SIAM J. MATH. ANAL.
Vol. 44, No. 4, pp. 2428–2458

ASYMPTOTIC EXPANSIONS FOR REGULARIZED STATE-DEPENDENT NEUTRAL DELAY EQUATIONS*

NICOLA GUGLIELMI† AND ERNST HAIRER‡

Abstract. Singularly perturbed delay differential equations arising from the regularization of state-dependent neutral delay equations are considered. Asymptotic expansions of their solutions are constructed and their limit for $\varepsilon \to 0^+$ is studied. Due to discontinuities in the derivative of the solution of the neutral delay equation and the presence of different time scales when crossing breaking points, new difficulties have to be managed. A two-dimensional dynamical system is presented which characterizes whether classical or weak solutions are approximated by the regularized problem. A new type of expansion (in powers of $\sqrt{\varepsilon}$) turns out to be necessary for the study of the transition from weak to classical solutions. The techniques of this article can also be applied to the study of general singularly perturbed delay equations.

Key words. neutral delay differential equations, breaking points, termination, generalized solutions, singularly perturbed delay differential equations, asymptotic expansions

AMS subject classifications. 34K40, 34K26, 34E05

DOI. 10.1137/100801238

1. Introduction.

We start by considering systems of neutral delay equations of the form

$$\begin{aligned} \dot y(t) &= f\Big(y(t), \dot y\big(\alpha(y(t))\big)\Big) \qquad \text{for} \quad t > 0, \\ y(t) &= \varphi(t) \qquad \text{for} \quad t \le 0 \end{aligned} \tag{1.1}$$

with vector functions $f(y, z)$, $\varphi(t)$ and scalar deviating argument $\alpha(y)$ satisfying $\alpha(y(t)) < t$ (nonvanishing delay). For convenience, we assume that $f(y, z)$ is defined for all $y, z \in \mathbb{R}^n$, $\alpha(y)$ for all $y \in \mathbb{R}^n$, and $\varphi(t)$ for all $t \in \mathbb{R}$ and that these functions are sufficiently differentiable. All results and techniques presented in this paper carry over straightforwardly to situations where $f(y, z)$ also depends on t and on $y(\alpha(y(t)))$ and where several different nonvanishing delays are present. Neutral delay equations arise in several applications, for example, in the two-body problem of classical electrodynamics (see, e.g., [Dri65, Dri84]), in optimal control problems (see, e.g., [Kis91]), in the modeling of transmission lines (see, e.g., [RH92]), and in classical light dispersion theory [MCG07].

If we introduce the derivative $\dot y(t) = z(t)$ as a new variable, we obtain

$$\begin{aligned} \dot y(t) &= z(t), \\ 0 &= f\Big(y(t), z\big(\alpha(y(t))\big)\Big) - z(t) \end{aligned} \tag{1.2}$$

with $y(t) = \varphi(t)$ and $z(t) = \dot\varphi(t)$ for $t \le 0$. Collecting $y(t)$ and $z(t)$ in one vector $Y(t)$, this system can be written as $M\dot Y(t) = \mathcal{F}(Y(t), Y(\alpha(MY(t))))$ with a constant,

*Received by the editors July 6, 2010; accepted for publication (in revised form) May 21, 2012; published electronically July 10, 2012. This work was supported by the Fonds National Suisse, project 200020-126638, and the Italian INdAM-G.N.C.S. and M.I.U.R.

http://www.siam.org/journals/sima/44-4/80123.html

†Dipartimento di Matematica Pura e Applicata, Università dell'Aquila, via Vetoio (Coppito), I-67010 L'Aquila, Italy (guglielm@univaq.it).

‡Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, CH-1211 Genève 4, Switzerland (Ernst.Hairer@unige.ch).

singular matrix M (in our case $MY = M\binom{y}{z} = \binom{y}{0}$). This is the general form of a differential-algebraic delay equation. Codes that are written for such systems (like RADAR5 [GH01, GH08]) can therefore be applied to neutral state-dependent delay equations.

The problem (1.1) typically has a solution with jump discontinuities in its first derivative, and a classical solution can cease to exist there [ÈN73, BZ03, GH07]. If $\dot\varphi(0)$ is different from the right-hand side of (1.1) at $t = 0$, the first derivative of the solution $y(t)$ has a jump discontinuity at 0. Consequently, the right-hand side of (1.1) becomes in general discontinuous at points $\xi > 0$, where $\alpha(y(\xi)) = 0$, and this happens also with the first derivative of the solution $y(t)$. The same situation arises at points $\tilde\xi > \xi$ satisfying $\alpha(y(\tilde\xi)) = \xi$. Points $\xi, \tilde\xi, \ldots$ are called breaking points in the literature.

A standard way of avoiding these discontinuities is by regularization, where the differential-algebraic equation is turned into an ordinary differential equation. The special form (1.2) suggests we consider for a small positive parameter ε the system of singularly perturbed (nonneutral) delay differential equations

$$\begin{aligned} \dot y(t) &= z(t), \\ \varepsilon\, \dot z(t) &= f\Big(y(t), z\big(\alpha(y(t))\big)\Big) \;-\; z(t) \end{aligned} \tag{1.3}$$

with $y(t) = \varphi(t)$ and $z(t) = \dot\varphi(t)$ for $t \le 0$. The fact that $z(t)$ appears with a minus sign in the right-hand side lets us expect that in the limit $\varepsilon \to 0^+$, the solution of (1.3) will be close to that of (1.2). Any code for (nonneutral but stiff) delay equations can then be used to solve the problem. Among further possibilities of regularizing the problem (1.1), let us mention the recent articles [FG11] and [GH11].

With such a kind of regularization one is naturally confronted with the following questions: (i) Given a neutral delay equation (1.1), does the solution of the regularized delay equation (1.3) approximate the solution of (1.1)? Which solution is approximated in the absence of a classical solution of (1.1)? (ii) Given a singularly perturbed delay equation (1.3), what does the solution look like for small positive $\varepsilon > 0$? Our paper tries to answer these questions and some surprising results occur. We should mention that the techniques of this paper extend straightforwardly to more general singularly perturbed delay equations, in particular to problems where $\dot y(t)$ is some nonlinear function of $y(t)$ and $z(t)$.

An important application of the regularization of neutral delay equations occurs when exploring numerically the presence of periodic orbits (see [BG09]). In fact, if the initial datum does not lie close to the periodic orbit, the numerical integration of the neutral system might terminate after the integration has overcome a certain number of breaking points. The use of regularized equations is a convenient means to avoid such terminations.

Neutral state-dependent delay equations have very interesting dynamics, due to the presence of breaking points. In section 2 we discuss the situation where a classical solution ceases to exist. There are several possibilities of defining weak (or generalized) solutions beyond such a point, and we discuss in detail the generalization that is relevant for our regularization. Section 3 summarizes the main results of the article. The rest of the paper deals with the singularly perturbed delay equation (1.3). We study asymptotic expansions in powers of ε for the solution up to the first breaking point (section 4) and beyond it (section 5). We have two different expansions—one approximating a classical solution and the other a generalized solution. In most situations the asymptotic expansions and the exact solution of (1.3) approach the solution of (1.1) in the limit $\varepsilon \to 0$. However, there are exceptional situations where a classical solution exists beyond the first breaking point, but the solution of (1.3) approaches a weak solution (in the sense of section 2.1) for small $\varepsilon > 0$. It is possible to characterize this situation with the help of a two-dimensional dynamical system (section 6). Section 7 gives rigorous estimates for the defect and the remainder of truncated asymptotic expansions. Finally we discuss in section 8 the situation when a weak solution turns again into a classical solution. This requires a subtle analysis and leads to scaled asymptotic expansions in powers of $\sqrt{\varepsilon}$. The case of an emerging classical solution is always correctly reproduced by the regularized problem (1.3).

2. Features of neutral delay equations. By the method of steps, the problem (1.1) represents an ordinary differential equation between breaking points. The solution $y(t)$ is continuous at $t = 0$ (by definition), but its derivative has a jump discontinuity at $t = 0$ if

$$\dot\varphi(0) \neq f\big(\varphi(0), \dot\varphi\big(\alpha(\varphi(0))\big)\big). \tag{2.1}$$

We shall assume this throughout the article, and we use the notation

$$\dot y_0^+ = f\big(\varphi(0), \dot\varphi\big(\alpha(\varphi(0))\big)\big) \qquad \text{and} \qquad \dot y_0^- = \dot\varphi(0). \tag{2.2}$$

2.1. Weak solutions. The first breaking point t_0 is reached when $\alpha(y(t)) = 0$ for the first time. Since $\alpha(y(t)) < 0$ for $t < t_0$, the left-hand derivative satisfies (assuming $y(t)$ enters transversally the manifold defined by $\{y\,;\,\alpha(y) = 0\}$)

$$\frac{d}{dt}\alpha\big(y(t)\big)\Big|_{t=t_0^-} = \alpha'\big(y(t_0)\big) f\big(y(t_0), \dot y_0^-\big) > 0. \tag{2.3}$$

If the right-hand derivative of $\alpha(y(t))$ is also positive, then the solution leaves the manifold in the opposite direction and, by the method of steps, a classical solution continues to exist. If, however,

$$\alpha'\big(y(t_0)\big) f\big(y(t_0), \dot y_0^+\big) < 0, \tag{2.4}$$

we arrive at a contradiction. The solution cannot leave the manifold into the region $\{y\,;\,\alpha(y) > 0\}$ because of (2.4) and it cannot return into $\{y\,;\,\alpha(y) < 0\}$ because of (2.3). In this situation, the solution terminates at the first breaking point $t = t_0$.

The reason for this termination is the fact that for $\alpha(y(t)) = 0$ we require

$$\dot y\big(\alpha(y(t))\big) \in \big\{\dot y_0^+, \dot y_0^-\big\}, \qquad \text{two particular values.}$$

If we relax this condition and require only that

$$\dot y\big(\alpha(y(t))\big) \in \big[\dot y_0^+, \dot y_0^-\big], \qquad \text{a whole segment,}^1$$

the solution can be continued in a weak sense. We introduce a scalar variable $u(t)$ and assume that for $\alpha(y(t)) = 0$,

$$\dot y\big(\alpha(y(t))\big) = u(t)\, \dot y_0^- + (1 - u(t))\, \dot y_0^+,$$

where $0 \le u(t) \le 1$. This yields

$$\begin{aligned} \dot y(t) &= f\big(y(t), u(t)\, \dot y_0^- + (1 - u(t))\, \dot y_0^+\big), \\ 0 &= \alpha\big(y(t)\big), \end{aligned} \tag{2.5}$$

which is a differential-algebraic equation. Differentiating the algebraic constraint with

[1] The segment is the set $[\dot y_0^+, \dot y_0^-] = \{\theta \dot y_0^- + (1-\theta)\dot y_0^+\,;\, 0 \le \theta \le 1\}$.

respect to t yields the relation

$$\alpha'(y(t))f\big(y(t),u(t)\,\dot y_0^- + (1-u(t))\,\dot y_0^+\big) = 0 \tag{2.6}$$

that has to be satisfied by the scalar function $u(t)$. The condition (2.3) and the termination assumption (2.4) guarantee the existence of $u(t_0) \in (0,1)$ satisfying (2.6). If we assume in addition

$$\alpha'(y(t))f_z\big(y(t),u(t)\,\dot y_0^- + (1-u(t))\,\dot y_0^+\big)(\dot y_0^- - \dot y_0^+) \neq 0, \tag{2.7}$$

the implicit function theorem permits us to express $u(t)$ as a function of $y(t)$, and (2.5) can be solved. This assumption makes the differential-algebraic equation a problem of index 2 [HW96, section VII.1]. The solution of (2.5) is called a *weak* or *generalized* or *ghost* solution [ÈN73, BZ03].

Remark. If the function $f(y,z)$ is nonlinear in z, the relation (2.6) can have several solutions $u(t_0)$ in the open interval $(0,1)$. Consequently, a weak solution of the problem (1.1) need not be unique. Moreover, it is possible that the problem has a classical solution and weak solutions at the same time. This is the case when $\alpha'\big(y(t_0)\big)f\big(y(t_0),\dot y_0^+\big) > 0$ and there exist $u(t_0) \in (0,1)$ satisfying (2.6).

Remark. Our definition of weak solutions corresponds to a sliding mode[2] in the sense of Utkin [Utk92]. It is closely related to differential inclusions and Filippov solutions [Fil88]; see also [HNW93, p. 199]. Recall that a Filippov solution is defined by

$$\begin{aligned} \dot y(t) &= u(t)\,f\big(y(t),\dot y_0^-\big) + (1-u(t))\,f\big(y(t),\dot y_0^+\big), \\ 0 &= \alpha\big(y(t)\big), \end{aligned}$$

which coincides with (2.5) only if $f(y,z)$ is linear in z. The Filippov solution has the advantage of being unique. However, it will turn out that for $\varepsilon \to 0^+$ the regularized problem (1.3) approaches a weak solution (2.5) in the sense of Utkin rather than a Filippov solution.

2.2. Solution escaping from the manifold. As long as the solution $u(t)$ of (2.5) satisfies $0 < u(t) < 1$ we are concerned with a weak solution of the neutral delay equation. If it leaves this interval at time $t = t_1$, we have (generically) the following two possibilities:

- $u(t_1) = 1$, $\dot u(t_1) > 0$: solution returns to the region $\{y\,;\,\alpha(y) < 0\}$;
- $u(t_1) = 0$, $\dot u(t_1) < 0$: solution passes through the manifold into the region $\{y\,;\,\alpha(y) > 0\}$.

In both situations we switch again to the neutral delay differential equation $\dot y(t) = f(y(t),\dot y(\alpha(y(t))))$ and continue the solution in the classical sense.

Figure 2.1 illustrates this with two examples. The left picture shows the solution of

$$\dot y(t) = 4 - 2\,t - \dot y\big(y(t) - 3\big), \qquad t > 0,$$

with $y(t) = 0$ for $t \le 0$. Until the first breaking point $t_0 = 1$ it is given by $y(t) = 4t - t^2$, then it follows the manifold $y(t) = 3$ until $t_1 = 2$, and it leaves it along $y(t) = -1 + 4t - t^2$. The right picture of Figure 2.1 shows the solution of

$$\dot y(t) = 2 + 2\,t - 3\,\dot y\big(y(t) - 3\big), \qquad t > 0,$$

with $y(t) = 0$ for $t \le 0$. This time it is given by $y(t) = 2t + t^2$ until $t_0 = 1$, stays in the manifold $y(t) = 3$ until $t_1 = 2$, and passes through it as $y(t) = (41 + 6t + \mathrm{e}^{-6(t-2)})/18$.

[2]In control theory, one speaks of "sliding mode control" if the control function (here $u(t)$) forces the system to "slide" along the manifold $\{y\,;\,\alpha(y) = 0\}$.

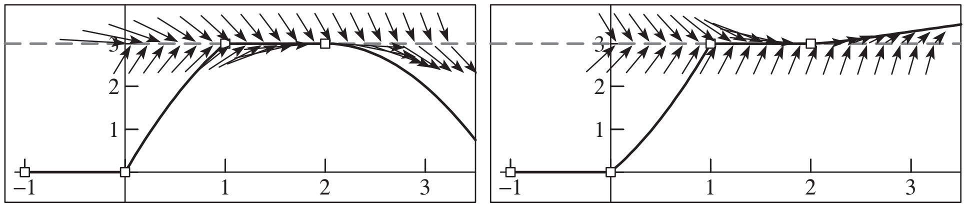

FIG. 2.1. *Solution $y(t)$ of neutral delay differential equations having a weak solution on the interval $[1,2]$. Left: solution returns into $\{y\,;\,\alpha(y)<0\}$; right: solution passes through the manifold.*

The arrows from below the manifold $\{y\,;\,\alpha(y)=y-3=0\}$ indicate the slopes $\alpha'(y(t))\,f(t,y(t),\dot y_0^-)$; those from above the slopes, $\alpha'(y(t))\,f(t,y(t),\dot y_0^+)$. (Note the t-dependence of f in this example.) A change of sign of one of these slopes permits the solution to escape from the manifold. This is equivalent to the above discussion of conditions on $u(t)$.

3. Main results. The aim of this article is to study the structure of the solution of (1.3) when $\varepsilon\to 0^+$ and to investigate the relationship between the limit solution and that of the neutral delay equation (1.1). This section gives an overview of the main results. Details are given in sections 4 through 8.

3.1. Until the first breaking point. If the neutral delay equation (1.1) has its first breaking point at t_0, then the assumption (2.3) implies that the regularized delay equation (1.3) has a breaking point at $t_0(\varepsilon)=t_0+\mathcal{O}(\varepsilon)$. We let $y(t)$ (and $z(t)=\dot y(t)$) be the solution of (1.1) and $y_\varepsilon(t)$, $z_\varepsilon(t)$ that of (1.3). For t between 0 and the first breaking point we then have

$$y_\varepsilon(t)-y(t)=\mathcal{O}(\varepsilon),\qquad z_\varepsilon(t)-z(t)=\mathcal{O}(\varepsilon)+\mathcal{O}(\mathrm{e}^{-t/\varepsilon}).$$

This part follows from singular perturbation theory for ordinary differential equations and is explained in section 4. The precise form of the exponentially decaying transient is important for the solution after the first breaking point. It is obtained from the study of asymptotic expansions.

