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# Energy behaviour of the Boris method for charged-particle dynamics

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**Abstract** The Boris algorithm is a widely used numerical integrator for the motion of particles in a magnetic field. This article proves near-conservation of energy over very long times in the special cases where the magnetic field is constant or the electric potential is quadratic. When none of these assumptions is satisfied, it is illustrated by numerical examples that the numerical energy can have a linear drift or its error can behave like a random walk. If the system has a rotational symmetry and the magnetic field is constant, then also the momentum is approximately preserved over very long times, but in a spatially varying magnetic field this is generally not satisfied.

*Keywords.* Boris algorithm, charged particle, magnetic field, energy conservation, backward error analysis, modified differential equation.

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## 1 Introduction

For a particle with position $x(t) \in \mathbf{R}^3$ moving in an electro-magnetic field, Newton's second law together with Lorentz's force equation gives the second order differential equation (assuming suitable units)

$$\ddot{x} = \dot{x} \times B(x) + F(x). \tag{1.1}$$

Here, $F(x) = -\nabla U(x)$ is an electric field with the scalar potential $U(x)$, and $B(x) = \nabla \times A(x)$ is a magnetic field with the vector potential $A(x) \in \mathbf{R}^3$. We assume throughout this paper that the forces are smooth functions of $x$. Denoting by $v = \dot{x}$ the velocity of the particle, the energy is given by

$$E(x, v) = \frac{1}{2}|v|^2 + U(x) \tag{1.2}$$

and it is conserved along solutions of (1.1).

The simplest discretization of (1.1) is the *Boris method* [1]

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} = \frac{x_{n+1} - x_{n-1}}{2h} \times B(x_n) - \nabla U(x_n). \tag{1.3}$$

It is a symmetric, second-order numerical integrator. Its popularity is mostly due to the fact that it is essentially explicit and has shown a good long-time behaviour in many examples. It just requires the solution of a 3-dimensional linear system which can be efficiently solved by a Rodriguez formula. In the absence of the magnetic field, it reduces to the Störmer–Verlet scheme (see [5, Section I.1.4] or [4]).

The two-step formulation (1.3) is very sensitive to round-off errors. For a practical implementation one introduces a velocity approximation for the difference $x_{n+1} - x_n$, so that the Boris method becomes

$$\begin{aligned} x_{n+1} &= x_n + h\, v_{n+1/2} \\ v_{n+1/2} &= v_{n-1/2} + h\, v_n \times B(x_n) - h\, \nabla U(x_n), \end{aligned} \tag{1.4}$$

where $v_n = \frac{1}{2}(v_{n+1/2} + v_{n-1/2}) = \frac{1}{2h}(x_{n+1} - x_{n-1})$ is taken as the velocity approximation at the grid points. Given the initial values $(x_0, v_0)$, the method is started with $v_{1/2} = v_0 + \frac{h}{2}\, v_0 \times B(x_0) - \frac{h}{2}\, \nabla U(x_0)$.

It is known from [9] that the map $(x_n, v_{n-1/2}) \mapsto (x_{n+1}, v_{n+1/2})$ is volume-preserving. Moreover, if the magnetic field $B$ is constant, then with the momenta $p_n = v_n + A(x_n)$, the map $(x_n, p_n) \mapsto (x_{n+1}, p_{n+1})$ is symplectic. This follows from the fact that the Boris method is a variational integrator if and only if $B$ is constant [2].

In Section 2 we describe the backward error analysis (modified differential equation) for the Boris method. We use this to show long-time near-conservation of the energy for two particular cases:

- if the magnetic field $B$ is constant (Section 3),
- if the electric potential $U$ is quadratic (Section 4).

In Section 5 we then illustrate by numerical experiments that in the general situation the numerical energy can have a linear drift or its error can behave like a random walk. In Section 6 we discuss the long-time near-conservation of angular momentum for systems with a rotational symmetry, which holds for constant $B$, but not in general.

