ON CONJECTURES OF STENGER IN THE THEORY OF ORTHOGONAL POLYNOMIALS

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Dedicated to Gradimir V. Milovanović on his 70th birthday

Abstract

The conjectures in the title deal with the zeros x_j, $j = 1, 2, \ldots, n$, of an orthogonal polynomial of degree $n > 1$ relative to a nonnegative weight function w on an interval $[a, b]$ and with the respective elementary Lagrange interpolation polynomials $\ell_k^{(n)}$ of degree $n - 1$ taking on the value 1 at the zero x_k and the value 0 at all the other zeros x_j. They involve matrices of order n whose elements are integrals of $\ell_k^{(n)}$, either over the interval $[a, x_j]$ or the interval $[x_j, b]$, possibly containing w as a weight function. The claim is that all eigenvalues of these matrices lie in the open right half of the complex plane. This is proven to be true for Legendre polynomials and a special Jacobi polynomial. Ample evidence for the validity of the claim is provided for a variety of other classical, and nonclassical, weight functions when the integrals are weighted, but not necessarily otherwise. Even in the case of weighted integrals, however, the conjecture is found by computation to be false for a piecewise constant positive weight function. Connections are mentioned with the theory of collocation Runge–Kutta methods in ordinary differential equations.

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1   Introduction

Let w be a nonnegative weight function on $[a, b]$, $-\infty \leq a < b \leq \infty$, and p_n the orthonormal polynomial of degree n relative to the weight function w. Let $\{x_j\}_{j=1}^n$ be the zeros of p_n and

$$(1) \qquad \ell_k^{(n)}(x) = \prod_{\substack{1 \leq j \leq n \\ j \neq k}} \frac{x - x_j}{x_k - x_j}, \quad k = 1, 2, \ldots, n,$$

the elementary Lagrange interpolation polynomial of degree $n-1$ having the value 1 at x_k and 0 at all the other zeros x_j. The Stenger conjectures relate to the eigenvalues of matrices of order n whose elements are certain integrals involving the elementary Lagrange polynomials (1), the claim being that the real part of all eigenvalues is positive. We distinguish between the *restricted Stenger conjecture* ([8, §2.3, Remark 2.2]), in which the matrices are

$$(2) \qquad \begin{aligned} U_n &= [u_{jk}^{(n)}], \quad u_{jk}^{(n)} = \int_a^{x_j} \ell_k^{(n)}(x)\mathrm{d}x, \\ & \qquad\qquad\qquad\qquad\qquad\qquad j, k = 1, 2, \ldots, n, \\ V_n &= [v_{jk}^{(n)}], \quad v_{jk}^{(n)} = \int_{x_j}^b \ell_k^{(n)}(x)\mathrm{d}x, \end{aligned}$$

and the *extended Stenger conjecture* (called "new conjecture" in [8, §2.4]), in which the matrices are

$$(3) \qquad \begin{aligned} U_n &= [u_{jk}^{(n)}], \quad u_{jk}^{(n)} = \int_a^{x_j} \ell_k^{(n)}(x)w(x)\mathrm{d}x, \\ & \qquad\qquad\qquad\qquad\qquad\qquad j, k = 1, 2, \ldots, n, \\ V_n &= [v_{jk}^{(n)}], \quad v_{jk}^{(n)} = \int_{x_j}^b \ell_k^{(n)}(x)w(x)\mathrm{d}x, \end{aligned}$$

where w is assumed to be positive a.e. on $[a, b]$. (For the fact that this assumption is essential, see Sections 7 and 8.) Thus, in the latter conjecture the elements of U_n, V_n depend on the weight function w not only through the polynomials $\ell_k^{(n)}$, but also by virtue of w being part of the integration process. Note that, unlike for the extended conjecture, the restricted conjecture requires $[a, b]$ to be a finite interval, at least for one of the two matrices U_n, V_n.

We also note that the order in which the x_j are arranged is immaterial since a permutation of $j = \{1, 2, 3, \ldots, n\}$ implies the same permutation of $k = \{1, 2, 3, \ldots, n\}$, which amounts to a similarity transformation of U_n resp. V_n, and therefore leaves the eigenvalues unchanged.

The weight function $w(x) = 1$ on $[-1.1]$ is special in the sense that the extended conjecture is the same as the restricted one and will be simply called the *Stenger conjecture*. Its proof will be given in Section 4. In Section 2 we will prove that the eigenvalues of U_n and V_n in the restricted as well as in the extended Stenger conjecture are the same if w is a symmetric weight function. In Section 3 we show that, both in the restricted and extended conjecture, the matrix $U_n^{(\alpha,\beta)}$ belonging to the Jacobi weight function $w(x) = (1 - x)^\alpha (1 + x)^\beta$ on $[-1, 1]$ with parameters α, β is the same as the matrix $V_n^{(\beta,\alpha)}$ with the Jacobi parameters interchanged. Section 5, devoted to the restricted Stenger conjecture, shows, partly by numerical computation, that the conjecture may be true for large classes of weight functions, but can also be false for other classes of weight functions. In contrast, Section 6 provides ample computational support for the validity of the extended Stenger conjecture for a variety of classical and nonclassical weight functions. Discrete weight functions are considered in Section 7. In Section 8 the extended Stenger conjecture is challenged in the case of a piecewise constant positive weight function. Related work on collocation Runge–Kutta methods is mentioned in the Appendix.

2 Symmetric weight functions

We assume here the weight function $w(x)$ to be symmetric, i.e., $w(-x) = w(x)$ on $[-b, b]$, $0 < b \leq \infty$, and the zeros x_j of the corresponding orthonormal polynomial p_n ordered increasingly,

$$-b < x_1 < x_2 < \cdots < x_n < b.$$

We then have, by symmetry,

(4) $$x_j + x_{n+1-j} = 0, \quad j = 1, 2, \ldots, n.$$

Theorem 1. *If w is symmetric, the eigenvalues of V_n are the same as those of U_n, both in the case of the restricted* (where $b < \infty$) *and the extended Stenger conjecture.*

Proof. We present the proof for the extended conjecture, the one for the restricted conjecture being the same (just drop the factor $w(t)$ in all integrals). From the definition of V_n in (3), we have

$$v_{jk} = \int_{x_j}^{b} \ell_k^{(n)}(x)w(x)\mathrm{d}x = \int_{-b}^{-x_j} \ell_k^{(n)}(-t)w(t)\mathrm{d}t,$$

and, therefore, by (4),

$$v_{jk} = \int_{-b}^{x_{n+1-j}} \ell_k^{(n)}(-t)w(t)\mathrm{d}t.$$

Since $\ell_k^{(n)}(-t) = 1$ if $-t = x_k$, that is, $t = -x_k = x_{n+1-k}$, and $\ell_k^{(n)}(-t) = 0$ if $t = x_j$, $j \neq n+1-k$, we get

$$v_{jk} = \int_{-b}^{x_{n+1-j}} \ell_{n+1-k}^{(n)}(x)w(x)\mathrm{d}x,$$

thus, by (3) (with $a = -b$),

$$v_{jk} = u_{n+1-j,n+1-k}.$$

In matrix form, this can be written as

$$V_n = \begin{bmatrix} & & & 1 \\ & & \cdot^{\displaystyle\cdot^{\displaystyle\cdot}} & \\ & 1 & & \\ 1 & & & \end{bmatrix} U_n \begin{bmatrix} & & & 1 \\ & & \cdot^{\displaystyle\cdot^{\displaystyle\cdot}} & \\ & 1 & & \\ 1 & & & \end{bmatrix},$$

which is a similarity transformation of U_n. Hence, V_n and U_n have the same eigenvalues. $\square$

3   Jacobi weight functions

In this section we look at Jacobi weight functions

(5) $$w^{(\alpha,\beta)}(x) = (1-z)^{\alpha}(1+x)^{\beta} \quad \text{on } [-1,1],$$

where α, β are greater than -1.

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Switching Jacobi parameters has the effect of turning a U-matrix into a V-matrix and vice versa. More precisely, we have

Theorem 2. *Let $U_n^{(\alpha,\beta)}$ be the matrix U_n for Jacobi polynomials with parameters α, β, and $V_n^{(\beta,\alpha)}$ the matrix V_n for Jacobi polynomials with parameters β, α. Then*

$$
(6) \qquad\qquad U_n^{(\alpha,\beta)} = V_n^{(\beta,\alpha)},
$$

both in the restricted and extended Stenger conjecture.

Proof. We give the proof for the restricted Stenger conjecture. It is the same for the extended conjecture, using $w^{(\alpha,\beta)}(-x) = w^{(\beta,\alpha)}(x)$.

We denote quantities x related to Jacobi parameters α, β by x^* after interchange of the parameters. Since the Jacobi polynomial satisfies $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ (cf. [9, Eq. (4.1.3)]), we can take $x_j^* = x_j^{(\beta,\alpha)} = -x_j = -x_j^{(\alpha,\beta)}$ for the zeros of $P_n^{(\beta,\alpha)}$. Noting that

$$
\ell_k^{(n)}(x;\alpha,\beta) = \prod_{j\neq k} \frac{x - x_j}{x_k - x_j} = -\prod_{j\neq k} \frac{x + x_j^*}{x_k^* - x_j^*} = \prod_{j\neq k} \frac{(-x) - x_j^*}{x_k^* - x_j^*} = \ell_k^{(n)}(-x;\beta,\alpha),
$$

we get

$$
u_{jk}^{(\alpha,\beta)} = \int_{-1}^{x_j} \ell_k^{(n)}(t;\alpha,\beta)\mathrm{d}t = \int_{-1}^{x_j} \ell_k^{(n)}(-t;\beta,\alpha)\mathrm{d}t = \int_{x_j^*}^{1} \ell_k^{(n)}(x;\beta,\alpha))\mathrm{d}x = v_{jk}^{(\beta,\alpha)}.
$$

$\square$

4 Proof of the Stenger conjecture for Legendre polynomials

By virtue of Theorem 1, it suffices to consider the matrix U_n.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of U_n and $y = [y_1, y_2, \ldots, y_n]^T \in \mathbb{C}^n$ a corresponding eigenvector,

$$
(7) \qquad\qquad U_n y = \lambda y, \quad y \neq [0, 0, \ldots, 0]^T,
$$

so that

$$
(8) \qquad \int_{-1}^{x_i} \left(\sum_{j=1}^{n} \ell_j^{(n)}(x) y_j \right) \mathrm{d}x = \lambda y_i, \quad i = 1, 2, \ldots, n.
$$

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Let $y(x) \in \mathbb{P}_{n-1}$ be the unique polynomial of degree $\leq n - 1$ interpolating to y_j at x_j, $j = 1, 2, \ldots, n$. By the Lagrange interpolation formula, and (8), we then have

$$(9) \qquad \int_{-1}^{x_i} y(t)\mathrm{d}t = \lambda y(x_i), \quad i = 1, 2, \ldots, n.$$

With w_i, $i = 1, 2, \ldots, n$, denoting the weights of the n-point Gauss–Legendre quadrature formula, multiply (9) by $w_i \overline{y(x_i)}$ and sum over i to get

$$\sum_{i=1}^{n} w_i \overline{y(x_i)} \int_{-1}^{x_i} y(t)\mathrm{d}t = \lambda \sum_{i=1}^{n} w_i |y(x_i)|^2.$$

