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Regularization of neutral delay differential equations with several delays

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Summary For neutral delay differential equations the right-hand side can be multi-valued, when one or several delayed arguments cross a breaking point. This article studies a regularization via a singularly perturbed problem, which smooths the vector field and removes the discontinuities in the derivative of the solution. A low-dimensional dynamical system is presented, which characterizes the kind of generalized solution that is approximated. For the case that the solution of the regularized problem has high frequency oscillations around a codimension-2 weak solution of the original problem, a new stabilizing regularization is proposed and analyzed.

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1 Introduction

We consider systems of neutral delay differential equations

$$\begin{aligned} &\dot y(t) = f(y(t), \dot y(\alpha_1(y(t))), \ldots, \dot y(\alpha_m(y(t)))) \qquad \text{for} \quad t>0 \\ &y(t) = \varphi(t) \qquad \text{for} \quad t \le 0 \end{aligned} \tag{1}$$

with smooth functions $f(y, z_1, \ldots, z_m)$, $\varphi(t)$ and $\alpha_j(y)$. More general situations, where f also depends on t and on $y(\alpha_j(y(t)))$, can be treated as well without any further difficulties. We consider time intervals where the solution satisfies $\alpha_j(y(t)) < t$ for all j.

The solution $y(t)$ is continuous at $t = 0$, but its derivative has a jump discontinuity at $t = 0$ if

$$\dot\varphi(0) \neq f(\varphi(0), \dot\varphi(\alpha_1(\varphi(0))), \ldots, \dot\varphi(\alpha_m(\varphi(0)))). \tag{2}$$

This article focusses on this situation, and we use the notation

$$\dot y_0^- = \dot\varphi(0), \qquad \dot y_0^+ = f(\varphi(0), \dot\varphi(\alpha_1(\varphi(0))), \ldots, \dot\varphi(\alpha_m(\varphi(0)))).$$

As long as $\alpha_j(y(t)) < 0$ for all j we are concerned with the ordinary differential equation

$$\dot y(t) = f(y(t), \dot\varphi(\alpha_1(y(t))), \ldots, \dot\varphi(\alpha_m(y(t)))) \tag{3}$$

and the classical theory can be applied. An interesting situation arises when for the first time one of the lag terms becomes zero, for example, $\alpha_1(y(t_1)) = 0$. Such a time instant is called breaking point. Because of (2) the vector field has a jump discontinuity along the manifold $\mathcal{M}_1 = \{y; \alpha_1(y) = 0\}$ and we have to distinguish two situations. Either, a classical solution continues to exist in the region $\alpha_1(y) > 0$, or the vector field points towards the manifold $\mathcal{M}_1$ from both sides. In the second case it is possible to define a weak solution evolving in the manifold (see Section 2).

Consider such a weak solution and assume that at some time t_2 the second lag term becomes zero: $\alpha_2(y(t_2)) = 0$. We then encounter several different situations. An interesting case is when $\alpha_1(y(t)) = 0$ and $\alpha_2(y(t)) = 0$ on a nonempty interval starting at t_2. This corresponds to a weak solution in the codimension-2 manifold $\mathcal{M}_1 \cap \mathcal{M}_2$, where $\mathcal{M}_2 = \{y; \alpha_2(y) = 0\}$. The study of all possible solutions and their regularization is the main topic of the present work.

In Section 2 we give a rigorous definition of generalized (classical and weak) solutions relating them to differential-algebraic systems of index 2. Weak solutions can be interpreted as so-called Utkin solutions. It is shown that for the linear case they are equivalent to Filippov solutions. In Section 3 we discuss a regularization via a singularly perturbed delay differential equation. We recall some results of [7] that concern the codimension-1 case, and we extend them to the codimension-2 situation. In particular we present a 4-dimensional dynamical system for which, near a breaking point in $\mathcal{M}_1 \cap \mathcal{M}_2$, the stationary points characterize the kind of solution (classical or weak) that is approximated by the regularization. In most situations the solution of the regularized problem is smooth after a short transient phase at breaking points, and codes for non-neutral stiff delay differential equations (such as Radar5 of [5]) can be applied to obtain an accurate approximation of the original problem in an efficient way.

However, it may happen that the regularized solution has high frequency oscillations, so that a numerical solution becomes inefficient. In Section 4 we propose a stabilizing regularization which, in many situations, eliminates the high oscillations. Some technical proofs are collected in an appendix (Section 5), and numerical experiments illustrate and verify our theoretical investigation.

2 Differential-algebraic systems of index 2

We concentrate on the case $m = 2$ of two delays, and we give a precise meaning to what we call a classical or a weak solution of (1). Whenever $\alpha_j(y(t)) \neq 0$ for all j (more precisely, $\alpha_j(y(t)) \neq t_k$, where the derivative of $y(t)$ has a jump discontinuity at t_k), the problem (1) is well-posed and a locally unique solution exists by the method of steps. We call it *classical solution.*

Due to the jump discontinuity at 0 (or at t_k) the vector field of (1) is not well-defined if $\alpha_j(y(t)) = 0$ for some j. In this case it is common to consider the right-hand side as a multi-valued function, where $\dot y(0)$ can take any value of the segment connecting $\dot y_0^-$ with $\dot y_0^+$, see the approach of Utkin [10]. Depending on whether one or two lag terms equal zero, we distinguish the following situations.

Codimension-1 *weak solution.* As long as $\alpha_2(y(t)) \neq 0$ (more precisely, if $\alpha_2(y(t))$ does not take a value, where $\dot y(t)$ has a jump discontinuity) we consider the differential-algebraic system

$$\begin{aligned} \dot y(t) &= f(y(t), \dot y_0^+ + \theta_1(t)(\dot y_0^- - \dot y_0^+), \dot y(\alpha_2(y(t)))) \\ 0 &= \alpha_1(y(t)) \end{aligned} \tag{4}$$

Differentiation of the algebraic relation with respect to time yields

$$\alpha_1'(y) f(y, \dot y_0^+ + \theta_1(\dot y_0^- - \dot y_0^+), \dot y(\alpha_2(y))) = 0$$

along the solution. We assume that this scalar equation permits to express θ_1 as a function of y. Inserted into (4) a solution can be obtained provided that the initial value satisfies the algebraic relation.

A second type of codimension-1 solutions is obtained from the differential-algebraic system

$$\begin{aligned} \dot y(t) &= f(y(t), \dot y(\alpha_1(y(t))), \dot y_0^+ + \theta_2(t)(\dot y_0^- - \dot y_0^+)) \\ 0 &= \alpha_2(y(t)) \end{aligned} \tag{5}$$

where the roles of $\alpha_1(y)$ and $\alpha_2(y)$ are exchanged.

Codimension-2 weak solution. On intervals, where both lag terms vanish, we consider the system

$$\begin{aligned}\dot y(t) &= f(y(t), \dot y_0^+ + \theta_1(t)(\dot y_0^- - \dot y_0^+), \dot y_0^+ + \theta_2(t)(\dot y_0^- - \dot y_0^+))\\ 0 &= \alpha_1(y(t))\\ 0 &= \alpha_2(y(t)).\end{aligned} \tag{6}$$

In this case we obtain the necessary conditions

$$\begin{aligned}\alpha_1'(y)f(y, \dot y_0^+ + \theta_1(\dot y_0^- - \dot y_0^+), \dot y_0^+ + \theta_2(\dot y_0^- - \dot y_0^+)) = 0\\ \alpha_2'(y)f(y, \dot y_0^+ + \theta_1(\dot y_0^- - \dot y_0^+), \dot y_0^+ + \theta_2(\dot y_0^- - \dot y_0^+)) = 0.\end{aligned}$$

We assume that this system determines (locally uniquely) θ_1 and θ_2 as a function of y, so that (6) becomes an ordinary differential equation for $y(t)$ provided the two algebraic relations are satisfied by the initial values.

Definition 1 *A continuous function $y(t)$ is a generalized solution of the neutral delay differential equation (1), if there exists a sequence $0 = t_0 < t_1 < t_2 < \ldots$, such that the restriction of $y(t)$ to the intervals $[t_{k-1}, t_k]$ is differentiable (admitting one-sided derivatives at the endpoints), and one of the following situations occurs in the open interval (t_{k-1}, t_k):*

(a) $\alpha_1(y(t)) \neq t_j$, $\alpha_2(y(t)) \neq t_j$ *for* $j < k$, *and (1) is satisfied;*
(b1) $\alpha_2(y(t)) \neq t_j$ *for* $j < k$, *and (4) holds with* $0 < \theta_1(t) < 1$*;*
(b2) $\alpha_1(y(t)) \neq t_j$ *for* $j < k$, *and (5) holds with* $0 < \theta_2(t) < 1$*;*
(c) *equation (6) holds with* $0 < \theta_1(t) < 1$ *and* $0 < \theta_2(t) < 1$.

The time instant t_k is called breaking point, if the length of the subinterval $[t_{k-1}, t_k]$ is maximal.

The first interval $(0, t_1)$ is always of type (a) because $\alpha_j(y(0)) < 0$ for all j. At the first breaking point t_1 we have $\alpha_j(y(t_1)) = 0$ for at least one subscript j. If after t_1 we have a sliding mode, i.e., a situation of type (b1) or (b2), it often happens that we have $\alpha_j(y(t_2)) = 0$ for both j in the following breaking point. This is the situation that interests us most.