3.2. Classical or weak solution. Beyond the first breaking point we can be concerned with a classical solution of (1.1) or with a weak solution, and the solution need not be unique. On the other hand, the solution of the singularly perturbed delay equation (1.3) exists beyond this point and is there uniquely defined. To study which solution is approximated in the limit $\varepsilon\to 0^+$, we introduce the scalar function

$$g(\theta)=\begin{cases}\alpha'\big(y(t_0)\big)\,f\big(y(t_0),\theta\,\dot y_0^-+(1-\theta)\,\dot y_0^+\big) & \text{for}\ \ \theta\le 1,\\ \alpha'\big(y(t_0)\big)\,f\big(y(t_0),\dot y_0^-\big) & \text{for}\ \ \theta\ge 1,\end{cases}\tag{3.1}$$

and we notice that $g(1)>0$ by (2.3). Geometrically, $g(\theta)$ represents the magnitude of the vector projection of $f\big(y(t_0),\theta\,\dot y_0^-+(1-\theta)\,\dot y_0^+\big)$ onto the normal to the manifold defined by $\{y\,;\,\alpha(y)=0\}$. This function determines the behavior of the solution for the neutral delay equation (1.1) as well as that for the singularly perturbed problem. For (1.1) we have that

- $g(0)>0$ implies the existence of a classical solution,
- $g(0)<0$ implies termination of a classical solution (see (2.4)),
- the existence of $\theta_0\in(0,1)$ satisfying $g(\theta_0)=0$ and $g'(\theta_0)\neq 0$ implies the presence of a weak solution with $\dot y(t_0^+)=f(y(t_0),\theta_0\,\dot y_0^-+(1-\theta_0)\,\dot y_0^+)$; see (2.5).

The question of whether the solution of the regularized problem (1.3) approximates a classical or weak solution of (1.1) beyond the first breaking point is determined by the solution of the following two-dimensional dynamical system:

$$\begin{aligned} \theta' &= -\theta\,\zeta, & \theta(0) &= 1, \\ \zeta' &= -\,\zeta + g(\theta), & \zeta(0) &= g(1). \end{aligned} \tag{3.2}$$

Its stationary points are $(0, g(0))$, which is attractive for $g(0) > 0$, and $(\theta_0, 0)$ with $g(\theta_0) = 0$, $\theta_0 \in (0,1)$, which is attractive when $g'(\theta_0) > 0$. The following theorem is a summary of the results proved in sections 5 through 7.

THEOREM 3.1. *Consider the solutions of the neutral delay equation* (1.1) *and its regularization* (1.3) *beyond the first breaking point* t_0.

(a) *If the solution of* (3.2) *converges to* $(0, g(0))$ *with* $g(0) > 0$*, then the solution of the regularized delay equation* (1.3) *is* $\mathcal{O}(\varepsilon)$*-close to the classical solution of* (1.1).

(b) *If the solution of* (3.2) *converges to* $(\theta_0, 0)$*, where* $g(\theta_0) = 0$ *and* $g'(\theta_0) > 0$*, then the solution of the regularized delay equation* (1.3) *is* $\mathcal{O}(\varepsilon)$*-close to the weak solution of* (1.1) *satisfying* $\dot y(t_0^+) = f\big(y(t_0), \theta_0\, \dot y_0^- + (1-\theta_0)\, \dot y_0^+\big)$.

It comes as a surprise to us that even when a classical solution exists beyond the first breaking point, the solution of the regularized equation can converge to a weak solution. A concrete example will be presented in section 6.2. Fortunately, Theorem 3.1 gives a precise characterization of this situation. In particular, if the function $g(\theta)$ does not admit a zero in the interval $(0,1)$ (no weak solution), then the solution of the regularized problem correctly approaches the classical solution. Theorem 3.1 also tells us which weak solution is selected by the regularization in the presence of several weak solutions.

Similar to the initial point $t = 0$ we also have a transient layer right after the first breaking point. On an ε-independent compact interval starting at the first breaking point t_0 we shall prove in section 7 the estimates

$$y_\varepsilon(t) - y(t) = \mathcal{O}(\varepsilon), \qquad z_\varepsilon(t) - z(t) = \mathcal{O}(\varepsilon) + \mathcal{O}(\mathrm{e}^{-(t-t_0)/\varepsilon}).$$

As before, $y_\varepsilon(t)$, $z_\varepsilon(t)$ denotes the unique solution of (1.3), and $y(t)$, $z(t)$ is the solution of (1.1) according to Theorem 3.1.

3.3. Escaping from sliding mode. At first glance, the transition from a weak solution (sliding mode) to a classical solution seems to be more delicate. It is interesting that the regularization (1.3) always correctly approximates such a transition. To be more precise, let us distinguish the two situations discussed in section 2.2.

If the solution of (1.1) returns to the region $\{y\,;\, \alpha(y) < 0\}$ at $t = t_1$, we can prove

$$y_\varepsilon(t) - y(t) = \mathcal{O}(\varepsilon), \qquad z_\varepsilon(t) - z(t) = \mathcal{O}(\varepsilon)$$

in an ε-independent neighborhood of t_1. An exponentially decaying transient phase is still present, but it is multiplied by ε^2 for the y-component and by ε for the z-component, so that they are dominated by the smooth perturbation terms.

If the solution of (1.1) leaves the sliding mode at $t = t_1$ into the opposite region $\{y\,;\, \alpha(y) > 0\}$, the analysis is much more involved. We shall prove that in an ε-independent neighborhood of t_1 we have

$$y_\varepsilon(t) - y(t) = \mathcal{O}(\varepsilon \ln \sqrt{\varepsilon}), \qquad z_\varepsilon(t) - z(t) = \mathcal{O}(\sqrt{\varepsilon}),$$

which still tends to zero for $\varepsilon \to 0$. Remarkably, the expression $\alpha(y_\varepsilon(t))$ for the solution $y_\varepsilon(t)$ of (1.3) satisfies at $t = t_1$

$$\alpha\big(y_\varepsilon(t_1)\big) = -\varepsilon \ln \sqrt{\varepsilon} + \mathcal{O}(\varepsilon)$$

and has a leading error term that does neither depend on $f(y,z)$ nor on $\alpha(y)$. These statements are obtained by patching together three different asymptotic expansions: in powers of ε for $t \le t_1 - \varepsilon^{1/3}$ and a different one for $t \ge t_1 + \varepsilon^{1/3}$, and an expansion in powers of $\sqrt{\varepsilon}$ on the interval $[t_1 - \varepsilon^{1/3}, t_1 + \varepsilon^{1/3}]$. A rigorous formulation of the results and detailed proofs are given in sections 8.2 (Theorem 8.5) and 8.3.

3.4. Subsequent breaking points. One can ask whether similar results hold also at breaking points $\widetilde{t}$ that are induced by t_0 and not by 0, i.e., for which we have $\alpha(y(\widetilde{t})) = t_0$. The main difference is that the deviated argument is no longer close to zero but is now close to t_0. This question is addressed in the recent article [GH11]. There it is shown that the present analysis can be extended straightforwardly and, although the function $g(\theta)$ in the dynamical system (3.2) becomes slightly more complicated, the same conclusions can be drawn.

4. Asymptotic expansion up to the first breaking point. As long as the solution of (1.3) satisfies $\alpha\big(y(t)\big) \le 0$, we are concerned with a singularly perturbed ordinary differential equation

$$\begin{aligned} \dot y(t) &= z(t) \\ \varepsilon\, \dot z(t) &= F\big(y(t)\big) \,-\, z(t) \end{aligned} \qquad \text{with} \qquad F(y) = f\big(y, \dot\varphi(\alpha(y))\big), \tag{4.1}$$

and we can apply standard techniques for the study of its solution; see [O'M91], [HW96, section VII.3]. This theory tells us that the solution can be split into a smooth and transient part (or outer and inner solution or smooth and nonsmooth) and expanded into a series in powers of ε as follows:

$$\begin{aligned} y(t) &= \sum_{j=0}^{N} \varepsilon^j\, y_j(t) + \varepsilon \sum_{j=0}^{N-1} \varepsilon^j\, \eta_j(t/\varepsilon) + \mathcal{O}(\varepsilon^{N+1}), \\ z(t) &= \sum_{j=0}^{N} \varepsilon^j\, z_j(t) + \sum_{j=0}^{N} \varepsilon^j\, \zeta_j(t/\varepsilon) + \mathcal{O}(\varepsilon^{N+1}). \end{aligned} \tag{4.2}$$

Here, $y_j(t)$ and $z_j(t)$—called smooth coefficient functions—are defined on an ε-independent interval $[0,T]$. The functions $\eta_j(\tau)$ and $\zeta_j(\tau)$—called transient coefficient functions—are defined for all $\tau \ge 0$, and they decay exponentially fast to zero for $\tau \to \infty$, i.e., they are bounded by $c\,\mathrm{e}^{-\gamma\tau}$ with some $c > 0$ and $\gamma > 0$. The integer N is an arbitrarily chosen truncation index.

These expansions have to match the initial values, which means that

$$\begin{aligned} y_0(0) &= \varphi(0), \qquad & y_j(0) + \eta_{j-1}(0) &= 0 \qquad \text{for } \ j \ge 1, \\ z_0(0) + \zeta_0(0) &= \dot\varphi(0), \qquad & z_j(0) + \zeta_j(0) &= 0 \qquad \text{for } \ j \ge 1. \end{aligned} \tag{4.3}$$

The coefficient functions are obtained by inserting the expansion (4.2) into (4.1), separating smooth and transient parts, and comparing like powers of ε. The smooth part yields for $\varepsilon = 0$ the relations

$$\dot y_0(t) = z_0(t) = F(y_0(t)), \qquad z_0(t) = F(y_0(t)).$$

The initial value $y_0(0) = \varphi(0)$ is given from (4.3). The algebraic relation determines the initial value $z_0(0) = F(\varphi(0))$ and by (4.3) also that for $\zeta_0(0)$. The transient part (here, prime denotes the derivative with respect to τ) gives

$$\eta_0'(\tau) = \zeta_0(\tau), \qquad \zeta_0'(\tau) = -\zeta_0(\tau),$$

which yields $\zeta_0(\tau) = \zeta_0(0)\,\mathrm{e}^{-\tau}$ and $\eta_0(\tau) = C - \zeta_0(0)\,\mathrm{e}^{-\tau}$. Since the transient coefficient functions have to decay exponentially for $\tau \to \infty$, it follows $C = 0$ so that the initial value satisfies $\eta_0(0) = -\zeta_0(0)$.

In a next step we have to solve the differential-algebraic system

$$\dot{y}_1(t) = z_1(t), \qquad \dot{z}_0(t) = F'(y_0(t))\, y_1(t) - z_1(t)$$

with initial value $y_1(0)$ given from $y_1(0) + \eta_0(0) = 0$. This is a linear differential equation for $y_1(t)$ and gives an explicit formula for $z_1(t)$. The initial value $\zeta_1(0)$ is then determined from $z_1(0) + \zeta_1(0) = 0$. The transient functions are defined by

$$\eta_1'(\tau) = \zeta_1(\tau), \qquad \zeta_1'(\tau) = F'(y_0(0))\, \eta_0(\tau) - \zeta_1(\tau).$$

This shows that $\zeta_1(\tau)$ and $\eta_1(\tau)$ are polynomials of degree one multiplied by $\mathrm{e}^{-\tau}$. We continue this procedure to compute further terms in the ε-expansion (4.2). The construction of the smooth coefficient functions is straightforward. For the transient coefficient functions we notice that they are defined by differential equations

$$\eta_j'(\tau) = \zeta_j(\tau), \qquad \zeta_j'(\tau) = -\,\zeta_j(\tau) + F'(y_0(0))\, \eta_{j-1}(\tau) + \cdots,$$

where the dots represent a linear combination of products $\tau^{j_0} \prod_{i=1}^{m} \eta_{j_i - 1}(\tau)$ with $m \geq 1$, $j_0 \geq 0$, $j_1 \geq 1, \ldots, j_m \geq 1$, and $j_0 + j_1 + \cdots + j_m = j$. It follows by induction on j that $\zeta_j(\tau)$ and $\eta_j(\tau)$ are of the form

$$p_{j1}(\tau)\mathrm{e}^{-\tau} + p_{j2}(\tau)\mathrm{e}^{-2\tau} + \cdots + p_{jj}(\tau)\mathrm{e}^{-j\tau} \tag{4.4}$$

with polynomials $p_{jk}(\tau)$ of degree $\leq j$ for $k = 1$ and of degree $\leq j - k$ for $k = 2, \ldots, j$. The case $j = 1$ has been treated before. Assume the statement to be true up to j. The inhomogeneity of the differential equation for $\zeta_{j+1}(\tau)$ is then a linear combination of terms $q_k(\tau)\mathrm{e}^{-k\tau}$, where $q_k(\tau)$ denotes a polynomial of degree $\leq j - k + 1$. The solution $\zeta_{j+1}(\tau)$ is then of the same form with the exception that, due to resonance, the degree of $q_1(\tau)$ is increased by one. This proves the statement for $\zeta_{j+1}(\tau)$. The function $\eta_{j+1}(\tau)$ is obtained by integration of $\zeta_{j+1}(\tau)$ and has the stated form, because $\eta_{j+1}(\tau) \to 0$ for $\tau \to \infty$.

The breaking point $t_0(\varepsilon)$ of the system (1.3) is the time instant t for which

$$a(t, \varepsilon) := \alpha\big(y_0(t) + \varepsilon y_1(t) + \cdots\big) = 0.$$

Recall that $\varphi(t)$ is defined and smooth for all $t \in \mathbb{R}$ and that $F(y) = f(y, \dot{\varphi}(\alpha(y)))$ is defined for all $y \in \mathbb{R}^n$. Consequently, the smooth coefficient functions $y_j(t)$ are defined also beyond the point t_0. Since $\frac{\partial a}{\partial t}(t_0, 0) = \alpha'(y_0(t_0))\dot{y}_0(t_0) > 0$, which is equivalent to (2.3), the implicit function theorem guarantees the existence of $t_0(\varepsilon) = t_0 + \mathcal{O}(\varepsilon)$, such that $a(t_0(\varepsilon), \varepsilon) = 0$. The smoothness of the appearing functions implies that $t_0(\varepsilon)$ can be expanded into powers of ε up to errors of size $\mathcal{O}(\varepsilon^{N+1})$. Finally, Theorem 3.2 of [HW96, p. 391] shows that the remainder in (4.2) is bounded uniformly for $0 \leq t \leq t_0(\varepsilon)$.

5. Asymptotic expansion beyond the first breaking point. For $t > t_0(\varepsilon)$ until the following breaking point, the problem (1.3) becomes

$$\begin{aligned} \dot{y}(t) &= z(t), \\ \varepsilon\, \dot{z}(t) &= f\Big(y(t), \widetilde{z}\big(\alpha(y(t))\big)\Big) \,-\, z(t), \end{aligned} \tag{5.1}$$

where

$$\widetilde{z}(t) = s(\varepsilon, t) \, + \, p(\varepsilon, t, \varepsilon \, \mathrm{e}^{-\tau}) \, \mathrm{e}^{-\tau} + \mathcal{O}(\varepsilon^{N+1}) \qquad \text{with} \quad \tau = t/\varepsilon \tag{5.2}$$

is the solution expansion (4.2) on the interval $[0, t_0(\varepsilon)]$. Here, the function $s(\varepsilon, t) = s_0(t) + \varepsilon s_1(t) + \cdots$ is the smooth part[3] of the expansion, and $p(\varepsilon, t, u)$ is a polynomial of degree at most N in ε and t and of degree at most $N-1$ in u. This special structure follows from the fact that the transient coefficient functions $\zeta_j(\tau)$ are of the form (4.4). Using the notation (2.2) we have $s(0,0) = \dot{y}_0^+$ and $p(0,0,0) = \dot{y}_0^- - \dot{y}_0^+$ (the jump discontinuity of the derivative at $t = 0$; see (2.1)).

Initial values for (5.1) are the solution values of the system (4.1) at $t = t_0(\varepsilon)$. Since $t_0(\varepsilon)$ admits an expansion in powers of ε (see end of section 4), the smooth coefficient functions $y_j(t_0(\varepsilon))$ also have such an expansion. Furthermore, the transient functions are all dominated by an $\mathcal{O}(\varepsilon^{N+1})$ error term. Therefore, we have at the breaking point expansions of the form

$$\begin{aligned} y\big(t_0(\varepsilon)\big) &= a_0 + a_1\varepsilon + \cdots + a_N\varepsilon^N + \mathcal{O}(\varepsilon^{N+1}), \\ z\big(t_0(\varepsilon)\big) &= b_0 + b_1\varepsilon + \cdots + b_N\varepsilon^N + \mathcal{O}(\varepsilon^{N+1}) \end{aligned} \tag{5.3}$$

with $a_0 = y_0(t_0)$ and $b_0 = z_0(t_0)$. These initial values satisfy

$$\alpha'(a_0)\, a_1 = 0, \qquad \alpha'(a_0)\, b_0 > 0. \tag{5.4}$$

The first relation is obtained by computing the first derivative of $\alpha(y(t_0(\varepsilon))) = 0$ with respect to ε at $\varepsilon = 0$, and the second one is equivalent to (2.3).