While we explain here that the Boris method, despite some remarkable properties, is not fully satisfactory with regard to energy and momentum conservation (unless the magnetic field is constant), we mention that recently several classes of explicit methods for (1.1) were developed that have long-time near-conservation of energy and momentum also for general non-constant magnetic fields [3, 7, 10, 11].

## 2 Backward error analysis

The idea of backward error analysis consists of searching for a modified differential equation (as a formal series in powers of $h$) such that its solution $y(t)$ formally satisfies $y(nh) = x_n$, where $x_n$ represents the numerical solution obtained by the Boris algorithm. Such a function has to satisfy

$$\frac{y(t+h) - 2y(t) + y(t-h)}{h^2} = \frac{y(t+h) - y(t-h)}{2h} \times B\big(y(t)\big) - \nabla U\big(y(t)\big).$$

Expanding all appearing functions into powers of $h$ yields (omitting the obvious argument $t$)

$$\ddot{y} + \frac{h^2}{12}\ddddot{y} + \ldots = \Big(\dot{y} + \frac{h^2}{6}\dddot{y} + \ldots\Big) \times B(y) - \nabla U(y). \tag{2.1}$$

For $h = 0$, the equation reduces to (1.1). The third and higher derivatives can be recursively eliminated by repeatedly differentiating the equation and setting $h$ to 0. This then gives the modified differential equation, which is a second-order differential equation whose right-hand side is a formal series in even powers of the step size $h$ with coefficient functions that depend on $(y, \dot{y})$:

$$\ddot{y} = \dot{y} \times B(y) - \nabla U(y) + h^2 F_2(y, \dot{y}) + h^4 F_4(y, \dot{y}) + \ldots \ .$$

To get a system of first order modified differential equations that is consistent with (1.4) we introduce the velocity approximation $w(t)$ by

$$y(t+h) - y(t-h) = 2h\, w(t),$$

so that formally $w(nh) = v_n$. Expanding into powers of $h$ yields the relation

$$w = \dot{y} + \frac{h^2}{3!}\dddot{y} + \frac{h^4}{5!} y^{(5)} + \ldots \ . \tag{2.2}$$

The third and higher derivatives of $y$ can be expressed with the help of (2.1) in terms of $(y, \dot{y})$. This then permits us to express $\dot{y}$ in terms of $(y, w)$.

Taking the inner product of (2.1) with $\dot{y}$ or with $w$, we will prove in the next two sections that the Boris method nearly preserves the energy over very long times if $B$ is constant or if $U$ is quadratic, as stated in the following theorem.

**Theorem 2.1** *Let the magnetic field $B(x)$ and the scalar potential $U(x)$ be arbitrarily differentiable functions of $x$, and suppose that the numerical solution $(x_n, v_n)$ of the Boris method stays in a compact set that is independent of the step size $h$. In the case that one of the following two conditions is satisfied:*

- *the magnetic field $B(x) = B$ is constant, or*
- *the scalar potential $U(x) = \frac{1}{2}x^\top Qx + q^\top x$ is quadratic,*

*then for arbitrary truncation number $N \geq 2$, the deviation of the energy $E(x,v) = \frac{1}{2}v^\top v + U(x)$ along the numerical solution is bounded by*

$$|E(x_n, v_n) - E(x_0, v_0)| \leq C_N h^2 \quad \text{for} \quad nh \leq h^{-N}, \tag{2.3}$$

*where $C_N$ is independent of $n$ and $h$ as long as $nh \leq h^{-N}$.*

*Remark 2.1* If the level sets of the energy $E(x,v)$ are compact, then the energy bound ensures, via an induction argument, that the numerical solution of the Boris method stays in a fixed compact set over times $nh \leq h^{-N}$: if (2.3) holds true for $n$, then the numerical solution at step $n+1$ has a distance to the energy surface of at most $\mathcal{O}(h)$ and hence stays in a compact set, so that by Theorem 2.1 the bound (2.3) holds also for $n+1$ as long as $(n+1)h \leq h^{-N}$.