More generally, let us assume that at some breaking point $t^* = t_k$ we have simultaneously $\alpha_1(y^*) = 0$ and $\alpha_2(y^*) = 0$ for $y^* = y(t^*)$. For $0 \le \theta_1, \theta_2 \le 1$ we then consider the scalar functions

$$g_j(\theta_1, \theta_2) = \alpha_j'(y^*)f(y^*, \dot y_0^+ + \theta_1(\dot y_0^- - \dot y_0^+), \dot y_0^+ + \theta_2(\dot y_0^- - \dot y_0^+)). \tag{7}$$

These functions determine the kind of solution beyond the breaking point t^*. We can have the following generic[1] situations:

Existence of classical solutions:

(a1) if $g_1(0,0) > 0$, $g_2(0,0) > 0$, there exists a solution in the region $\{y\,;\,\alpha_1(y) > 0, \alpha_2(y) > 0\}$;
(a2) if $g_1(0,1) > 0$, $g_2(0,1) < 0$, there exists a solution in the region $\{y\,;\,\alpha_1(y) > 0, \alpha_2(y) < 0\}$;
(a3) if $g_1(1,0) < 0$, $g_2(1,0) > 0$, there exists a solution in the region $\{y\,;\,\alpha_1(y) < 0, \alpha_2(y) > 0\}$;
(a4) if $g_1(1,1) < 0$, $g_2(1,1) < 0$, there exists a solution in the region $\{y\,;\,\alpha_1(y) < 0, \alpha_2(y) < 0\}$;

*Existence of codimension-*1 *weak solutions:*

(b1) if there is a $\theta_1 \in (0,1)$ with $g_1(\theta_1,0) = 0$, $\partial_1 g_1(\theta_1,0) \neq 0$, and $g_2(\theta_1,0) > 0$, a solution of (4) exists in $\{y\,;\,\alpha_1(y) = 0, \alpha_2(y) > 0\}$;
(b2) if there is a $\theta_2 \in (0,1)$ with $g_2(0,\theta_2) = 0$, $\partial_2 g_2(0,\theta_2) \neq 0$, and $g_1(0,\theta_2) > 0$, a solution of (5) exists in $\{y\,;\,\alpha_1(y) > 0, \alpha_2(y) = 0\}$;
(b3) if there is a $\theta_1 \in (0,1)$ with $g_1(\theta_1,1) = 0$, $\partial_1 g_1(\theta_1,1) \neq 0$, and $g_2(\theta_1,1) < 0$, a solution of (4) exists in $\{y\,;\,\alpha_1(y) = 0, \alpha_2(y) < 0\}$;
(b4) if there is a $\theta_2 \in (0,1)$ with $g_2(1,\theta_2) = 0$, $\partial_2 g_2(1,\theta_2) \neq 0$, and $g_1(1,\theta_2) < 0$, a solution of (5) exists in $\{y\,;\,\alpha_1(y) < 0, \alpha_2(y) = 0\}$;

*Existence of codimension-*2 *weak solutions:*

(c1) if there exist $\theta_1, \theta_2 \in (0,1)$ with $g_1(\theta_1,\theta_2) = 0$, $g_2(\theta_1,\theta_2) = 0$, and invertible $(\partial_j g_i(\theta_1,\theta_2))_{i,j=1}^2$, there is a solution of (6) in the codimension-2 manifold $\{y\,;\,\alpha_1(y) = 0, \alpha_2(y) = 0\}$.

In the situations (a1)-(a4) we have a solution of (1) in the classical sense. The sign conditions guarantee that the vector field points into the correct orthant. Under the assumption (b1) a locally unique solution of (4) exists. Consistent initial values are $y(t^*) = y^*$ and $\theta_1(t^*) = \theta_1$, and the condition on the derivative permits an application of the implicit function theorem, so that θ_1 can be expressed in terms of y by the differentiated algebraic relation. The situations (b2)-(b4) have a similar interpretation. The condition (c1) is such that (6) possesses a locally unique solution.

The solution beyond the breaking point t^* need not be unique. It is for example possible to have a bifurcation into more than one classical solutions. Also the co-existence of classical and weak solutions is possible (see Example 3).

[1] With the word “generic” we mean that both, $g_1(\theta_1,\theta_2)$ and $g_2(\theta_1,\theta_2)$ are nonzero at the corners of the unit square, and that $g_1(\theta_1,\theta_2) = g_2(\theta_1,\theta_2) = 0$ can occur only inside the unit square.

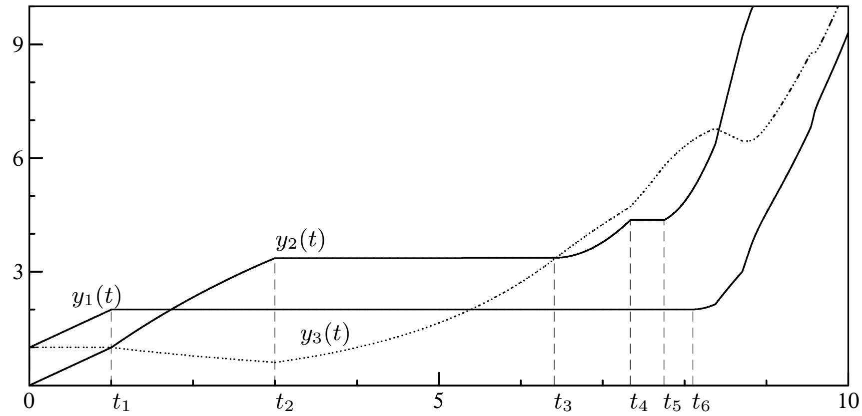

Fig. 1. Various regimes of the generalized solution of problem (8).

Example 1 Consider the system of neutral delay differential equations (for $t \geq t_0 = 0$)

$$\begin{aligned} \dot y_1(t) &= c_1 y_3(t) + a_{11}\, \dot y_1(y_1(t) - r_1) + a_{12}\, \dot y_2(y_2(t) - r_2) \\ \dot y_2(t) &= c_2 y_3(t) + a_{21}\, \dot y_1(y_1(t) - r_1) + a_{22}\, \dot y_2(y_2(t) - r_2) \\ \dot y_3(t) &= c_3 y_3(t) + a_{31}\, \dot y_1(y_1(t) - r_1) + a_{32}\, \dot y_2(y_2(t) - r_2) \end{aligned} \tag{8}$$

where c_i, r_j and a_{ij} $(i = 1,2,3,\ j = 1,2)$ are given constants and the initial conditions are $y_1(t) = y_3(t) = 1$ and $y_2(t) = 0$ for $t \leq 0$. For the parameters we choose the values $c_1 = 1, c_2 = 1, c_3 = 0$, $r_1 = 2, r_2 = 1 + 6\,(1 - \mathrm{e}^{-1/2}) \approx 3.36$, and $a_{11} = -4, a_{12} = -2, a_{21} = 2, a_{22} = -4, a_{31} = -1, a_{32} = 2$.

The generalized solution is plotted in Figure 1 and is analyzed step by step. We observe that classical, codimension-1 and codimension-2 weak solution regimes alternate in a quite generic way.

Interval $[0, t_1]$ *with* $t_1 = 1$. We have an ordinary differential equation with (classical) solution given by $y_1(t) = 1 + t$, $y_2(t) = t$, $y_3(t) = 1$. The time instant t_1 is a breaking point ($y_1(t_1) = 2$) and one can check that a classical solution ceases to exist.

Interval $[t_1, t_2]$ *with* $t_2 = 3$. We have a codimension-1 weak solution in the manifold $\{y\,;\, y_1 = r_1\}$. We replace the term $\dot y_1(y_1(t) - r_1)$ by the new variable $1 - \theta_1(t)$ and solve the system (4). This yields $\theta_1(t) = 1 - \frac{1}{4}\, y_3(t)$ and the solution $y_1(t) = 2$, $y_2(t) = 1 + 6(1 - \exp(-\frac{1}{4}(t-1)))$, and $y_3(t) = \exp(-\frac{1}{4}(t-1))$. A new breaking point t_2 appears when $y_2(t_2) = r_2$.

Interval $[t_2, t_3]$ *with* $t_3 = 4 + 2\ln(10/3) \approx 6.41$. We have a codimension-2 weak solution in the manifold $\{y\,;\, y_1 = r_1, y_2 = r_2\}$. We replace $\dot{y}_1(y_1(t)-r_1)$ by $1-\theta_1(t)$ and $\dot{y}_2(y_2(t)-r_2)$ by $1-\theta_2(t)$ and consider the system (6). This gives the conditions $\theta_1(t)=1-\frac{1}{10}\,y_3(t)$, $\theta_2(t)=1-\frac{3}{10}\,y_3(t)$ for the new variables. The solution is then seen to be $y_1(t)=r_1=2$, $y_2(t)=r_2=1+6(1-\mathrm{e}^{-1/2})$, and $y_3(t)=\mathrm{e}^{(t-4)/2}$. This solution persists till t_3 when $\theta_2(t_3)=0$ so that it switches to a codimension-1 weak regime.

Interval $[t_3,t_4]$ *with* $t_4\approx 7.34$. We have a solution in the codimension-1 manifold $\{y\,;\,y_1=r_1\}$. We have $\theta_1(t)=\frac{1}{4}(6-y_3(t))$ and the differential equations $\dot{y}_2(t)=\frac{3}{2}\,y_3(t)-5$, $\dot{y}_3(t)=-\frac{1}{4}\,y_3(t)+\frac{5}{2}$, whose analytic solutions are not reported for brevity. At t_4 we have $y_2(t_4)=t_1+r_2$ so that a further breaking point appears and the solution enters into the codimension-2 manifold $\{y\,;\,y_1=r_1, y_2=t_1+r_2\}$, which is different from the one encountered on the interval $[t_2,t_3]$.

Interval $[t_4,t_5]$ *with* $t_5\approx 7.75$. We have again a codimension-2 sliding mode. The component $y_3(t)$ increases and at t_5 the solution leaves the manifold $\{y\,;\,y_2=t_1+r_2\}$.

Interval $[t_5,t_6]$ *with* $t_6\approx 8.12$. We have a sliding mode in the manifold $\{y\,;\,y_1=r_1\}$. The continued increase of $y_3(t)$ implies at t_6 the condition $\theta_1(t_6)=0$, so that the solution leaves the manifold and enters into a classical regime.

2.1 Connection with Filippov solutions

There is an alternative for defining weak solutions. For the codimension-2 case we consider the four vector fields

$$f^{++}(y)=f(y,\dot{y}_0^+,\dot{y}_0^+),\qquad f^{+-}(y)=f(y,\dot{y}_0^+,\dot{y}_0^-),$$
$$f^{-+}(y)=f(y,\dot{y}_0^-,\dot{y}_0^+),\qquad f^{--}(y)=f(y,\dot{y}_0^-,\dot{y}_0^-)$$

in the manifold defined by $\alpha_1(y)=0$ and $\alpha_2(y)=0$. They are the limit of the vector field in (1) for $\alpha_1(y)\to 0$ and $\alpha_2(y)\to 0$, depending from which side the limit is taken (for example, $f^{++}(y)$ is obtained when only values with $\alpha_1(y)>0$ and $\alpha_2(y)>0$ are considered).