Since $\overline{y(x)} \int_{-1}^{x} y(t)\mathrm{d}t$ is a polynomial of degree $2n - 1$, and n-point Gauss quadrature is exact for any such polynomial, and since $|y(x)|^2$ is a polynomial of degree $2n - 2$, we have

$$(10) \qquad \int_{-1}^{1} \overline{y(x)} \left(\int_{-1}^{x} y(t)\mathrm{d}t \right) \mathrm{d}x = \lambda \int_{-1}^{1} |y(x)|^2 \mathrm{d}x.$$

Integration by parts on the left yields the identity

$$(11) \quad \int_{-1}^{1} \overline{y(x)} \left(\int_{-1}^{x} y(t)\mathrm{d}t \right) \mathrm{d}x + \int_{-1}^{1} y(x) \left(\int_{-1}^{x} \overline{y(t)}\mathrm{d}t \right) \mathrm{d}x = \left| \int_{-1}^{1} y(t)\mathrm{d}t \right|^2.$$

The real part of the left-hand side of (10) is

$$\frac{1}{2} \left[\int_{-1}^{1} \overline{y(x)} \left(\int_{-1}^{x} y(t)\mathrm{d}t \right) \mathrm{d}x + \int_{-1}^{1} y(x) \left(\int_{-1}^{x} \overline{y(t)}\mathrm{d}t \right) \mathrm{d}x \right],$$

which, by (11), equals $\frac{1}{2} \left| \int_{-1}^{1} y(t)\mathrm{d}t \right|^2$. Therefore, taking the real part on the right of (10) yields

$$(12) \qquad \mathrm{Re}\,\lambda \int_{-1}^{1} |y(x)|^2 \mathrm{d}x = \frac{1}{2} \left| \int_{-1}^{1} y(t)\mathrm{d}t \right|^2.$$

From this, it follows that $\mathrm{Re}\,\lambda \geq 0$.

To prove strict positivity of $\mathrm{Re}\,\lambda$, we have to show that the integral on the right of (12) does not vanish. To do this, we look at $\int_{-1}^{x} y(t)\mathrm{d}t - \lambda y(x)$,

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which is a polynomial of degree n vanishing at x_i, $i = 1, 2, \ldots, n$, by (9). Therefore,

$$(13) \qquad \int_{-1}^{x} y(t) \mathrm{d}t - \lambda y(x) = \mathrm{const}\, P_n(x),$$

where P_n is the Legendre polynomial of degree n. We now multiply (13) by $(1 - x)^{k-1}$, $1 \le k \le n$, and integrate over $[-1, 1]$. Then, by orthogonality, we get

$$\int_{-1}^{1} (1 - x)^{k-1} \left(\int_{-1}^{x} y(t) \mathrm{d}t \right) \mathrm{d}x = \lambda \int_{-1}^{1} (1 - x)^{k-1} y(x) \mathrm{d}x.$$

On the left, integrating by parts, letting

$$u(x) = \int_{-1}^{x} y(t) \mathrm{d}t, \quad v'(x) = (1 - x)^{k-1},$$

$$u'(x) = y(x), \quad v(x) = \int_{1}^{x} (1 - t)^{k-1} \mathrm{d}t = -(1 - x)^{k}/k,$$

and noting that $u(-1) = v(1) = 0$, we get

$$(14) \qquad \int_{-1}^{1} \frac{(1 - x)^k}{k} y(x) \mathrm{d}x = \lambda \int_{-1}^{1} (1 - x)^{k-1} y(x) \mathrm{d}x, \quad 1 \le k \le n.$$

Now suppose that $\int_{-1}^{1} y(x) \mathrm{d}x = 0$. Then (14) for $k = 1$ implies that $y(x)$ is orthogonal to all linear functons. Putting $k = 2$ in (14) then implies orthogonality of $y(x)$ to all quadratic functions. Proceding in this manner up to $k = n - 1$, we conclude that $y(x)$ is orthogonal to all polynomials of degree $n - 1$, in particular to itself, so that $\int_{-1}^{1} y^2(x) \mathrm{d}x = 0$, hence $y(x) \equiv 0$. This contradicts (7). Thus, by (12), $\mathrm{Re}\,\lambda > 0$. $\square$

5 The restricted Stenger conjecture

5.1 Proof of the restricted Stenger conjecture for a special Jacobi polynomial

Here we consider the weight function $w(x) = 1 - x$ on $[-1, 1]$, that is, the Jacobi weight function $(1 - x)^\alpha (1 + x)^\beta$ with parameters $\alpha = 1$, $\beta = 0$, and

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denote by x_i, $i = 1, 2, \ldots, n$, the zeros of the Jacobi polynomial $P_n^{(1,0)}$ and by U_n the matrix in (2) formed with these zeros x_i. As is well known, the x_i are the internal nodes of the $(n+1)$-point Gauss–Radau quadrature formula

$$(15) \qquad \int_{-1}^{1} f(x)\mathrm{d}x = \sum_{i=1}^{n} w_i f(x_i) + w_{n+1} f(x_{n+1}), \quad f \in \mathbb{P}_{2n},$$

where $x_{n+1} = 1$.

Let again $\lambda \in \mathbb{C}$ be an eigenvalue of U_n and $y = [y_1, y_2, \ldots, y_n] \in \mathbb{C}^n$ a corresponding eigenvector, and $y(x)$ as defined in Section 4. Multiplying (9) now by $w_i(1 - x_i)\overline{y(x_i)}$ and summing over $i = 1, 2, \ldots, n+1$, we obtain

$$\sum_{i=1}^{n+1} w_i(1 - x_i)\overline{y(x_i)} \int_{-1}^{x_i} y(t)\mathrm{d}t = \lambda \sum_{i=1}^{n+1} w_i(1 - x_i)|y(x_i)|^2.$$

(The last term in the sums on the left and right, of course, are zero.) Therefore, by (15), since $(1 - x)\overline{y(x)} \int_{-1}^{x} y(t)\mathrm{d}t$ is a polynomial of degree $\leq 2n$, and $(1 - x)|y(x)|^2$ a polynomial of degree $\leq 2n - 1$,

$$(16) \qquad \int_{-1}^{1} (1 - x)\overline{y(x)} \left(\int_{-1}^{x} y(t)\mathrm{d}t \right) \mathrm{d}x = \lambda \int_{-1}^{1} (1 - x)|y(x)|^2\mathrm{d}x.$$

The real part of the left-hand side of (16) is
(17)
$$\frac{1}{2} \left[\int_{-1}^{1} (1 - x)\overline{y(x)} \left(\int_{-1}^{x} y(t)\mathrm{d}t \right) \mathrm{d}x + \int_{-1}^{1} (1 - x)y(x) \left(\int_{-1}^{x} \overline{y(t)}\mathrm{d}t \right) \mathrm{d}x \right]$$
$$= \frac{1}{2} \int_{-1}^{1} (1 - x)\frac{\mathrm{d}}{\mathrm{d}x} \left| \int_{-1}^{x} y(t)\mathrm{d}t \right|^2 \mathrm{d}x,$$

having used the product rule of differentiation on the right. Integration by parts then yields

$$\frac{1}{2} \int_{-1}^{1} \left| \int_{-1}^{x} y(t)\mathrm{d}t \right|^2 \mathrm{d}x = \operatorname{Re} \lambda \int_{-1}^{1} (1 - x)|y(x)|^2\mathrm{d}x.$$

Since the integral on the right is positive, and so is the integral on the left, there follows $\operatorname{Re} \lambda > 0$. $\square$

It may be thought that the same kind of proof might work also for Jacobi weight functions with parameters $\alpha = 0$, $\beta = 1$ or $\alpha = \beta = 1$, using

Gauss–Radau quadrature with fixed node -1 or Gauss–Lobatto quadrature, respectively. The last step in the proof (integration by parts of the integral on the right of (17)), however, fails to produce the desired conclusion, the first factor in that integral being $1 + x$, resp. $1 - x^2$.

5.2   A Counterexample

The simplest conterexample we came across involves a Gegenbauer polynomial of small degree.

Counterexample.

$$(18) \qquad p_n(x) = C_n^{(\alpha)}(x), \quad n = 5, \quad \alpha = 10,$$

where $C_n^{(\alpha)}$ is the Gegenbauer polynomial of degree n.

From [1, Eq. 22.3.4] one finds

$$C_5^{(\alpha)}(x) = \alpha(\alpha + 1)(\alpha + 2)\, x \left[\frac{4}{15}(\alpha + 3)(\alpha + 4)\, x^4 - \frac{4}{3}(\alpha + 3)\, x^2 + 1 \right].$$

One zero of $C_5^{(\alpha)}$, of course, is 0, while the other four are the zeros of the polynomial P in brackets. When $\alpha = 10$, one finds

$$P(x) = \frac{1}{3} \left(\frac{728}{5}\, x^4 - 52\, x^2 + 3 \right).$$

This is a quadratic polynomial in x^2, the zeros of which could be found explicitly. However, we proceed computationally, using Matlab, since eventually, to obtain eigenvalues, one has to compute anyway.

The Matlab routine doing the computations is `counterex.m`.[1] It computes the elements of U_n in (2) (where $n = 5$) exactly by 3-point Gauss–Legendre quadrature of the last integral in

$$(19)\ u_{jk} = \int_{-1}^{x_j} \ell_k^{(5)}(x)\mathrm{d}x = \frac{1}{2}(1 + x_j) \int_{-1}^{1} \ell_k^{(5)} \left(\frac{1}{2}(1 + x_j)\, t - \frac{1}{2}(1 - x_j) \right) \mathrm{d}t$$

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[1]All Matlab routines referenced in this paper, and all textfiles used, can be accessed at CONJS of the website `https://www.cs.purdue.edu/archives/2002/wxg/codes`.

and uses a routine `lagrange.m` for calculating the elementary Lagrange interpolation polynomials as well as the `OPQ` routines `r_jacobi.m`, `gauss.m`. For the latter, see [4, pp. 301, 304].

The output, showing the five eigenvalues `d` of U_5, is

```
>> counterex

d =

   .431796388637445 + 0.000000000000000i
   .285123529721968 + .272861054932517i
   .285123529721968 - .272861054932517i
  -.001021724040688 + .286723270044925i
  -.001021724040688 - .286723270044925i

>>
```

The last pair of eigenvalues has negative real part, disproving, at least computationally, the restricted Stenger conjecture. The extended conjecture, however, seems to be valid for this example; see Section 6.2, Example 1.

5.3 Conjectures

The counterexample in Section 5.2 is symptomatic for more general counterexamples, not only regarding Gegenbauer, but also many other weight functions. They are formulated here as separate conjectures, all firmly rooted in computational evidence.

