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Solutions leaving a codimension-2 sliding

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Summary Piece-wise smooth differential equations (their regularization, numerical integration, and classification of solutions) is the topic of the present work. The behaviour close to one discontinuity surface and also the entering into the intersection of two discontinuity surfaces is well understood. Here, we study the solutions that exit a codimension-2 sliding mode. Some results are expected, others come as a surprise. We are able to explain situations, where difficulties in numerical computations are reported in the recent literature.

The analysis is based on asymptotic expansions for singularly perturbed problems and on the study of a time-parameterized two-dimensional dynamical system (hidden dynamics). Various situations are illustrated by examples.

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1 Introduction

In this article we study the situation, where the solution of a discontinuous dynamical system leaves a codimension-2 sliding mode. Following the notation of [15] we consider in the n-dimensional phase space the hypersurfaces

$$\Sigma_\alpha = \{y \in \mathbb{R}^n \,;\, \alpha(y) = 0\} \qquad \Sigma_\beta = \{y \in \mathbb{R}^n \,;\, \beta(y) = 0\}, \qquad (1)$$

where $\alpha(y)$ and $\beta(y)$ are scalar smooth functions. We assume that both hypersurfaces intersect transversally, so that in a neighbourhood of $y \in \Sigma = \Sigma_\alpha \cap \Sigma_\beta$, these hypersurfaces divide the phase space into the four regions $\mathcal{R}^{++} = \{y\,;\,\alpha(y) > 0,\, \beta(y) > 0\}$, $\mathcal{R}^{+-} = \{y\,;\,\alpha(y) > 0,\, \beta(y) < 0\}$, $\mathcal{R}^{-+} = \{y\,;\,\alpha(y) < 0,\, \beta(y) > 0\}$, and $\mathcal{R}^{--} = \{y\,;\,\alpha(y) < 0,\, \beta(y) < 0\}$. The discontinuous differential equation is given by

$$\dot y = \begin{cases} f^{++}(y) & y \in \mathcal{R}^{++} \\ f^{+-}(y) & y \in \mathcal{R}^{+-} \\ f^{-+}(y) & y \in \mathcal{R}^{-+} \\ f^{--}(y) & y \in \mathcal{R}^{--} \end{cases} \tag{2}$$

where the vector fields are individually assumed to be smoothly extendable to a neighbourhood of the adjacent discontinuity surfaces. For $y \in \Sigma_\alpha$ or $y \in \Sigma_\beta$ we adopt the concept of Filippov [13], and we consider solutions (sliding modes), for which the derivative $\dot y(t)$ is a convex combination of the adjacent vector fields (for further solution concepts we refer to [3,5], stochastically perturbed sliding motion is considered in [27]). In the intersection $\Sigma_\alpha \cap \Sigma_\beta$ we restrict our study to a two-parameter family of convex combinations (called 'blending' in [4], bilinear interpolation in [8,7], and convex canopy in [18], see also [11]).

Solutions evolving in one of the discontinuity surfaces (codimension-1 sliding) are well understood. The situation where the solution evolves in the intersection of two or more discontinuity surfaces arises in many practical applications. We mention multibody dynamics subject to Coulomb friction [25], gene regulatory networks describing the interactions of genes and proteins (see [12,17,23,2]), and systems in control engineering [30,26]. Further interesting applications and numerical approaches can be found in the monographs [1] and [6]. The study of the behaviour of solutions close to the intersection of discontinuity surfaces is much more challenging (mainly due to non-uniqueness). Recently in [15] we have given a classification of solutions entering the intersection $\Sigma_\alpha \cap \Sigma_\beta$, which can be interpreted as the limit of a regularized differential equation. Related results can be found in [9].

The present work can be seen as a follow-up of [15]. Here, we study the exit of solutions from a codimension-2 sliding. We know of two recent publications treating the same topic. The article [9] gives sufficient conditions guaranteeing that the regularized solution converges to a sliding solution on $\Sigma_\alpha \cap \Sigma_\beta$, and it illustrates through several examples the kind of "exiting" that can happen. The article [20, Section IV] presents examples that illustrate various ways of exiting from codimension-2 sliding. Our present study tends towards a classification of all possible ways of exiting a codimension-2 sliding mode. We give new information on when and how a sliding mode solution can leave $\Sigma_\alpha \cap \Sigma_\beta$. Several situations are rigorously proven with the help of asymptotic expansions before and after the exit point. New is also the study of the exit time from codimension-2 sliding.