The approach of Filippov [4] for weak codimension-2 solutions of (1) consists in searching a function $y(t)$ having its derivative in the convex combination of the four vector fields, i.e.,

$$\dot{y}=\lambda_1 f^{++}(y)+\lambda_2 f^{+-}(y)+\lambda_3 f^{-+}(y)+\lambda_4 f^{--}(y), \qquad (9)$$

where $\lambda_1+\lambda_2+\lambda_3+\lambda_4=1$ and $\lambda_j\ge 0$ for all j. There is an ambiguity, because three parameters among the λ_j have to be determined to satisfy the two conditions $\alpha_1'(y)\dot{y}=0$ and $\alpha_2'(y)\dot{y}=0$, and much research is devoted to this question, see [2,1].

In the case where $f(y,z_1,z_2) = a(y) + A_1(y)z_1 + A_2(y)z_2$ is linear in (z_1,z_2) (with a vector function $a(y)$ and matrix functions $A_j(y)$), the differential equation (6) can be rewritten as

$$\begin{aligned} \dot y = (1-\theta_1)(1-\theta_2) f^{++}(y) + (1-\theta_1)\theta_2 f^{+-}(y) \\ + \theta_1(1-\theta_2) f^{-+}(y) + \theta_1\theta_2 f^{--}(y), \end{aligned} \tag{10}$$

and it is seen to be a special case of (9). Among all Filippov solutions this choice has been advocated in [3]. We remark that for the linear case, for which the identity

$$f^{++}(y) - f^{+-}(y) - f^{-+}(y) + f^{--}(y) = 0$$

holds, every Filippov vector field can be written in the form (10).

3 Regularized problem

A common approach for solving numerically neutral delay differential equations is by regularization. There are several possibilities, but we focus our attention on the regularization via the singularly perturbed (non-neutral) delay equation

$$\begin{aligned} \dot y(t) &= z(t) \\ \varepsilon\, \dot z(t) &= f\big(y(t), z(\alpha_1(y(t))), z(\alpha_2(y(t)))\big) \,-\, z(t) \end{aligned} \tag{11}$$

with $y(t) = \varphi(t)$ and $z(t) = \dot\varphi(t)$ for $t \le 0$. Here, $\varepsilon > 0$ is a small positive parameter. We note that for $\varepsilon = 0$, the problem (11) reduces to (1) with $m = 2$. Standard codes for stiff, state-dependent delay equations (like Radar5 of [5]) can be applied to solve this problem for $\varepsilon > 0$. For example, solving the regularization of equation (8) with $\varepsilon = 10^{-3}$ by the code Radar5 yields an excellent approximation of the solution of Figure 1 without the need of switching between different differential and differential-algebraic equations.

The solution of (11) is unique also beyond breaking points, and the functions $y(t)$ and $z(t)$ are everywhere continuous. It is therefore of interest to study if, for $\varepsilon \to 0$, the solution of (11) approximates a solution of (1) and, in the case of non-uniqueness of the solution of (1), which of the solutions will be approximated by (11). These questions have been addressed in [7] for the case of one delay and for codimension-1 weak solutions. Our aim is to get insight into the more complicated case of codimension-2 weak solutions.

Until the first breaking point, where $\alpha_j(y(t)) < 0$ for $j = 1,2$, so that $z(\alpha_j(y(t)))$ is replaced by $\dot\varphi(\alpha_j(y(t)))$, we are concerned with a singularly perturbed ordinary differential equation. It is well-known (see for example, [9] or [8, Section VI.3]) that the solution of (11) can be split into a smooth and a transient part (outer and inner solution) and that, due to the special dependence on $z(t)$, it is of the form (for $t \geq 0$)

$$\begin{aligned} y(t) &= y_0(t) + \varepsilon y_1(t) - \varepsilon(\dot{y}_0^- - \dot{y}_0^+)\, \mathrm{e}^{-t/\varepsilon} + \mathcal{O}(\varepsilon^2) \\ z(t) &= z_0(t) + (\dot{y}_0^- - \dot{y}_0^+)\, \mathrm{e}^{-t/\varepsilon} + \mathcal{O}(\varepsilon), \end{aligned} \tag{12}$$

where $y_0(t)$ is the solution of (3), $z_0(t) = \dot{y}_0(t)$ with $z_0(0) = \dot{y}_0^+$, and $y_1(t)$ is a smooth function satisfying $y_1(0) = \dot{y}_0^- - \dot{y}_0^+$.

3.1 Approximation of weak solutions in codimension one

We distinguish the situations, where the solution enters transversally the manifold $\{y\,;\, \alpha_1(y) = 0\}$ from below (that is from the region $\{y\,;\, \alpha_1(y) < 0\}$) and where it enters from above. The first situation has been studied in [7] and we briefly present the results that are important for the present work. We then give a formal derivation of the possible behaviors in the second situation without giving rigorous error estimates.

From below. Let t_1 be a breaking point of (1), for which $\alpha_1(y_0(t_1)) = 0$ and $\alpha_2(y_0(t_1)) < 0$. If $\alpha_1'(y_0(t_1))\dot{y}_0(t_1) > 0$, the implicit function theorem guarantees the existence of a breaking point $t_1(\varepsilon)$ of (11), which depends smoothly on ε. For the solution $y(t)$ of (11) we thus have $\alpha_1(y(t_1(\varepsilon))) = 0$ and $\alpha_2(y(t_1(\varepsilon))) \leq c < 0$ (with c independent of ε). We denote $t^* = t_1(\varepsilon)$ and notice that the transient layer in (12) is negligible at $t = t^*$. This implies that $y(t^*) = y^* + \varepsilon v^* + \mathcal{O}(\varepsilon^2)$ and $z(t^*) = z^* + \mathcal{O}(\varepsilon)$ with ε-independent vectors y^*, v^*, and z^*. The condition $0 = \alpha_1(y(t_1(\varepsilon))) = \alpha_1(y^* + \varepsilon v^* + \mathcal{O}(\varepsilon^2))$ thus yields

$$\alpha_1(y^*) = 0, \qquad \alpha_1'(y^*)\, v^* = 0. \tag{13}$$

Since there is a jump discontinuity at t_1 in $z_0(t) = \dot{y}_0(t)$, but none in the solution $(y(t), z(t))$ of (11), we must have a transient part also right after $t_1(\varepsilon)$. This motivates the ansatz, for $t \geq t_1(\varepsilon)$ and $\tau = (t - t_1(\varepsilon))/\varepsilon$,

$$\begin{aligned} y(t) &= y_0(t) + \varepsilon y_1(t) + \varepsilon\, \tilde{\eta}(\tau) + \mathcal{O}(\varepsilon^2) \\ z(t) &= z_0(t) + \tilde{\zeta}(\tau) + \mathcal{O}(\varepsilon), \end{aligned} \tag{14}$$

where $y_0(t)$ is a solution of (3), $z_0(t) = \dot{y}_0(t)$, and $\tilde{\eta}(\tau)$, $\tilde{\zeta}(\tau)$ are functions that converge exponentially fast to zero for $\tau \to \infty$. To achieve uniqueness of the coefficient functions we assume that they are of the form $y_j(t) = \tilde y_j(t - t_1(\varepsilon))$, $z_0(t) = \tilde z_0(t - t_1(\varepsilon))$ with ε-independent functions $\tilde y_j(s)$, $\tilde z_0(s)$, and $\tilde\eta(\tau)$, $\tilde\zeta(\tau)$. By continuity at t^*, the coefficient functions of (14) must therefore satisfy $y_0(t^*) = y^*$, $y_1(t^*) + \tilde\eta(0) = v^*$, and $z_0(t^*) + \tilde\zeta(0) = z^*$.

To obtain the equations that define $\tilde\eta(\tau)$ and $\tilde\zeta(\tau)$, we insert the ansatz (14) into (11), we replace t by $t^* + \varepsilon\tau$, and we consider the ε-independent part. Using $\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\varepsilon}$, the first relation of (13), and

$$\frac{\alpha_1(y(t))}{\varepsilon} = \alpha_1'(y^*)\Big(z_0(t^*)\,\tau + y_1(t^*) + \tilde\eta(\tau)\Big) + \mathcal{O}(\varepsilon),$$

yield the differential equations $\tilde\eta'(\tau) = \tilde\zeta(\tau)$ and

$$\begin{aligned}\tilde\zeta'(\tau) = f\Big(y^*, \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\,\mathrm{e}^{-\alpha_1'(y^*)\big(z_0(t^*)\,\tau + y_1(t^*) + \tilde\eta(\tau)\big)}, \dot\varphi(\alpha_2(y^*))\Big)\\ - (z_0(t^*) + \tilde\zeta(\tau)).\end{aligned}$$

For the functions

$$\begin{aligned}\eta(\tau) &= \alpha_1'(y^*)(z_0(t^*)\,\tau + y_1(t^*) + \tilde\eta(\tau)),\\ \zeta(\tau) &= \alpha_1'(y^*)(z_0(t^*) + \tilde\zeta(\tau)),\end{aligned} \tag{15}$$

we thus obtain the two-dimensional dynamical system

$$\begin{aligned}\eta' &= \zeta, & \eta(0) &= 0,\\ \zeta' &= -\,\zeta + g(\mathrm{e}^{-\eta}), & \zeta(0) &= \alpha_1'(y^*)z^*,\end{aligned} \tag{16}$$

where

$$g(\theta) = \alpha_1'(y^*) f\Big(y^*, \dot y_0^+ + \theta\,(\dot y_0^- - \dot y_0^+), \dot\varphi(\alpha_2(y^*))\Big) \quad \text{for} \quad 0 \le \theta \le 1,$$

and $g(\theta) = g(1)$ for $\theta \ge 1$. The initial condition $\eta(0) = 0$ follows from (13), and $\zeta(0) = \alpha_1'(y^*)z^* = g(1) > 0$ is a consequence of the assumption that the solution of (11) enters transversally the manifold $\alpha_1(y) = 0$, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\alpha_1(y(t))\big|_{t=t^*_-} > 0$ (the subscript of t^* indicates the left derivative of the function).