For the solution of (5.1) we make the ansatz (with coefficient functions different from those of section 4)

$$\begin{aligned} y(t_0(\varepsilon) + t) &= \sum_{j=0}^{N} \varepsilon^j \, y_j(t) + \varepsilon \sum_{j=0}^{N-1} \varepsilon^j \, \eta_j(t/\varepsilon) + \mathcal{O}(\varepsilon^{N+1}), \\ z(t_0(\varepsilon) + t) &= \sum_{j=0}^{N} \varepsilon^j \, z_j(t) + \sum_{j=0}^{N} \varepsilon^j \, \zeta_j(t/\varepsilon) + \mathcal{O}(\varepsilon^{N+1}), \end{aligned} \tag{5.5}$$

where, similar to section 4, $y_j(t), z_j(t)$ are smooth coefficient functions defined on a compact interval $[0, T]$, and $\eta_j(\tau), \zeta_j(\tau)$ are transient coefficient functions defined for all $\tau \geq 0$ and converging exponentially fast to zero for $\tau \to \infty$. For $t = 0$ these expansions have to match (5.3), i.e.,

$$y_0(0) = a_0, \quad y_{j+1}(0) + \eta_j(0) = a_{j+1}, \quad z_j(0) + \zeta_j(0) = b_j \qquad \text{for} \quad j \geq 0. \tag{5.6}$$

We insert the expansions (5.5) into the singularly perturbed problem (5.1) and compare like powers of ε in the smooth as well as transient parts of the system. This

[3] In fact, we have $s_j(t) = y_j(t)$, where $y_j(t)$ are the smooth coefficient functions of (4.2). We change the notation to avoid a confusion with the coefficient functions of (5.5).

yields

$$\dot y_j(t) = z_j(t), \qquad \eta_j'(\tau) = \zeta_j(\tau) \qquad \text{for} \quad j \ge 0 \tag{5.7}$$

for the first (trivial) equation. Putting

$$A = y_1(t) + \eta_0(\tau) + \varepsilon\,\big(y_2(t) + \eta_1(\tau)\big) + \cdots \qquad \text{and} \qquad B = \varepsilon^{-1}\,\alpha\big(y_0(t) + \varepsilon\,A\big),$$

we obtain for the nontrivial part

$$\begin{aligned} &\sum_{j\ge 0} \varepsilon^{j+1} \dot z_j(t) \;+\; \sum_{j\ge 0} \varepsilon^j \zeta_j'(\tau) \;+\; \sum_{j\ge 0} \varepsilon^j z_j(t) \;+\; \sum_{j\ge 0} \varepsilon^j \zeta_j(\tau) \\ &\quad = \; f\Big(y_0(t) + \varepsilon\,A,\; s\big(\varepsilon, \varepsilon\,B\big) + p\big(\varepsilon, \varepsilon\,B, \varepsilon\,\mathrm{e}^{-B}\big)\,\mathrm{e}^{-B}\Big) + \mathcal{O}(\varepsilon^{N+1}) \end{aligned} \tag{5.8}$$

whenever $B \ge 0$. This is the case between $t_0(\varepsilon)$ and the following breaking point. On intervals, where $B < 0$, the right-hand side of (5.8) has the simple form

$$\cdots \;=\; f\Big(y_0(t) + \varepsilon\,A,\; \dot\varphi\big(\alpha\big(y_0(t) + \varepsilon\,A\big)\big)\Big) + \mathcal{O}(\varepsilon^{N+1}). \tag{5.9}$$

For the construction of the coefficient functions of (5.5) we distinguish the following two cases:

- $\alpha(y_0(t)) = 0$ for $t \in [0, T]$: in this case the expression B is uniformly bounded in ε, and the exponential term in (5.8) gives a contribution to the smooth part of the system;
- $\alpha(y_0(t)) > 0$ for $t \in (0, T]$: in this case the exponential term will contribute only to the transient part of the system.

5.1. Expansion for a solution close to the manifold. In this section we construct coefficient functions of (5.5) such that $\alpha(y_0(t)) = 0$ for $t \in [0, T]$. The truncated expansion (5.5) will then be $\mathcal{O}(\varepsilon)$-close to the manifold $\{y\,;\,\alpha(y) = 0\}$. Together with (5.7), this implies that

$$\alpha'\big(y_0(t)\big) z_0(t) = 0. \tag{5.10}$$

Expanding $\alpha(y_0(t) + \varepsilon A)$ into a Taylor series around $y_0(t)$ and using $\alpha(y_0(t)) = 0$, the expression B in (5.8) is seen to become

$$B = \alpha'(y_0(t))\Big(y_1(t) + \eta_0(\tau) + \varepsilon\,\big(y_2(t) + \eta_1(\tau)\big)\Big) + \frac{\varepsilon}{2}\,\alpha''(y_0(t))\big(y_1(t) + \eta_0(\tau)\big)^2 + \cdots .$$

For the construction of the coefficient functions in (5.5), we expand the nonlinearity into powers of ε, we separate the smooth and transient parts in (5.8) and respectively (5.9), and we compare like powers of ε. The ε^0 term in (5.8), i.e., the equation obtained by putting $\varepsilon = 0$, yields

$$\zeta_0'(\tau) + z_0(t) + \zeta_0(\tau) = f\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\mathrm{e}^{\alpha'(y_0(t))(y_1(t) + \eta_0(\tau))}\big). \tag{5.11}$$

Its smooth term (i.e., τ-independent term) is

$$z_0(t) = f\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\mathrm{e}^{\alpha'(y_0(t)) y_1(t)}\big), \tag{5.12}$$

where $\dot y_0^+$ and $\dot y_0^-$ are as in (2.2). The construction of the coefficient functions is done in the following steps.

Step 1a. Multiplying (5.12) by $\alpha'(y_0(t))$ and using (5.10) yields

$$0 = \alpha'\big(y_0(t)\big)\, f\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\, u_0(t)\big) \quad \text{with} \quad u_0(t) = \mathrm{e}^{-\alpha'(y_0(t))y_1(t)}. \tag{5.13}$$

If there exists $c > 0$ such that $\alpha'(y_0(0))\, f(y_0(0), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\,\mathrm{e}^{-c}) = 0$ (for a justification see Step 1c), and if $\alpha'(y_0(0))\, f_z(y_0(0), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\,\mathrm{e}^{-c})(\dot y_0^- - \dot y_0^+) \neq 0$, an application of the implicit function theorem shows that the relation (5.13) permits us to express the scalar function $u_0(t)$ (satisfying $u_0(0) = \mathrm{e}^{-c}$) in terms of $y_0(t)$.

Step 1b. The relations (5.7) and (5.12) give the system

$$\dot y_0(t) = z_0(t), \qquad z_0(t) = f\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\, u_0(t)\big).$$

Inserting $u_0(t)$ from Step 1a, this yields a differential equation for $y_0(t)$ and an explicit formula for $z_0(t)$. The initial values $y_0(0)$ and $z_0(0) + \zeta_0(0)$ are available from (5.6). This therefore fixes $\zeta_0(0)$.

Step 1c. We obtain the transient part by subtracting the smooth part (5.12) from (5.11), then substituting $\varepsilon\tau$ for t, and finally taking the coefficient of ε^0. This yields

$$\begin{aligned} \zeta_0'(\tau) + \zeta_0(\tau) = {} & f\big(y_0(0), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\,\mathrm{e}^{-\alpha'(y_0(0))(y_1(0)+\eta_0(\tau))}\big) \\ & - f\big(y_0(0), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\,\mathrm{e}^{-c}\big), \end{aligned} \tag{5.14}$$

where $c = \alpha'(y_0(0))y_1(0)$. Introducing the scalar functions

$$\widehat\eta_0(\tau) = \alpha'\big(y_0(0)\big)\big(y_1(0) + \eta_0(\tau)\big), \qquad \widehat\zeta_0(\tau) = \alpha'\big(y_0(0)\big)\zeta_0(\tau)$$

leads to $\widehat\eta_0'(\tau) = \widehat\zeta_0(\tau)$, and left-multiplying (5.14) by $\alpha'(y_0(0))$ gives

$$\widehat\zeta_0'(\tau) + \widehat\zeta_0(\tau) \;=\; \alpha'(y_0(0))\, f\big(y_0(0), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\,\mathrm{e}^{-\widehat\eta_0(\tau)}\big). \tag{5.15}$$

The initial value $\widehat\eta_0(0) = 0$ is given by (5.4), because $y_0(0) = a_0$ and $y_1(0)+\eta_0(0) = a_1$ (see (5.3)), and $\widehat\zeta_0(0)$ is given by Step 1b. Section 5.2 discusses the situation when the expression B becomes negative for t in certain intervals. Section 6 studies conditions guaranteeing that the solution components $\widehat\eta_0(\tau)$ and $\widehat\zeta_0(\tau)$ of this system converge exponentially fast to c and 0, respectively.

A logical reasoning would start with the dynamical system for $(\widehat\eta_0(\tau), \widehat\zeta_0(\tau))$ and assuming that its solution converges exponentially fast to $(c, 0)$. This then provides the positive number c which was required in Step 1a.

Step 1d. The right-hand side of (5.14) converges exponentially fast to zero for $\tau \to \infty$ (i.e., it is bounded by a function $c\,\mathrm{e}^{-\gamma\tau}$ with positive c and γ), so that this is also true for the solutions of (5.14), in particular for that corresponding to the initial value given by Step 1b. The function $\eta_0(\tau)$ is obtained by integration of $\eta_0'(\tau) = \zeta_0(\tau)$. For a suitably chosen initial value, it converges exponentially fast to zero. This initial value then determines $y_1(0)$ by the continuity requirement (5.6) of the solution at the breaking point.

Step 2a. We next differentiate (5.13) with respect to t. Under the assumption

$$\alpha'\big(y_0(t)\big) f_z\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\, u_0(t)\big)(\dot y_0^- - \dot y_0^+) \neq 0, \tag{5.16}$$

the scalar function $\alpha'(y_0(t))\dot y_1(t)$ and hence also $\alpha'(y_0(t))z_1(t)$ can be expressed in terms of $y_1(t)$ and the known functions $y_0(t)$ and $\dot y_0(t)$. The smooth part of the coefficient of ε in (5.8) gives

$$\begin{aligned} \dot z_0(t) + z_1(t) = {} & f_z\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\, u_0(t)\big)(\dot y_0^- - \dot y_0^+) \\ & \cdot u_0(t)\, \alpha'(y_0(t))\, y_2(t) + \cdots, \end{aligned} \tag{5.17}$$

where the dots represent an expression depending only on $y_0(t)$ and $y_1(t)$. Premultiplication of this equation with $\alpha'(y_0(t))$ therefore implies that $\alpha'(y_0(t))y_2(t)$ can be expressed in terms of $y_1(t)$ and known functions.

Step 2b. As a consequence of Step 2a and formula (5.17), not only the function $\alpha'(y_0(t))y_2(t)$ but also $z_1(t)$ can be expressed in terms of $y_1(t)$ and known functions. Inserting the resulting formula for $z_1(t)$ into $\dot{y}_1(t) = z_1(t)$ yields a differential equation for $y_1(t)$. The initial value, already determined in Step 1d, thus gives the function $y_1(t)$. From the relation $z_1(t) = \dot{y}_1(t)$ this step thus yields $z_1(t)$ and, by (5.6), the initial value $\zeta_1(0)$.

Step 2c. The transient coefficient of ε in (5.8) yields $\eta_1'(\tau) = \zeta_1(\tau)$ and

$$\zeta_1'(\tau) + \zeta_1(\tau) = -\,G'\big(\mathrm{e}^{-\widehat{\eta}_0(\tau)}\big)\,\mathrm{e}^{-\widehat{\eta}_0(\tau)}\,\alpha'(y_0(0))(y_2(0)+\eta_1(\tau)) + r\big(\tau,\eta_0(\tau)\big), \tag{5.18}$$

where $G(\theta) := f(y_0(0), \dot{y}_0^+ + (\dot{y}_0^- - \dot{y}_0^+)\,\theta)$ for $\theta \in \mathbb{R}$, the functions $\widehat{\eta}_0(\tau)$ and $\eta_0(\tau)$ are given from Steps 1c and 1d, and $r(\tau,\eta)$ collects the remaining terms. The computation of the ε-coefficient in (5.8) shows that the inhomogeneity $r(\tau,\eta_0(\tau))$ depends at most polynomially on τ and contains $\eta_0(\tau)$ as a factor, so that it converges exponentially fast to zero for $\tau \to \infty$. Premultiplication of these equations by $\alpha'(y_0(0))$ gives a linear nonautonomous system for

$$\widehat{\eta}_1(\tau) = \alpha'(y_0(0))(y_2(0)+\eta_1(\tau)), \qquad \widehat{\zeta}_1(\tau) = \alpha'(y_0(0))\,\zeta_1(\tau).$$

We let $g(\theta) = \alpha'(y_0(0))\,G(\theta)$. Since $\widehat{\eta}_0(\tau) \to c$ exponentially fast (see Step 1c), the functions $g'(\mathrm{e}^{-\widehat{\eta}_0(\tau)})$ and $\alpha'(y_0(0))\,r(\tau,\eta_0(\tau))$ converge exponentially fast to $g'(\mathrm{e}^{-c})$ and to 0, respectively. Initial values are given by Steps 2a–2b. Assuming $g'(\mathrm{e}^{-c}) > 0$ (see Theorem 6.1(a) below), the linear system obtained by replacing $g'(\mathrm{e}^{-\widehat{\eta}_0(\tau)})$ with $g'(\mathrm{e}^{-c})$ is asymptotically stable. This implies that the solutions $\widehat{\eta}_1(\tau)$ and $\widehat{\zeta}_1(\tau)$ converge exponentially fast to zero.

Step 2d. The right-hand side of (5.18) converges exponentially fast to zero, so that this is also true for its solution with initial value given by Step 2b. The function $\eta_1(\tau)$ is obtained by integration of $\eta_1'(\tau) = \zeta_1(\tau)$. For a suitably chosen initial value, it converges exponentially fast to zero. This initial value then determines $y_2(0)$ by the continuity requirement (5.6) of the solution at the breaking point.

This analysis extends straightforwardly to further terms in the asymptotic expansion. The only difference is that in the differential equation for $\eta_k(\tau)$ and $\zeta_k(\tau)$, the function r in (5.18) will depend on $\eta_j(\tau)$ for $j = 0, 1, \ldots, k-1$.

5.2. Multiple breaking points. In the situation of section 5.1 it is possible that the solution $\widehat{\eta}_0(\tau)$ in Step 1c stays nonnegative for all $\tau > 0$. In this case the asymptotic expansion of section 5.1 is valid on an ε-independent nonempty interval.

It may also happen that $\widehat{\eta}_0(\tau)$ changes sign, i.e., there exists $\tau_1 > 0$ such that $\widehat{\eta}_0(\tau_1) = 0$ and $\widehat{\eta}_0{}'(\tau_1) = \widehat{\zeta}_0(\tau_1) \le 0$. (Equality can be excluded, because $\widehat{\eta}_0(\tau_1) = \widehat{\zeta}_0(\tau_1) = 0$ implies $\widehat{\eta}_0{}''(\tau_1) = \widehat{\zeta}_0{}'(\tau_1) > 0$ by (5.22) and (2.3), so that the function cannot change sign at τ_1.) In this situation the regularized problem (1.3) has a breaking point at $t_1(\varepsilon) = t_0(\varepsilon) + \varepsilon\,\tau_1 + \mathcal{O}(\varepsilon^2)$ (as a consequence of the implicit function theorem), and the differential equation (5.9) has to be considered beyond $t_1(\varepsilon)$. Therefore, the differential equations (5.14) and (5.15) have to be modified as follows: the first expression in the right-hand side of (5.14) is now the ε-independent term of (5.9) which is $f(y_0(0), \dot{y}_0^-)$ because of $\alpha(y_0(t)) = 0$. The second expression remains unchanged, because we do not touch the functions $y_0(t)$, $z_0(t)$, and $u_0(t)$. Beyond τ_1 and as long as $\widehat\eta_0(\tau)$ remains negative, the differential equations (5.14) and (5.15) thus have to be replaced by

$$\zeta_0'(\tau)+\zeta_0(\tau)=f\big(y_0(0),\dot y_0^-\big)-f\big(y_0(0),\dot y_0^+ +(\dot y_0^- -\dot y_0^+)\,\mathrm{e}^{-c}\big), \tag{5.19}$$

$$\widehat\zeta_0'(\tau)+\widehat\zeta_0(\tau)=\alpha'(y_0(0))\,f\big(y_0(0),\dot y_0^-\big). \tag{5.20}$$

The solution of (5.20) with initial value $\widehat\zeta_0(\tau_1)<0$ converges to the positive value $\alpha'(y_0(0))\,f(y_0(0),\dot y_0^-)>0$ (see (2.3)). Consequently, there exists $\tau_2>\tau_1$ for which the solution of $\widehat\eta_0'(\tau)=\zeta_0(\tau)$ satisfies $\widehat\eta_0(\tau_2)=0$ and $\widehat\eta_0'(\tau_2)>0$. This gives rise to a further breaking point $t_2(\varepsilon)=t_0(\varepsilon)+\varepsilon\,\tau_2+\mathcal{O}(\varepsilon^2)$ of the singularly perturbed problem (1.3). Beyond this breaking point we have to consider again (5.15). This situation may repeat itself, and we can be concerned with an odd number of breaking points that are all $\mathcal{O}(\varepsilon)$-close to t_0.