5.3.1 Gegenbauer polynomials

Conjecture 1. *The restricted Stenger conjecture for U_n (and, by Theorem 1, also for V_n) is true for all Gegenbauer polynomials $C_n^{(\alpha)}$ with $2 \leq n \leq 4$, but for $n \geq 5$ is true only for $\alpha > -1$ up to some $\alpha_n > 1$.*

The routine `Uconj_restr_jac.m` evaluates the matrix U_n (for Jacobi polynomials) in Matlab double-precision arithmetic and its eigenvalues in 32-digit variable-precision arithmetic. Since the eigenvalues become more ill-conditioned as n increases, we first make sure that they are accurate to at least four significant decimal digits by running the routine entirely in 32-digit

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arithmetic for selected values of α (and also of β) in $(-1, 1]$ and selected values of n, using the routine `sUconj_restr_jac.m`, and comparing the results with those obtained in double precision.

Conjecture 1 has then been confirmed for all $\alpha = -.9 : .1 : 10$, using the routine `run_Uconj_restr_jac.m`. Estimates of α_n have been obtained by a bisection-type procedure and are shown in Table 1. They are "estimates" in the sense that the conjecture is true for $\alpha \leq \alpha_n$, but false for $\alpha = \alpha_n + .001$.

n	α_n	n	α_n
5	9.000	25	1.025
10	1.264	30	1.017
15	1.081	35	1.012
20	1.041	40	1.009

Table 1. Estimates of α_n, $n = 5 : 5 : 40$

It appears that α_n converges monotonically down to 1 as $n \to \infty$.

5.3.2 Jacobi polynomials

Conjecture 2. *The restricted Stenger conjecture for U_n holds true in the case of Jacobi polynomials $P_n^{(\alpha,\beta)}$, for all $n > 1$, if $-1 < \alpha, \beta \leq 1$, but not necessarily otherwise.*

The positive part of the conjecture has been confirmed for $[\alpha, \beta] = -.9 : .1 : 1$, and in each case for $n = 2 : 40$, using the routine `run_Uconj_restr_jac.m`. The negative part follows from Conjecture 1, Table 1 (if true). By Theorem 2, the same conjecture can be made for the matrix V_n.

5.3.3 Algebraic/logarithmic weight functions

Here we first examine weight functions of the type

(20) $$w_\alpha(x) = x^\alpha \log(1/x) \text{ on } [0, 1] \text{ with } \alpha > -1.$$

Conjecture 3. *For the matrix U_n, the restricted Stenger conjecture holds true in the case of the weight function (20), for all $n > 1$, if $-1 < \alpha \leq \alpha_1$, where $1 < \alpha_1 < 2$, but not necessarily otherwise. For the matrix V_n, in contrast, the conjecture is true for all $\alpha > -1$.*

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In order to compute the zeros x_j of the required orthogonal polynomials (needed to obtain the Lagrange polynomials $\ell_k^{(n)}$), for degrees $2 \leq n \leq 40$ and arbitrary $\alpha > -1$, we need a routine that generates the respective recurrence coefficients for the orthogonal polynomials. This can be done by applying a multicomponent discretization procedure, using appropriate quadrature rules to discretize the integral $\int_0^1 f(x)x^\alpha \log(1/x)\mathrm{d}x$, where f is a polynomial of degree $\leq 2n - 1$. It was found to be helpful to split the integral in two integrals, one extended from 0 to ξ, and the other from ξ to 1, $0 < \xi < 1$, and use ξ to optimize the rate of convergence (that is, to minimize the parameter `Mcap` in the discretization routine `mcdis.m`). Using obvious changes of variables, one finds

$$
(21) \qquad \int_0^\xi f(x)x^\alpha \log(1/x)\mathrm{d}x = \xi^{\alpha+1}\left[\log(1/\xi)\int_0^1 f(t\xi)t^\alpha\mathrm{d}t \right.
$$
$$
\left. +\frac{1}{(1+\alpha)^2}\int_0^\infty f(\xi e^{-t/(1+\alpha)})te^{-t}\mathrm{d}t\right],
$$

$$
(22) \qquad \int_\xi^1 f(x)x^\alpha \log(1/x)\mathrm{d}x = (1-\xi)\int_0^1 f(x(t))[x(t)]^\alpha \log(1/x(t))dt,
$$

where in (22), $x(t) = (1-\xi)t + \xi$ maps the interval $[0,1]$ onto $[\xi, 1]$. In (21), the first integral on the right can be discretized (without error) by n-point Gauss–Jacobi quadrature on $[0,1]$ with Jacobi parameters 0 and α, and the second integral (with small error) by sufficiently high-order generalized Gauss–Laguerre quadrature with Laguerre parameter 1. The integral in (22) can be discretized by sufficiently high-order Gauss–Legendre quadrature on $[0,1]$. For the optimal ξ, one can use, as found empirically (using the routine `run_r_alglog1.m`),

$$
\xi = \begin{cases} [1 + 10(\alpha + .9)]/1000 & \text{if } -.9 \leq \alpha \leq 1, \\ .02 & \text{if } \alpha > 1. \end{cases}
$$

This is implemented in the routine `r_alglog1.m`.

The routine `sUconj_restr_log1.m`, run with `dig = 32`, generates the matrix U_n and its eigenvalues in 32-digit arithmetic. It relies on the global $n \times 2$ arrays `ab` and `ableg` containing the first n recurrence coefficients of the (monic) orthogonal polynomials relative to the weight functions w_α and 1, respectively (both supported on $[0,1]$). The array `ab`, when $\alpha =$

$-1/2, 0, 1/2, 1, 2$ is available, partly in [5, 2.3.1,2.41,2.4.3], to 32 digits for n at least as large as 100, whereas `ableg` can easily be generated by the routine `sr_jacobi01.m`. For these five values of α, we can therefore produce reference values to high precision for the eigenvalues of U_n.

The Matlab double-precision routine `Uconj_restr_log1.m`, also run with `dig = 32`, generates the matrix U_n in double-precision arithmetic and the eigenvalues in 32-digit arithmetic, for arbitrary values of $\alpha > -1$, its global array `ab` being produced by the routine `r_alglog1.m`. When the eigenvalues so obtained are compared with the reference values, for the above five values of α, it is found that for $n \leq 40$ they all are accurate to at least four decimal digits (cf. `test_Uconj_restr_log1.m`). This provides us with some confidence that the routine `Uconj_restr_log1.m`, when $n \leq 40$, will produce eigenvalues to the same accuracy, also when α is arbitrary in the range from $-1/2$ to 2.

The routine `run_Uconj_restr_log1.m` validates the restriced Stenger conjecture for the matrix U_n when $\alpha = -1/2, 0, 1/2, 1$, at least for all n between 2 and 40, but refutes it when $\alpha = 2$ and $n = 8$, producing a pair of eigenvalues with negative real part $-1.698\ldots(-3)$. This provides some indication that Conjecture 3 for the matrix U_n may be valid. We strengthen this expectation by running the routine for additional values of α, and at the same time try to estimate the value of α_1 in dependence of n by applying a bisection-type procedure. It is found that, when $n \leq 40$, Conjecture 3 for U_n is true with α_1 as shown in Table 2.

n	α_1
10	1.511
20	1.253
30	1.203
40	1.179

Table 2. The values of α_1 in Conjecture 2 in dependence of n

It appears that α_1 is monotonically decreasing. Since it is bounded below by 1, it would then have to converge to a limit value (perhaps $= 1$).

The routines dealing with the matrix V_n are `Vconj_restr_log1.m` and `run_Vconj_restr_log1.m`. They validate Conjecture 3 for the matrix V_n when $\alpha = -1/2, 0, 1/2, 1, 2, 5, 10$, in each case for $2 \leq n \leq 40$.
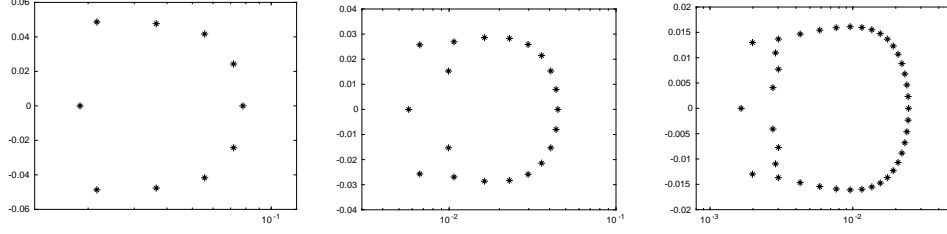
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Figure 1: Eigenvalues of the matrix U_n for a logarithmic weight function and $n = 10, 20, 40$ (from left to right)
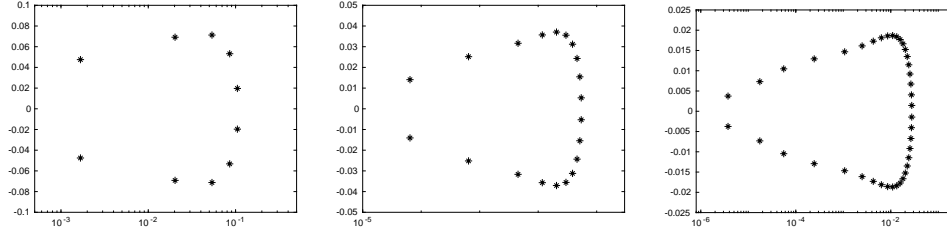

Figure 2: Eigenvalues of the matrix V_n for a logarithmic weight function and $n = 10, 20, 40$ (from left to right)

For illustration, the eigenvalues of U_n are shown in Fig. 1 for $\alpha = 0$ and $n = 10, 20, 40$, and those of V_n in Fig. 2 for the same α and n.

For the weight function

$$(23) \qquad w(x) = x^\alpha \log^2(1/x) \ \text{ on } [0, 1], \quad \text{with } \alpha > -1,$$

our conjecture for U_n is the same as the one in Conjecture 3, but not so for V_n.

CoNjecture 4. *For the matrix U_n, the restricted Stenger conjecture holds true in the case of the weight function* (23), *for all $n > 1$, if $-1 < \alpha < \alpha_2$, where α_2 is a number between 1 and 2, but not necessarily otherwise. For the matrix V_n, the conjecture is false for all $\alpha > -1$.*

The routines used to make this conjecture are the same as those used for Conjecture 3 but with "log1" replaced by "log2". The statements regarding the matrix U_n are arrived at in the same way as in Conjecture 3, the values of α_2 now being as shown in Table 3.

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n	α_2
10	1.852
20	1.480
30	1.394
40	1.353

Table 3. The values of α_2 in Conjecture 4 in dependence of n

With regard to V_n, the conjecture is found to be false for $\alpha = -1/2, 0, 1/2,$ $1, 2, 5$ and $n = 7$ in each case, there being a single pair of conjugate complex eigenvalues with negative real part.

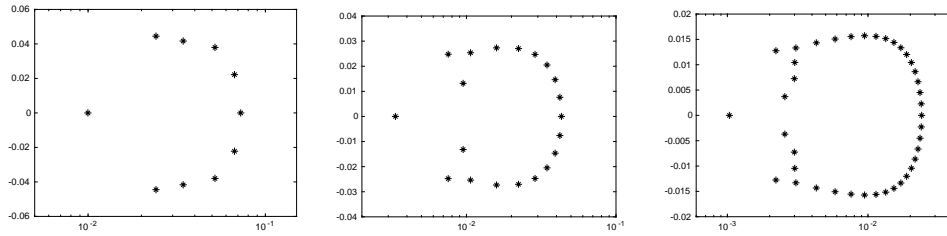We illustrate by showing in Fig. 3 the eigenvalues of U_n for $\alpha = 0$ and $n = 10, 20, 40$.