Up to now we do not know whether the smooth part of the solution (14) corresponds to a classical or to a weak solution. One of the main results of [7] tells us that one of the following two situations occurs:

(a) if the solution of (16) converges to a stationary point $(\eta, \zeta) = (c, 0)$ (that is $g(\mathrm{e}^{-c}) = 0$), then the solution (14) of (11) approximates a codimension-1 weak solution of (4) with $\theta_1(t_1) = \mathrm{e}^{-c}$.

(b) if the solution of (16) behaves like $(\eta, \zeta) \approx (c\tau, c)$ for $\tau \to \infty$ with $c = g(0) > 0$, then the solution of (11) approximates a classical solution of (1).

From above. If, at some further breaking point (we denote it again t_1) the solution of (1) enters the manifold from the opposite side, the situation is slightly different. The argument in $\dot y(\alpha_1(y(t)))$ approaches 0 from the right, so that the solution of the regularized problem (11) has a transient layer already when approaching the manifold. We again make an ansatz (14), but this time we consider positive and negative values of τ. Since $z_0(t)$ and $y_1(t)$ have jump discontinuities at $t = t^*$, the same will happen for $\tilde\zeta(\tau)$ and $\tilde\eta(\tau)$ at $\tau = 0$. We also introduce functions $\eta(\tau)$ and $\zeta(\tau)$ as in (15), where the left and right limits have to be taken for $z_0(t^*)$ and $y_1(t^*)$, depending on whether $\tau < 0$ or $\tau > 0$. These functions are smooth (also for $\tau = 0$) and they satisfy the differential equation (16).

Until now, we have not yet defined t^*, y^*, and we have not yet fixed initial values for (16). If the solution of (11) crosses transversally the manifold, there exists a $t_1(\varepsilon)$ close to t_1 such that $\alpha_1(y(t_1(\varepsilon))) = 0$. We then put $t^* = t_1(\varepsilon)$ and $y^* = y_0(t^*)$. It may also happen that the solution of (11) stays away from the manifold in a $\mathcal{O}(1)$-neighborhood of t_1. In this case we put $t^* = t_1$ and $y^* = y_0(t^*)$.

In the "from below" case we have seen that the solution of (16) behaves like $(\eta, \zeta) \approx (c\tau, c)$ for $\tau \to \infty$ with $c = g(0)$, if a classical solution is approximated by (11). Reversing time, the same reasoning of [7] shows that in the "from above" case the solution of (16) behaves like

$$\eta(\tau) \approx c\tau, \quad \zeta(\tau) = c \qquad \text{for} \quad \tau \to -\infty, \tag{17}$$

where $c = g(0) < 0$. This condition uniquely determines the solution of (16) up to a time shift. If t^* is such that $\alpha_1(y(t^*)) = 0$, then this shift is fixed by the condition $\eta(0) = 0$. This time shift does not influence the qualitative behavior of the solution. Similar to the previoius case exactly one of the following two situations occurs:

(a) if the solution of (16) converges to a stationary point $(\eta, \zeta) = (c, 0)$ (that is $g(\mathrm{e}^{-c}) = 0$), then the solution (14) of (11) approximates a codimension-1 weak solution of (4) with $\theta_1(t_1) = \mathrm{e}^{-c}$.
(b) if the solution of (16) behaves like $(\eta, \zeta) \approx (c\tau, c)$ for $\tau \to \infty$ with $c = g(1) < 0$, then the solution of (11) approximates a classical solution of (1) in the region $\{y\,;\, \alpha_1(y) < 0\}$.

Let us remark that in the "from below" situation the initial values of the dynamical system (16) could be replaced by (17) with $c = g(1) > 0$. Since $g(\theta) = g(1)$ for $\theta \ge 1$, these initial values and the choice of the time shift imply that $\zeta(\tau) = g(1)$ for $\tau \le 0$ and $\eta(0) = 0$. Consequently, the initial values (17) can be used in both situations.

3.2 Approximation of weak solutions in codimension two

At some other breaking point of (1), say t_2, it may happen that the generalized solution $(y_0(t), z_0(t))$ of (1) enters (transversally) a codimension-2 manifold at t_2, so that simultaneously $\alpha_1(y_0(t_2)) = 0$ and $\alpha_2(y_0(t_2)) = 0$.

A special situation. Assume for the moment that we are in a sliding mode in $\{y\,;\, \alpha_2(y) = 0,\, \alpha_1(y) < 0\}$ just before t_2, and that there exists a breaking point $t_2(\varepsilon)$ of (11), for which $\alpha_1(y(t_2(\varepsilon))) = 0$. From the analysis over the interval $[t_1(\varepsilon), t_2(\varepsilon)]$ (Section 3.1), we obtain with $t^* = t_2(\varepsilon)$ that $y(t^*) = y^* + \varepsilon v^* + \mathcal{O}(\varepsilon^2)$ and $z(t^*) = z^* + \mathcal{O}(\varepsilon)$ with ε-independent vectors y^*, v^*, and z^*. They satisfy

$$\alpha_1(y^*) = 0, \qquad \alpha_2(y^*) = 0, \qquad \alpha_1'(y^*)\, v^* = 0. \tag{18}$$

Due to the discontinuity at t_2 of the derivative of the solution of (1), the solution of (11) will have a transient layer also at this breaking point. Similar as in the previous section we consider the ansatz, for $t \ge t_2(\varepsilon)$ and $\tau = (t - t_2(\varepsilon))/\varepsilon$,

$$\begin{aligned} y(t) &= y_0(t) + \varepsilon y_1(t) + \varepsilon\, \tilde\eta(\tau) + \mathcal{O}(\varepsilon^2) \\ z(t) &= z_0(t) + \tilde\zeta(\tau) + \mathcal{O}(\varepsilon), \end{aligned} \tag{19}$$

where $y_0(t)$ is a solution of (1) in the sense of Definition 1, $z_0(t) = \dot y_0(t)$, and $\tilde\eta(\tau)$, $\tilde\zeta(\tau)$ are functions that (ideally) converge exponentially fast to zero for $\tau \to \infty$. To achieve uniqueness of the expansion we assume that $y_j(t) = \tilde y_j(t - t^*)$, $z_0(t) = \tilde z_0(t - t^*)$ with ε-independent functions $\tilde y_j(s)$ and $\tilde z_0(s)$. By continuity at t^*, the coefficient functions of (19) must satisfy $y_0(t^*) = y^*$, $y_1(t^*) + \tilde\eta_0(0) = v^*$, $z_0(t^*) + \tilde\zeta_0(0) = z^*$.

The smooth functions $y_0(t)$ and $z_0(t)$ satisfy the problem (1) for $t \ge t_2(\varepsilon)$. As discussed in Section 2, these functions need not be unique and it is of interest to study which solution is approximated by the regularization (11). An analysis, identical to that of Section 3.1, motivates to consider the scalar functions

$$\begin{aligned} \eta_j(\tau) &= \alpha_j'(y^*)(z_0(t^*)\,\tau + y_1(t^*) + \tilde\eta(\tau)), \\ \zeta_j(\tau) &= \alpha_j'(y^*)(z_0(t^*) + \tilde\zeta(\tau)). \end{aligned} \tag{20}$$

To obtain a small defect, when (19) is inserted into (11), these functions have to satisfy the four-dimensional dynamical system

$$\begin{aligned} \eta_1' &= \zeta_1, & \eta_1(0) &= \alpha_1'(y^*)\, v^*, \\ \eta_2' &= \zeta_2, & \eta_2(0) &= \alpha_2'(y^*)\, v^*, \\ \zeta_1' &= -\,\zeta_1 + g_1(\mathrm{e}^{-\eta_1}, \mathrm{e}^{-\eta_2}), & \zeta_1(0) &= \alpha_1'(y^*)\, z^*, \\ \zeta_2' &= -\,\zeta_2 + g_2(\mathrm{e}^{-\eta_1}, \mathrm{e}^{-\eta_2}), & \zeta_2(0) &= \alpha_2'(y^*)\, z^*, \end{aligned} \tag{21}$$

where, on the square $0 \le \theta_1, \theta_2 \le 1$,

$$g_j(\theta_1, \theta_2) = \alpha_j'(y^*) f\Big(y^*, \dot y_0^+ + \theta_1(\dot y_0^- - \dot y_0^+), \dot y_0^+ + \theta_2(\dot y_0^- - \dot y_0^+)\Big),$$

and $g_j(\theta_1, \theta_2) = g_j(\min(1, \theta_1), \min(1, \theta_2))$ for other values of $\theta_j \ge 0$. We have encountered these functions already in Section 2, where the different kinds of solutions of (1) have been discussed. The extension outside the unit square is due to the fact that only for $0 < \theta_j < 1$ the multi-valued expression $\dot y(0)$ has to be replaced by a convex combination of the right- and left-side derivatives, and for $\theta_j > 1$ (i.e., $\eta_j < 0$ and $\alpha_j(y(t)) < 0$) we are in the classical regime, where the left-side derivative has to be considered.

Since the breaking point is given by $\alpha_1(y(t_2(\varepsilon))) = 0$, it follows from (18) that $\eta_1(0) = 0$. Our assumption that we enter through a sliding in $\{y\,;\, \alpha_2(y) = 0\}$ implies that $\eta_2(0) > 0$ and that $\theta_2^* = \mathrm{e}^{-\eta_2(0)}$ is the value $\theta_2(t^*)$ of the solution of (5). We have $g_2(1, \theta_2^*) = 0$. The initial values for the ζ components are $\zeta_j(0) = g_j(1, \theta_2^*)$.

To analyze the behavior of the solutions of (21) it is convenient to introduce the functions $\theta_j(\tau) = \mathrm{e}^{-\eta_j(\tau)}$, which are closely connected to the variables of the differential-algebraic equations of Section 2. The system (21) thus becomes

$$\begin{aligned} \theta_1' &= -\theta_1\, \zeta_1, & \theta_1(0) &= \theta_1^*, \\ \theta_2' &= -\theta_2\, \zeta_2, & \theta_2(0) &= \theta_2^*, \\ \zeta_1' &= -\,\zeta_1 + g_1(\theta_1, \theta_2), & \zeta_1(0) &= g_1(\theta_1^*, \theta_2^*), \\ \zeta_2' &= -\,\zeta_2 + g_2(\theta_1, \theta_2), & \zeta_2(0) &= g_2(\theta_1^*, \theta_2^*), \end{aligned} \tag{22}$$

where $\theta_j^* = \mathrm{e}^{-\eta_j^*}$ with $\eta_j^* = \alpha_j'(y^*)\, v^*$.