The considerations of this section can be incorporated in the previous construction of the asymptotic expansion. All we have to do is to replace in (5.14) and (5.15) the function $\widehat\eta_0(\tau)$ with $\max(0,\widehat\eta_0(\tau))$. In this way the correct differential equation is chosen for positive and also for negative $\widehat\eta_0(\tau)$. The smooth part of the expansion is not influenced by the presence of several breaking points that are ε-close to the termination instant t_0.

5.3. Expansion for a solution transversal to the manifold. Here we consider the situation where $\alpha(y_0(0))=0$ at the breaking point, but soon after $\alpha(y_0(t))$ becomes positive. More precisely, opposed to (5.10), we assume that

$$\alpha'\big(y_0(t)\big)\,z_0(t)>0$$

at $t=0$, which by continuity implies the inequality also in a neighborhood of 0. We still have (5.8), but with B replaced by

$$\begin{aligned}B&=\varepsilon^{-1}\alpha\big(y_0(t)\big)+\alpha'(y_0(t))\Big(y_1(t)+\eta_0(\tau)+\varepsilon\,\big(y_2(t)+\eta_1(\tau)\big)\Big)+\cdots\\&=\alpha'(y_0(0))z_0(0)\,\tau+\alpha'(y_0(t))\Big(y_1(t)+\eta_0(\tau)+\varepsilon\,\big(y_2(t)+\eta_1(\tau)\big)\Big)+\cdots.\end{aligned}$$

This implies that the term e^{-B} in (5.8) no longer contributes to the smooth part.

Step 1a. Putting $\varepsilon=0$ we get

$$\dot y_0(t)=z_0(t),\qquad z_0(t)=f\big(y_0(t),s_0(\alpha(y_0(t)))\big)$$

with initial value $y_0(0)$ given by (5.6). Here, the function $s_0(t)$ is the leading smooth term in the expression (5.2). Recall that $s_0(0)=\dot y_0^+$.

Step 1b. Regarding the transient part, we obtain $\eta_0'(\tau)=\zeta_0(\tau)$ and

$$\begin{aligned}\zeta_0'(\tau)+\zeta_0(\tau)=\;&f\big(y_0(0),\dot y_0^+ +(\dot y_0^- -\dot y_0^+)\,\mathrm{e}^{-\alpha'(y_0(0))(z_0(0)\tau+y_1(0)+\eta_0(\tau))}\big)\\&-f\big(y_0(0),\dot y_0^+\big)\end{aligned} \tag{5.21}$$

with $\dot y_0^+$ and $\dot y_0^-$ given by (2.2). Introducing the scalar functions

$$\widehat\eta_0(\tau)=\alpha'\big(y_0(0)\big)\big(z_0(0)\tau+y_1(0)+\eta_0(\tau)\big),\qquad \widehat\zeta_0(\tau)=\alpha'\big(y_0(0)\big)\big(z_0(0)+\zeta_0(\tau)\big)$$

and left-multiplying the above equations by $\alpha'(y_0(0))$ gives $\widehat\eta_0'(\tau)=\widehat\zeta_0(\tau)$ and

$$\widehat\zeta_0'(\tau)+\widehat\zeta_0(\tau)\;=\;\alpha'\big(y_0(0)\big)\,f\big(y_0(0),\dot y_0^+ +(\dot y_0^- -\dot y_0^+)\,\mathrm{e}^{-\widehat\eta_0(\tau)}\big). \tag{5.22}$$

These are exactly the same differential equations as those obtained in (5.15). The difference is that here we are interested in solutions $\widehat{\eta}_0(\tau)$, $\widehat{\zeta}_0(\tau)$ that approach exponentially fast $d\tau + c$ and d (with $d = \alpha'(y_0(0))z_0(0) > 0$ and $c = \alpha'(y_0(0))y_1(0)$), respectively. Initial values $\widehat{\eta}_0(0)$, $\widehat{\zeta}_0(0)$ are given, because $y_1(0)+\eta_0(0)$ and $z_0(0)+\zeta_0(0)$ are determined by the matching condition (5.6). The stability investigation of section 6 studies conditions on the problem guaranteeing that $\widehat{\eta}_0(\tau)-d\tau-c$ and $\widehat{\zeta}_0(\tau)-d$ converge exponentially fast to zero.

Step 1c. The right-hand side of (5.21) converges exponentially fast to zero, so that this is also true for its solution $\zeta_0(\tau)$ with initial value given by Step 1a and by (5.6). The function $\eta_0(\tau)$ is obtained by integration of $\eta_0'(\tau) = \zeta_0(\tau)$. For a suitably chosen initial value, it converges exponentially fast to zero. This initial value then determines $y_1(0)$ by the continuity requirement (5.6).

This procedure can be repeated and gives further coefficient functions of the asymptotic expansion. The main difference is that the differential equation (5.22) will be linear and thus easier to analyze (as was the case for the expansion of section 5.1). The analysis in section 6 shows that $\widehat{\eta}_0(\tau)$ never becomes negative in the present situation. Therefore, considerations like those of section 5.2 are not necessary.

6. Global dynamics of transient coefficient functions. Both constructions of asymptotic expansions (in sections 5.1 and 5.3) have led to the same two-dimensional dynamical system,

$$\begin{aligned} \eta' &= \zeta, & \eta(0) &= 0, \\ \zeta' &= -\zeta + g(\mathrm{e}^{-\eta}), & \zeta(0) &= \zeta_0 > 0, \end{aligned} \tag{6.1}$$

with initial value $\zeta_0 = \alpha'(a_0)\, b_0 > 0$ (see (5.4)) and

$$g(\theta) = \begin{cases} \alpha'(a_0)\, f\big(a_0, \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\,\theta\big) & \text{for} \quad \theta \le 1, \\ \alpha'(a_0)\, f\big(a_0, \dot y_0^-\big) & \text{for} \quad \theta \ge 1. \end{cases} \tag{6.2}$$

To study its global dynamics, we introduce the new variable $\theta = \mathrm{e}^{-\eta}$, so that the system (6.1) becomes

$$\begin{aligned} \theta' &= -\theta\,\zeta, & \theta(0) &= 1, \\ \zeta' &= -\zeta + g(\theta), & \zeta(0) &= \zeta_0 > 0. \end{aligned} \tag{6.3}$$

The properties of the function $g(\theta)$ of (6.2) are as follows:

(G1) We always assume $g(1) > 0$; this is equivalent to (2.3).

(G2) If $g(0) < 0$, then the solution of (1.1) terminates at the breaking point t_0; this is the inequality of (2.4).

(G3) If $g(0) > 0$, then a classical solution exists beyond the breaking point t_0.

The properties of the flow of (6.3) (see Figures 6.1 and 6.2) are as follows:

(F1) The solution of (6.3) stays for all times in the half-plane $\theta > 0$.

(F2) Stationary points of (6.3) are $(\theta, \zeta) = (0, g_0)$ (with the abbreviation $g_0 = g(0)$) and $(\theta, \zeta) = (\mathrm{e}^{-c}, 0)$, where c is a root of $g(\mathrm{e}^{-c}) = 0$.

(F3) In the upper half-plane $\zeta > 0$ the flow is directed to the left, i.e., $\theta(\tau)$ is monotonically decreasing; in the lower half-plane it is directed to the right.

(F4) Above the curve $\zeta = g(\theta)$ the flow is directed downward, i.e., $\zeta(\tau)$ is monotonically decreasing; below this curve it is directed upward.

6.1. Discussion of the validity of the asymptotic expansions. We shall prove that the solution of the initial value problem (6.3) determines the behavior of the singularly perturbed delay equation (1.3) beyond the first breaking point.

THEOREM 6.1. *Suppose that the function $g(\theta)$ of* (6.2) *satisfies* (G1).

(a) *If the solution of* (6.3) *converges to a stationary point* $(\theta,\zeta) = (\mathrm{e}^{-c}, 0)$ *for which* $g'(\mathrm{e}^{-c}) > 0$*, and if the solution of the nonlinear equation* (5.13) *is chosen according to* $\alpha'(y_0(0))\, y_1(0) = c$*, then the asymptotic expansion of section* 5.1 *is such that* $\eta_0(\tau)$, $\zeta_0(\tau)$ *converge exponentially fast to zero for* $\tau \to \infty$.

(b) *If* $g_0 = g(0) > 0$ *and if the solution of* (6.3) *converges to the stationary point* $(\theta,\zeta) = (0, g_0)$*, then the asymptotic expansion of section* 5.3 *is such that* $\eta_0(\tau), \zeta_0(\tau)$ *converge exponentially fast to zero for* $\tau \to \infty$.

Proof. The Jacobian matrix of the dynamical system (6.3) is

$$\begin{pmatrix} -\zeta & -\theta \\ g'(\theta) & -1 \end{pmatrix}.$$

At a stationary point $(\mathrm{e}^{-c}, 0)$ its characteristic equation is $\lambda^2 + \lambda + \mathrm{e}^{-c} g'(\mathrm{e}^{-c}) = 0$, and at $(0, g_0)$ it is $(\lambda + 1)(\lambda + g_0) = 0$. Under the assumptions of the theorem the eigenvalues have negative real part, so that the stationary points are asymptotically stable. Backsubstitution via the relation $\theta = \mathrm{e}^{-\eta}$ gives information for the solution of system (6.1).

(a) In the situation (a) the solution $(\eta(\tau), \zeta(\tau))$ of (6.1) converges exponentially fast to $(c, 0)$. This is precisely the condition required in the end of Step 1c (section 5.1) for the functions $(\widehat{\eta}_0(\tau), \widehat{\zeta}_0(\tau))$. This guarantees that the functions $(\eta_0(\tau), \zeta_0(\tau))$ converge exponentially fast to 0 (Step 1d of section 5.1).

(b) In the situation (b) it follows by integration of $\eta'(\tau) = \zeta(\tau)$ that $\eta(\tau)$ approaches exponentially fast a function $g_0\tau + c$. This is the condition required in the end of Step 1b (section 5.3). ∎

It is of interest to study conditions on the original problem (1.1), which determine the kind of asymptotic expansion for the regularization (1.3). We expect that if (G1) and (G2) hold, so that the solution of (1.1) terminates at t_0, the solution of (1.3) has a weak solution beyond t_0, and it is given by the expansion of section 5.1. However, if (G1) and (G3) hold, so that a classical solution continues to exist beyond the breaking point, we expect a classical solution of (1.3) which is given by the expansion of section 5.3. The following two lemmas give sufficient conditions for this to be true.

LEMMA 6.2. *Suppose* (G1) *and* (G2)*, and the roots of* $g(\mathrm{e}^{-\eta}) = 0$ *are discrete (e.g.,* $g(\theta)$ *is strictly monotone for* $0 < \theta < 1$*). Then there exists a root* $c > 0$ *of* $g(\mathrm{e}^{-\eta}) = 0$ *such that the solution of* (6.3) *converges to* $(\theta,\zeta) = (\mathrm{e}^{-c}, 0)$*, which implies that the transient functions* $(\eta_j(\tau), \zeta_j(\tau))$ *of the asymptotic expansion of section* 5.1 *converge to* 0 *exponentially fast.*

Proof. Property (F3) and the fact that the only stationary point on the vertical axis is below the origin imply that the solution of (6.3), which starts in the upper half-plane, crosses the horizontal axis at a point $(d_0, 0)$ with $0 < d_0 < 1$. It therefore lies on the graph of a function $\zeta = \psi(\theta)$, which satisfies $\psi(d_0) = 0$, $\psi(1) = \zeta(0)$, and is positive between d_0 and 1. On a point (θ,ζ) of the reflected curve $\zeta = -\psi(\theta)$, the tangent vector is $(-\theta\zeta, \zeta + g(\theta))$, whereas the flow of (6.3) points in the direction $(-\theta\zeta, -\zeta + g(\theta))$. Consequently, the solution passing through $(d_0, 0)$ lies strictly above this reflected curve and crosses the horizontal axis at some point $(d_1, 0)$, where $d_1 > d_0$ can be larger than 1. We now consider the graph of the solution in the lower half-plane and denote it again by $\zeta = \psi(\theta)$. The same argumentation as before shows that the solution passing through $(d_1, 0)$ lies strictly below the reflected curve $\zeta = -\psi(\theta)$ and crosses the horizontal axis at d_2 which satisfies $d_0 < d_2 < d_1$. This procedure can be repeated. It implies that the solution is bounded for all times. Furthermore, it does not tend to a limit cycle. Indeed, if this were to happen, the limit cycle has to cross the horizontal axis at some d_0, and the above analysis shows that the solution cannot come back to this point. The Poincaré–Bendixson Theorem therefore proves that the solution converges to a stationary point of system (6.3). □

LEMMA 6.3. *Suppose* (G1) *and* (G3) *and* $g(\theta) > 0$ *for all* $0 < \theta < 1$. *Then the solution of* (6.3) *converges to* $(\theta, \zeta) = (0, g_0)$, *so that the transient functions* $(\eta_j(\tau), \zeta_j(\tau))$ *of the asymptotic expansion of section* 5.3 *converge to* 0 *exponentially fast.*

Proof. Since $g(\theta) > 0$ for $0 < \theta < 1$ the vector field points upward on the horizontal axis $\zeta = 0$. Therefore, the solution of (6.3) starting with positive $\zeta(0)$ stays in the first quadrant. By property (F4) it is bounded, and property (F3) implies that $\theta(\tau)$ is monotonically decreasing. This excludes the situation of a limit cycle and proves that the solution converges to the stationary point $(0, g_0)$. □

These two lemmas cover the most important situations, probably all of practical interest. But what happens when these sufficient conditions are not satisfied?

Let (G1) and (G2) be satisfied, which characterizes the situation of a terminating solution at the breaking point. In this case, $(0, g_0)$ is repulsive, so that the expansion of section 5.3 is not possible. Generically, we thus have the situation of Lemma 6.2 and, as expected, the expansion of section 5.1 describes the solution of (1.3).

Let (G1) and (G3) be satisfied, which characterizes the existence of a classical solution beyond the breaking point of (1.1). Typically, the solution of the regularization (1.3) will be given by the asymptotic expansion of section 5.3, but in exceptional cases it can be given by the expansion of section 5.1; see the example in the following section. This unexpected result shows that care has to be taken with the regularization (1.3) of (1.1).

6.2. An illustrative example. We consider the singularly perturbed delay equation with scalar nonlinearity independent of y and lag term $\alpha(y) = y - 1$:

$$\begin{aligned} \dot y(t) &= z(t), \\ \varepsilon\, \dot z(t) &= f\big(z\big(y(t)-1\big)\big) - z(t) \end{aligned}$$

with $y(t) = 0$ for $t \le 0$. As long as $y(t) \le 1$, the solution is given by

$$y(t) \;=\; f(0)\, t + \varepsilon\, f(0)\big(\mathrm{e}^{-t/\varepsilon} - 1\big), \qquad z(t) \;=\; f(0)\big(1 - \mathrm{e}^{-t/\varepsilon}\big).$$

We assume $f(0) > 0$ so that neglecting exponentially small terms, the first breaking point is at $t_0(\varepsilon) = f(0)^{-1} + \varepsilon$. The solution of (6.1), or equivalently of (6.3), with

$$g(\theta) = f\big(f(0)(1-\theta)\big) \qquad \text{for} \quad \theta \le 1,$$

and $g(\theta) = f(0)$ for $\theta > 1$, determines which asymptotic expansion is relevant beyond this breaking point. As a concrete example we consider

$$f(z) = \gamma\,(1 - \beta_1 z)(1 - \beta_2 z). \tag{6.4}$$

The phase portraits of various choices of the parameters are given in Figures 6.1 and 6.2, where stationary points are marked by circles, and the initial value for the essential solution curve is indicated by a black point.

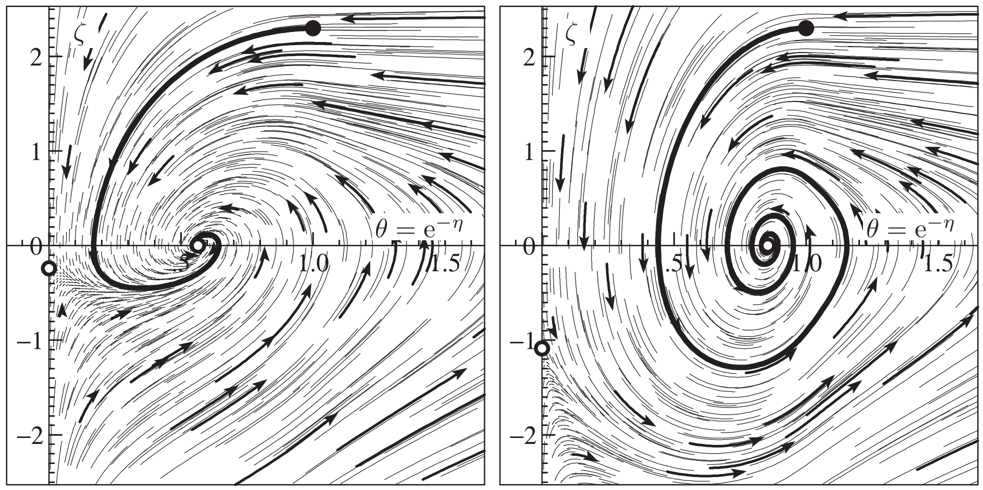

FIG. 6.1. *Phase portrait of the differential equation* (6.3) *for the problem of section* 6.2 *with function* $f(z) = \gamma(1-\beta_1 z)(1-\beta_2 z)$ *and parameters satisfying* (G1) *and* (G2). *Left:* $\gamma = 2.3$, $\beta_1 = 0.4$, $\beta_2 = 1$; *right:* $\gamma = 2.3$, $\beta_1 = 0.4$, $\beta_2 = 3$.