Figure 3: Eigenvalues of the matrix U_n for a square-logarithmic weight function and $n = 10, 20, 40$ (from left to right)

5.3.4 Laguerre and generalized Laguerre weight functions

For generalized Laguerre weight functions

$$(24) \qquad w(x) = x^\alpha e^{-x} \quad \text{on} \quad [0, \infty], \quad \alpha > -1,$$

it only makes sense to look at the U-conjecture.

Conjecture 5. *For the matrix U_n, the restricted Stenger conjecture is true in the case of the weight function (24), for all $n > 1$, if $-1 < \alpha \le \alpha_0$, where $1 < \alpha_0 < 2$, but not necessarily otherwise.*

The routines written for this conjecture are `Uconj_restr_lag.m` and `run_Uconj_restr_lag.m`. The latter, run for $\alpha = -.9 : .1 : 2$, $n = 2 : 40$,

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confirmes the conjecture up to, and including, $\alpha = 1.2$, but refutes it when $\alpha = 1.3$ and $n = 40$, producing a single pair of conjugate complex eigenvalues with negative real part. The case $\alpha = 1.3$ was checked by running the routine `run_sUconj_restr_lag.m` in 32-digit arithmetic, which produced eigenvalues agreeing with those obtained in double precision to at least 12 digits. (This check may take as many as five hours to run.) A bisection-type procedure, run in double precision, yields the values of α_0 shown in Table 4 in dependence of n.

n	α_0
10	2.475
20	1.522
30	1.317
40	1.228

Table 4. The values of α_0 in Conjecture 5 in dependence of n

Fig. 4 shows the eigenvalues of U_n when $\alpha = 0$ and $n = 10, 20, 40$.
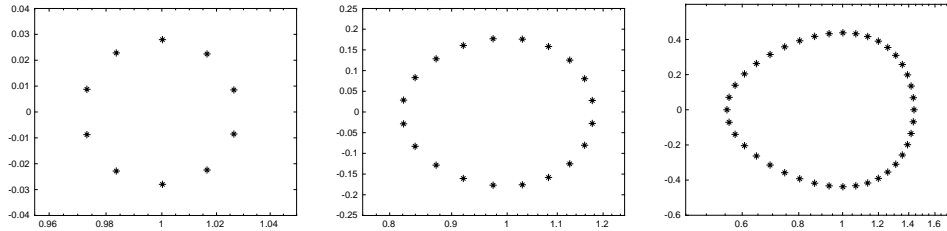
Figure 4: Eigenvalues of the matrix U_n for the Laguerre weight function and $n = 10, 20, 40$ (from left to right)

6   The extended Stenger conjecture

To avoid extensive and time-consuming Matlab variable-precision computations, we restrict ourselves in Sections 6.2 – 6.6 to values of n that are less than, or equal to, 30. Also note that in all figures of this section the horizontal axis carries a logarithmic scale.

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6.1 Proof of a weak form of the extended Stenger conjecture for a special Jacobi polynomial

We consider here, as in Section 5.1, the Jacobi weight function $w(x) = (1 - x)^\alpha (1 + x)^\beta$ on $[-1, 1]$, with $\alpha = 1$, $\beta = 0$, and continue using the same notations as in that section. In particular, we again use the $(n + 1)$-point Gauss–Radau quadrature formula,

$$\text{(25)} \qquad \int_{-1}^{1} f(x)\mathrm{d}x = \sum_{i=1}^{n+1} w_i f(x_i) + R_n(f),$$

where $x_{n+1} = 1$, but this time include the remainder term,

$$\text{(26)} \qquad R_n(f) = -\gamma_n \frac{f^{(2n+1)}(\xi)}{(2n+1)!}, \quad \gamma_n = 2^{2n+1} \frac{(n+1)n!^4}{(2n+1)!^2}$$

(cf. [3, top of p. 158, where γ^b should read γ_n^b]). In place of (9), we now have

$$\text{(27)} \qquad \int_{-1}^{x_i} y(t)(1 - t)\mathrm{d}t = \lambda\, y(x_i), \quad i = 1, 2, \ldots, n.$$

Multiplying this, as in Section 5.1, by $w_i(1 - x_i)\overline{y(x_i)}$ and summing over $i = 1, 2, \ldots, n + 1$, we obtain

$$\text{(28)} \qquad \sum_{i=1}^{n+1} w_i(1 - x_i)\overline{y(x_i)} \int_{-1}^{x_i} y(t)(1 - t)\mathrm{d}t = \lambda \sum_{i=1}^{n+1} w_i(1 - x_i)|y(x_i)|^2.$$

Since

$$\text{(29)} \qquad f(x) := (1 - x)\overline{y(x)} \int_{-1}^{x} y(t)(1 - t)\mathrm{d}t$$

is a polynomial of degree $2n + 1$ and the left-hand side of (28) is equal to the quadrature sum on the right of (25) with f as in (29), we get

$$\sum_{i=1}^{n+1} w_i(1 - x_i)\overline{y(x_i)} \int_{-1}^{x_i} y(t)(1 - t)\mathrm{d}t$$

$$= \int_{-1}^{1} (1 - x)\overline{y(x)} \left(\int_{-1}^{x} y(t)(1 - t)\mathrm{d}t \right) \mathrm{d}x + \gamma_n \frac{f^{(2n+1)}(\xi)}{(2n+1)!},$$

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where $f^{(2n+1)}$ is a nonnegative constant, namely

$$f^{(2n+1)}(\xi) = \frac{(2n+1)!}{n+1}|a_{n-1}|^2,$$

with a_{n-1} the leading coefficient (of x^{n-1}) of the polynomial $y(x)$. Thus,

$$
(30) \qquad
\begin{aligned}
&\sum_{i=1}^{n+1} w_i(1-x_i)\overline{y(x_i)}\int_{-1}^{x_i} y(t)(1-t)\mathrm{d}t \\
&= \int_{-1}^{1}(1-x)\overline{y(x)}\left(\int_{-1}^{x} y(t)(1-t)\mathrm{d}t\right)\mathrm{d}x + C_n,
\end{aligned}
$$

where

$$C_n = \frac{\gamma_n}{n+1}|a_{n-1}|^2.$$

Now the real part of the left-hand side of (28), by (30), is

$$
\begin{aligned}
\frac{1}{2}&\left[\int_{-1}^{1}(1-x)\overline{y(x)}\left(\int_{-1}^{x} y(t)(1-t)\mathrm{d}t\right)\mathrm{d}x\right.\\
&\left.\qquad + \int_{-1}^{1}(1-x)y(x)\left(\int_{-1}^{x}\overline{y(t)}(1-t)\mathrm{d}t\right)\mathrm{d}x\right] + C_n\\
&= \frac{1}{2}\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left|\int_{-1}^{x} y(t)(1-t)\mathrm{d}t\right|^2\mathrm{d}x + C_n\\
&= \frac{1}{2}\left|\int_{-1}^{1} y(t)(1-t)\mathrm{d}t\right|^2 + C_n,
\end{aligned}
$$

so that, by (28),

$$
(31) \qquad
\frac{1}{2}\left|\int_{-1}^{1} y(t)(1-t)\mathrm{d}t\right|^2 + C_n = \operatorname{Re}\lambda\int_{-1}^{1}(1-x)|y(x)|^2\mathrm{d}x.
$$

the integrand on the right being a polynomial of degree $2n-1$. From this, it follows that $\operatorname{Re}\lambda \geq 0$. $\square$

Strict positivity of $\operatorname{Re}\lambda$ holds if $|a_{n-1}| > 0$, that is, if $y(x)$ is a polynomial of exact degree $n-1$, or if the integral on the left of (31) does not vanish. Computation, using the routines check_pos.m and run_check_pos.m, confirms that both are indeed the case, at least for $n \leq 40$. Table 5 shows, for selected values of n, the minimum values of $\left|\int_{-1}^{1} y(t)(1-t)\mathrm{d}t\right|$ and $|a_{n-1}|$, the minimum being taken over all eigenvalues/vectors. For checking purposes, the computations have also been carried out entirely in 32-digit arithmetic.

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n	min int	min $|a_{n-1}|$	n	min int	min $|a_{n-1}|$
5	2.273(−1)	4.425(−1)	25	4.906(−1)	1.088(−1)
10	4.228(−1)	2.062(−1)	30	4.809(−1)	1.126(−1)
15	4.200(−1)	1.358(−1)	35	4.595(−1)	1.162(−1)
20	4.966(−1)	1.193(−1)	40	4.435(−1)	1.295(−1)

Table 5. The minimum values min int of $\left| \int_{-1}^{1} y(t)(1-t)\mathrm{d}t \right|$ and $|a_{n-1}|$

6.2 Jacobi weight functions

The element $u_{jk}^{(n)}$ of the matrix U_n in (3) for the Jacobi weight function $w(x) = (1-x)^\alpha (1+x)^\beta$ on $[-1,1]$ is

$$u_{jk}^{(n)} = \int_{-1}^{x_j} \ell_k^{(n)}(x) w(x) \mathrm{d}x = \frac{1}{2}\left(1 + x_j\right) \int_{-1}^{1} \ell_k^{(n)}(x(t)) w(x(t)) \mathrm{d}t,$$

where

$$x(t) = \frac{1}{2}\left(1 + x_j\right) t - \frac{1}{2}\left(1 - x_j\right)$$

maps $[-1,1]$ onto $[-1, x_j]$. An elementary computation yields

$$(32) \qquad u_{jk}^{(n)} = \left(\frac{1+x_j}{2}\right)^{\alpha+\beta+1} \int_{-1}^{1} \ell_k^{(n)}(x(t)) \left[\frac{3-x_j}{1+x_j} - t\right]^\alpha (1+t)^\beta \mathrm{d}t.$$

Although the second factor in the integrand of (32) may be algebraically singular at a point close to, but larger than, 1 (when $x_j < 1$ is close to 1), we simply apply Gauss–Jacobi quadrature with Jacobi parameters 0 and β to the integral in (32) and choose the number of quadrature points large enough so as to produce eigenvalues of U_n accurate to at least 4 decimal places (which is good enough for plotting purposes). This is implemented by the Matlab function `Uconj_ext_jac.m` and can be run with the Matlab script `run_Uconj_ext_jac.m`.

Example 1. Gegenbauer weight function $w(x) = (1 - x^2)^\alpha$ on $[-1,1]$ with $\alpha = 10$.

This is the weight function for which the restricted Stenger conjecture is false already for $n = 5$ (cf. Section 5.2). The extended conjecture, however, is found to be true for all $2 \le n \le 30$; see Fig. 5 for the cases $n = 5, 15, 30$.
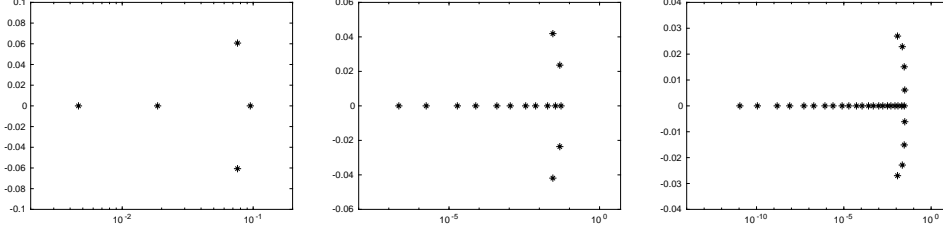
Figure 5: Eigenvalues of the matrix U_n for a special Gegenbauer polynomial of degrees $n = 5, 15, 30$ (from left to right)

Example 2. Jacobi weight function with parameters $(\alpha, \beta) = [-.9 : .6 : .9, \; 1.7 : .7 : 3.8, \; 4.7 : .9 : 7.4]$.