## 3 Modified energy – constant magnetic field

To prove near-conservation of energy for the Störmer–Verlet method (in the absence of the magnetic field), one takes the scalar product of (2.1) with $\dot{y}$ and thus finds a modified energy that is close to $E(x,v)$ [4]. In the present situation, this procedure yields the following result.

**Theorem 3.1** *There exist $h$-independent functions $E_{2j}(x,v)$ such that the function*

$$E_h(x,v) = E(x,v) + h^2 E_2(x,v) + h^4 E_4(x,v) + \ldots,$$

*truncated at the $\mathcal{O}(h^N)$ term, satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_h(y,\dot{y}) = h^2\, \dot{y}^\top \Big( \Big( \frac{1}{3!}\dddot{y} + \frac{h^2}{5!} y^{(5)} + \ldots \Big) \times B(y) \Big) + \mathcal{O}(h^N) \tag{3.1}$$

*along solutions of the modified differential equation (2.1).*

*Proof* Multiplied with $\dot{y}^T$, even-order derivatives of $y$ give a total derivative:

$$\dot{y}^T y^{(2m)} \;=\; \frac{d}{dt}\Big(\dot{y}^T y^{(2m-1)} - \ddot{y}^T y^{(2m-2)} + \ldots \mp (y^{(m-1)})^T y^{(m+1)} \pm \frac{1}{2}\,(y^{(m)})^T y^{(m)}\Big).$$

Multiplication of (2.1) with $\dot{y}^\top$ therefore yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\dot{y}^\top\dot{y} + U(y) + \frac{h^2}{12}\big(\dot{y}^\top\dddot{y} - \frac{1}{2}\ddot{y}^\top\ddot{y}\big) + \dots\Big) = \frac{h^2}{6}\dot{y}^\top\Big(\dddot{y}\times B(y)\Big) + \dots .$$

The fact that $\dot{y}$ is orthogonal to $\dot{y}\times B(y)$ is used on the right-hand side. Expressing second and higher derivatives of $y$ with the help of the modified differential equation proves the statement of the theorem. □

**Corollary 3.1** *If the magnetic field $B(x) = B$ is constant, there exist $h$-independent functions $\widetilde{E}_{2j}(x, v)$, such that the function*

$$\widetilde{E}_h(x, v) = E(x, v) + h^2\widetilde{E}_2(x, v) + h^4\widetilde{E}_4(x, v) + \dots,$$

*truncated at the $\mathcal{O}(h^N)$ term, satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_h(y, \dot{y}) = \mathcal{O}(h^N) \tag{3.2}$$

*along solutions of the modified differential equation (2.1).*

*Proof* This follows from Theorem 3.1, because the expression $\dot{y}^\top\big(y^{(k)}\times B\big)$ can be written as a total derivative for odd values $k = 2m + 1$, namely as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{y}^\top\big(y^{(2m)}\times B\big) - \ddot{y}^\top\big(y^{(2m-1)}\times B\big) + \dots \pm (y^{(m-1)})^\top\big(y^{(m+1)}\times B\big)\Big),$$

since $(y^{(m)})^\top\big(y^{(m)}\times B\big) = 0$. This total derivative can be moved to the other side of (3.1) to obtain (3.2). □

If the numerical solution stays in a compact set that is independent of $h$, then Corollary 3.1 implies the long-time near-conservation of energy (2.3) by the standard argument of writing the deviation of the modified energy as a telescoping sum,

$$\widetilde{E}_h(x_n, v_n) - \widetilde{E}_h(x_0, v_0) = \sum_{j=1}^{n}\big(\widetilde{E}_h(x_j, v_j) - \widetilde{E}_h(x_{j-1}, v_{j-1})\big),$$

where each term in the sum is of size $\mathcal{O}(h^{N+1})$, uniformly for $(x_j, v_j)$ in the compact set.