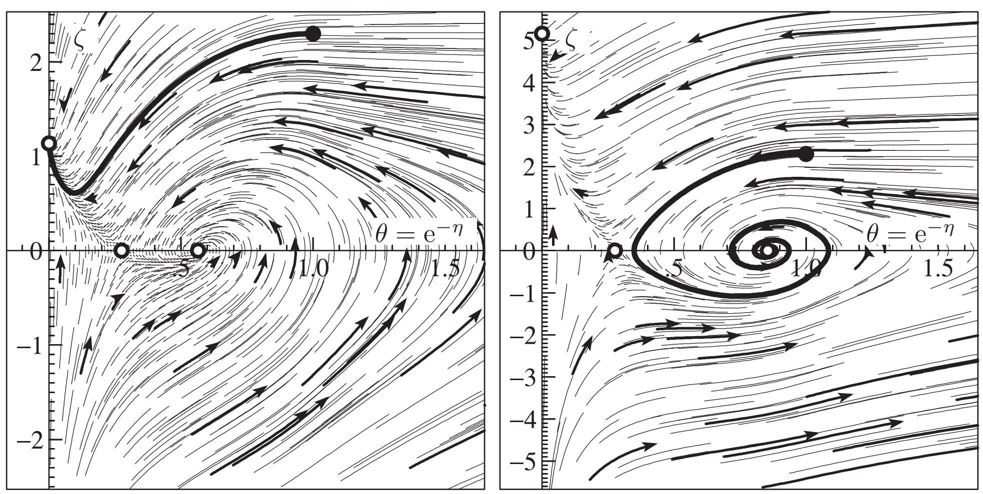

FIG. 6.2. *Phase portrait of the differential equation* (6.3) *for the problem of section* 6.2 *with function* $f(z) = \gamma(1-\beta_1 z)(1-\beta_2 z)$ *and parameters satisfying* (G1) *and* (G3). *Left:* $\gamma = 2.3$, $\beta_1 = 0.6$, $\beta_2 = 1$; *right:* $\gamma = 2.3$, $\beta_1 = 0.6$, $\beta_2 = 3$.

Case 1: *Asymptotic expansion of section* 5.1. For this special situation we obtain $y_0(t) = 1$, $z_0(t) = 0$, and $y_1(t) = c$, where $c > 0$ is such that $g(\mathrm{e}^{-c}) = 0$, i.e., $c = -\ln(1-(\beta_1\gamma)^{-1})$. The transient functions $\eta_0(\tau)$ and $\zeta_0(\tau)$ are given from the solution of (6.1) (resp., (6.3)). By Theorem 6.1 this expansion is relevant for both problems of Figure 6.1 and for the problem corresponding to the right picture of Figure 6.2. This is expected for the problems of Figure 6.1, because there $g(0) = f(f(0)) < 0$ and the limit problem for $\varepsilon = 0$ does not have a solution beyond the breaking point t_0. The first terms of the asymptotic expansion yield an excellent approximation to the solution of (1.3). In the right picture of Figure 6.1 the function $\theta(\tau)$ is seen to become larger than one on a nonempty time interval (i.e., $\widehat{\eta}_0(\tau)$ is negative on this interval), which implies that (1.3) has three breaking points $\mathcal{O}(\varepsilon)$-close to t_0 (cf. section 5.2).

Case 2: *Unexpected asymptotic expansion of section* 5.1. For the parameters corresponding to the right picture of Figure 6.2, the phase portrait shows that the solution of (6.3) converges to the stationary point $(\mathrm{e}^{-c}, 0)$ and not to $(0, g_0)$, which is also stable. Theorem 6.1 therefore proves the validity of the asymptotic expansion of section 5.1. This is a rather surprising phenomenon: on the one hand the limit problem for $\varepsilon = 0$ has a classical solution $y(t) = 1 + f(f(0))\,t$ on a nonempty interval beyond the breaking point t_0. On the other hand the solution of (1.3) remains for small $\varepsilon > 0$ close to the manifold $y = 1$.

Case 3: *Asymptotic expansion of section* 5.3. By Theorem 6.1 the construction of section 5.3 is relevant if $g(0) = f(f(0)) > 0$ and if the solution of (6.3) converges to the stationary point $(0, g_0)$. This happens in the situation of the left picture of Figure 6.2. We have $z_0(t) = f(f(0))$, $y_0(t) = 1 + f(f(0))\,t$, and the transient functions $\eta_0(\tau)$ and $\zeta_0(\tau)$ are given by (6.3).

Summarizing our findings of these examples we conclude as follows: for $g(0) < 0$ (termination of the solution for the limit problem) the expansion of section 5.1 is always relevant (the stationary point $(0, g_0)$ is unstable); however, for $g(0) > 0$ (existence of classical solution beyond the first breaking point for the limit problem) the stationary point of (6.3) determines which of the expansions, that of section 5.3 or that of section 5.1, is relevant.

7. Estimation of the defect and remainder. We consider the asymptotic expansion (5.5) corresponding to the situation of section 5.1. We truncate the series, and we define

$$\widehat{y}\big(t_0(\epsilon) + t\big) = \sum_{j=0}^{N} \varepsilon^j\, y_j(t) + \varepsilon \sum_{j=0}^{N-1} \varepsilon^j\, \eta_j(t/\varepsilon),$$

$$\widehat{z}\big(t_0(\epsilon) + t\big) = \sum_{j=0}^{N} \varepsilon^j\, z_j(t) + \sum_{j=0}^{N-1} \varepsilon^j\, \zeta_j(t/\varepsilon).$$

By construction of the coefficient functions we have uniformly on compact ε-independent intervals (and neglecting $\mathcal{O}(\varepsilon^N)$ terms)

$$\begin{aligned} \dot{\widehat{y}}(t) &= \widehat{z}(t), \\ \varepsilon\, \dot{\widehat{z}}(t) &= f\Big(\widehat{y}(t), s\big(\varepsilon, \alpha(\widehat{y}(t))\big) + p\big(\varepsilon, \alpha(\widehat{y}(t)), \varepsilon\widehat{u}(t)\big)\, \widehat{u}(t)\Big) - \widehat{z}(t), \\ \varepsilon \ln \widehat{u}(t) &= -\alpha\big(\widehat{y}(t)\big), \end{aligned} \tag{7.1}$$

where the last line should be considered as a definition of $\widehat{u}(t)$. Recall that for the dominant transient terms $\eta_0(\tau)$, $\zeta_0(\tau)$, the expressions $\eta(\tau) = \alpha'(y_0(0))(y_1(0) + \eta_0(\tau))$ and $\zeta(\tau) = \alpha'(y_0(0))\zeta_0(\tau)$ are a solution of the two-dimensional dynamical system (6.1) A stability assumption on this system permits us to prove the following asymptotic expansion for the solution.

THEOREM 7.1. *Consider the regularized neutral delay equation* (5.1) *beyond the breaking point* $t_0(\varepsilon)$. *Suppose that the solution of* (6.1) *with initial values* $\eta(0) = 0$, $\zeta(0) = \alpha'(y_0(0)) f(y_0(0), \dot{y}_0^-) > 0$ *converges to a stationary point* $\zeta = 0$, $\eta = c > 0$, *where*

$$\alpha'\big(y_0(0)\big) f_z\big(y_0(0), \dot{y}_0^+ + (\dot{y}_0^- - \dot{y}_0^+)\, \mathrm{e}^{-c}\big)\, (\dot{y}_0^- - \dot{y}_0^+) > 0. \tag{7.2}$$

For sufficiently small ε, there then exists an interval $t_0(\varepsilon) \le t \le T$ (with $T > t_0$ independent of ε), where the problem (5.1) *admits a unique solution which satisfies*

$$y(t) = \widehat{y}(t) + \mathcal{O}(\varepsilon^N), \qquad z(t) = \widehat{z}(t) + \mathcal{O}(\varepsilon^N). \tag{7.3}$$

Proof. Similar to (7.1) we introduce the function $u(t)$ by $\varepsilon \ln u(t) = -\alpha(y(t))$ for the system (5.1). We differentiate this algebraic relation (index reduction), so that (5.1) becomes equivalent to the singularly perturbed ordinary differential equation (again neglecting the $\mathcal{O}(\varepsilon^{N+1})$ terms)

$$\begin{aligned} \dot y(t) &= z(t), \\ \varepsilon\, \dot z(t) &= f\Big(y(t), s\big(\varepsilon, \alpha(y(t))\big) + p\big(\varepsilon, \alpha(y(t)), \varepsilon u(t)\big)\, u(t)\Big) \;-\; z(t), \\ \varepsilon\, \dot u(t) &= -\, u(t)\, \alpha'\big(y(t)\big)\, z(t). \end{aligned} \tag{7.4}$$

This permits us to apply techniques of the standard theory for ordinary differential equations; see, for example, [HW96, Chap. VI.3].

(a) The asymptotic stability of the system (6.1) implies that for an arbitrarily given $\delta > 0$ there exists a $T_0 > 0$ such that its solution with initial values specified in the theorem satisfies $|\eta(\tau) - c| \le \delta$ and $|\zeta(\tau)| \le \delta$ for $\tau > T_0$. We treat the solution of our problem separately on the interval $[t_0(\varepsilon), t_0(\varepsilon) + \varepsilon\, T_0]$ and for $t \ge t_0(\varepsilon) + \varepsilon\, T_0$.

(b) We divide the second and third equations in (7.4) by ε and obtain an ordinary differential equation satisfying a Lipschitz condition with a Lipschitz constant of size $\mathcal{O}(\varepsilon^{-1})$. A standard application of Gronwall's lemma implies that the estimate (7.3) holds on the interval $[t_0(\varepsilon), t_0(\varepsilon) + \varepsilon\, T_0]$, which is of length $\mathcal{O}(\varepsilon)$.

(c) It remains to investigate time intervals with $t \ge t_0(\varepsilon) + \varepsilon T_0$. To study the stability of the system (7.4), we consider the Jacobian of the second and third equations with respect to (z, u) at $(y, z, u) = (y_0(0), z_0(0), \mathrm{e}^{-c})$ and $\varepsilon = 0$. It is given by

$$\begin{pmatrix} -\, I & d \\ -\, \mathrm{e}^{-c} \alpha'\big(y_0(0)\big) & 0 \end{pmatrix} \quad \text{with} \quad d := f_z\big(y_0(0), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\, \mathrm{e}^{-c}\big)\, (\dot y_0^- - \dot y_0^+).$$

If n denotes the dimension of y, this matrix has $n-1$ eigenvalues equal to -1, and the remaining two eigenvalues are the roots of the equation $\lambda^2 + \lambda + \mu = 0$, where $\mu = \mathrm{e}^{-c}\alpha'(y_0(0))d > 0$ by (7.2). Hence, all eigenvalues of this matrix have negative real part. By diagonalization it is possible to find an inner product for which the matrix has a strictly negative logarithmic norm. A continuity argument shows that there exists an ε-independent neighborhood of $(y, z, u) = (y_0(0), z_0(0), \mathrm{e}^{-c})$, where the matrix has a logarithmic norm smaller than a negative constant. Consequently, for sufficiently small δ and ε, there exists an ε-independent T_1 such that $y = y_0(t) + \mathcal{O}(\varepsilon)$ and $u = \exp(-\alpha'(y_0(t))(y_1(t) + \eta_0(t/\varepsilon)) + \mathcal{O}(\varepsilon))$ are in this neighborhood for all t in the interval $\varepsilon\, T_0 \le t \le T_1$. On this interval the theory of asymptotic expansions for singularly perturbed ordinary differential equations proves the statement (see, e.g., [HW96, Chap. VI.3, pp. 388–392]). ∎

It is of interest to study how far the validity of the asymptotic expansion and of the estimate (7.3) can be extended. Recall that the dominating smooth functions of the expansion are given by the system

$$\dot y_0(t) = z_0(t), \qquad z_0(t) = f\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\, u_0(t)\big), \tag{7.5}$$

where the function $u_0(t)$ is defined by the relation

$$\alpha'\big(y_0(t)\big) f\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\, u_0(t)\big) = 0; \tag{7.6}$$

see (5.13). This is a differential-algebraic system. As long as

$$\alpha'\big(y_0(t)\big)\, f_z\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)\, u_0(t)\big)(\dot y_0^- - \dot y_0^+) > 0, \tag{7.7}$$

which reduces to (7.2) for $t = 0$, the implicit function theorem guarantees that (7.6) can be solved for $u_0(t)$. Together with the definition of $z_0(t)$ this then leads to an ordinary differential equation for $y_0(t)$. We assume throughout this article that the functions $y_0(t)$, $z_0(t)$, and $u_0(t)$ exist as far as we interested and that there the assumption (7.7) holds. To extend the interval of validity of the asymptotic expansion, we consider the system (7.4) for $t \geq T_1$ (with T_1 as in the proof of Theorem 7.1). We adapt the inner product norm to the new arguments of the Jacobian and prove that the estimate (7.3) is valid on an interval $[T_1, T_2]$. This procedure can be iterated as long as $u_0(t) \in [c_0, 1]$ with $c_0 > 0$. If $u_0(t)$ crosses the value 1, the system (5.1) will have a new breaking point; if $u_0(t)$ approaches 0, the function $y_1(t)$ will become unbounded and the asymptotic expansion is no longer valid. Both situations will be studied in detail in section 8.

8. Asymptotic expansion of emerging classical solution. We consider the neutral delay equation (1.1) with terminating solution at $t = t_0$ (cf. condition (2.4)). Beyond this point, a weak solution $\big(y_0(t), z_0(t)\big)$ is defined by (7.5)–(7.6). As long as, with $\dot y_0^+, \dot y_0^-$ defined in (2.2),

$$\alpha'\big(y_0(t)\big) f\big(y_0(t), \dot y_0^+\big) < 0, \qquad \alpha'\big(y_0(t)\big) f\big(y_0(t), \dot y_0^-\big) > 0, \tag{8.1}$$

a classical solution cannot exist. However, a solution emerges tangentially from the manifold at a point $t = t_1$, when one of the expressions in (8.1) changes sign. If the first expression changes sign, we have $u_0(t_1) = 0$ for the function defined in (7.6), and the solution continues in the region $\{y\,;\, \alpha(y) > 0\}$. If the second expression in (8.1) changes sign, we have $u_0(t_1) = 1$, and the solution goes back to the region $\{y\,;\, \alpha(y) < 0\}$.

In this section we are interested to see whether the regularization (1.3) can correctly reproduce this behavior. We consider the situation of section 5.1, where the regularized solution remains close to the manifold $\{y\,;\, \alpha(y) = 0\}$ on a nonempty interval beyond the first breaking point. In fact, it lies in the region $\{y\,;\, \alpha(y) > 0\}$.

8.1. Solution, escaping through a breaking point. In this section we assume that the function $\alpha'(y_0(t)) f(y_0(t), \dot y_0^-)$ changes sign (from positive to negative) at $t = t_1$. Because of (7.7) this is equivalent to $u_0(t_1) = 1$ and $\dot u_0(t_1) > 0$ for the function given by (7.6). The discussion at the end of section 7 shows that the asymptotic expansion of section 5.1 does not blow up in a neighborhood of t_1. In fact, there will be a breaking point close to t_1. To see this we observe that

$$\alpha\big(y(t)\big) = \varepsilon\, \alpha'\big(y_0(t)\big)\, y_1(t) + \mathcal{O}(\varepsilon^2) = -\varepsilon \ln u_0(t) + \mathcal{O}(\varepsilon^2),$$

so that the existence of a breaking point $t_1(\varepsilon) = t_1 + \mathcal{O}(\varepsilon)$ is a consequence of the implicit function theorem. Until this breaking point, the expansion of section 5.1 is valid. Beyond it we are concerned with the ordinary differential equation (4.1), and the analysis of section 4 yields an asymptotic expansion for the solution of (1.3) on an interval $t_1(\varepsilon) \leq t \leq T$ (with T independent of ε). Initial values are given by continuity as an expansion in powers of ε. Since for $t = t_1(\varepsilon)$ we have $z(t_1(\varepsilon)) = f(y(t_1(\varepsilon)), \dot y_0^-)$ and $\dot\varphi(0) = \dot y_0^-$ by (2.2), the transient parts of the expansion will be of size $\mathcal{O}(\varepsilon^2)$ for the y-component and of size $\mathcal{O}(\varepsilon)$ for the z-component.