The general situation. If the solution enters the codimension-2 manifold through the sliding in $\{y\,;\, \alpha_1 = 0,\, \alpha_2 < 0\}$, we have $\theta_2^* = 1$ and θ_1^* is some value between 0 and 1. In the exceptional case, where it enters as classical solution in $\{y\,;\, \alpha_1 < 0,\, \alpha_2 < 0\}$ we have $\theta_1^* = \theta_2^* = 1$, and the system (22) with initial values for $\tau = 0$ is still relevant.

Before we consider further situations we notice that, for the special situation discussed before, the initial values in (22) can also be replaced by

$$\theta_1(\tau) \approx \mathrm{e}^{-c_1\tau}, \quad \theta_2(\tau) = \theta_2^*, \quad \zeta_1(\tau) = c_1, \quad \zeta_2(\tau) = 0 \quad \text{for} \quad \tau \to -\infty,$$

where $c_1 = g_1(1, \theta_2^*)$. Indeed, with a correct definition of the time shift, we have $\zeta_2(\tau) = 0$, $\theta_2(\tau) = \theta_2^*$ (because of $g_2(1, \theta_2^*) = 0$), and $\zeta_1(\tau) = g_1(1, \theta_2^*)$ for all $\tau \le 0$, and also $\theta_1(0) = 1$.

From the discussion of the "from above" case in Section 3.1 it is clear that in the general case we have to consider initial values at $-\infty$. They are

$$\begin{array}{ll} \theta_j(\tau) = \theta_j^* & \text{if sliding along } \alpha_j(y) = 0 \\ \theta_j(\tau) \approx \mathrm{e}^{-c_j\tau} & \text{if no sliding along } \alpha_j(y) = 0 \end{array} \qquad \text{for} \quad \tau \to -\infty.$$

In the first case the values of θ_1^* and θ_2^* are obtained from the solution at t_2 of the differential-algebraic system (4) and (5), respectively. For the second case we put $\theta_j^* = 0$ if the solution enters from above ($\alpha_j(y) > 0$) and $\theta_j^* = 1$ if it enters from below ($\alpha_j(y) < 0$). The value of c_j is $c_j = g_j(\theta_1^*, \theta_2^*)$

We notice that the pair (θ_1^*, θ_2^*) lies on the border of the unit square. It informs us from where the solution $(y_0(t), z_0(t))$ of (1) enters the codimension-2 manifold at $t = t_2$. In the exceptional case, where (θ_1^*, θ_2^*) is a corner of the square, it enters as a classical solution, otherwise through a codimension-1 sliding.

Properties of the flow of (22)

(F1) The solution stays in the region $\theta_1 > 0, \theta_2 > 0$ for all $\tau \ge 0$.

(F2) Above the surface $\zeta_1 = g_1(\theta_1, \theta_2)$ the flow is directed downwards, i.e., $\zeta_1(\tau)$ is monotonically decreasing; below this surface it is directed upwards; an analogous property holds true for $\zeta_2(\tau)$. This implies that the functions $\zeta_1(\tau)$ and $\zeta_2(\tau)$ are bounded as follows:

$$\min_{0 \le \theta_1, \theta_2 \le 1} g_j(\theta_1, \theta_2) \le \zeta_j(\tau) \le \max_{0 \le \theta_1, \theta_2 \le 1} g_j(\theta_1, \theta_2).$$

(F3) In the region $\zeta_1 > 0$ the solution component $\theta_1(\tau)$ is monotonically decreasing; it is monotonically increasing in the region $\zeta_1 < 0$; an analogous property holds for $\theta_2(\tau)$. This, however, does not imply the boundedness of the functions $\theta_1(\tau)$ and $\theta_2(\tau)$.

(F4) For large values of θ_j we consider the transformation $\nu_j = 1/\theta_j$. The differential equation for θ_j in (22) thus turns into $\nu_j' = \nu_j \zeta_j$. For example, in the region $\theta_2 > 1$, where $g_j(\theta_1, \theta_2) = g_j(\theta_1, 1)$, the differential equation (22) becomes

$$\begin{aligned} \theta_1' &= -\theta_1\, \zeta_1, \qquad & \zeta_1' &= -\,\zeta_1 + g_1(\theta_1, 1), \\ \nu_2' &= \nu_2\, \zeta_2, \qquad & \zeta_2' &= -\,\zeta_2 + g_2(\theta_1, 1). \end{aligned} \tag{23}$$

We say that $(\theta_1, \theta_2 = \infty, \zeta_1 = g_1(\theta_1, 1), \zeta_2 = g_2(\theta_1, 1))$ is a stationary point of (22), if $(\theta_1, \nu_2 = 0, \zeta_1 = g_1(\theta_1, 1), \zeta_2 = g_2(\theta_1, 1))$ is a stationary point of (23). A similar modification has to be done in the region $\theta_1 > 1$ and in the intersection of both regions.

Stationary points of (22)

(A1) $\theta_1 = 0$, $\theta_2 = 0$, $\zeta_1 = g_1(0, 0)$, $\zeta_2 = g_2(0, 0)$,
(A2) $\theta_1 = 0$, $\theta_2 = \infty$, $\zeta_1 = g_1(0, 1)$, $\zeta_2 = g_2(0, 1)$,
(A3) $\theta_1 = \infty$, $\theta_2 = 0$, $\zeta_1 = g_1(1, 0)$, $\zeta_2 = g_2(1, 0)$,
(A4) $\theta_1 = \infty$, $\theta_2 = \infty$, $\zeta_1 = g_1(1, 1)$, $\zeta_2 = g_2(1, 1)$,
(B1) $\theta_2 = 0$, $\zeta_1 = 0$, θ_1 satisfies $g_1(\theta_1, 0) = 0$, and $\zeta_2 = g_2(\theta_1, 0)$,
(B2) $\theta_1 = 0$, $\zeta_2 = 0$, θ_2 satisfies $g_2(0, \theta_2) = 0$, and $\zeta_1 = g_1(0, \theta_2)$,
(B3) $\theta_2 = \infty$, $\zeta_1 = 0$, θ_1 satisfies $g_1(\theta_1, 1) = 0$, and $\zeta_2 = g_2(\theta_1, 1)$,
(B4) $\theta_1 = \infty$, $\zeta_2 = 0$, θ_2 satisfies $g_2(1, \theta_2) = 0$, and $\zeta_1 = g_1(1, \theta_2)$,
(C1) $\zeta_1 = 0$, $\zeta_2 = 0$, θ_1, θ_2 satisfy $g_1(\theta_1, \theta_2) = 0$, $g_2(\theta_1, \theta_2) = 0$.

The following theorem shows how the solution of the 4-dimensional dynamical system (22) characterizes which generalized solution of (1) is approximated by the solution of (11) when $\varepsilon \to 0$. We still consider only generic situations as defined in the footnote of Section 2.

Theorem 1 *If the solution of (22) converges to one of the stationary points* (A1)-(A4)*, then the regularized solution of (11) approximates the corresponding classical solution among* (a1)-(a4) *of Section 2.*

If it converges to one of the stationary points (B1)-(B4)*, then the regularized solution of (11) approximates the corresponding codimension-*1 *weak solution among* (b1)-(b4) *of Section 2.*

If is converges to the stationary point (C1)*, then the regularized solution of (11) approximates a codimension-*2 *weak solution* (c1)*.*

Proof Consider the solution of (11) for $t \geq t_2(\varepsilon)$, represented by the truncated asymptotic expansion (19). After a transient phase of length $\mathcal{O}(\varepsilon)$ the solution of the dynamical system (22) will be close to a stationary point. We give here a formal discussion for the different cases. Rigorous error estimates can be obtained in the same way as for the codimension-1 situation, which has been treated in detail in the article [7].

For the case (A2) we have $\theta_1(\tau) \to 0$ and $\theta_2(\tau) \to +\infty$ for $\tau \to \infty$, so that $\eta_1(\tau) \to +\infty$ and $\eta_2(\tau) \to -\infty$. Since the function $\tilde{\eta}(\tau)$ of (19) is assumed to converge to zero, it follows from (20) that $\alpha_1'(y^*)\, z_0(t^*) > 0$ and $\alpha_2'(y^*)\, z_0(t^*) < 0$. Because of $z_0(t) = \dot{y}_0(t)$, this implies that the solution $y_0(t)$ of (1) enters the region $\mathcal{R}_{+-} = \{y\,;\, \alpha_1(y) > 0,\, \alpha_2(y) < 0\}$, and the solution of (11) approximates the classical solution of type (a2), see Section 2. The cases (A1), (A3), and (A4) are treated in the same way.

For the case (B1) we have $\theta_2(\tau) \to 0$ and $\theta_1(\tau) \to \mathrm{e}^{-c} < 1$, so that $\eta_2(\tau) \to +\infty$ and $\eta_1(\tau) \to c > 0$. This time, it follows from (20) that $\alpha_2'(y^*)\, z_0(t^*) > 0$ and $\alpha_1'(y^*)\, z_0(t^*) = 0$. Moreover, the relation $g_1(\mathrm{e}^{-c}, 0) = 0$ implies that $y_0(t)$ is a solution of (4) with $\theta_1(t^*) = \mathrm{e}^{-c}$. Consequently, the solution of (11) approximates a codimension-1 weak solution of type (b1). The cases (B2)-(B4) and (C1) are treated similarly. □

3.3 Extension to weak solutions in manifolds of higher codimension

All that is said until now permits a straight-forward extension to the situation of more than two delayed arguments, which then can lead to weak solutions in manifolds of codimension higher than 2. The definition of weak solutions is again via systems of differental-algebraic delay equations.

For the study of the solution of the regularized problem (11), extended to the case $m > 2$, we again use the technique of asymptotic expansions. If at some breaking point t_r the solution enters transversally a codimension-r manifold (with $r \geq 3$), then the above analysis will lead to a $2r$-dimensional dynamical system for the variables $\theta_j, \zeta_j, j = 1, \ldots, r$. Its stationary points can readily be computed, and an extension of the statement of Theorem 1 is still true. The study of asymptotic stability of the stationary solutions is more involved when r becomes large. The following investigation is therefore restricted to the case $r \leq 2$.