An important ingredient of our analysis is a regularization (see [28, 29, 22, 10, 24, 14]) of the discontinuous differential equation (2), which takes the form

$$\begin{aligned}\dot y = \Big(&\big(1+\pi(u)\big)\big(1+\pi(v)\big)\, f^{++}(y) + \big(1+\pi(u)\big)\big(1-\pi(v)\big)\, f^{+-}(y) \\ &+ \big(1-\pi(u)\big)\big(1+\pi(v)\big)\, f^{-+}(y) + \big(1-\pi(u)\big)\big(1-\pi(v)\big)\, f^{--}(y)\Big)\Big/4,\end{aligned} \tag{3}$$

where $u = \alpha(y)/\varepsilon$ and $v = \beta(y)/\varepsilon$. Here, $\pi(u)$ is a continuous, monotone scalar function that takes the value -1 for $u \le -1$, and the value $+1$ for $u \ge 1$, e.g., $\pi(u) = \min\big(1, \max(-1, u)\big)$. By definition of the function $\pi(u)$, the differential equation (3) agrees with (2) whenever $|\alpha(y)| \ge \varepsilon$ and $|\beta(y)| \ge \varepsilon$. In the ε-stripes along Σ_α and Σ_β it is a linear interpolation of the vector fields, and in $\Sigma_\alpha \cap \Sigma_\beta$ a bilinear interpolation. Because of the division by ε in the definition of u and v, the differential equation (3) is of singular perturbation type. In the following, we abbreviate the right-hand side of (3) by $f\big(y, \pi(u), \pi(v)\big)$, so that the regularized differential equation becomes

$$\dot y = f\Big(y, \pi\big(\alpha(y)/\varepsilon\big), \pi\big(\beta(y)/\varepsilon\big)\Big). \tag{4}$$

The study of the regularized differential equation (4) is important because of at least two reasons. First of all, it replaces a discontinuous differential equation by a continuous one, so that standard software for ordinary differential equations can be applied for their numerical integration. Since the system is of singular perturbation type, the use of integrators for stiff differential equations is advised. Secondly, the solution of (4) is unique as long as it exists. In the limit $\varepsilon \to 0$ it selects the most natural solution of the discontinuous system in the case where (2) has more than one solutions (classical solutions and sliding modes).

In the analysis of [15], where a classification of solutions entering the intersection $\Sigma_\alpha \cap \Sigma_\beta$ is given, a two-dimensional autonomous dynamical system on the unit square $[-1,1] \times [-1,1]$ plays a crucial role (hidden dynamics). Its solution determines the fast transition between the incoming and continuing solutions. In the present work, a similar two-dimensional differential equation is central to the analysis, but here the system depends on the slow time t. By an abuse of notation, we continue to speak about an equilibrium (depending on time t), if the right-hand side of this system vanishes. The aim of the present work is to get new insight into situations, where a solution leaves the intersection $\Sigma_\alpha \cap \Sigma_\beta$ from a codimension-2 sliding mode.

In Section 2, solutions evolving in the intersection $\Sigma_\alpha \cap \Sigma_\beta$ (so-called codimension-2 sliding modes) are studied. Using singular perturbation techniques it is proved (Theorem 1 below) that as long as the 'time-dependent equilibrium' is asymptotically stable, the solution of the regularized differential equation (4) stays close to $\Sigma_\alpha \cap \Sigma_\beta$. This happens even in the situation, when a classical solution or a codimension-1 sliding mode co-exist with the codimension-2 sliding mode. There are essentially two situations, where Theorem 1 is no longer applicable. Either the 'equilibrium' leaves the unit square (case A) or it becomes unstable or disappears (case B).

Section 3 is devoted to case A. Leaving the unit square provokes a transient layer in the solution that is scaled by ε^2. A study of this transient layer gives information on whether the solution continues as a codimension-1 sliding mode or as a classical solution. The transient between solutions takes place on a time interval of length $\mathcal{O}(\varepsilon)$.

Section 4 is devoted to case B. Precise statements are more difficult to obtain, so that we decided to present the typical situations by examples. Several unexpected behaviours arise in case B. For the situation, where the 'equilibrium' turns into an unstable one, the exit point depends on the time instant, when the solution entered the codimension-2 sliding mode. Its numerical approximation is strongly affected by the accuracy of the time integrator and by round-off errors. The situation, where the 'equilibrium' disappears at some time instant t is the most difficult to analyse. Numerical experiments at a typical example indicate that here the exit time is proportional to $\mathcal{O}(\varepsilon^\kappa)$ with $\kappa \approx 0.67$.