Alternatively to this proof of long-time near-conservation of energy in the case of constant $B$, (2.3) follows also from the known theory of symplectic integrators, e.g. [5, Chap. IX], since the Boris method is known to be symplectic in the case of constant $B$.

## 4 Modified energy – quadratic electric potential

Instead of $\dot y^\top$, the equation (2.1) is now pre-multiplied with the expression in front of $\times B(y)$. This implies that the term with the magnetic field disappears.

**Theorem 4.1** *There exist $h$-independent functions $E_{2j}(x,v)$ (different from those of the previous section), such that the function*

$$E_h(x,v) = E(x,v) + h^2 E_2(x,v) + h^4 E_4(x,v) + \dots,$$

*truncated at the $\mathcal{O}(h^N)$ term, satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_h(y,\dot y) = -h^2\Big(\frac{1}{3!}\dddot y + \frac{h^2}{5!} y^{(5)} + \dots\Big)\nabla U(y) + \mathcal{O}(h^N) \tag{4.1}$$

*along solutions of the modified differential equation (2.1).*

*Proof* The proof is similar to that of Theorem 3.1. One uses the fact that the scalar product ${y^{(k)}}^\top y^{(l)}$ is a total differential whenever $k+l$ is odd. □

**Corollary 4.1** *If the scalar potential is quadratic, $U(x) = \frac{1}{2} x^\top Q x + q^\top x$ with a symmetric matrix $Q$, then there exist $h$-independent functions $\widehat E_{2j}(x,v)$ such that the function*

$$\widehat E_h(x,v) = E(x,v) + h^2 \widehat E_2(x,v) + h^4 \widehat E_4(x,v) + \dots,$$

*truncated at the $\mathcal{O}(h^N)$ term, satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} \widehat E_h(y,\dot y) = \mathcal{O}(h^N) \tag{4.2}$$

*along solutions of the modified differential equation (2.1).*

*Proof* This follows from Theorem 4.1, because the expression $y^{(k)\top}\nabla U(y) = y^{(k)\top}(Qy+q)$ can be written as a total differential for odd values of $k$. □

This result shows that for a quadratic potential $U(x) = \frac{1}{2} x^\top Q x + q^\top x$ the energy is nearly preserved independently of the form of the magnetic field $B(x)$, with the same relation as in (2.3). An even stronger result holds.

**Theorem 4.2** *Let the scalar potential be quadratic, $U(x) = \frac{1}{2} x^\top Q x + q^\top x$ with a symmetric matrix $Q$. Then, for every vector field $B(x)$, the Boris method (1.4) exactly conserves the discrete modified energy*

$$E_h(x,v) = \frac{1}{2} v^T v + U(x) - \frac{h}{2} v^T \nabla U(x)$$

*in the sense that*

$$E_h(x_{n+1}, v_{n+1/2}) = E_h(x_n, v_{n-1/2}).$$

*Proof* Taking the scalar product of the lower relation of (1.4) with $v_n = (v_{n+1/2} + v_{n-1/2})/2$ yields

$$\frac{1}{2} v_{n+1/2}^\top v_{n+1/2} - \frac{1}{2} v_{n-1/2}^\top v_{n-1/2} = -\frac{h}{2} (v_{n+1/2} + v_{n-1/2})^\top (Qx_n + q).$$

Adding the identity

$$\begin{aligned} U(x_{n+1}) - U(x_n) &= \frac{1}{2} x_{n+1}^\top Q x_{n+1} - \frac{1}{2} x_n^\top Q x_n + q^\top (x_{n+1} - x_n) \\ &= h v_{n+1/2}^\top \Big( \frac{1}{2} Q\,(x_{n+1} + x_n) + q \Big) \end{aligned}$$

and observing that the terms with $h v_{n+1/2}^\top Q x_n$ cancel, this then proves the statement of the theorem. □