3.4 Stability analysis of stationary points of (22)

The Jacobian matrix of the vector field in (22) is given by

$$\begin{pmatrix} -\zeta_1 & 0 & -\theta_1 & 0 \\ 0 & -\zeta_2 & 0 & -\theta_2 \\ \partial_1 g_1(\theta_1,\theta_2) & \partial_2 g_1(\theta_1,\theta_2) & -1 & 0 \\ \partial_1 g_2(\theta_1,\theta_2) & \partial_2 g_2(\theta_1,\theta_2) & 0 & -1 \end{pmatrix}. \tag{24}$$

If the equation $\theta_j' = -\theta_j \zeta_j$ is replaced by $\nu_j' = \nu_j \zeta_j$ for an investigation close to $\theta_j = \infty$ (see (F4)), then the corresponding row has to be modified: $-\zeta_j$ has to be replaced by ζ_j, and $-\theta_j$ by ν_j. For example, in the region $\theta_2 > 1$, where instead of (22) the transformed differential equation (23) has to be considered, the Jacobian matrix is

$$\begin{pmatrix} -\zeta_1 & 0 & -\theta_1 & 0 \\ 0 & \zeta_2 & 0 & \nu_2 \\ \partial_1 g_1(\theta_1, 1) & 0 & -1 & 0 \\ \partial_1 g_2(\theta_1, 1) & 0 & 0 & -1 \end{pmatrix}. \tag{25}$$

A similar modification is necessary in the region $\theta_1 > 1$ and in the intersection of both regions.

Asymptotic stability of stationary points (A1)-(C1) *of (22).* The eigenvalues of the Jacobian are

(A1) $-1,\ -1,\ -g_1(0,0),\ -g_2(0,0)$;
they are negative if $g_1(0,0) > 0$ and $g_2(0,0) > 0$;
(A2) $-1,\ -1,\ -g_1(0,1),\ g_2(0,1)$;
they are negative if $g_1(0,1) > 0$ and $g_2(0,1) < 0$;
(A3) $-1,\ -1,\ g_1(1,0),\ -g_2(1,0)$;
they are negative if $g_1(1,0) < 0$ and $g_2(1,0) > 0$;
(A4) $-1,\ -1,\ g_1(1,1),\ g_2(1,1)$;
they are negative if $g_1(1,1) < 0$ and $g_2(1,1) < 0$;
(B1) $-1, -g_2(\theta_1, 0)$, and the roots of the equation $\lambda^2+\lambda+\theta_1\, \partial_1 g_1(\theta_1, 0)$;
they have negative real part if $g_2(\theta_1, 0) > 0$ and $\partial_1 g_1(\theta_1, 0) > 0$;
(B2) $-1, -g_1(0, \theta_2)$, and the roots of the equation $\lambda^2+\lambda+\theta_2\, \partial_2 g_2(0, \theta_2)$;
they have negative real part if $g_1(0, \theta_2) > 0$ and $\partial_2 g_2(0, \theta_2) > 0$;
(B3) $-1, g_2(\theta_1, 1)$, and the roots of the equation $\lambda^2+\lambda+\theta_1\, \partial_1 g_1(\theta_1, 1)$;
they have negative real part if $g_2(\theta_1, 1) < 0$ and $\partial_1 g_1(\theta_1, 1) > 0$.
(B4) $-1, g_1(1, \theta_2)$, and the roots of the equation $\lambda^2+\lambda+\theta_2\, \partial_2 g_2(1, \theta_2)$;
they have negative real part if $g_1(1, \theta_2) < 0$ and $\partial_2 g_2(1, \theta_2) > 0$;
(C1) With the matrices

$$\Theta = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \qquad G = \begin{pmatrix} \partial_1 g_1(\theta_1, \theta_2) & \partial_2 g_1(\theta_1, \theta_2) \\ \partial_1 g_2(\theta_1, \theta_2) & \partial_2 g_2(\theta_1, \theta_2) \end{pmatrix} \tag{26}$$

the characteristic equation for the matrix (24) becomes

$$\det(\Theta\, G - \mu\, I) = 0, \qquad \mu = -\lambda(1+\lambda). \tag{27}$$

This shows that the eigenvalues λ of (24) are in the negative half-plane if and only if the eigenvalues μ of $\Theta\, G$ lie in the parabolic region

$$\{\mu\, ;\, -\Re\mu + (\Im\mu)^2 < 0\}. \tag{28}$$

This condition is satisfied if and only if

$$\begin{aligned} &\det(\Theta G) > 0, \qquad \mathrm{trace}(\Theta G) > 0, \\ &\det(\Theta G) < \frac{\mathrm{trace}(\Theta G)}{2}\Bigl(1 + \frac{\mathrm{trace}(\Theta G)}{2}\Bigr). \end{aligned} \tag{29}$$

Even in the linear case, where $f(y, z_1, z_2) = a(y) + A_1(y)z_1 + A_2(y)z_2$, it may happen that all stationary points are unstable. This is undesirable for numerical reasons, because the solution $y(t)$ of the regularized problem will have high frequency oscillations of size $\mathcal{O}(\varepsilon)$.

However, we shall prove that in the case, where all stationary points (A1)-(A4) and (B1)-(B4) are unstable, we have $\det(\Theta G) > 0$, and we give sufficient conditions that also $\mathrm{trace}(\Theta G) > 0$. In Section 4 below we shall present a stabilization technique for the case that only the last condition of (29) is violated.

Proposition 1 *Consider the situation, where a unique stationary point of type* (C1) *exists, and where* $g(\theta_1, \theta_2)$ *is an affine map on the square* $0 \le \theta_1, \theta_2 \le 1$. *Assume that all equilibria of types* (A1)-(A4) *and* (B1)-(B4) *are unstable solutions of the system (22). Then, the matrix* G *of (26) satisfies* $\det G > 0$.

The manifolds, given by $\alpha_1(y) = 0$ and $\alpha_2(y) = 0$, divide the space into four orthants. We denote $\mathcal{R}_{++} = \{y\,;\, \alpha_1(y) > 0,\, \alpha_2(y) > 0\}$, and simlarly $\mathcal{R}_{-+}, \mathcal{R}_{--}, \mathcal{R}_{+-}$ for other choices of the signs.

Proposition 2 *Let the assumptions of Proposition 1 hold. Assume further that there exist two neighboring orthants such that in each of them there is a solution of (1), which enters the codimension-*2 *manifold without any codimension-*1 *sliding phase. Then, the matrix* G *of (26) has positive diagonal elements and we have* $\mathrm{trace}(\Theta G) > 0$.

The proofs of these two propositions, which are technical, are postponed to Section 5.

Example 2 Consider the system of neutral delay equations

$$\begin{aligned} \dot y_1(t) &= c_1 + a_{11}\,\dot y_1(y_1(t) - 1) + a_{12}\,\dot y_2(y_2(t) - 1), \\ \dot y_2(t) &= c_2 + a_{21}\,\dot y_1(y_1(t) - 1) + a_{22}\,\dot y_2(y_2(t) - 1), \end{aligned} \tag{30}$$

which is the same as (8) with $c_3 = a_{31} = a_{32} = 0$, and $r_1 = r_2 = 1$. We assume constant initial functions $y_1(t) = \varphi_1$, $y_2(t) = \varphi_2$ for $t \le 0$. For the parameters we choose the three sets of values, given in Table 1. The solution is $y_1(t) = \varphi_1 + c_1\,t$, $y_2(t) = \varphi_2 + c_2\,t$ until the first breaking point, which is at $t_1 = (1 - \varphi_2)/c_2$ for the chosen

problem	c_1	c_2	a_{11}	a_{12}	a_{21}	a_{22}	φ_1	φ_2	θ_1	θ_2
(P1)	1/2	1/2	−4	2	−2	−4	0	1/2	7/10	9/10
(P2)	2	2	−4	2	−2	−4	0	1/2	7/10	9/10
(P3)	2	1/4	1	−24	1/4	−4	0	9/10	1/2	1/2

Table 1. Parameters for the delay equation of Example 2.

parameters. After this point we have a codimension-1 sliding in the manifold $\{y\,;\,y_2=1\}$ until the second breaking point, which is at $t_2=t_1+(1-y_1(t_1))/\gamma$ with $\gamma=(c_1a_{22}-c_2a_{12})/a_{22}$. We then have a codimenion-2 sliding in the manifold $\{y\,;\,y_1=1,\,y_2=1\}$ for all three cases.

For the study of the regularization we compute the functions

$$g_1(\theta_1,\theta_2)=c_1+a_{11}c_1(1-\theta_1)+a_{12}c_2(1-\theta_2),$$
$$g_2(\theta_1,\theta_2)=c_2+a_{21}c_1(1-\theta_1)+a_{22}c_2(1-\theta_2).$$

One can check that for all three problems none of the stationary points of cases (A1)-(A4) and (B1)-(B4) in the previous stability analysis is stable. The stationary point (C1) is obtained as the solution of $g_1(\theta_1,\theta_2)=g_2(\theta_1,\theta_2)=0$. It is given by

$$\theta_1=1+\frac{c_2(c_1a_{22}-c_2a_{12})}{\det G},\qquad \theta_2=1+\frac{c_1(c_2a_{11}-c_1a_{21})}{\det G},\qquad G=-\begin{pmatrix}a_{11}c_1 & a_{12}c_2\\ a_{21}c_1 & a_{22}c_2\end{pmatrix},$$

and it is indicated as a small circle in Figure 2. The initial value for the system (22) is indicated as a black dot (it corresponds to the values $\theta_1^*=1$ and $\theta_2^*=0.75$).

For problem (P1) the eigenvalues of the matrix ΘG are $\mu_{12}=(16\pm \mathrm{i}\sqrt{59})/10$. They lie in the parabolic region (28) which implies that the stationary point (C1) of (22) is asymptotically stable. This corresponds to the left pictures in Figure 2, where the projection of the solution of (22) onto the (η_1,ζ_1) and (η_2,ζ_2) planes (note that $\theta_j=\mathrm{e}^{-\eta_j}$) are drawn.