We used the script `run_Uconj_ext_jac.m` to check the extended U-conjecture for all these Jacobi weight functions, separately for $n = 5, 15, 30$, and found in every case that the conjecture is valid. By Theorem 2, the same is true for the matrix V_n.

To illustrate, we show in Fig. 6 the eigenvalues of U_n for the three parameter choices $\alpha = \beta = -.9$, $\alpha = -.3$, $\beta = -.9$, and $\alpha = 5.6$, $\beta = 1.7$, in each case with $n = 30$.
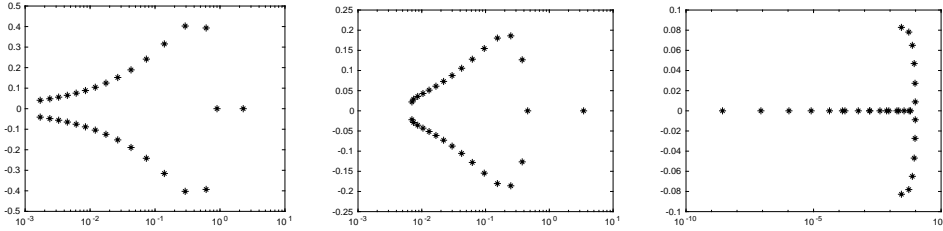
Figure 6: Eigenvalues of the matrix U_n, $n = 30$, for selected Jacobi polynomials

6.3   Algebraic/logarithmic weight functions

6.3.1   The weight function $w(x) = x^\alpha \log(1/x)$ on $[0, 1]$

Here, for the matrix U_n, we use the change of variables $x = x_j t$ in

$$u_{jk}^{(n)} = \int_0^{x_j} \ell_k^{(n)}(x)\, x^\alpha \log(1/x)\mathrm{d}x = x_j^{\alpha+1} \int_0^1 \ell_k^{(n)}(x_j t)\, t^\alpha \log(1/(x_j t))\mathrm{d}t$$

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to get

$$(33) \quad u_{jk}^{(n)} = x_j^{\alpha+1} \left[\log(1/x_j) \int_0^1 \ell_k^{(n)}(x_j t)\, t^\alpha \mathrm{d}t + \int_0^1 \ell_k^{(n)}(x_j t)\, t^\alpha \log(1/t) \mathrm{d}t \right].$$

Both integrals can be evaluated exactly, the first by m-point Gauss–Jacobi quadrature on $[0,1]$ with Jacobi parameters 0 and α, where $m = \lceil n/2 \rceil$, and the second by m-point Gauss quadrature relative to the weight function $w(t) = t^\alpha \log(1/t)$ on $[0,1]$. For the latter, the recurrence coefficients for the relevant orthogonal polynomials (when $\alpha = 0, -1/2, 1/2, 1, 2, 5$) are available to 32 decimal digits, partly in [5, 2.3.1, 2.4.1, 2.4.3], which allow us to generate the Gaussian quadrature rule in a well-known manner (cf., e.g., [3, §3.1.1]), using the OPQ routine gauss.m (see [4, p. 304]). This is implemented by the Matlab function Uconj_ext_log1.m and can be run with the Matlab script run_Uconj_ext_log1.m.

Alternatively, when $n \leq 40$, we may compute the recurence coefficients for arbitrary $\alpha > -1$ as described in Section 5.3.3. This is implemented by the routines r_alglog1.m, Uconj_ext_log1.m, and run0_Uconj_ext_log1.m.

Example 3. Algebraic/logarithmic weight function $w(x) = x^\alpha \log(1/x)$ on $[0,1]$ with $\alpha = (-.9 : .1 : 5)(5.2 : .2 : 7)(7.5 : .5 : 10)$.

Our routines validate the extended Stenger conjecture for all these values of α and $2 \leq n \leq 30$. The eigenvalues of U_n are shown in the case $\alpha = 0$ in Fig. 7, and in the cases $\alpha = -1/2, 1/2$ in Figs. 8 resp. 9 for $n = 5, 15, 30$. They are similar when $\alpha = 1, 2, 5$.
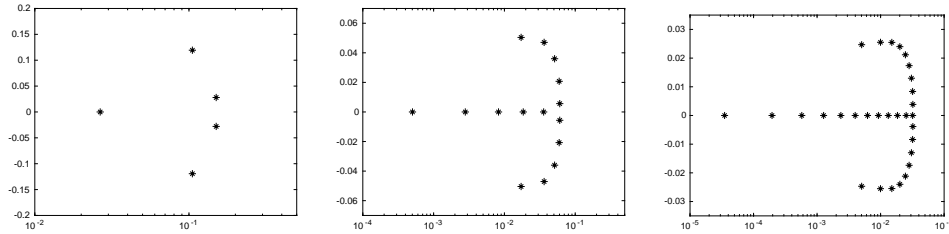

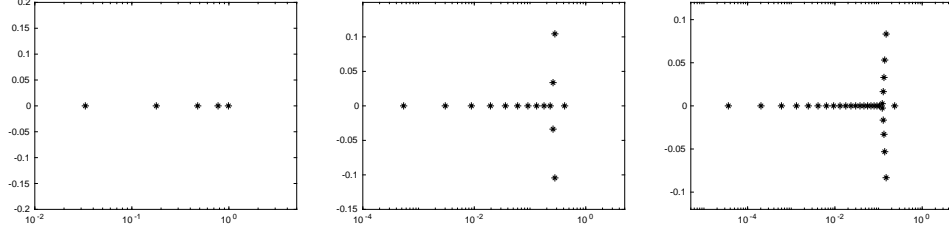Figure 7: Eigenvalues of U_n in the case of a logarithmic weight function for $n = 5, 15, 30$ (from left to right)

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Figure 8: Eigenvalues of U_n in the case of an algebraic/logarithmic weight function with parameter $\alpha = -1/2$ for $n = 5, 15, 30$ (from left to right)
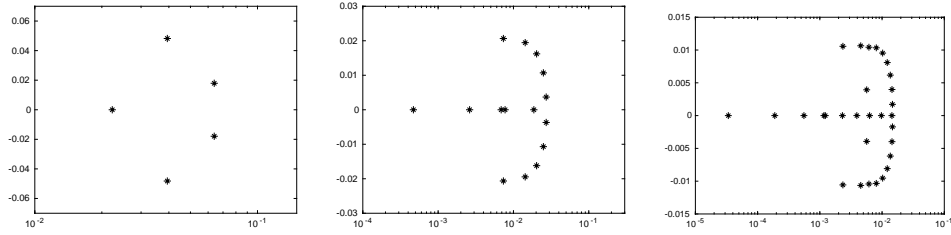

Figure 9: Eigenvalues of U_n in the case of an algebraic/logarithmic weight function with parameter $\alpha = 1/2$ for $n = 5, 15, 30$ (from left to right)

With regard to V_n, the conjecture has been similarly validated, using the routines `Vconj_ext_log1` and `run_Vconj_ext_log1.m`, for the same values of n and α as in Example 3. To compute the matrix V_n we have used

$$(34) \qquad v_{jk}^{(n)} = \int_0^1 \ell_k^{(n)}(x) x^\alpha \log(1/x)\mathrm{d}x - u_{jk}^{(n)}$$

with $u_{jk}^{(n)}$ as in (33) and the integral evaluated by $\lceil n/2 \rceil$-point Gaussian quadrature relative to the weight function $w(x)$. The eigenvalues of V_n are found to be similar to those for U_n shown in Figs. 7 – 9.

6.3.2 Algebraic/square-logarithmic weight function $w(x) = x^\alpha \log^2(1/x)$ on $[0,1]$, $\alpha > -1$

Similarly as in Section 6.3.1, one finds

(35)
$$u_{jk}^{(n)} = x_j^{\alpha+1} \left[\log^2(1/x_j) \int_0^1 \ell_k^{(n)}(x_j t) \, t^\alpha \mathrm{d}t + 2\log(1/x_j) \int_0^1 \ell_k^{(n)}(x_j t) \, t^\alpha \log(1/t)\mathrm{d}t \right. $$
$$\left. + \int_0^1 \ell_k^{(n)}(x_j t) \, t^\alpha \log^2(1/t)\mathrm{d}t \right],$$

where again the integrals can be evaluated exactly and some of the required recurrence coeffiicients taken from [5, 2.3.2], [5, 2.4.5], [5, 2.4.7]. This is implemented by the Matlab function `Uconj_ext_log2.m` and driver `run_Uconj_ext_log2.m`.

Example 4. Algebraic/square-logarithmic weight function $w(x) = x^\alpha \log^2(1/x)$ on $[0,1]$ with $\alpha = 0, -1/2, 1/2, 1, 2, 5$.

Our routines validate the extended Stenger conjecture for all these values of α and $2 \leq n \leq 30$. The eigenvalues of U_n in the case $\alpha = 0$ are found to be similar to those depicted in Fig. 7 for the weight function $\log(1/x)$. For the cases $\alpha = -1/2, 1/2, 5$ they are shown respectively in Figs. 10–12 for $n = 5, 15, 30$. Interestingly, all eigenvalues appear to be real when $\alpha - = -1/2$.
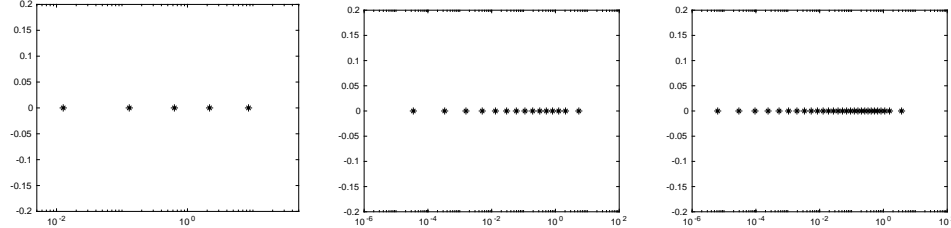

Figure 10: Eigenvalues of U_n in the case of an algebraic/square-logarithmic weight function, with exponent $\alpha = -1/2$, for $n = 5, 15, 30$ (from left to right)