*Remark 4.1* An interesting example (penning trap with asymmetric magnetic field), for which the Boris algorithm shows an unbounded energy, is given in [8] (see also [7]) by

$$U(x) = -5\Big(x_1^2 + x_2^2 - 2\,x_3^2\Big), \qquad B(x) = 100 \begin{pmatrix} 1/3 \\ 0 \\ 1 \end{pmatrix} + 50 \begin{pmatrix} x_2 - x_3 \\ x_1 + x_3 \\ x_2 - x_1 \end{pmatrix}$$

with initial values $x(0) = \big(1/3, 0, 1/2\big)^\top$, $v(0) = \big(0, 1, 0\big)^\top$. Since the potential $U(x)$ is quadratic, Theorem 4.2 applies. This shows that the energy can be unbounded only if the numerical solution does not stay in a compact set. Note that the level sets of the energy $E(x, v)$ are not compact for this example.

## 5 Numerical experiments

This section studies the long-time behaviour of the numerical energy in the situation where the electric potential is non-quadratic and the magnetic field is non-constant. Inspired by the work [6], where a symmetric, non-symplectic, but volume-preserving integrator for molecular dynamics simulations is studied, we expect that the following two situations can arise:

- a linear energy-drift of size $\mathcal{O}(h^2 t)$;
- a random walk behaviour for the error of the numerical energy.

Before passing to the numerical experiments we write the modified energies of Sections 3 and 4 in terms of $y$ and $w$. We then have formally $x_n = y(nh)$ and $v_n = w(nh)$, where $(x_n, v_n)$ is the numerical solution obtained by (1.4). For this we use the relation (2.2) connecting $w$ with $\dot y$. We have

$$E(y, \dot y) = E(y, w) + \mathcal{O}(h^2).$$

As a consequence of Theorems 3.1 and 4.1 the solution $\big(y(t), w(t)\big)$ of the modified differential equation is seen to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(E(y, w) + h^2 F(y, w)\Big) = h^2 G(y, w) + \mathcal{O}(h^4). \tag{5.1}$$

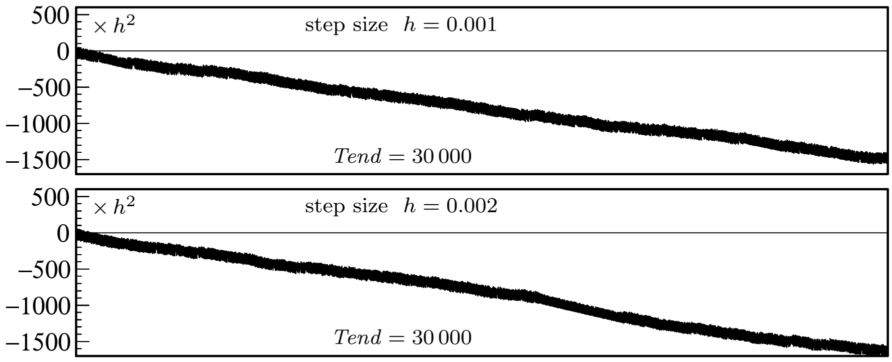


**Fig. 5.1** Energy error $E(x_n, v_n) - E(x_0, v_0)$ along numerical solution, for Example 5.1

In the following numerical experiments we plot $E(x_n, v_n)$.

For both examples of this section we consider the scalar potential

$$U(x) = x_1^3 - x_2^3 + \frac{1}{5}x_1^4 + x_2^4 + x_3^4 \tag{5.2}$$

and initial values

$$x(0) = (0.0, 1.0, 0.1)^\top, \quad v(0) = (0.09, 0.55.0.30)^\top. \tag{5.3}$$

The quartic terms imply that the level sets of the energy are compact, so that the exact solution of the problem exists and remains bounded for all times.