For problem (P2) the eigenvalues of ΘG are $\mu_{12}=2(16\pm \mathrm{i}\sqrt{59})/5$. We have $\det(\Theta G)>0$ and $\mathrm{trace}(\Theta G)>0$, but the eigenvalues lie outside the parabolic region (28). The stationary point (C1) is thus unstable, which can be observed in the middle pictures of Figure 2.

For problem (P3) we have $\det(\Theta G)>0$ (by Proposition 1), but for this choice of parameters $\mathrm{trace}(\Theta G)<0$. The stationary point (C1) is unstable as can be seen in the right pictures of Figure 2.

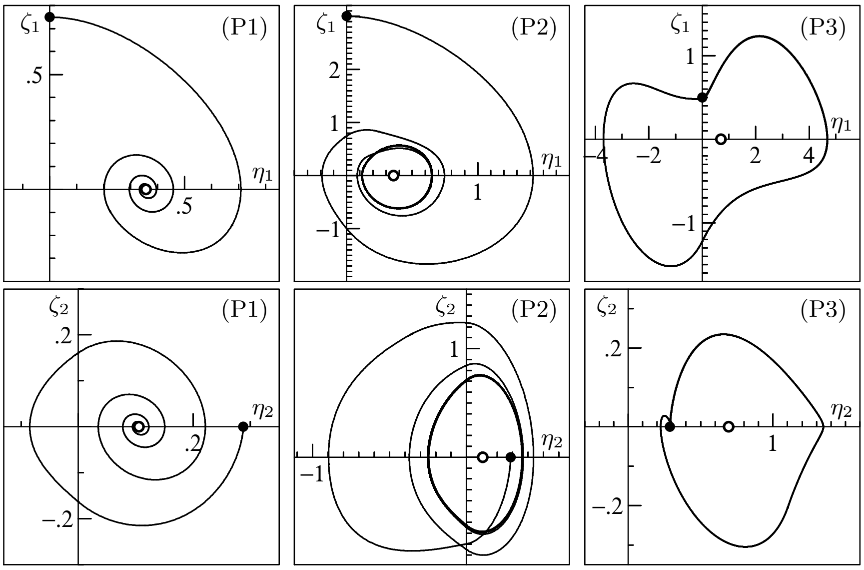

Fig. 2. Solution of (22) for problems (P1), (P2), (P3) of Example 2

Since the solution of the regularized problem beyond the second breaking point $t_2(\varepsilon) \approx t_2 + \mathcal{O}(\varepsilon)$ satisfies (19), the above figures give a good impression of the solution. In particular, for problem (P1) the solution has two additional breaking points (where the function $\eta_2(\tau)$ crosses the vertical axis) in the left lower picture of Figure 2, then it converges rapidly to a stationary solution. For problems (P2) and (P3) the behavior is completely different. The solution oscillates around an unstable stationary point, and it has many breaking points for which the distance between two of them is of size $\mathcal{O}(\varepsilon)$.

Example 3 We consider again the neutral delay equation (30), but this time with parameters $\varphi_1 = 0$, $\varphi_2 = 1/2$, $c_1 = 1/2$, $c_2 = 1/2$, $a_{12} = -2$, $a_{21} = 3$, $a_{22} = -3$, and a_{11} is for the moment not specified. As in Example 2 the solution enters the codimension-2 manifold through a sliding along $\mathcal{M}_2 = \{y\,;\,\alpha_2(y) = 0\}$ (see right picture of Figure 3). The interest of this example is that for certain parameters the solution then bifurcates into three solutions. We have

– a classical solution of type (a1) if $a_{11} > 1$,
– a codimension-1 weak solution of type (b1) if $1 < a_{11} < 3/2$,
– a codimension-2 weak solution (c1) if $a_{11} < 3/2$.

The dynamical system (22) determines which of these solutions is approximated by the regularized equation (11). The stationary point, which corresponds to (a1), is asymptotically stable. The one corresponding to (b1) is unstable because $\partial_1 g_1(\theta_1, 0) < 0$ for $a_{11} > 0$. For

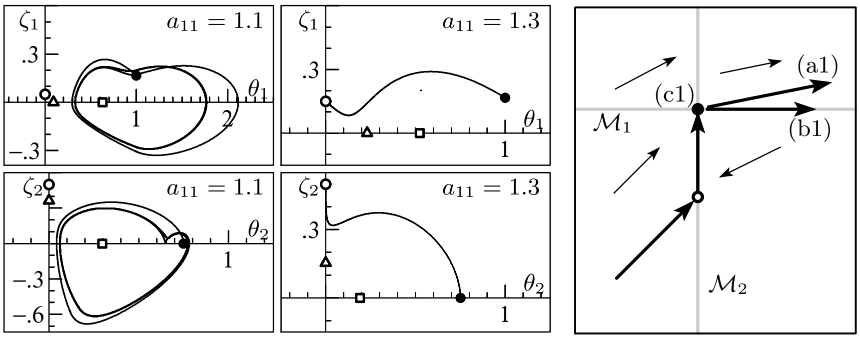

Fig. 3. Problem of Example 3: solution of (22) (left and middle), solutions in the (y_1, y_2)-space together with the vector fields in the four orthants (right).

the stationary point corresponding to (c1) we have $\det(\Theta G) > 0$ for $a_{11} < 2$ and $\mathrm{trace}(\Theta G) > 0$ for $a_{11} < (11 - \sqrt{13})/6 \approx 1.23$. The third condition of (29) is not satisfied for $a_{11} \in (1, 3/2)$. In Figure 3 the stationary points for (a1), (b1), and (c1) are represented by a circle, a triangle, and a square, respectively.

One could expect that the solution of (11) approximates the classical solution (a1). However, it turns out that for values of a_{11} that are close to 1 (see the left pictures of Figure 3) the solution of (22) approaches a limit cycle turning around the stationary point for (c1). This implies that the solution of (11) oscillates around the codimension-2 weak solution with a period of size $\mathcal{O}(\varepsilon)$. Only for values of a_{11} close to 1.5 (see the middle pictures of Figure 3), the solution of (22) converges to the stationary point corresponding to (a1), so that the classical solution is approximated by (11).

4 Stabilizing regularization

For problem (P1) of Example 2, where the transient components are rapidly damped out, we are concerned with a smooth solution soon after the first breaking point. The regularized problem (11) can therefore be solved very efficiently with stiff integrators such as Radar5 (see [5,6]).

However, for problem (P2), the solution is highly oscillatory (frequency proportional to ε^{-1}) with amplitude of size $\mathcal{O}(\varepsilon)$ for the y-component and of size $\mathcal{O}(1)$ for the z-component. It has many breaking points in distances of the order of ε. Every numerical integrator will take step sizes proportional to ε, and it is extremely expensive to obtain an accurate approximation of (1) with the help of (11). Taking a smaller ε does not improve the situation, because the solution of the system (22) is independent of ε. As a remedy we propose the following algorithm, if there are undamped oscillations after a breaking point $t_2(\varepsilon)$ of (11).

Algorithm 1 (stabilizing regularization) *Integrate the regularized system (11) with a fixed small $\varepsilon > 0$ until the breaking point $t_2(\varepsilon)$, and then replace ε with $\kappa\varepsilon$, where $\kappa < 1$ is suitably chosen.*

The solution, obtained with this algorithm, can also be written in the form (19), and the same analysis can be performed as in Section 3.2. Here, the transient functions $\tilde\eta(\tau)$ and $\tilde\zeta(\tau)$ are obtained from the dynamical system, where the only difference to (21) is that the factor κ has to be added in front of ζ_1' and ζ_2'. With the variables $\theta_j = \mathrm{e}^{-\eta_j}$ this system becomes

$$\begin{aligned} \theta_1' &= -\theta_1\,\zeta_1, & \theta_1(0) &= \theta_1^*,\\ \theta_2' &= -\theta_2\,\zeta_2, & \theta_2(0) &= \theta_2^*,\\ \kappa\,\zeta_1' &= -\,\zeta_1 + g_1(\theta_1,\theta_2), & \zeta_1(0) &= g_1(\theta_1^*,\theta_2^*),\\ \kappa\,\zeta_2' &= -\,\zeta_2 + g_2(\theta_1,\theta_2), & \zeta_2(0) &= g_2(\theta_1^*,\theta_2^*), \end{aligned} \tag{31}$$

with the same initial values as in (22). Stationary points are not changed. For a stability analysis we have to consider the matrix, where the 3rd and 4th rows of (24) are divided by κ. This leads to a characteristic equation of the form (27) with μ replaced by $\mu = -\lambda(1+\kappa\lambda)$. Consequently, the parabilic region (28) is enlarged to

$$\{\mu\,;\, -\Re\mu + \kappa\,(\Im\mu)^2 < 0\}, \tag{32}$$

and the stationary point (C1) is asymptotically stable if the eigenvalues μ of ΘG are in (32). We thus have proved the following result.

Proposition 3 *Assume that the problem (1) admits a stationary point* (C1)*, for which the first two conditions of (29) are satisfied, but the third one is not. Then there exists $0 < \kappa^* < 1$ such that for all $0 < \kappa < \kappa^*$ the stationary point* (C1) *is an asymptotically stable solution of (31).* □

In the situation of problem (P2) of Example 2, we put $\kappa = 1/3$. The solution of the system (31) is plotted in Figure 4. In contrast to the situation in Figure 2 (look at the pictures corresponding to (P2)), the solution converges rapidly to the stationary point (C1), which implies a fast damping of transient components and an immense reduction of breaking points. Figure 5 shows the solution after the breaking point $t_2(\varepsilon)$. The left picture is without stabilization. We have high oscillations and many breaking points, which is in perfect

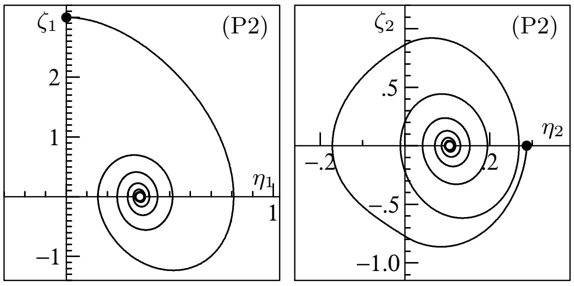

Fig. 4. Solution of the system (31) for problem (P2) of Example 2 with $\kappa = 1/3$.