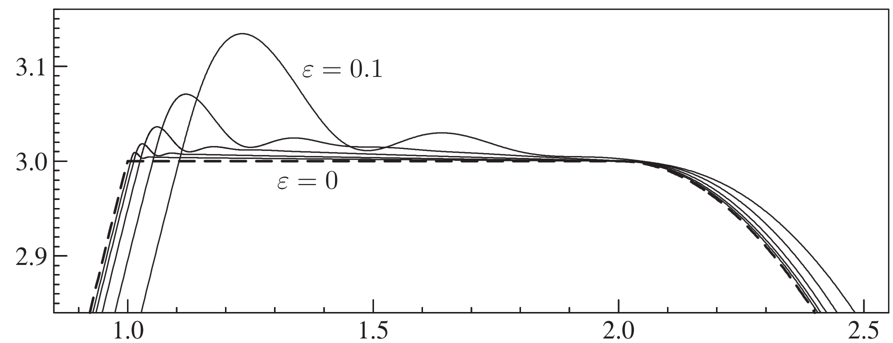

FIG. 8.1. *Solution $y(t)$ of the singularly perturbed delay equation* (8.2) *with $\varepsilon = 0.1 \cdot 2^{-n}$ for $n = 0, 1, 2, \ldots$, and with $\varepsilon = 0$ (broken thick line).*

Example. Consider the singularly perturbed delay equation

$$\begin{aligned} &\dot y(t) = z(t), \\ &\varepsilon\, \dot z(t) = d(t) - z\big(\alpha(y(t))\big) - z(t), \qquad \alpha(y) = y - 3, \quad d(t) = 4 - 2t, \end{aligned} \tag{8.2}$$

with initial functions $y(t) = z(t) = 0$ for $t \le 0$. For $\varepsilon = 0$, the solution until the first breaking point $t_0 = 1$ is $z(t) = 4 - 2t$, $y(t) = 4t - t^2$. We have a weak solution $y(t) = 3$, $z(t) = 0$ on the interval $[1, 2]$, and after the point $t_1 = 2$ a classical solution emerges from the manifold $y = 3$, which is given by $z(t) = 4 - 2t$, $y(t) = -1 + 4t - t^2$.

For small $\varepsilon > 0$, the solution until the first breaking point $t_0(\varepsilon) = 1 + \varepsilon + \mathcal{O}(\varepsilon^2)$ is $z(t) = 4 + 2\varepsilon - 2t - (4 + 2\varepsilon)\mathrm{e}^{-t/\varepsilon}$ and $y(t) = (4 + 2\varepsilon)t - t^2 + \varepsilon(4 + 2\varepsilon)(\mathrm{e}^{-t/\varepsilon} - 1)$. Beyond this breaking point, the smooth part of the asymptotic expansion is

$$y(t) = 3 - \varepsilon\, \ln(t/2) + \mathcal{O}(\varepsilon^2), \qquad z(t) = -\varepsilon/t + \mathcal{O}(\varepsilon^2),$$

and we see that there exists a further breaking point near $t_1 = 2$. For various choices of ε, the solution component $y(t)$ is plotted in Figure 8.1. The nonsmooth transients are well visible at the first breaking point; they are by a factor ε smaller (and not visible) at the second breaking point.

8.2. Refined asymptotic expansion close to the breaking point. More challenging is the situation where the function $\alpha'(y_0(t)) f(y_0(t), \dot y_0^+)$ changes sign at $t = t_1$ (this time from negative to positive). This is equivalent to $u_0(t_1) = 0$ and $\dot u_0(t_1) < 0$ for the function $u_0(t)$ defined by (7.6). We have the following asymptotic behavior for t close to t_1.

LEMMA 8.1. *For the asymptotic expansion of section* 5.1 *consider the situation where the function $u_0(t)$ of* (7.6) *is positive for $t_0 \le t < t_1$ and satisfies $u_0(t) = a(t_1 - t) + \mathcal{O}((t_1 - t)^2)$ for $t \to t_1$ with $a > 0$. The smooth coefficient functions of the expansion then satisfy asymptotically for $t \to t_1$ $(t < t_1)$*

$$\varepsilon^j y_j(t) \sim \begin{cases} \varepsilon \ln(t_1 - t) & \text{for} \quad j = 1, \\ \varepsilon\Big(\dfrac{\varepsilon}{(t_1 - t)^2}\Big)^{j-1} & \text{for} \quad j \ge 2 \end{cases} \tag{8.3}$$

and

$$\varepsilon^j z_j(t) \sim (t_1 - t)\Big(\frac{\varepsilon}{(t_1 - t)^2}\Big)^j. \tag{8.4}$$

Here, the symbol $\sim$ means that the left-hand function divided by the right-hand function converges to a nonzero vector.

Proof. We follow the recursive construction of the smooth coefficient functions (section 5.1) and analyze their behavior close to t_1. The definition of $u_0(t)$ in (5.13) implies that $\alpha'(y_0(t))y_1(t) = -\ln u_0(t) \sim -\ln(t_1 - t)$. Differentiating this relation with respect to time and using $\dot{y}_1(t) = z_1(t)$ proves that $\alpha'(y_0(t))z_1(t)$ is asymptotically equal to $1/(t_1 - t)$ plus a smooth function multiplied by $y_1(t)$. From Step 2a (section 5.1) we obtain that

$$u_0(t)\alpha'(y_0(t))y_2(t) = a(t)y_1(t) + \frac{1}{t_1 - t} + \cdots \tag{8.5}$$

is of the same structure. The differential equation that determines $y_1(t)$ is linear with an inhomogeneity that contains a summand proportional to $1/(t_1 - t)$. This proves (8.3) for $j = 1$ and by differentiation (8.4).

For the next step we divide (8.5) by $u_0(t) \sim (t_1 - t)$ and we differentiate the relation with respect to time. Using $z_2(t) = \dot{y}_2(t)$ this proves that $\alpha'(y_0(t))z_2(t)$ is asymptotically equal to $1/(t_1 - t)^3$ and contains a summand that depends linearly on $y_2(t)$. The function $y_2(t)$ is thus determined by a linear differential equation with an inhomogeneity containing a summand proportional to $1/(t_1 - t)^3$. This proves the statement for $j = 2$ and, by an induction argument, for all j. Note that we lose two powers of $(t_1 - t)$ when increasing j by one. $\square$

Due to the factor $\varepsilon/(t_1 - t)^2$ in (8.3)–(8.4), the asymptotic expansion of section 5.1 does not give any information for $t_1 - \sqrt{\varepsilon} < t < t_1$. (All terms in the expansion are of comparable size.) We therefore need a refined analysis close to the point t_1. Since we need to cover an interval of length $\mathcal{O}(\sqrt{\varepsilon})$, we consider the two-scale ansatz

$$y(t) = y_0(t) + \varepsilon\,\eta(\tau), \quad z(t) = z_0(t) + \sqrt{\varepsilon}\,\zeta(\tau), \quad u(t) = u_0(t) + \sqrt{\varepsilon}\,\nu(\tau), \tag{8.6}$$

where the variable τ is given by $t = t_1 + \sqrt{\varepsilon}\,\tau$ and the functions $y_0(t)$, $z_0(t)$, and $u_0(t)$ are those of (7.5)–(7.6). The functions $\eta(\tau)$, $\zeta(\tau)$, and $\nu(\tau)$ are to be determined so that $y(t)$, $z(t)$, and $u(t)$ satisfy the system (7.4), which is equivalent to (5.1). The perturbations then have to satisfy the system (with $t = t_1 + \sqrt{\varepsilon}\,\tau$)

$$\begin{aligned} \eta'(\tau) &= \zeta(\tau), \\ \sqrt{\varepsilon}\,\zeta'(\tau) &= f_z\big(y_0(t), \dot{y}_0^+ + (\dot{y}_0^- - \dot{y}_0^+)u_0(t)\big)(\dot{y}_0^- - \dot{y}_0^+)\,\nu(\tau) - \zeta(\tau) \\ &\quad + \sqrt{\varepsilon}\,R\big(\varepsilon, t, \eta(\tau), \nu(\tau)\big), \end{aligned} \tag{8.7}$$

where the smooth function εR collects $-\varepsilon\dot{z}_0(t)$ as well as linear terms in $\varepsilon\eta(\tau)$ and $\varepsilon\sqrt{\varepsilon}\nu(\tau)$, and quadratic and higher order terms in $(\varepsilon\eta(\tau), \sqrt{\varepsilon}\nu(\tau))$ with coefficients depending smoothly on ε and t. The third equation of (7.4), which is responsible for the singularity in the previous expansion, now becomes (again with $t_1 + \sqrt{\varepsilon}\,\tau$)

$$\varepsilon\left(\dot{u}_0(t) + \nu'(\tau)\right) = -\big(u_0(t) + \sqrt{\varepsilon}\,\nu(\tau)\big)\,\alpha'\big(y_0(t) + \varepsilon\,\eta(\tau)\big)\big(z_0(t) + \sqrt{\varepsilon}\,\zeta(\tau)\big). \tag{8.8}$$

Since $u_0(t_1 + \sqrt{\varepsilon}\,\tau) = \sqrt{\varepsilon}\,\tau\,\dot{u}_0(t_1) + \mathcal{O}(\varepsilon)$ and $\alpha'(y_0(t))z_0(t) = 0$ by (5.10), the right-hand expression of (8.8) contains a factor ε and we are concerned with a regular differential equation for $\nu(\tau)$. To solve the system (8.7)–(8.8) we expand the functions in powers of $\sqrt{\varepsilon}$,

$$\begin{aligned} \eta(\tau) &= \eta_0(\tau) + \sqrt{\varepsilon}\,\eta_1(\tau) + \varepsilon\,\eta_2(\tau) + \varepsilon\,\sqrt{\varepsilon}\,\eta_3(\tau) + \cdots, \\ \zeta(\tau) &= \zeta_0(\tau) + \sqrt{\varepsilon}\,\zeta_1(\tau) + \varepsilon\,\zeta_2(\tau) + \varepsilon\,\sqrt{\varepsilon}\,\zeta_3(\tau) + \cdots, \\ \nu(\tau) &= \nu_0(\tau) + \sqrt{\varepsilon}\,\nu_1(\tau) + \varepsilon\,\nu_2(\tau) + \varepsilon\,\sqrt{\varepsilon}\,\nu_3(\tau) + \cdots. \end{aligned} \tag{8.9}$$

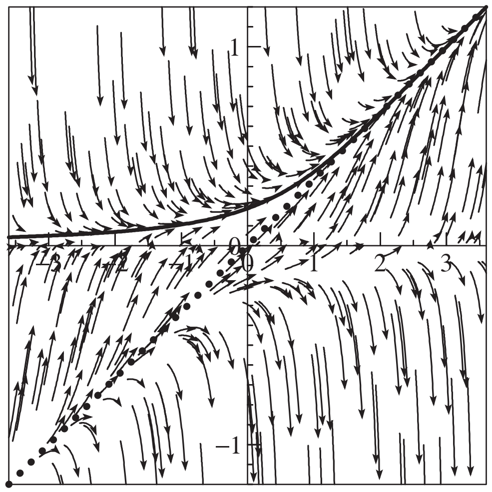

FIG. 8.2. *Solutions* $\nu_0(\tau)$ *of* (8.11) *with parameters* $c = -1/3$ *and* $d = 6$.

For the leading terms we get the system

$$
\begin{aligned}
\eta_0'(\tau) &= \zeta_0(\tau), \\
0 &= f_z\big(y_0(t_1), \dot y_0^+\big)(\dot y_0^- - \dot y_0^+)\,\nu_0(\tau) - \zeta_0(\tau), \qquad (8.10)\\
\nu_0'(\tau) &= -\big(\dot u_0(t_1)\,\tau + \nu_0(\tau)\big)\alpha'\big(y_0(t_1)\big)\zeta_0(\tau) - \dot u_0(t_1).
\end{aligned}
$$

Inserting $\zeta_0(\tau)$ from the second relation into the third equation, we are led to the nonautonomous scalar differential equation

$$\nu_0'(\tau) = -d\,\big(c\,\tau + \nu_0(\tau)\big)\nu_0(\tau) - c, \qquad (8.11)$$

where $c = \dot u_0(t_1) < 0$ and $d = \alpha'(y_0(t_1)) f_z(y_0(t_1), \dot y_0^+)(\dot y_0^- - \dot y_0^+) > 0$ by our assumption (7.7). Solutions of this differential equation are drawn in Figure 8.2.

LEMMA 8.2. *The differential equation* (8.11) *with fixed parameters* $c < 0$ *and* $d > 0$ *possesses a unique solution satisfying* $\nu_0(\tau) \to 0$ *for* $\tau \to -\infty$. *It is defined for all* $\tau \in \mathbb{R}$, *it is positive and monotonic increasing, and it asymptotically behaves like*

$$
\begin{aligned}
\nu_0(\tau) &\sim -(d\tau)^{-1} \quad &&\textit{for} \quad \tau \to -\infty,\\
\nu_0(\tau) &\sim -c\tau &&\textit{for} \quad \tau \to \infty.
\end{aligned}
$$

Proof. The slope of the vector field equals $-c > 0$ on the straight lines $\nu_0 = 0$ and $\nu_0 = -c\tau$. It is zero on the hyperbola given by $-d(c\tau + \nu_0)\nu_0 - c = 0$. Therefore, the set $A = \{(\tau, \nu)\,;\, \nu > 0,\ 0 < -d(c\tau + \nu)\nu - c < -c\}$ is positively invariant for the differential equation (8.11). Moreover, the function $h(\tau, \nu) = d\,(c\,\tau + \nu_0)\nu_0 - c$ satisfies $(h(\tau, \nu) - h(\tau, \hat\nu))(\nu - \hat\nu) < 0$ on A, so that solutions are contractive there. This implies the existence of a unique solution satisfying $\nu_0(\tau) \to 0$ for $\tau \to -\infty$.

The asymptotic behavior at $-\infty$ follows from neglecting the quadratic term in the differential equation, and that at $+\infty$ can be seen from the differential equation for the function $\nu_0(\tau) + c\tau$. ◻

Once the function $\nu_0(\tau)$ is known, $\zeta_0(\tau)$ is uniquely given by the second relation of (8.10), and the function $\eta_0(\tau)$ is obtained by integration of the first relation. The integration constant can be used for matching initial values.

To obtain further coefficient functions, we insert the expansion (8.9) into (8.7)–(8.8), we substitute $t = t_1 + \sqrt{\varepsilon}\tau$, and we expand everything into powers of $\sqrt{\varepsilon}$. Comparing like powers of $\sqrt{\varepsilon}$ then yields the equations

$$\begin{aligned}
&\eta_k'(\tau) = \zeta_k(\tau),\\
&0 = f_z\big(y_0(t_1), \dot y_0^+\big)(\dot y_0^- - \dot y_0^+)\,\nu_k(\tau) - \zeta_k(\tau) + e_\zeta(\tau),\\
&\nu_k'(\tau) = -\big(\dot u_0(t_1)\,\tau + \nu_0(\tau)\big)\alpha'\big(y_0(t_1)\big)\zeta_k(\tau) - \nu_k(\tau)\alpha'\big(y_0(t_1)\big)\zeta_0(\tau) + e_\nu(\tau).
\end{aligned} \tag{8.12}$$

Here, the remainder term $e_\zeta(\tau)$ is a linear combination of τ^{k-1}, of $\zeta_{k-1}'(\tau)$, of $\tau^{l_1}\eta_{l_2}(\tau)$ with $l_1+l_2 \le k-1$, of $\tau^{l_1}\nu_{l_2}(\tau)$ with $l_1+l_2 \le k-2$, and of $\tau^{l_1}\prod_{i=1}^{l_2}\eta_{r_i}(\tau)\prod_{i=1}^{l_3}\nu_{s_i}(\tau)$ with $l_1 + \sum_{i=1}^{l_2}(r_i+2) + \sum_{i=1}^{l_3}(s_i+1) \le k+1$ and $l_2 + l_3 \ge 2$. This follows from the fact that the function εR of (8.7) is a linear combination of terms $\varepsilon(\sqrt{\varepsilon}\tau)^l$, $l \ge 0$, of $\varepsilon^{l_0}(\sqrt{\varepsilon}\tau)^{l_1}\varepsilon\eta(\tau)$ with $l_0, l_1 \ge 0$, of $\varepsilon^{l_0}(\sqrt{\varepsilon}\tau)^{l_1}\varepsilon\sqrt{\varepsilon}\nu(\tau)$ with $l_0, l_1 \ge 0$, and of[4] $\varepsilon^{l_0}(\sqrt{\varepsilon}\tau)^{l_1}(\varepsilon\eta(\tau))^{l_2}(\sqrt{\varepsilon}\nu(\tau))^{l_3}$ with nonnegative l_i satisfying $l_2 + l_3 \ge 2$. The remainder term $e_\nu(\tau)$ is a linear combination of τ^k, of $\tau^{l_0}(\tau^{l_1}+\nu_{l_1-1})\prod_{i=1}^{l_3}\eta_{r_i}(\tau)$ with $l_1 \ge 1$, $l_3 \ge 1$, and $l_0 + l_1 + \sum_{i=1}^{l_3}(r_i+2) \le k+2$, and of $\tau^{l_0}(\tau^{l_1}+\nu_{l_1-1})\zeta_{l_2}\prod_{i=1}^{l_3}\eta_{r_i}(\tau)$ with $l_1 \ge 1$, $l_2 \ge 0$, $l_3 \ge 0$, and $l_0 + l_1 + l_2 + \sum_{i=1}^{l_3}(r_i+2) \le k+1$.