*Example 5.1 (Linear drift in the energy)* In addition to the scalar potential (5.2) and the initial values (5.3) we consider the magnetic field

$$B(x) = \nabla \times \frac{1}{3}\begin{pmatrix} -x_2\sqrt{x_1^2+x_2^2} \\ x_1\sqrt{x_1^2+x_2^2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{x_1^2+x_2^2} \end{pmatrix}. \tag{5.4}$$

We apply the Boris algorithm in its stabilized form (1.4) with step sizes $h = 0.001$ and $h = 0.002$. The error in the energy $E(x_n, v_n) - E(x_0, v_0)$ along the numerical solution is plotted in Figure 5.1 as a function of time. The vertical axis is scaled by $h^2$. Since the figures for both step sizes are similar, we conclude that there is a linear drift of slope $\mathcal{O}(h^2)$ in the numerical energy, which is superposed by high oscillations of size $\mathcal{O}(h^2)$.

This drift in the energy is somewhat expected. Integrating the relation (5.1) shows that, in general, we have a linear drift bounded by $th^2M$, where $M$ is an upper bound of $G(y, w)$. The term $h^2F(y, w)$ gives rise to bounded, high oscillations of size $\mathcal{O}(h^2)$.

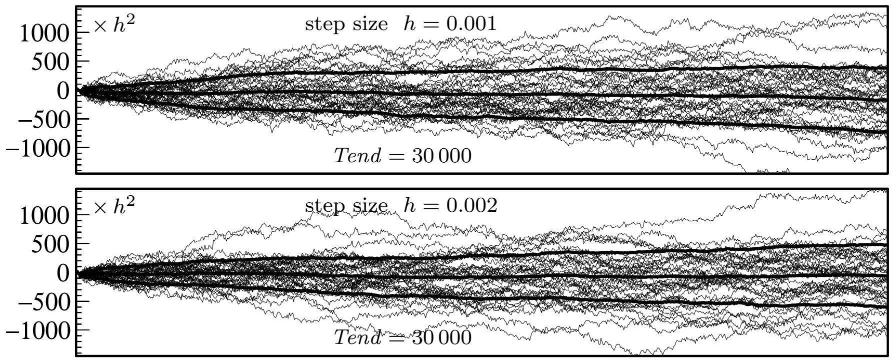


**Fig. 5.2** Energy error $E(x_n, v_n) - E(x_0, v_0)$ along numerical solution, for Example 5.2

*Example 5.2 (Random walk behaviour of the numerical energy)* This time we consider the magnetic field

$$B(x) = \nabla \times \frac{1}{4} \begin{pmatrix} x_3^2 - x_2^2 \\ x_3^2 - x_1^2 \\ x_2^2 - x_1^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_2 - x_3 \\ x_1 + x_3 \\ x_2 - x_1 \end{pmatrix} . \tag{5.5}$$

in addition to (5.2) and (5.3). We apply the Boris algorithm (1.4) with step sizes $h = 0.001$ and $h = 0.002$ on an interval of length $T_{end} = 30\,000$. To better appreciate the random walk behaviour, we compute the numerical solution for initial values, where $v_3(0)$ is replaced by $v_3(0) + \theta 10^{-13}$. Here, $\theta$ is randomly chosen in the interval $[0, 1]$. Figure 5.2 shows the energy error of 41 trajectories. Thick lines indicate average and variance taken over 101 trajectories.

This behaviour can be explained as in [6]. Let us assume that the solution of the modified differential equation is ergodic on an invariant set $A$ with respect to an invariant measure $\mu$. Then we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t G\big(y(s), w(s)\big)\, \mathrm{d}s = \int_A G(x, v)\, \mu\big(\mathrm{d}(x, v)\big)$$

for the function $G(x, v)$ of (5.1). If the integral to the right is non-zero, we will have a linear drift of size $\mathcal{O}(th^2)$. However, if this integral vanishes, the error of the modified energy will behave like a random walk. In addition to the $\mathcal{O}(h^2)$ term $h^2 F(x, v)$ of (5.1) there will be a drift of size $\sqrt{t}\, h^2$.