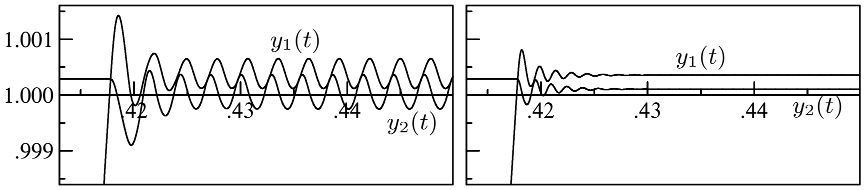

Fig. 5. Solution of the regularization of problem (P2) of Example 2 on the interval $0.4130 \le t \le 0.4499$; regularization parameter is $\varepsilon = 10^{-3}$; without stabilization (left picture), with stabilization parameter $\kappa = 1/3$ (right picture).

agreement with the pictures (in the middle) of Figure 2. The right picture of Figure 5 shows the result of Algorithm 1 with stabilization parameter $\kappa = 1/3$. Again a perfect agreement can be seen with the prediction of Figure 4. Whereas the breaking point $t_2(\varepsilon)$ is given by $y_1(t_2(\varepsilon)) = 0$, we see that the other component $y_2(t)$ gives rise to 4 additional breaking points that are $\mathcal{O}(\varepsilon)$-close to $t_2(\varepsilon)$. This corresponds to the four time instants, where the curve in the right picture of Figure 4 crosses the imaginary axis.

Problem (P3) of Example 2 has been chosen such that all stationary points (A1)-(A4) and (B1)-(B4) are unstable, and that the condition $\mathrm{trace}(\Theta G) > 0$ of (29) for the stability of (C1) is violated. The parameter κ in Algorithm 1 does not influence this condition. Therefore, it cannot have a stabilizing effect for this problem.

5 Appendix: proof of Propositions 1 and 2

Proposition 3 is a motivation for studying situations, where the first two conditions of (29) are satisfied, so that Algorithm 1 can be successfully applied.

Proof of Proposition 1. We assume that $g_j(\theta_1, \theta_2)$ are affine functions for $j = 1, 2$, and that the two lines $g_j(\theta_1, \theta_2) = 0$ intersect at some

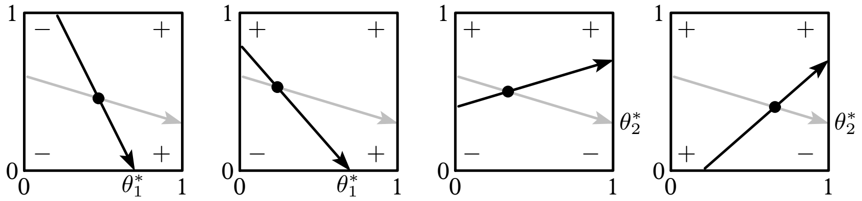

Fig. 6. Sign patterns of $g_1(\theta_1,\theta_2)$ at the corners of the unit square together with the lines $g_1(\theta_1,\theta_2)=0$ (black) and $g_2(\theta_1,\theta_2)=0$ (grey).

point $(\theta_1,\theta_2)\in(0,1)\times(0,1)$. In a graphical representation we use the convention that an arrow along the line $g_j(\theta_1,\theta_2)=0$ is such that $g_j(\theta_1,\theta_2)>0$ on the left hand side and $g_j(\theta_1,\theta_2)<0$ on the right-hand side. In Figure 6 the black arrow represents $g_1(\theta_1,\theta_2)=0$, and the grey arrow represents $g_2(\theta_1,\theta_2)=0$.

The gradient of $g_j(\theta_1,\theta_2)$ is obtained by a rotation of 90° (in the positive sense) of the arrow representing $g_j(\theta_1,\theta_2)=0$. Since the row vectors of the matrix G are the gradients of the functions $g_j(\theta_1,\theta_2)$, the condition $\det G>0$ is satisfied if and only if the arrow for $g_2(\theta_1,\theta_2)=0$ points into the region $g_1(\theta_1,\theta_2)>0$.

For the proof of Proposition 1 we consider all possible sign-patterns of $(g_1(\theta_1,\theta_2),g_2(\theta_1,\theta_2))$ at the corners of the unit square. Theoretically we have $4^4=256$ of such patterns, if we do not exclude patterns for which an intersection inside the square does not exist. This number can be drastically reduced by exploiting symmetries in the problem. We consider involutions that transform the mapping $(g_1(\theta_1,\theta_2),g_2(\theta_1,\theta_2))$ into one of the following mappings (or into compositions of them):

$$\begin{array}{ccc} -g_1(1-\theta_1,\theta_2) & g_1(\theta_1,1-\theta_2) & g_2(\theta_2,\theta_1) \\ g_2(1-\theta_1,\theta_2), & -g_2(\theta_1,1-\theta_2), & g_1(\theta_2,\theta_1). \end{array} \tag{33}$$

These involutions have the property that, whenever all equilibria of types (A1)-(A4) and (B1)-(B4) are unstable solutions of the system (22) for $(g_1(\theta_1,\theta_2),g_2(\theta_1,\theta_2))$, the same is true for the transformed mappings. Moreover, they leave the sign of $\det G$ invariant. These involutions thus introduce equivalence classes, and it is sufficient to prove the statement of Proposition 1 only for one representative out of each equivalence class.

As a consequence, the number of signs "+" for the function g_1 at the four corners of the unit square can be restricted to 2 and 3 (4 is not possible because of the existence of a stationary point (C1) and 1 is equivalent to 3 by means of the first mapping in (33)). Similarly, by the second mapping in (33)), the number of signs "+" for the function g_2 at the four corners can also be restricted to 2 and 3. The instability of the stationary point (A4) implies that either $g_1(1,1) > 0$ or $g_2(1,1) > 0$. As a consequence of the third symmetry in (33) we can further assume that $g_1(1,1) > 0$. There remain four cases to be analyzed (see Figure 6). We indicate every case by a 4-tuple **sp** of signs of $g_1(\theta_1, \theta_2)$ at the four corners $(0,0), (0,1), (1,1), (1,0)$.

sp $= (-, \pm, +, +)$: There exists $\theta_1^* \in (0,1)$ such that $g_1(\theta_1^*, 0) = 0$ (the first two cases of Figure 6). The assumed instability of the stationary point (B1) implies that $g_2(\theta_1^*, 0) < 0$. Consequently, the arrow for $g_2(\theta_1, \theta_2) = 0$ points into the region $g_1(\theta_1, \theta_2) > 0$ and we have $\det G > 0$.

sp $= (\pm;\ +;\ +;\ -)$: By the assumed instability of (A3) we have $g_2(1,0) < 0$. If the arrow for $g_2(\theta_1, \theta_2) = 0$ would point into the region $g_1(\theta_1, \theta_2) < 0$, then there exists θ_2^* such that $g_2(1, \theta_2^*) = 0$ and $g_1(1, \theta_2^*) < 0$ (third and fourth pictures of Figure 6). This contradicts the instability of (B4) and proves $\det G > 0$. □

In the above proof we did not exploit the fact that $g_j(\theta_1, \theta_2)$ are affine functions. If $g_j(\theta_1, \theta_2) \neq 0$ on the corners of the unit square, if the curves defined by $g_j(\theta_1, \theta_2) = 0$ intersect the border of the unit square at exactly two points and at most once at every edge, and if there is exactly one stationary point of type (C1) at which the matrix G is invertible, then all arguments of the above proof can be applied to the nonlinear situation and Proposition 1 still holds.

Proof of Proposition 2. The property that the diagonal elements of the matrix G are positive is invariant with respect to the transformations given by (33). It is therefore sufficient to prove the statement for the four cases considered in the proof of Proposition 1.

Let us first study under which condition a classical solution of (1) can enter the codimension-2 manifold without any codimension-1 sliding. By reversing time in the discussion of the existence of classical solutions after a breaking point in a codimension-2 manifold (see (a1)-(a4) of Section 2), we find that a solution can enter the codimension-2 manifold within an orthant (without codimension-1 sliding) only if

$$\begin{array}{lll} g_1(0,0) < 0, & g_2(0,0) < 0 & \text{for entering within } \mathcal{R}_{++}, \\ g_1(0,1) < 0, & g_2(0,1) > 0 & \text{for entering within } \mathcal{R}_{+-}, \\ g_1(1,1) > 0, & g_2(1,1) > 0 & \text{for entering within } \mathcal{R}_{--}, \\ g_1(1,0) > 0, & g_2(1,0) < 0 & \text{for entering within } \mathcal{R}_{-+}. \end{array} \tag{34}$$

We now distinguish the four situations of Figure 6.

sp $= (-, -, +, +)$: The signs of $g_1(0,0)$ and $g_1(1,0)$ imply that $\partial_1 g_1 > 0$. If condition (34) holds either for the neighboring orthants

$\mathcal{R}_{++}, \mathcal{R}_{+-}$ or for $\mathcal{R}_{--}, \mathcal{R}_{-+}$, then the sign pattern implies $\partial_2 g_2 > 0$. If it holds for $\mathcal{R}_{+-}, \mathcal{R}_{--}$, then $g_2(\theta_1^*, 1) > 0$ for θ_1^* of the proof of Proposition 1 and we also have $\partial_2 g_2 > 0$. For the remaining case $\mathcal{R}_{-+}, \mathcal{R}_{++}$ we consider the point where $g_1(\theta_1^{**}, 1) = 0$ and $g_2(\theta_1^{**}, 1) > 0$ and conclude $\partial_2 g_2 > 0$ also in this case.

sp $= (-, +, +, +)$: As before we have $\partial_1 g_1 > 0$. In this case the condition (34) is violated for $\mathcal{R}_{+-}$. From the assumed instability of (A2) we have $g_2(0,1) > 0$. If the condition (34) is satisfied for $\mathcal{R}_{-+}, \mathcal{R}_{++}$, we have $g_2(0,1) - g_2(0,0) > 0$. If it is satisfied for $\mathcal{R}_{--}, \mathcal{R}_{-+}$ have $g_2(1,1) - g_2(1,0) > 0$. We thus have $\partial_2 g_2 > 0$ in all situations.

sp $= (\pm, +, +, -)$: In these two cases the condition (34) is violated for $\mathcal{R}_{+-}$ as well as for $\mathcal{R}_{-+}$. There are no neighboring orthants satisfying (34), so that nothing has to be proved. □

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