Inserting $\zeta_k(\tau)$ from the second into the third relation of (8.12) and using (8.10) yields

$$\nu_k'(\tau) = -d\,\big(c\,\tau + 2\nu_0(\tau)\big)\nu_k(\tau) + e(\tau), \tag{8.13}$$

where $e(\tau) = e_\nu(\tau) - (\dot u_0(t_1)\,\tau + \nu_0(\tau))\alpha'(y_0(t_1))e_\zeta(\tau)$, and the constants c, d are as in (8.11). This is a linear differential equation with inhomogeneity depending on previously computed coefficient functions.

LEMMA 8.3 (negative τ). *With the function $\nu_0(\tau)$ of Lemma* 8.2 *the linear differential equations* (8.13) *possess unique solutions $\nu_k(\tau)$ that grow at most polynomially for $\tau \to -\infty$. The system* (8.12) *then uniquely determines $\zeta_k(\tau)$, and $\eta_k(\tau)$ up to an additive constant. These functions admit, for $\tau \to -\infty$, an asymptotic expansion of the form*

$$\nu_k(\tau),\ \zeta_k(\tau) \sim \sum_{j \le k-1} c_{0j}\,\tau^j + \sum_{l=1}^{k}(\ln|\tau|)^l \sum_{j \le k-l} c_{lj}\,\tau^j,$$
$$\eta_k(\tau) \sim \sum_{j \le k} d_{0j}\,\tau^j + \sum_{l=1}^{k}(\ln|\tau|)^l \sum_{j \le k-l+1} d_{lj}\,\tau^j + d\,(\ln|\tau|)^{k+1}.$$

Proof. The function $\nu_0(\tau)$ of Lemma 8.2 has, for $\tau \to -\infty$, an asymptotic expansion of the form

$$\nu_0(\tau) \sim \frac{c_{-1}}{\tau} + \frac{c_{-3}}{\tau^3} + \frac{c_{-5}}{\tau^5} + \cdots.$$

The coefficients c_j can be computed recursively after inserting the expansion into the differential equation (8.11). This expansion can also be obtained from the explicit

[4] For vector valued $\eta(\tau)$ the product $\eta(\tau)^l$ has to be understood as the value of an l-linear function applied with all arguments equal to $\eta(\tau)$.

representation (8.23), given below. By (8.10), the function $\zeta_0(\tau)$ then also has such an expansion, and the asymptotic expansion for $\eta_0(\tau)$ is obtained by integration,

$$\eta_0(\tau) \sim d \,\ln|\tau| + d_0 + \frac{d_{-2}}{\tau^2} + \frac{d_{-4}}{\tau^4} + \cdots .$$

This proves the statement for $k = 0$. For the moment the integration constant d_0 is a free parameter, but we shall see later that it has to be chosen as $d_0 = \mathcal{O}(\ln\varepsilon)$.

We next consider the linear differential equation (8.13) for $k \geq 1$. Denoting $a(\tau) = -d(c\tau + 2\nu_0(\tau)) = 2\gamma^2\tau - 2d\nu_0(\tau)$ and $A(\tau) = \int_0^\tau a(\sigma)\,\mathrm{d}\sigma$ its integral, the variation of constants formula yields

$$\nu_j(\tau) = C\mathrm{e}^{A(\tau)} + \mathrm{e}^{A(\tau)} \int_{-\infty}^{\tau} \mathrm{e}^{-A(\sigma)} e(\sigma)\,\mathrm{d}\sigma.$$

The only solution that does not increase exponentially fast for $\tau \to -\infty$ is the one given by $C = 0$. Assume that the inhomogeneity $e(\tau)$ has an asymptotic expansion of the form

$$e(\tau) \sim \sum_{j\leq n} e_{0j}\tau^j + \sum_{l=1}^{m} (\ln|\tau|)^l \sum_{j\leq n-l+1} e_{lj}\,\tau^j; \tag{8.14}$$

then the quotient $q(\tau) = e(\tau)/a(\tau)$ has a similar expansion with different coefficients and n replaced by $n-1$, and the derivative $q'(\tau)$ has a similar expansion with n replaced by $n-2$. Integration by parts shows that

$$\begin{aligned}\int_{-\infty}^{\tau} \mathrm{e}^{-A(\sigma)} e(\sigma)\,\mathrm{d}\sigma &= \int_{-\infty}^{\tau} \mathrm{e}^{-A(\sigma)} A'(\sigma) \frac{e(\sigma)}{a(\sigma)}\,\mathrm{d}\sigma \\ &= -\,\mathrm{e}^{-A(\tau)} \frac{e(\tau)}{a(\tau)} + \int_{-\infty}^{\tau} \mathrm{e}^{-A(\sigma)} \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{e(\sigma)}{a(\sigma)}\right)\mathrm{d}\sigma.\end{aligned}$$

Recursively applying integration by parts proves that the solution (with $C = 0$) has a similar asymptotic expansion as the inhomogeneity, but with n replaced by $n-1$.

The proof of the lemma now proceeds by induction on k. Assume that the asymptotic expansion is true for $\eta_l(\tau)$, $\zeta_l(\tau)$, $\nu_l(\tau)$ with $l < k$. The inhomogeneity $e(\tau)$ of (8.13) then satisfies (8.14) with $m = k$ and $n = k$. Consequently, the solution $\nu_k(\tau)$ is of the stated form. Since $e_\zeta(\tau)$ satisfies (8.14) with $m = k$ and $n = k-1$, we also obtain the statement for $\zeta_k(\tau)$. The asymptotic expansion for $\eta_k(\tau)$ is obtained by integration of that of $\zeta_k(\tau)$. ◻

Choice of the transition point. An asymptotic expansion is useful only if the summands decrease (in absolute value) up to a sufficiently large truncation index. Because of the asymptotic behavior (8.3)–(8.4), the expansion of section 5.1 is meaningful only for $t < t_1$ such that $\varepsilon(t_1 - t)^{-2}$ is some positive power of ε. For the expansion (8.9), whose asymptotic behavior is given in Lemma 8.3, the summands decrease if $\sqrt{\varepsilon}\,|\tau| = t_1 - t$ is some positive power of ε. Both expressions contain the same power of ε if we fix the transition from one asymptotic expansion to the other at the point $t^* = t_1 - \varepsilon^{1/3} = t_1 - \sqrt{\varepsilon}\tau^*$ with $\tau^* = \varepsilon^{-1/6}$. This motivates the consideration of the interval $[t_1 - \sqrt{\varepsilon}\tau^*, t_1 + \sqrt{\varepsilon}\tau^*]$ in Theorem 8.5 below.

Matching initial values. At the point $t^* = t_1 - \varepsilon^{1/3}$ the solution of (1.3) does not have any transient layer. This means that the solution is completely determined by imposing the value of $y(t^*)$. Due to the asymptotic behavior (8.3) this value is of the form $y_0(t^*) + \varepsilon\eta^*$ with $\eta^* = \mathcal{O}(\ln\varepsilon)$. To get a smooth transition we put $\eta(-\tau^*) = \eta^*$. In the expansion (8.9) we arbitrarily fix $\eta_0(-\tau^*) = \eta^*$ and $\eta_j(-\tau^*) = 0$ for $j \geq 1$. Note that the above construction of "smooth" coefficient functions does not need any initial values for $\zeta_j(\tau)$ and $\nu_j(\tau)$.

LEMMA 8.4 (positive τ). *The coefficient functions of the expansion* (8.9), *determined in Lemmas* 8.2 *and* 8.3, *are bounded for $\tau \to \infty$ by*

$$\nu_k(\tau) = \mathcal{O}(\tau^{k+1}), \qquad \zeta_k(\tau) = \mathcal{O}(\tau^{k+1}), \qquad \eta_k(\tau) = \mathcal{O}(\tau^{k+2}).$$

Proof. We consider the functions $\omega_j(\tau) = c_j\tau^{j+1} + \nu_j(\tau)$, where $c_0 = \dot{u}_0(t_1)$, $c_1 = \frac{1}{2}\ddot{u}_0(t_1), \ldots$ are the Taylor coefficients of $u_0(t)$ such that

$$u(t_1 + \sqrt{\varepsilon}\,\tau) = u_0(t_1 + \sqrt{\varepsilon}\,\tau) + \sqrt{\varepsilon}\,\nu(\tau) = \sqrt{\varepsilon}\,\omega_0(\tau) + \varepsilon\,\omega_1(\tau) + \varepsilon\sqrt{\varepsilon}\,\omega_2(\tau) + \cdots.$$

From (8.11) we see that the coefficient function $\omega_0(\tau)$ satisfies the differential inequality $\omega_0'(\tau) = -d\,\omega_0(\tau)(\omega_0(\tau) - c\,\tau) \leq d\,c\,\tau\,\omega_0(\tau)$ (recall that $c = c_0 < 0$ and $d > 0$). This implies, for $\tau \geq 0$,

$$0 < \omega_0(\tau) < \omega_0(0)\,\mathrm{e}^{-\gamma^2\tau^2} \qquad \text{with} \qquad \gamma^2 = -\frac{d\,c}{2} > 0, \tag{8.15}$$

and $\nu_0(\tau) = \omega_0(\tau) - c_0\tau$ approaches exponentially fast the line $-c\,\tau$. By (8.10), the components of $\zeta_0(\tau)$ behave like $\nu_0(\tau)$, and after integration the function $\eta_0(\tau)$ is seen to be of the form $C_1 + C_2\tau^2 + \mathcal{O}(\tau^{-1}\mathrm{e}^{-\gamma^2\tau^2})$; cf. [HW97, p. 133]. To study higher order coefficient functions, we notice that (8.8) and (8.7) yield a differential equation

$$\omega_k'(\tau) = a(\tau)\,\omega_k(\tau) + h_k(\tau), \qquad a(\tau) = d\,c\,\tau - 2\,d\,\omega_0(\tau) < -\,2\,\gamma^2\,\tau, \tag{8.16}$$

where the inhomogeneity is a linear combination of terms, each of which contains one factor $\omega_j(\tau)$ with $0 \leq j \leq k-1$. By induction on k we shall prove that $\omega_k(\tau) = \mathcal{O}(\tau^{3k}\,\mathrm{e}^{-\gamma^2\tau^2})$ and that the stated estimates in the lemma are true.

Assume that these estimates hold up to the level k. The dominant terms of the inhomogeneity $h_{k+1}(\tau)$ are $\omega_k(\tau)\alpha'(y_0(t_1))\zeta_1(\tau)$ and $\omega_k(\tau)\alpha''(y_0(t_1))(\tau\,\dot{y}_0(t_1), \zeta_0(\tau))$ and $\omega_k(\tau)\alpha''(y_0(t_1))(\eta_0(\tau), z_0(t_1))$, and they are all bounded by $\mathcal{O}(\tau^{3k+2}\,\mathrm{e}^{-\gamma^2\tau^2})$. The variation of constants formula thus yields $\omega_{k+1}(\tau) = \mathcal{O}(\tau^{3k+3}\,\mathrm{e}^{-\gamma^2\tau^2})$. The estimates for the coefficient functions $\nu_{k+1}(\tau)$, $\zeta_{k+1}(\tau)$, and $\eta_{k+1}(\tau)$ then follow at once from their definition. ∎

We are now able to prove the validity of the ansatz (8.6) and of the asymptotic expansion (8.9) close to the breaking point t_1.

THEOREM 8.5. *We denote (with $t = t_1 + \sqrt{\varepsilon}\tau$)*

$$\widehat{y}(t) = y_0(t) + \varepsilon\,\widehat{\eta}(\tau), \qquad \widehat{z}(t) = z_0(t) + \sqrt{\varepsilon}\,\widehat{\zeta}(\tau), \tag{8.17}$$

where the functions $y_0(t)$ and $z_0(t)$, defined by (7.5)–(7.6), *are assumed to exist for t in an open ε-independent interval centered at t_1. The functions $\widehat{\eta}(\tau)$ and $\widehat{\zeta}(\tau)$ (and also $\widehat{\nu}(\tau)$) denote the series* (8.9) *truncated after the terms with factor $(\sqrt{\varepsilon})^{N+1}$. For sufficiently small ε, there then exists on the interval $[t_1 - \sqrt{\varepsilon}\tau^*, t_1 + \sqrt{\varepsilon}\tau^*]$ with $\tau^* = \varepsilon^{-1/6}$ a solution of* (5.1) *which satisfies*

$$y(t) = \widehat{y}(t) + \mathcal{O}(\varepsilon \cdot \varepsilon^{N/3}), \qquad z(t) = \widehat{z}(t) + \mathcal{O}(\sqrt{\varepsilon} \cdot \varepsilon^{N/3}). \tag{8.18}$$

Proof. We start by estimating the *defect for negative τ*. By Lemma 8.3 the coefficient functions satisfy $\nu_k(\tau), \zeta_k(\tau) = \mathcal{O}(|\tau|^{k-1}\ln|\tau|)$ and $\eta_k(\tau) = \mathcal{O}(|\tau|^k\ln|\tau|)$

for $\tau \to -\infty$. Inserting the truncated series $\widehat{\eta}(\tau)$, $\widehat{\zeta}(\tau)$, and $\widehat{\nu}(\tau)$ into the differential equation yields a defect of size $\mathcal{O}((\sqrt{\varepsilon}\,\tau)^N)$ in the second equation of (8.7) and a defect of size $\mathcal{O}(\varepsilon(\sqrt{\varepsilon}\,\tau)^N)$ in (8.8). Notice that the presence of $\ln|\tau|$ in the above estimates and the presence of $\ln\varepsilon$ in the initial value of $\eta_0(\tau)$ are compensated with a factor $\sqrt{\varepsilon}$. On the interval $[-\tau^*, 0]$ we have $\sqrt{\varepsilon}|\tau| \le \varepsilon^{1/3}$, so that the defects are bounded by $\mathcal{O}(\varepsilon^{N/3})$ and $\mathcal{O}(\varepsilon \cdot \varepsilon^{N/3})$, respectively.

We next estimate the *defect for positive* τ. By Lemma 8.4 the coefficient functions satisfy $\nu_k(\tau), \zeta_k(\tau) = \mathcal{O}(|\tau|^{k+1})$, and $\eta_k(\tau) = \mathcal{O}(|\tau|^{k+2})$ for $\tau \to +\infty$. Inserting the functions $\widehat{\eta}(\tau)$, $\widehat{\zeta}(\tau)$, and $\widehat{\nu}(\tau)$ into the system (8.7)–(8.8) yields a defect of size $\mathcal{O}(\tau(\sqrt{\varepsilon}\tau)^{N+1})$ in (8.7) and of size $\mathcal{O}((\sqrt{\varepsilon}\tau)^{N+3})$ in (8.8). On the interval $[0, \tau^*]$, the defects are bounded by $\mathcal{O}(\varepsilon^{N/3})$ and $\mathcal{O}(\varepsilon \cdot \varepsilon^{N/3})$, respectively.

Before proceeding with the proof of the theorem we note that the second equation of (8.7) is of the form

$$\sqrt{\varepsilon}\,\zeta'(\tau) = -\zeta(\tau) + g(\tau), \tag{8.19}$$

where $g(\tau) = f_z\big(y_0(t), \dot y_0^+ + (\dot y_0^- - \dot y_0^+)u_0(t)\big)(\dot y_0^- - \dot y_0^+)\,\nu(\tau) + \sqrt{\varepsilon}\,R(\varepsilon, t, \eta(\tau), \nu(\tau))$ with $t = t_1 + \sqrt{\varepsilon}\tau$. It follows from the variation of constants formula together with integration by parts that this equation has a unique solution without transient layer. It has an asymptotic expansion given by

$$\zeta(\tau) = g(\tau) - \sqrt{\varepsilon}\,g'(\tau) + \varepsilon\,g''(\tau) - \varepsilon\sqrt{\varepsilon}\,g'''(\tau) + \cdots. \tag{8.20}$$

We continue with the proof of (8.18). Since neither the solution of (5.1) nor the approximation (8.17) has a transient phase at $t^* = t_1 - \sqrt{\varepsilon}\,\tau^*$ (with $\tau^* = \varepsilon^{-1/6}$), we can use the representation (8.20) for the function $\zeta(\tau)$. Inserting it into the first equation of (8.7) and into (8.8), we obtain a regular differential equation

$$\begin{aligned} \eta'(\tau) &= S\big(\tau, \eta(\tau), \nu(\tau)\big),\\ \nu'(\tau) &= -d\big(c\tau + \nu(\tau)\big)\nu(\tau) + T\big(\tau, \eta(\tau), \nu(\tau)\big), \end{aligned} \tag{8.21}$$

where S and T are smooth functions and satisfy (for $|\tau| \le \varepsilon^{-1/6}$)

$$\begin{aligned} \|S(\tau,\eta,\nu) - S(\tau,\widehat{\eta},\widehat{\nu})\| &\le \sqrt{\varepsilon}\,L_1\|\eta - \widehat{\eta}\,\| + L_2|\nu - \widehat{\nu}|,\\ |T(\tau,\eta,\nu) - T(\tau,\widehat{\eta},\widehat{\nu})| &\le \varepsilon^{1/3}L_3\|\eta - \widehat{\eta}\,\| + \varepsilon^{1/6}L_4|\nu - \widehat{\nu}|. \end{aligned} \tag{8.22}$$

After Taylor expansion and division by ε, the right-hand side of (8.8) contains terms $\sqrt{\varepsilon}\tau\eta(\tau)$ and $\sqrt{\varepsilon}\tau^2\nu(\tau)$, which give rise to the factors $\sqrt{\varepsilon}|\tau| \le \varepsilon^{1/3}$ and $\sqrt{\varepsilon}|\tau|^2 \le \varepsilon^{1/6}$ in the second estimate of (8.22) (notice that $|\tau| \le \varepsilon^{-1/6}$).