Similar results and validations, using the routines `Vconj_ext_log2.m` and `run_Vconj_ext_log2.m`, are obtained for the matrix V_n, which, as in (34), is
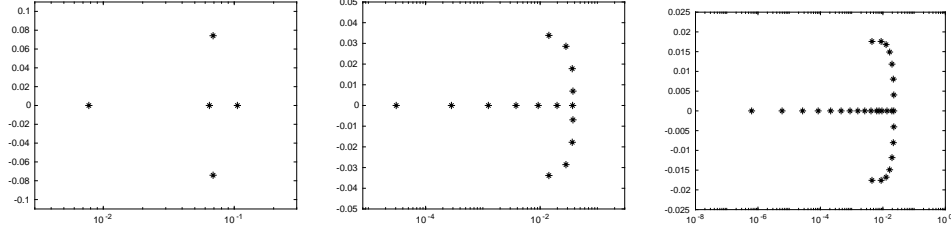
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Figure 11: Eigenvalues of U_n in the case of an algebraic/square-logarithmic weight function, with exponent $\alpha = 1/2$, for $n = 5, 15, 30$ (from left to right)
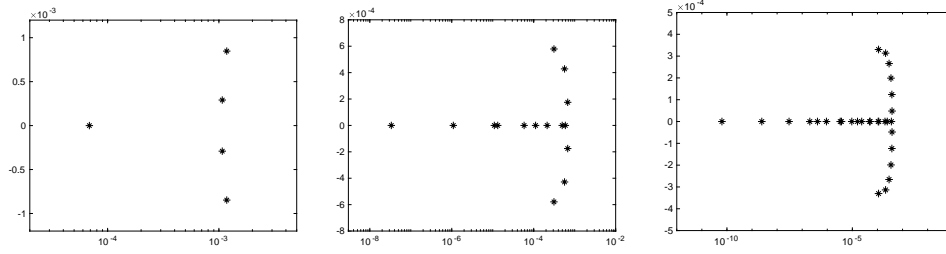

Figure 12: Eigenvalues of U_n in the case of an algebraic/square-logarithmic weight function, with exponent $\alpha = 5$, for $n = 5, 15, 30$ (from left to right)

computed exactly by

$$(36) \qquad v_{jk}^{(n)} = \int_0^1 \ell_k^{(n)}(x) x^\alpha \log^2(1/x) \mathrm{d}x - u_{jk}^{(n)}$$

with $u_{jk}^{(n)}$ as in (35).

6.4 Laguerre and generalized Laguerre weight functions

Here, the weight function is assumed to be $w(x) = x^\alpha \mathrm{e}^{-x}$ on $[0, \infty]$, where $\alpha > -1$. We write

$$u_{jk}^{(n)} = \int_0^\infty \ell_k^{(n)}(x) x^\alpha \mathrm{e}^{-x} \mathrm{d}x - \int_{x_j}^\infty \ell_k^{(n)}(x) x^\alpha \mathrm{e}^{-x} \mathrm{d}x$$

and, in the second integral, make the change of variables $x = x_j + t$ to get

$$(37) \qquad u_{jk}^{(n)} = \int_0^\infty \ell_k^{(n)}(x) x^\alpha \mathrm{e}^{-x} \mathrm{d}x - \mathrm{e}^{-x_j} \int_0^\infty \ell_k^{(n)}(x_j + t)(x_j + t)^\alpha \mathrm{e}^{-t} \mathrm{d}t.$$

The first integral can be evaluated exactly by $\lceil n/2 \rceil$-point generalized Gauss–Laguerre quadrature. The second integral, similarly as in (32) for Jacobi weight functions, has an algebraic singularity close to, and to the left of, the origin when x_j is close to zero (and α not an integer). As in Section 6.2, we ignore this and simply apply Gauss-Laguerre quadrature of sufficiently high order so as to obtain plotting accuracy for all the eigenvalues of U_n. However, there is yet another complication: Around $n = 25$, the Gauss–Laguerre weights, in Matlab double precision, start becoming increasingly inaccurate (in terms of relative accuracy) and adversely affect the accuracy of the second integral in (37). For this reason, we use 32-digit variable-precision arithmetic to compute these weights and convert them to Matlab double precision, once computed. At the same time we lower the accuracy requirement from 4- to 3-digit accuracy.

Example 5. Generalized Laguerre weight function $w(x) = x^\alpha \mathrm{e}^{-x} \mathrm{d}x$ on $[0, \infty]$ for the same values of α and n as in Example 2.

The Matlab routines implementing this and validating the conjecture in each case are `Uconj_ext_lag.m` and `run_Uconj_ext_lag.m`. They may take several hours to run because of the extensive variable-precision work involved. The accuracy achieved for the eigenvalues is consistently of the order of 10^{-4} or better, but the necessary number of quadrature points is found to be as large as 440 (for $\alpha = -.9$ and $n = 30$).

For illustration, we show in Fig. 13 the eigenvalues obtained in the case of the ordinary Laguerre weight function ($\alpha = 0$) and for $n = 5, 15, 30$. Notice the extremely small real eigenvalues when $n = 30$, the smallest being of the order 10^{-43}.

Using

$$(38) \qquad\qquad v_{jk}^{(n)} = \int_0^\infty \ell_k^{(n)}(x) x^\alpha \mathrm{e}^{-x} \mathrm{d}x - u_{jk}^{(n)}$$

with $u_{jk}^{(n)}$ as in (37), the conjecture has been similarly validated with the help of the routines `Vconj_ext_lag.m`, `run_Vconj_ext_lag.m`.
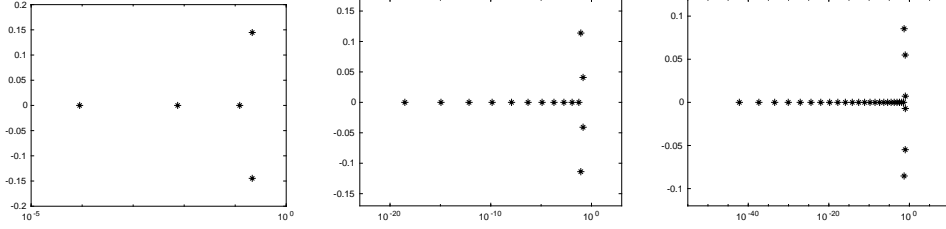
Figure 13: Eigenvalues of U_n in the case of the Laguerre weight function for $n = 5, 15, 30$ (from left to right)

6.5 Hermite and generalized Hermite weight functions

These are the weight functions $w(x) = |x|^{2\mu}e^{-x^2}$ on $[-\infty, \infty]$, $\mu > -1/2$. Since they are symmetric, it suffices, by Theorem 1, to consider U_n. To simplify matters, we assume 2μ to be a nonnegative integer.

For the evaluation of $u_{jk}^{(n)}$, we distinguish the cases $x_j < 0$ and $x_j \geq 0$. In the former case, by the change of variables $x = x_j - t$, one gets

$$(39) \qquad u_{jk}^{(n)} = e^{-x_j^2} \int_0^\infty \ell_k^{(n)}(x_j - t)(t - x_j)^{2\mu}e^{2x_jt}e^{-t^2}dt, \quad x_j < 0.$$

Here, half-range Gauss–Hermite quadrature (cf. [5, 2.9.1]) is expected to converge rapidly. When $x_j \geq 0$, breaking up the first integral in (3) (with $a = -\infty$) into two parts, one extended from $-\infty$ to 0 and the other from 0 to x_j, and making appropriate changes of variables in each, yields

$$(40) \quad u_{jk}^{(n)} = \int_0^\infty \ell_k^{(n)}(-t)t^{2\mu}e^{-t^2}dt + x_j^{2\mu+1} \int_0^1 \ell_k^{(n)}(x_jt)e^{-x_j^2t^2}t^{2\mu}dt, \quad x_j \geq 0.$$

The first integral can be evaluated exactly by $\lceil(n + 2\mu)/2\rceil$-point half-range Gauss–Hermite quadrature. The second integral may be approximated by Gauss–Jacobi quadrature on $[0, 1]$ with Jacobi parameters 0 and 2μ. This, too, is expected to converge quickly.

Example 6. Generalized Hermite weight function $w(x) = |x|^{2\mu}e^{-x^2}$ on $[-\infty, \infty]$, $\mu = 0 : 1/2 : 25$ and $n = 5, 15, 30$.

The conjecture has been validated in all cases, using the routines Uconj_ext_herm.m, run_Uconj_ext_herm.m. For illustration, the eigenvalues of U_n are shown in Fig. 14 for the case $\mu = 0$.
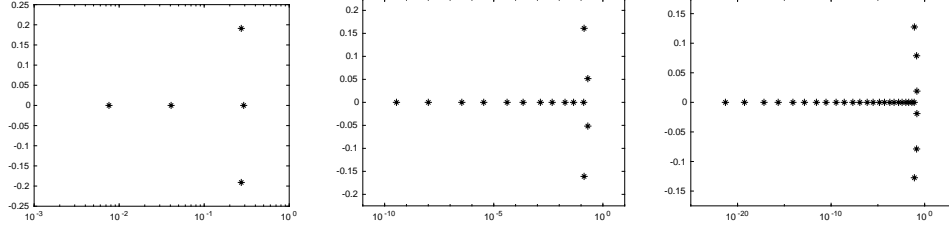
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Figure 14: Eigenvalues of U_n in the case of the Hermite weight function for $n = 5, 15, 30$ (from left to right)

6.6 A weight function supported on two disjoint intervals

We now consider a weight function which is not positive a.e.,

$$(41) \qquad w(x) = \begin{cases} |x|(x^2 - \xi^2)^p(1 - x^2)^q & \text{if } x \in [-1, -\xi] \cup [\xi, 1], \\ \\ 0 & \text{otherwise}, \end{cases}$$

where $0 < \xi < 1$, $p > -1$, $q > -1$. This weight function, of interest in theoretical chemistry when $p = q = -1/2$, has been studied in [2]. In our present context, we assume, for simplicity, that p and q are nonnegative integers. Then only integrations of polynomials are required, which, as before, can be done exactly.

Since the weight function w is symmetric, it suffices, by Theorem 1, to look at the matrices U_n only.

Any polynomial π_n orthogonal with respect to w can have at most one zero in the interval $[-\xi, \xi]$ where w is zero ([3, Theorem 1.20]). By symmetry, therefore, all zeros of π_n are located in the intervals $(-1, -\xi)$ or $(\xi, 1)$, except when n is odd, in which case there is a zero at the origin.

The recurrence coefficients α_k, β_k for the (monic) polynomials π_n are known explicitly ([2, Eq. (4.1)]): All $\alpha_k = 0$, by symmetry, and

$$\beta_0 = (1 - \xi^2)^{p+q+1}\Gamma(p+1)\Gamma(q+1)/\Gamma(p+q+2),$$
$$\beta_1 = \tfrac{1}{2}(1 - \xi^2)\alpha_0^J + \tfrac{1}{2}(1 + \xi^2),$$

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$$\left.\begin{array}{l} \beta_{2k} = (\tfrac{1}{2}(1 - \xi^2))^2 \beta_k^J / \beta_{2k-1} \\[2mm] \beta_{2k+1} = \tfrac{1}{2}(1 - \xi^2)\alpha_k^J + \tfrac{1}{2}(1 + \xi^2) - \beta_{2k} \end{array}\right\} \quad k = 1, 2, 3, \dots ,$$

where α_k^J, β_k^J are the recurrence coefficients of the monic Jacobi polynomials with paramters $\alpha = q$, $\beta = p$. Therefore, the zeros of π_n are easily computed by the OPQ routine `gauss.m` (see [4, p. 304]).

