the maximum number of pieces one can get is 13! How?

An article by Martin Gardner in the “Scientific American” contains the general result for a torus in n dimensions and m cuts along hyperplanes.

Suppose you are allowed to move the pieces between the cuts, what is now the maximum number of pieces you can get?
Circular Billiard

In which direction do you have to push the red ball so that it bounces off the rim exactly once and then hits the blue ball?

Is there more than one solution?
Circular Billiard: Algebraic Solution

\[
f(\theta) = (1 + c \cos \theta) \sqrt{(a - \cos \theta)^2 + (b - \sin \theta)^2} \\
- (1 - a \cos \theta - b \sin \theta) \sqrt{(c + \cos \theta)^2 + (\sin \theta)^2}
\]
Are there always four solutions?
We need to find an ellipse which touches the circle tangentially.
An example with 4 solutions and one with 2 only.

\[ Q(u) = (m_2 - m_2 m_1) u^4 + (2m_1 - 2m_1^2 + 2e^2 + 2m_2^2) u^3 + 6u^2 m_2 m_1 
+ (-2m_2^2 + 2m_1^2 + 2m_1 - 2e^2) u - m_2 m_1 - m_2 = 0 \]

where \( \theta = 2 \arctan(u) \). \( Q(u) \) is a 4th degree polynomial, there can not be more than four solutions.
Dependence on the Ball Position

A numerical experiment: fixing the position of one ball and varying the position of the second ball. Counting the number of solutions gives the gray shade:
Dependence on the Ball Position

The separatrix \((x,y)\) can be computed analytically (\(t\) parameter):

\[
x(t) = -\frac{c}{\hat{h}} \left[ (1 + c)t^6 + 3(1 + 3c)t^4 + 3(1 - 3c)t^2 + (1 - c) \right],
\]

\[
y(t) = \frac{16}{\hat{h}} c^2 t^3, \quad \hat{h} = (1 + 3c + 2c^2)t^6 + 3(1 + c + 2c^2)t^4 +
\]

\[+ 3(1 - c + 2c^2)t^2 + (1 - 3c + 2c^2) \]

where \(c\) is the position of the fixed ball on the \(x\) axis.
Dependence on the Ball Position

Top view of a coffee mug with a point source of light emulating the circular billiard game (Drexler and G, SIAM review Vol. 40, No. 2, 1998)
A periodic signal $f(t)$ can be decomposed into its Fourier components:

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikt}.$$  

Discrete version: for the vector $f$ of length $n$, where $f_j := f(t_j)$, $t_j = j \Delta t$, $\Delta t = 2\pi/n$, we have

$$f_j = \sum_{k=0}^{n-1} \hat{f}_k e^{ikt_j}.$$  

What if the signal is an image, or a piece of music?
Population Dynamics

Joint work with Antonio Steiner, Il Volterriano
1997-2003
The Lotka Volterra System

We consider rabbits and foxes living in a common habitat. If $x$ denotes the rabbit population and $y$ the fox population, the Lotka Volterra model states

\[
\begin{align*}
\dot{x} &= x - xy \\
\dot{y} &= -y + xy
\end{align*}
\]

Approximating the derivative using its definition:

\[
\begin{align*}
\frac{x(t_{n+1}) - x(t_n)}{\Delta t} &= x(t_n) - x(t_n)y(t_n) \\
\frac{y(t_{n+1}) - y(t_n)}{\Delta t} &= -y(t_n) + x(t_n)y(t_n)
\end{align*}
\]

We get a discrete dynamical system.
Solutions

The exact solutions are cycles, but in general exact solutions can not be found. Numerical solutions spiral:
A Symplectic Method

A very small change in the original method, instead of

\[
\frac{x(t_{n+1}) - x(t_n)}{\Delta t} = x(t_n) - x(t_n)y(t_n)
\]
\[
\frac{y(t_{n+1}) - y(t_n)}{\Delta t} = -y(t_n) + x(t_n)y(t_n)
\]

changing in the second line \( t_n \) to \( t_{n+1} \),

\[
\frac{x(t_{n+1}) - x(t_n)}{\Delta t} = x(t_n) - x(t_n)y(t_n)
\]
\[
\frac{y(t_{n+1}) - y(t_n)}{\Delta t} = -y(t_n) + x(t_{n+1})y(t_n)
\]

leads to a method for which one can prove that the approximate solution is cyclic like the exact solution.
The approximate solutions are also cycles, like the mathematically exact or “biological” ones.
and Failure of the new method

But here, in spite of the proof of cyclic behavior, the method failed. On the right one can see why with the zoom!
So When does the Method Work?

The boundary between where the method works and where it does not is very complicated: it is a fractal.
Room Temperature in Montreal

\[ \frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + \frac{\partial^2 u}{\partial y^2}(x, t) + \frac{\partial^2 u}{\partial z^2}(x, t) + f(x, t) \]

Room temperature in our living room in Montreal (outside temperature up to -47 degrees): the heat equation.

Insulated walls, not well insulated windows and doors.
Why a Turntable in the Microwave?

The physical model is Maxwell’s equation

\[ \nabla \times E = -\mu H_t, \]
\[ \nabla \times H = \varepsilon E_t + \sigma E \]
Why a Turntable in the Microwave?

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Noise Levels in a VOLVO S90

Noise simulation on a parallel computer to improve passenger comfort
For cancer treatment it is important to have a very precise model of the body part that will be exposed to radiation.
Aircraft Industry: B-747 in Flight

Simulation of a B-747 flying through a thunderstorm, computation of the ice accumulation on the wings and around the engine intake.
The Fundamental Role of CAM

Physics:
- Fluid Dynamics
- Aero Dynamics
- Radiation
- Particle Physics

Biology:
- Population Dynamics
- Antibiotics
- Micromachines

Engineering:
- Circuits Simulation
- Active Noise Cancellation
- Cellular Phone Systems

Pure Mathematics:
- Geometry
- Asymptotics
- Existence

See also: www.math.mcgill.ca/mgander