2 Codimension-2 sliding modes

Codimension-2 sliding modes are solutions of (2) that evolve in the intersection $\Sigma_\alpha \cap \Sigma_\beta$. There, the vector field is defined by bilinear interpolation of the neighbour vector fields. We first recall results from [15] on solutions entering $\Sigma_\alpha \cap \Sigma_\beta$, and then we study the remainder in the asymptotic expansions of the sliding solutions. To simplify the presentation we assume throughout this section that $\pi(u) = u$ for $|u| \le 1$.

2.1 Hidden dynamics

We are interested in the situation, where the solution of (2) approaches $\Sigma_\alpha \cap \Sigma_\beta$ and stays there as a codimension-2 sliding mode. Much insight into the regularized problem (4) is obtained by a two-scale asymptotic expansion with slow time t and fast time $\tau = t/\varepsilon$. Beyond the time instant $t^*(\varepsilon) = t^* + \mathcal{O}(\varepsilon)$, when the solution of (4) enters the set $\{y\,; |\alpha(y)| \le \varepsilon,\, |\beta(y)| \le \varepsilon\}$ at $y\big(t^*(\varepsilon)\big) = y^* + \mathcal{O}(\varepsilon)$, $y^* \in \Sigma_\alpha \cap \Sigma_\beta$, the solution can be written as (see [15])

$$y\big(t^*(\varepsilon) + t\big) = y_0(t) + \varepsilon\big(y_1(t) + \eta_0(\tau)\big) + \varepsilon^2\big(y_2(t) + \eta_1(\tau)\big) + \ldots \quad (5)$$

Here, $y_0(t)$ is the codimension-2 sliding mode of the non-regularized problem (2). The transient term $\eta_0(\tau)$ converges exponentially fast to zero, and is visible only on an interval of length $\mathcal{O}(\varepsilon)$. It is determined by $u(\tau) = \alpha'(y^*)\big(y_1(0) + \eta_0(\tau)\big)$ and $v(\tau) = \beta'(y^*)\big(y_1(0) + \eta_0(\tau)\big)$, which are the solution of the two-dimensional dynamical system (called 'hidden dynamics' in [19])

$$\begin{aligned} u' &= \alpha'(y^*) f(y^*, u, v) \\ v' &= \beta'(y^*) f(y^*, u, v). \end{aligned} \quad (6)$$

Here, 'prime' indicates the derivative with respect to τ. Initial values are determined by the incoming solution. It is explained in [15] that the solution of (4) turns at $t^*(\varepsilon)$ into a codimension-2 sliding mode, if the solution of (6) converges for $\tau \to \infty$ to a stationary point (u^*, v^*) in the unit square $(-1, 1) \times (-1, 1)$. Throughout this article we assume that this stationary point is asymptotically stable.

2.2 Smooth asymptotic expansion

Assume that the solution of (2) enters a codimension-2 sliding mode. After a short time the transient is damped out and the solution of (4) can be approximated by a smooth asymptotic expansion

$$y\big(t^*(\varepsilon) + t\big) = y_0(t) + \varepsilon y_1(t) + \ldots + \varepsilon^N y_N(t) + \mathcal{O}(\varepsilon^{N+1}). \quad (7)$$

Since for a codimension-2 sliding the solution remains $\mathcal{O}(\varepsilon)$-close to $\Sigma_\alpha \cap \Sigma_\beta$, we have

$$\alpha\big(y_0(t)\big) = 0, \qquad \beta\big(y_0(t)\big) = 0. \quad (8)$$

The coefficient functions of (7) can be computed as follows.

We first note that, as a consequence of (8), we have ε-expansions

$$\frac{\alpha\big(y(t)\big)}{\varepsilon} = u_0(t)+\varepsilon u_1(t)+\ldots, \qquad \frac{\beta\big(y(t)\big)}{\varepsilon} = v_0(t)+\varepsilon v_1(t)+\ldots, \tag{9}$$

where $u_0 = \alpha'(y_0)y_1$, $u_1 = \alpha'(y_0)y_2 + \frac{1}{2}\alpha''(y_0)y_1^2$, etc., and the same relations for v_j, where α is replaced by β. Inserting the expansion (7) into (4) and then putting $\varepsilon = 0$ yields a differential equation for $y_0(t)$ which, together with (8), gives the differential-algebraic system

$$\begin{aligned} \dot y_0 &= f(y_0, u_0, v_0) \\ 0 &= \alpha(y_0) \\ 0 &= \beta(y_0) \end{aligned} \tag{10}$$

as long as $-1 \le u_0, v_0 \le 1$. Note that we assume $\pi(u) = u$ for $|u| \le 1$. Differentiating the algebraic relations with respect to t yields[1]

$$\begin{aligned} 0 &= \alpha'(y_0) f(y_0, u_0, v_0) \\ 0 &= \beta'(y_0) f(y_0, u_0, v_0). \end{aligned} \tag{11}$$

We assume that this 2-dimensional system permits us to express locally u_0 and v_0 in terms of y_0 (index 1), so that the first equation of (10) becomes an ordinary differential equation for $y_0(t)$.