The computation of $u_{jk}^{(n)}$ is different, depending on where the zero x_j is located. In fact,

$$u_{jk}^{(n)} = -\frac{1 + x_j}{2} \int_{-1}^{1} \ell_k^{(n)}(x_1(t))x_1(t)(x_1^2(t) - \xi^2)^p (1 - x_1^2(t))^q \, dt \quad \text{if } x_j < -\xi,$$

where $x_1(t) = \frac{1+x_j}{2}t + \frac{x_j-1}{2}$ maps $[-1, 1]$ onto $[-1, x_j]$;

$$u_{jk}^{(n)} = -\frac{1 - \xi}{2} \int_{-1}^{1} \ell_k^{(n)}(x_2(t))x_2(t)(x_2^2(t) - \xi^2)^p (1 - x_2^2(t))^q \, dt \quad \text{if } x_j = 0,$$

where $x_2(t) = \frac{1-\xi}{2}t - \frac{1+\xi}{2}$ maps $[-1, 1]$ onto $[-1, -\xi]$; and

$$u_{jk}^{(n)} = \left(u_{jk}^{(n)}\right)_{x_j=0} + \frac{x_j - \xi}{2} \int_{-1}^{1} \ell_k(x_3(t))x_3(t)(x_3^2(t) - \xi^2)^p (1 - x_3^2(t))^q \, dt \quad \text{if } x_j > \xi,$$

where $x_3(t) = \frac{x_j-\xi}{2}t + \frac{x_j+\xi}{2}$ maps $[-1, 1]$ onto $[\xi, x_j]$.

All integrals can be computed exactly by $(\lceil (n + 1)/2 \rceil + p + q)$-point Gauss–Legendre quadrature.

Example 7. The weight function (41) with $\xi = .1 : .2 : .9$ and $p, q = 0 : 5$ for $n = 5, 15, 30$.

The routines `Uconj_ext_twoint.m`, `run_Uconj_ext_twoint.m` can be used to validate the conjecture in all cases, even though the weight function is not in the class of weight functions assumed in the conjecture. (For another such example, see Example 9 with $N = 1$.)

To illustrate, we show in Fig. 15 the eigenvalues of U_n, $n = 5, 15, 30$, in the case $\xi = 1/2$, $p = q = 0$, i.e., for the weight function $w(x)$ on $[-1, 1]$ equal to $|x|$ outside of $[-1/2, 1/2]$ and 0 inside.
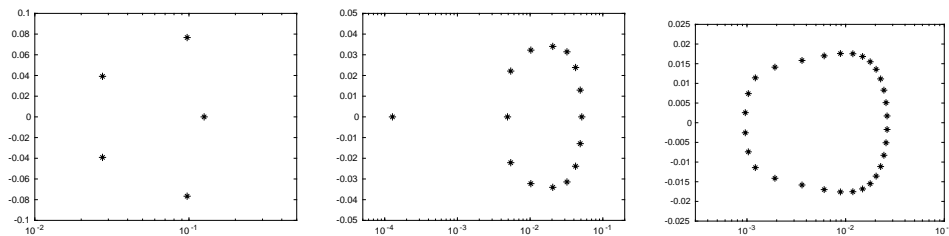
Figure 15: Eigenvalues of U_n in the case of a two-interval weight function for $n = 5, 15, 30$ (from left to right)

7   Discrete weight functions

To demonstrate that an assuption about the weight function like the one made for the extended Stenger conjecture is called for, we now consider a discrete measure $d\lambda_{N+1}$ supported on $N+1$ points $0, 1, 2, \ldots, N$ with jumps $w_k > 0$ at the points k, $k = 0, 1, \ldots, N$. The corresponding orthogonal polynomials, now $N+1$ in number, are again denoted by p_n, $n = 0, 1, \ldots, N$. If $w_0 = w_1 = \cdots = w_N = 1$, we are dealing eith the classical discrete orthogonal polynomials attributed to Chebyshev [3, Example 1.15]). They are the special case $\alpha = \beta = 0$ of Hahn polynomials with parameters α, β (cf. [3, last entry of Table 1.2]). Both the weight function and the zeros of p_n are symmetric about the midpoint $N/2$. In particular, when N is even and n odd, one of the zeros is equal to $N/2$, hence an integer.

For the elements of U_n we have

$$(42) \qquad u_{j,k}^{(n)} = \sum_{i=0}^{i_j} w_i \, \ell_k^{(n)}(i), \quad i_j = \lfloor x_j \rfloor,$$

where x_j are the zeros of p_n (assumed in increasing order). These can be generated by the functions `r_hahn.m` and `gauss.m`.

Example 8. The measure $d\lambda_{N+1}$, $N \geq 2$, with $w_0 = w_1 = \cdots = w_N = 1$, and p_n with $2 \leq n \leq N$.

It is important to note that when the zeros of p_n are computed by the routine `gauss.m`, and when N is even and n odd, the integer zero $x_j = N/2$ may end up becoming slightly less than $N/2$, in which case $\lfloor x_j \rfloor$ in (42) will

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yield an incorrect result. Similarly, the smallest zero, when computed, may turn out to become negative, or the largest zero equal to N. To avoid these pitfalls, we overwrite the zero, once computed, by $N/2$ or reset $\lfloor x_j \rfloor$, $j = 1, n$, by 0 resp. $N - 1$.

On running the script `run_Uconj_ext_hahn.m`, using `Uconj_ext_hahn.m`, to copmpute U_n and its eigenvalues, we found that the extended Stenger conjecture is still true for all $N \leq 10$ and all $2 \leq n \leq N$, but no longer when $N > 10$. The values of N and n for which eigenvalues with negative real parts appear are shown in Table 6 for $11 \leq N \leq 30$.

N	n	N	n
11	11	21	18–21
12	12	22	18–22
13	13	23	19–22 23*
14	13 14	24	16 19–23 24*
15	14 15	25	16 18 20–24 25*
16	15 16	26	18 20–25 26*
17	15–17	27	17 21* 22* 23* 24–26 27*
18	16–18	28	20–25 26* 27* 28*
19	16–19	29	18 19 22* 23 24 25* 26* 27 28* 29*
20	17–20	30	20 22–27 28* 29* 30*

Table 6. The presence of delinquent eigenvalues of U_n in the case of a discrete weight function

Asterisks indicate the presence of two pairs of delinquent complex conjugate eigenvalues rather than the usual single pair. (48-digit arithmetic was used for the last two entries in Table 6.)

Since the weight function is symmetric (with respect to the midpoint $N/2$), by Theorem 1 the same pattern of validity and nonvalidity holds also for the V-conjecture.

We illustrate by showing in Fig. 16 the eigenvalues of U_n, $n = N$, for $N = 11, 15, 30$.

Since there are no approximations involved, the results obtained should be quite accurate. In fact, we reran Example 8 in 48-digit arithmetic and found the double-precision eigenvalues accurate to 13, 12, and 10 digits for, resp., $n = 11, 15, 30$.

With regard to the restricted Stenger conjecture, the routines used are `run_Uconj_restr_hahn.m` and `Uconj_restr_hahn.m`. They, too, confirm the
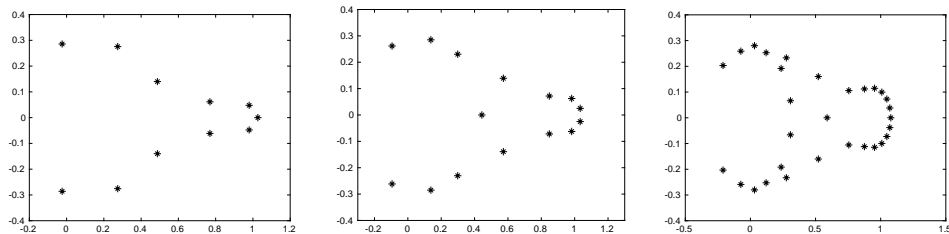
Figure 16: Eigenvalues of U_n in the case of discrete weight functions for $n = N = 11, 15, 30$ (from left to right)

validity of the conjecture for $N \leq 10$ and $2 \leq n \leq N$. But for $N > 11$, there are now more values of n than shown in Table 6 for which there are eigenvalues with negative real parts, and there can be as many as four pairs of delinquent eigenvalues.

8 Block-discrete and ε-block-discrete weight functions

It may be interesting to see whether the eigenvalues of U_n behave similarly as in Example 8 when the weight function is not $(N+1)$-discrete, but $(N+1)$-block-discrete, that is, of the form

$$(43) \qquad w(x; N+1) = \begin{cases} w_\nu & \text{if } 2\nu \leq x \leq 2\nu + 1, \quad \nu = 0, 1, \ldots, N, \\ 0 & \text{otherwise,} \end{cases}$$

where $w_0, w_1, \ldots, w_N$, $N \geq 1$, are positive numbers. Thus, the weight function is made up of $N+1$ "blocks" with base 1 and heights w_ν, $\nu = 0, 1, \ldots, N$, any two consecutive blocks being separated by a zero-block. More generally, we may consider $(N+1)$-ε-block-discrete weight functions, where the separating zero-blocks are replaced by ε-blocks, that is,

$$(44) \qquad w(x; N+1, \varepsilon) = \begin{cases} w_\nu & \text{if } 2\nu \leq x < 2\nu + 1, \quad \nu = 0, 1, \ldots, N, \\ \varepsilon & \text{if } 2\nu - 1 \leq x < 2\nu, \quad \nu = 1, 2, \ldots N, \\ 0 & \text{otherwise.} \end{cases}$$

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The orthogonal polynomials p_n associated with the weight function $w(x; N+1, \varepsilon)$ can be generated from their three-term recurrence relation, which in turn can be computed (exactly) by a $(2N+1)$-component discretization procedure (cf. [3, §2.2.4]) using $\lceil n/2 \rceil$-point Gauss–Legendre quadrature on $[0, 1]$. This is implemented in Matlab double and variable precision by the routines `ab_blockhahn.m`, `sab_blockhahn.m`. (For checking purposes, the same recurrence relation was also computed by a moment-based routine in sufficiently high precision.)

The elements u_{jk} of the matrix U_n,

$$u_{jk} = \int_0^{x_j} \ell_k^{(n)}(x)\, w(x; N+1, \varepsilon)\mathrm{d}x,$$

where x_j are the zeros of p_n, can be computed (exactly) as follows. Let $m = \lfloor x_j \rfloor$.