The approximations $\widehat{\eta}(\tau)$ and $\widehat{\nu}(\tau)$ of (8.17) satisfy (8.21) with a defect of size $\mathcal{O}(\varepsilon^{N/3})$. The function $h(\tau,\nu) = -d(c\tau+\nu)\nu$, appearing in (8.21), satisfies $h(\tau,\nu) - h(\tau,\widehat{\nu}) = -d(c\tau+\nu+\widehat{\nu})(\nu-\widehat{\nu})$. Since $c\tau+\nu_0(\tau) > 0$ for all τ, we obtain the differential inequalities (with a slightly increased value of L_4)

$$\begin{aligned} D_+\|\eta(\tau) - \widehat{\eta}(\tau)\| &\le \sqrt{\varepsilon}\,L_1\|\eta(\tau) - \widehat{\eta}(\tau)\| + L_2|\nu(\tau) - \widehat{\nu}(\tau)| + C_1\varepsilon^{N/3},\\ D_+|\nu(\tau) - \widehat{\nu}(\tau)| &\le \varepsilon^{1/3}L_3\|\eta(\tau) - \widehat{\eta}(\tau)\| + \varepsilon^{1/6}L_4|\nu(\tau) - \widehat{\nu}(\tau)| + C_2\varepsilon^{N/3}, \end{aligned}$$

where we make use of Dini derivatives as in [HW96, p. 392]. Solving this inequality for the scaled variables $(\eta, \varepsilon^{-1/6}\nu)$ yields the bounds $\|\eta(\tau) - \widehat{\eta}(\tau)\| + \varepsilon^{-1/6}|\nu(\tau) - \widehat{\nu}(\tau)| \le C\mathrm{e}^{\varepsilon^{1/6}(\tau+\tau^*)}\varepsilon^{N/3-1/6}$ for $\tau \ge -\tau^*$. This yields the estimates $|\nu(\tau) - \widehat{\nu}(\tau)| \le C\varepsilon^{N/3}$

for $\tau \in [-\tau^*, \tau^*]$. Inserting them into the differential inequality for $\|\eta(\tau) - \widehat{\eta}(\tau)\|$ and into (8.20) yields the estimates (8.18). ◻

A few explicit formulas. The solution of (8.11) can be expressed in terms of the error function. The change of variables $\nu_0(\tau) + c\,\tau = \kappa(\tau)\,\mathrm{e}^{-\gamma^2\tau^2}$ leads to a differential equation for $\kappa(\tau)$ which can be solved by separation of variables. This gives

$$\nu_0(\tau) + c\,\tau = \frac{\gamma\,\mathrm{e}^{-\gamma^2\tau^2}}{C + d\int_{-\infty}^{\gamma\tau}\mathrm{e}^{-\sigma^2}\mathrm{d}\sigma}, \qquad \gamma^2 = -\frac{d\,c}{2} > 0. \tag{8.23}$$

The only solution satisfying $\nu_0(\tau) \to 0$ for $\tau \to -\infty$ (or more precisely $\nu_0(\tau) \sim -(d\tau)^{-1}$) is obtained when the integration constant is chosen as $C = 0$. Using the fact that $\int_{-\infty}^{\infty}\mathrm{e}^{-\sigma^2}\mathrm{d}\sigma = \sqrt{\pi}$, we get the representation

$$\nu_0(\tau) + c\,\tau = \frac{\gamma\,\mathrm{e}^{-\gamma^2\tau^2}}{d\,(\sqrt{\pi} - \int_{\gamma\tau}^{\infty}\mathrm{e}^{-\sigma^2}\mathrm{d}\sigma)},$$

which is more suitable for large positive τ. This shows that at the critical point $t = t_1$, i.e., $\tau = 0$, we have asymptotically for $\varepsilon \to 0$ that

$$\alpha\big(y(t_1)\big) = -\,\varepsilon\,\ln u(t_1) = -\,\varepsilon\,\ln\big(\sqrt{\varepsilon}\,(c\,\tau + \nu_0(\tau) + \cdots\big) = -\,\varepsilon\,\ln\sqrt{\varepsilon} - \varepsilon\,\ln\sqrt{\tfrac{2|c|}{\pi d}} + \cdots$$

and the dominant term of the distance of the solution $y(t_1)$ of (1.3) to the manifold $\{y\,;\,\alpha(y) = 0\}$ is independent of the problem. For $t = t_1 + \sqrt{\varepsilon}\,\tau$ and large positive τ this analysis shows that

$$\alpha\big(y(t)\big) = -\,\varepsilon\,\ln u(t) = \gamma^2\,(t - t_1)^2 -\,\varepsilon\,\ln\sqrt{\varepsilon} - \varepsilon\,\ln\sqrt{\tfrac{|c|}{2\pi d}} + \cdots$$

and the solution is seen to leave the manifold like a parabola.

8.3. Escaping solution without breaking point. We finally arrive at the study of the solution of (1.3), when it leaves the manifold $\{y\,;\,\alpha(y) = 0\}$ opposite the side where it entered. The proof of Lemma 8.4 shows that for $t^* = t_1 + \sqrt{\varepsilon}\,\tau^*$ with $\tau^* = \varepsilon^{-1/6}$ we have

$$0 \le u(t^*) \le \sqrt{\varepsilon}\,\mathrm{e}^{-\gamma^2\tau^{*2}} = \sqrt{\varepsilon}\,\mathrm{e}^{-\gamma^2\varepsilon^{-1/3}},$$

which is smaller than every power of ε. Therefore, the term $u(t)$ in (7.4) does not contribute to the smooth part of the solution of (1.3). Since there is no reason for a transient phase at t^*, we can replace the function $u(t)$ by zero, and we are concerned with a singularly perturbed differential equation

$$\begin{aligned} \dot{y}(t) &= z(t), \\ \varepsilon\,\dot{z}(t) &= f\Big(y(t), s\big(\varepsilon, \alpha(y(t))\big)\Big) \;-\; z(t). \end{aligned} \tag{8.24}$$

The standard well-developed theory for singularly perturbed ordinary differential equations can be applied. We are in the lucky situation, where we know that a transient is absent, so that the asymptotic expansion is of the form

$$y(t) = \sum_{j=0}^{N}\varepsilon^j\,y_j(t) + \mathcal{O}(\varepsilon^{N+1}), \qquad z(t) = \sum_{j=0}^{N}\varepsilon^j\,z_j(t) + \mathcal{O}(\varepsilon^{N+1}). \tag{8.25}$$

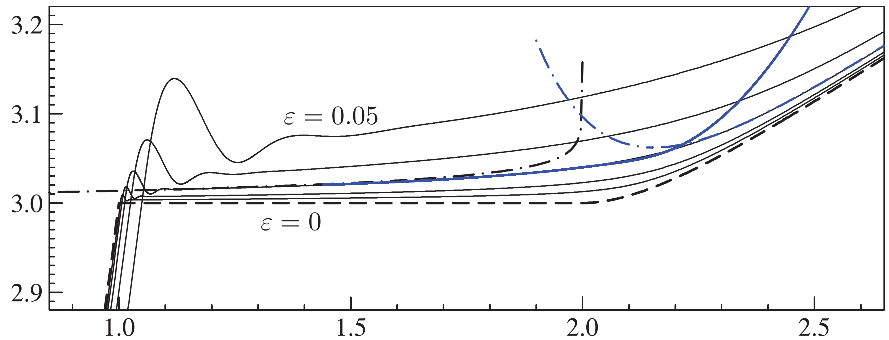

FIG. 8.3. *Solution $y(t)$ of the singularly perturbed delay equation* (8.26) *with $\varepsilon = 0.05 \cdot 2^{-n}$, $n = 0, \dots, 4$; for $\varepsilon = 0.0125$, the truncated asymptotic expansions are included for three situations.*

We only have to match the initial value for $y(t)$ at $t = t^*$. The value for $z(t)$ is automatically correct (because of the absence of a transient layer). We put $y_j(t^*) = 0$ for $j \geq 1$, and $y_0(t^*) = y^*$, where $y^* = y^*(\varepsilon)$ is the approximation at t^* obtained with the truncated asymptotic expansion (8.6)–(8.9).

The coefficient functions of (8.25) are obtained by inserting the series into (8.24) and by comparing like powers of ε as follows: express $z_0(t)$ in terms of $y_0(t)$, then solve an ordinary differential equation for $y_0(t)$, express $z_1(t)$ in terms of $y_1(t)$ and known functions, then solve an ordinary differential equation for $y_1(t)$, etc.

Example. This time we consider the singularly perturbed delay differential equation

$$\begin{aligned} \dot y(t) &= z(t), \\ \varepsilon\, \dot z(t) &= d(t) - 3\, z\big(\alpha(y(t))\big) - z(t), \qquad \alpha(y) = y - 3, \quad d(t) = 2 + 2t, \end{aligned} \tag{8.26}$$

with initial functions $y(t) = z(t) = 0$ for $t \leq 0$. For $\varepsilon = 0$, the solution until the first breaking point $t_0 = 1$ is $z(t) = 2 + 2t$, $y(t) = 2t + t^2$. As in the previous section, we have a weak solution $y(t) = 3$, $z(t) = 0$ on the interval $[1, 2]$, but after the point $t_1 = 2$ a classical solution emerges the manifold $y = 3$ in the opposite direction. There, the solution is given by $z(t) = (1 - \mathrm{e}^{-6(t-2)})/3$, $y(t) = 41/18 + t/3 + \mathrm{e}^{-6(t-2)}/18$.

For positive ε we have the solution $z(t) = 2t + 2(1-\varepsilon)(1 - \mathrm{e}^{-t/\varepsilon})$, $y(t) = t^2 + 2(1-\varepsilon)(t - \varepsilon(1 - \mathrm{e}^{-t/\varepsilon}))$ until the first breaking point $t_0(\varepsilon) = 1 + \varepsilon + \mathcal{O}(\varepsilon^2)$. Beyond this breaking point, the smooth part of the asymptotic expansion is

$$y(t) = 3 - \varepsilon \, \ln\Big(\frac{2-t}{3}\Big) + \mathcal{O}(\varepsilon^2), \qquad z(t) = \frac{\varepsilon}{2-t} + \mathcal{O}(\varepsilon^2),$$

and we see that already the ε-term has a singularity at $t_1 = 2$ (see Figure 8.3). The asymptotic expansion of section 5.1 approximates the solution only on intervals $[t_0, T]$ with $T < t_1$. Close to $t_1 = 2$ the solution satisfies

$$y\big(t_1 + \sqrt{\varepsilon}\,\tau\big) = 3 - \varepsilon\, \ln\Big(\sqrt{\varepsilon}\,\omega_0(\tau) + \varepsilon\,\omega_1(\tau) + \varepsilon\,\sqrt{\varepsilon}\,\omega_2(\tau) + \cdots\Big),$$

where the coefficient functions $\omega_j(\tau)$ are the solution of $\omega_0' = -(6\omega_0 + 2\tau)\omega_0$, $\omega_1' = -(12\omega_0 + 2\tau)\omega_1 - \omega_0\,(4 - 6\,\eta_0 - 6\,\omega_0')$, $\omega_2' = -(12\omega_0 + 2\tau)\omega_2 - \omega_1\zeta_1 - \omega_0(\zeta_1' - 6\eta_1 - 6\omega_0)$, with bounded initial values at $-\infty$, and $\eta_0 = -\ln(\sqrt{\varepsilon}\,\omega_0)$, $\eta_1 = -\omega_2/\omega_1$, $\zeta_1 = 6\omega_1 - 6\eta_0 + 4 - 6\omega_0'$.

Away from the critical point t_1 the solution admits the expansion (8.25), where $y_0(t)$ is the solution for $\varepsilon = 0$, and $y_1(t) = 1 + (C - 2t)\mathrm{e}^{-6(t-2)}$ with C chosen to match the previous expansion at $t^* = 2 + \varepsilon^{1/3}$.

Figure 8.3 shows the exact solution of (8.26) for several values of ε. For the particular value $\varepsilon = 0.0125$, the approximations obtained by truncated asymptotic expansions are included. On the interval $(1, 2)$ we see the smooth part of the expansion (5.5). Away from the initial transient phase it is an excellent approximation as long as one is not close to $t_1 = 2$, where this expansion has a singularity. The scaled expansion (8.6)–(8.9) approximates the solution on a $\mathcal{O}(\varepsilon^{1/3})$ neighborhood of $t_1 = 2$. Surprisingly it is also an excellent approximation on the whole interval $(1, 2)$. This can be explained by the fact that the functions $y_0(t), z_0(t), u_0(t)$ are polynomials, so that no error is introduced by replacing them with their Taylor polynomials. Finally, the expansion (8.25) approximates the solution from the instant (close to $2 + \varepsilon^{1/3}$) where the previous expansion loses its value.

We remark that the given analysis presents interesting analogies—we refer in particular to the time scales for the several observed phenomena—to the results given in [MPN11] for a singularly perturbed scalar state-dependent delay differential equation.

Acknowledgments. We thank the referees for their critical reading of our article and their useful suggestions.

REFERENCES

[BG09] A. Bellen and N. Guglielmi, *Solving neutral delay differential equations with state-dependent delays*, J. Comput. Appl. Math., 229 (2009), pp. 350–362.

[BZ03] A. Bellen and M. Zennaro, *Numerical methods for delay differential equations*, in Numerical Mathematics and Scientific Computation, in Clarendon Press, New York, 2003.

[Dri65] R. D. Driver, *Existence and continuous dependence of solutions of a neutral functional differential equation*, Arch. Ration. Mech. Anal., 19 (1965), pp. 149–166.

[Dri84] R. D. Driver, *A neutral system with state-dependent delay*, J. Differential Equations, 54 (1984), pp. 73–86.

[ÈN73] L. E. Èl'sgol'ts and S. B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, New York, 1973.

[FG11] G. Fusco and N. Guglielmi, *A regularization for discontinuous differential equations with application to state-dependent delay differential equations of neutral type*, J. Differential Equations, 250 (2011), pp. 3230–3279.

[Fil88] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Math. Appl. (Soviet Series) 18, Kluwer Academic, Dordrecht, The Netherlands, 1988.

[GH01] N. Guglielmi and E. Hairer, *Implementing Radau* II*A methods for stiff delay differential equations*, Computing, 67 (2001), pp. 1–12.

[GH07] N. Guglielmi and E. Hairer, *Stiff delay equations*, Scholarpedia, 2 (2007), p. 2850.

[GH08] N. Guglielmi and E. Hairer, *Computing breaking points in implicit delay differential equations*, Adv. Comput. Math., 29 (2008), pp. 229–247.

[GH11] N. Guglielmi and E. Hairer, *Numerical approaches for state-dependent neutral delay equations with discontinuities*, Math. Computers in Simulation, to appear.

[HNW93] E. Hairer, S. P. Nørsett, and G. Wanner, *Solving Ordinary Differential Equations* I. *Nonstiff Problems*, 2nd ed., Springer Series in Computational Mathematics 8. Springer, Berlin, 1993.

[HW96] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations* II. *Stiff and Differential-Algebraic Problems*, 2nd ed., Springer Ser. Comput. Math. 14, Springer-Verlag, Berlin, 1996.

[HW97] E. Hairer and G. Wanner, *Analysis by Its History*, Undergraduate Texts Math., Springer-Verlag, New York, 1997.

[Kis91] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Math. Appl. (East European Series) 44, Kluwer Academic, Dordrecht, The Netherlands, 1991.

[MCG07] M. Marino, A. Carati, and L. Galgani, *Classical light dispersion theory in a regular lattice*, Ann. Phys., 322 (2007), pp. 799–823.

[MPN11] J. Mallet-Paret and R. D. Nussbaum, *Superstability and rigorous asymptotics in singularly perturbed state-dependent delay-differential equations*, J. Differential Equations, 250 (2011), pp. 4037–4084.

[O'M91] R. E. O'Malley, Jr., *Singular Perturbation Methods for Ordinary Differential Equations*, Appl. Math. Sci. 89, Springer-Verlag, New York, 1991.

[RH92] A. E. Ruehli and H. Heeb, *Circuit models for three dimensional geometries including dielectrics*, IEEE Trans. Microwave Theory and Techniques, 40 (1992), pp. 1507–1516.

[Utk92] V. I. Utkin, *Sliding Modes in Control and Optimization*, Comm. Control Engrg. Ser., Springer-Verlag, Berlin, 1992.