## 6 Momentum

If the scalar and vector potentials have the invariance properties

$$U(e^{\tau S} x) = U(x) \quad \text{and} \quad e^{-\tau S} A(e^{\tau S} x) = A(x) \quad \text{for all real } \tau \tag{6.1}$$

with a skew-symmetric matrix $S$, then the momentum

$$M(x,v) = \big(v + A(x)\big)^\top Sx \tag{6.2}$$

is conserved along solutions of the differential equation (1.1). This can be shown by scalarly multiplying (1.1) with $Sx$ and noting that the skew-symmetry of $S$ yields $x^\top S\ddot{x} = \frac{d}{dt}(x^T S\dot{x})$ and the invariance properties (6.1) imply $S\nabla U(x) = 0$ and $x^\top S(\dot{x} \times B(x)) = -\frac{d}{dt}(x^\top SA(x))$.

**Theorem 6.1** *If the vector and scalar potentials have the invariance properties* (6.1)*, then there exist h-independent functions* $M_{2j}(x,v)$ *such that the function*

$$M_h(x,v) = M(x,v) + h^2 M_2(x,v) + h^4 M_4(x,v) + \ldots,$$

*truncated at the* $\mathcal{O}(h^N)$ *term, satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} M_h(y,\dot{y}) = h^2\, y^\top S\, \Big(\Big(\frac{1}{3!}\dddot{y} + \frac{h^2}{5!} y^{(5)} + \ldots\Big) \times B(y)\Big) + \mathcal{O}(h^N) \tag{6.3}$$

*along solutions of the modified differential equation (2.1).*

*Proof* We multiply (2.1) with $y^\top S$ and note that $y^\top S\, y^{(k)}$ can be written as a total differential for even values of $k$, and $y^\top S\, \nabla U(y) = 0$. This yields the stated result in the same way as in Theorem 3.1. □

When $B(x) = B$ does not depend on $x$, then $A(x) = -\frac{1}{2} x \times B$ (up to a constant vector) and the above invariance properties are satisfied if $S$ is the skew-symmetric matrix that embodies the cross product with $B$, i.e., $Sv = v \times B$, and $U$ is invariant under rotations with the axis $B$, so that $\nabla U(x) \times B = 0$ for all $x$. The conserved momentum then reads

$$M(x,v) = v^\top (x \times B) - \frac{1}{2}\, |x \times B|^2.$$

**Corollary 6.1** *If the magnetic field* $B(x) = B$ *is constant and* $\nabla U(x) \times B = 0$ *for all x, then there exist h-independent functions* $\widetilde{M}_{2j}(x,v)$*, such that the function*

$$\widetilde{M}_h(x,v) = M(x,v) + h^2 \widetilde{M}_2(x,v) + h^4 \widetilde{M}_4(x,v) + \ldots,$$

*truncated at the* $\mathcal{O}(h^N)$ *term, satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{M}_h(y,\dot{y}) = \mathcal{O}(h^N) \tag{6.4}$$

*along solutions of the modified differential equation (2.1).*

*Proof* This follows from Theorem 6.1, since for constant $B$ we have with $Sy = y \times B$ that $(y \times B)^\top (y^{(k)} \times B)$ can be written as a total differential for odd values of $k$. □

If the numerical solution stays in a compact set that is independent of $h$, then Corollary 3.1 implies, for arbitrary positive integers $N$, that

$$M(x_n, v_n) = M(x_0, v_0) + \mathcal{O}(h^2) \quad \text{for } nh \le Ch^{-N}, \tag{6.5}$$

where the constant symbolized by the $\mathcal{O}$-notation is independent of $n$ and $h$ with $nh \le Ch^{-N}$.

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