If $m = 0$,

$$u_{jk}^{(n)} = w_0 \int_0^{x_j} \ell_k^{(n)}(x)\mathrm{d}x = w_0\, x_j \int_0^1 \ell_k^{(n)}(x_j t)\mathrm{d}t;$$

if $m = 1$,

$$
\begin{aligned}
u_{jk} &= w_0 \int_0^1 \ell_k^{(n)}(x)\mathrm{d}x + \varepsilon \int_1^{x_j} \ell_k^{(n)}(x)\mathrm{d}x \\
&= \int_0^1 \Big[w_0\, \ell_k^{(n)}(t) + \varepsilon\, (x_j - 1)\ell_k^{(n)}((x_j - 1)t + 1) \Big];
\end{aligned}
$$

if $m > 0$ is even,

$$
\begin{aligned}
u_{jk} &= \sum_{\nu=0}^{(m-2)/2} w_\nu \int_{2\nu}^{2\nu+1} \ell_k^{(n)}(x)\mathrm{d}x + w_{m/2} \int_m^{x_j} \ell_k^{(n)}(x)\mathrm{d}x + \varepsilon \sum_{\nu=1}^{m/2} \int_{2\nu-1}^{2\nu} \ell_k^{(n)}(x)\mathrm{d}x \\
&= \int_0^1 \Bigg(\sum_{\nu=0}^{(m-2)/2} w_\nu\, \ell_k^{(n)}(2\nu + t) + w_{m/2}(x_j - m)\ell_k^{(n)}((x_j - m)t + m) \\
&\quad + \varepsilon \sum_{\nu=1}^{m/2} \ell_k^{(n)}(2\nu - 1 + t) \Bigg) \mathrm{d}t;
\end{aligned}
$$

If $m > 1$ is odd,

$$u_{jk} = \sum_{\nu=0}^{(m-1)/2} w_\nu \int_{2\nu}^{2\nu+1} \ell_k^{(n)}(x)\mathrm{d}x + \varepsilon \sum_{\nu=1}^{(m-1)/2} \int_{2\nu-1}^{2\nu} \ell_k^{(n)}(x)\mathrm{d}x + \varepsilon \int_m^{x_j} \ell_k^{(n)}(x)\mathrm{d}x$$

$$= \int_0^1 \left(w_0\, \ell_k^{(n)}(t) + \sum_{\nu=1}^{(m-1)/2} \left[w_\nu \ell_k^{(n)}(2\nu + t) + \varepsilon\, \ell_k^{(n)}(2\nu - 1 + t) \right] \right.$$

$$\left. + \varepsilon\, (x_j - m)\ell_k^{(n)}((x_j - m)t + m) \right) \mathrm{d}t.$$

All integrals on the far right of these equations can be computed exactly by $\lceil n/2 \rceil$-point Gauss–Legendre quadrature on $[0, 1]$. The first pitfall mentioned in Example 8, associated with computing the floor of x_j, is no longer an issue since the midpoint is now $N + 1/2$, a half-integer, not an integer.

Example 9. The $(N+1)$-block-discrete Hahn weight function with parameters $\alpha = \beta = 0$ and p_n with $2 \leq n \leq N$.

This is the weight function (43) with $w_0 = w_1 = \cdots = w_N = 1$. To check the behavior of the eigenvalues in this case, we have run the script run_Uconj_ext_blockhahn.m, using the function Uconj_ext_blockhahn.m and epsilon= 0, for $N = 1 : 10$ and $2 \leq n \leq 30$ for each N. It was found that the extended Stenger conjecture is still true for $2 \leq n \leq 30$ (and

N	n
2	9 17 18 20–22 25 28–30
3	11 14 21 22 24 27 28 30
4	13 14 16 18 20–22 24* 25* 29* 30
5	14 17 18 20 22–26 27* 28* 30
6	10 15–17 20–25 26* 27 29 30
7	11 17 20 21 23–28
8	14 18 20 23* 24 25* 27 28* 29* 30
9	15 19 21 23 24 26 27* 29* 30
10	13 14 16 18 20 22 26* 28* 29*

Table 7. The presence of delinquent eigenvalues of U_n in the case of a block-discrete weight function

probably for all $n \geq 2$) when $N = 1$, i.e., for a 2-block-discrete Hahn weight function. When $N > 1$, however, eigenvalues with negative real parts again

show up, starting from some $n \geq 9$, and frequently, but not always, thereafter. The values of N and n for which this occurs are shown in Table 7. There is usually one pair of delinquent complex conjugate eigenvalues, but in some cases there are two such pairs. These are identified by an asterisk in Table 7.

The validity of the extended Stenger conjecture for $N = 1$ is interesting. It may well be for the same (unknown) reason that validates the conjecture in the case of the two-interval weight function of Section 6.6; cf. Example 7.

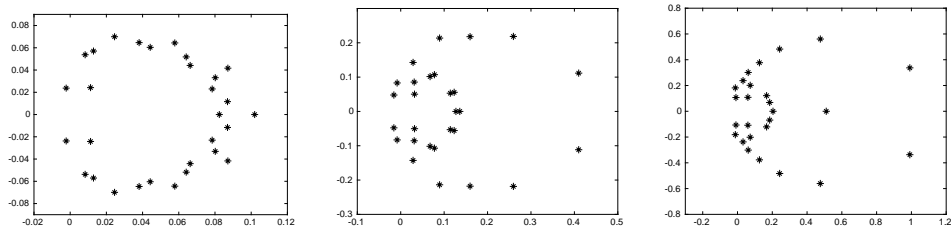To illustrate, we show in Fig. 17 the eigenvalues in the cases $(N, n) = (2, 30), (5, 28), (10, 26)$.

Figure 17: Eigenvalues of U_n in the case of the $(N+1)$-block-discrete Hahn weight functions with $(N, n) = (2, 30), (5, 28), (10, 26)$ (from left to right)

The restricted Stenger conjecture, in this example, fares much better, though failing also in a few cases. Using the routines `run_Uconj_restr_block hahn.m` and `Uconj_restr_blockhahn.m` for $N = 1:10, 2 \leq n \leq 30$, we found the conjecture to be true for $N = [1, 2, 3, 4, 9], 2 \leq n \leq 30$, and false in only the five cases, $(N, n) = (5, 30), (6, 28), (7, 30), (8, 28), (10, 25)$. To rule out the presence of severe numerical instabilities as a cause for this unexpected behavior, all cases have been rerun, and confirmed, in 32-digit arithmetic. The double-precision eigenvalues were compared with those obtained in 32-digit precision and found to agree to 5–15 digits, the delinquent ones always to at least 11 digits.

For illustration, we show in Fig. 18 the eigenvalues in the cases $(N, n) = (2, 30), (6, 28), (10, 25)$, the last two containing a pair of eigenvalues with negative real part.

The presence of delinquent eigenvalues in this example, strictly speaking, does not invalidate the extended Stenger conjecture, since the weight function (43) does not satisfy the positivity a.e. condition imposed by Stenger.
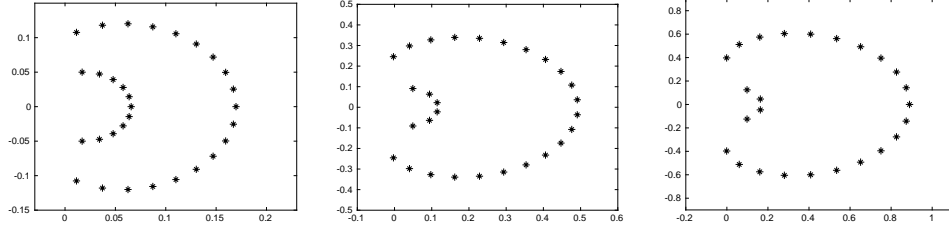
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Figure 18: Eigenvalues of U_n in the case of the $(N+1)$-block-discrete Hahn weight functions with $(N, n) = (2, 30), (6, 28), (10, 25)$ (from left to right)

However, the matrix U_n associated with the weight function (44), depending on the potitive parameter ε, by a continuity argument will have the same pattern of delinquent eigenvalues as the matrix U_n associated with the weight function (43) when ε is sufficiently small. This then shows that the extended Stenger conjecture cannot be valid for all admissible weight functions. We illustrate this with the final example,

Example 10. The $(N + 1)$-ε-block-discrete weight function (44) for $N = 2$, $\varepsilon = 1/100$, and $n = 9$.

This relates to the first item in Table 7. The routine `run_Uconj_ext_epsi lon_blockhahn_N2_n9.m`, using `r_blockhahn` to generate the required recurrence coefficients by an $(N+1)$-component discretization procedure $(N = 2)$ implemented by the routines `mcdis.m` and `quad_blockhahn.m`, computes the eigenvalues of U_n for $n = 9$. They are shown in Table 8.

k	λ_k	k	λ_k
1	$.269543881598 + .100451106056\,\mathrm{i}$	6	$.113959909084 + .146440180631\,\mathrm{i}$
2	$.269543881598 - .100451106056\,\mathrm{i}$	7	$.113959909084 - .146440180631\,\mathrm{i}$
3	$.257834699637$	8	$-.000421036373 + .156050111474\,\mathrm{i}$
4	$.242999895873 + .190318565957\,\mathrm{i}$	9	$-.000421036373 - .156050111474\,\mathrm{i}$
5	$.242999895873 - .190318565957\,\mathrm{i}$		

Table 8. The eigenvalues λ_k of U_n, $n = 9$, for the weight function of Example 10

Recomputing them in 32-digit arithmetic proves them correct to all digits shown.

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Appendix: Relation to Runge–Ktta methods Let $x_1, x_2, \ldots, x_n$ be distinct real numbers (typically in the interval $[0, 1]$). The corresponding (collocation) Runge–Kutta method (see [7, Theorem II. 7.7] is then given by the coefficients

$$(45) \qquad a_{jk} = \int_0^{x_j} \ell_k(x)\mathrm{d}x, \quad b_k = \int_0^1 \ell_k(x)\mathrm{d}x,$$

where $\ell_k(x)$ is the kth elementary Lagrange interpolation polynomial of degree $n - 1$. We collect the coefficients in the $n \times n$ matrix $A = (a_{jk})_{j,k=1}^n$, in the column vector $b = (b_k)_{k=1}^n$, and we denote the column vector with all elements equal to 1 by $\mathbb{1}$.

An application of the Runge–Kutta method with step size h to the Dahlquist test equation $\dot{y} = \lambda y$ yields (with $z = h\lambda$)

$$(46) \qquad y_1 = R(z)y_0, \quad R(z) = 1 + zb^{\mathrm{T}}(I - zA)^{-1}\mathbb{1},$$

where $R(z)$ is the stability function of the method. Note that for an invertible matrix A, its eigenvalues are the reciprocal of the poles of the rational function $R(z)$.

The adjoint method of (45) is given by the coefficients (cf. [7, Theorem II. 8.3])

$$(47) \qquad a^*_{n+1-j,n+1-k} = b_k - a_{jk} = \int_{x_j}^1 \ell_k(x)\mathrm{d}x, \quad b^*_{n+1-k} = b_k.$$

Its stability function is related to that of (45) by $R^*(z) = 1/R(-z)$.

Connection to the Stenger conjecture. The $n \times n$ matrix with coefficients a_{jk} of (45) is equal to the matrix U_n (with $a = 0$) of (2) in Section 1, and the matrix with coefficients a^*_{jk} of (47) is equal to V_n (with $b = 1$). Since the nonzero eigenvalues of A are the reciprocal of the poles of the stability function (46), there is a close connection between the Stenger conjecture and A-stability of a Runge–Kutta method.

The (shifted) Legendre polynomials are orthogonal with respect to the constant weight function $w(x) = 1$ on $[0, 1]$. The corresponding collocation Runge–Kutta method is the so-called Gauss method of order $2n$, which is A-stable (see [6, Section IV.5]). Its stability function is the diagonal Padé approximation $R_{n,n}(x)$, for which all poles are in the right half of the complex

plane. This provides another proof of the Stenger conjecture for Legendre polynomials.

Acknowledgments. The authors thank Martin J. Gander for having alerted them to the sensitivity, when n is large, of the eigenvalues of the matrices U_n, V_n to small changes in their elements, and they acknowlege helpful correspondence with Frank Stenger.

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