We next compare the coefficient of ε, when the expansion (7) together with (9) is inserted into (4). Augmented by the definition of u_0 and v_0 this yields the differential-algebraic system for (y_1, u_1, v_1),

$$\begin{aligned} \dot y_1 &= \partial_y f(y_0, u_0, v_0)y_1 + \partial_u f(y_0, u_0, v_0)u_1 + \partial_v f(y_0, u_0, v_0)v_1 \\ 0 &= \alpha'(y_0)y_1 - u_0 \\ 0 &= \beta'(y_0)y_1 - v_0, \end{aligned} \tag{12}$$

where $\partial_u f$ and $\partial_v f$ denote partial derivatives (one-sided derivatives at the end points of the interval $[-1, 1]$). As before, we differentiate the algebraic relations with respect to t. This time we obtain a linear system, which permits us to express u_1 and v_1 in terms of y_1 and the known functions y_0, u_0, v_0. This procedure can be continued to compute further coefficient functions in the expansion (7).

We consider initial values for $y_j(t)$ in (10), (12), etc., at the ε-independent time $t^* = t^*(0)$ (see Section 2.1). They are given by the smooth part of the expansion (5).

[1] There is an interesting connection to the dynamical system (6) of the hidden dynamics. Considering t as a fixed parameter, the values $u_0(t)$ and $v_0(t)$ can be interpreted as an equilibrium of (6) corresponding to $y^* = y_0(t)$.

2.3 Estimation of the remainder

Consider the truncated asymptotic expansions

$$\begin{aligned}\hat{y}(t) &= y_0(t) + \varepsilon y_1(t) + \ldots + \varepsilon^N y_N(t),\\ \hat{u}(t) &= u_0(t) + \varepsilon u_1(t) + \ldots + \varepsilon^N u_N(t),\\ \hat{v}(t) &= v_0(t) + \varepsilon v_1(t) + \ldots + \varepsilon^N v_N(t),\end{aligned} \tag{13}$$

where the coefficient functions $y_j(t), u_j(t), v_j(t)$ are those of Section 2.2. Our aim is to prove that the function $\hat{y}(t)$ is $\mathcal{O}(\varepsilon^{N+1})$-close to the solution of (4).

To be able to apply singular perturbation techniques we differentiate $u = \alpha(y)/\varepsilon$ and $v = \beta(y)/\varepsilon$ with respect to time, so that the equation (4) becomes (for $-1 \le u, v \le 1$) the system

$$\begin{aligned}\dot{y} &= f(y,u,v)\\ \varepsilon\dot{u} &= \alpha'(y) f(y,u,v)\\ \varepsilon\dot{v} &= \beta'(y) f(y,u,v).\end{aligned} \tag{14}$$

The solutions of (4) and of (14) are the same provided that the initial values are consistent with $u(0) = \alpha\big(y(0)\big)/\varepsilon$ and $v(0) = \beta\big(y(0)\big)/\varepsilon$. The stability of this system is governed by the 2-dimensional matrix

$$G(y,u,v) = \begin{pmatrix} \alpha'(y)\partial_u f(y,u,v) & \alpha'(y)\partial_v f(y,u,v) \\ \beta'(y)\partial_u f(y,u,v) & \beta'(y)\partial_v f(y,u,v)\end{pmatrix}. \tag{15}$$

Theorem 1 *Consider the problem (14) together with initial values $y(0) = y^0 = y_0^0 + \varepsilon y_1^0 + \ldots + \varepsilon^N y_N^0 + \mathcal{O}(\varepsilon^{N+1})$, $u(0) = \alpha(y^0)/\varepsilon$, $v(0) = \beta(y^0)/\varepsilon$.*

- *Assume that the reduced problem (10) with initial value $y_0(0) = y_0^0$ has a solution $y_0(t), u_0(t), v_0(t)$ on the compact interval $[0,T]$, which satisfies $-1 \le u_0(t), v_0(t) \le 1$ on this interval.*
- *Suppose that both eigenvalues $\lambda_j = \lambda_j(y,u,v)$, $j = 1,2$ of the matrix $G(y,u,v)$ of (15) satisfy $\lambda_j \le \mu < 0$ on an ε-independent neighbourhood $\mathcal{U}$ of $\big\{\big(y_0(t), u_0(t), v_0(t)\big)\,;\, t \in [0,T]\big\}$.*

Then, the considered problem (14) has, for sufficiently small ε, a unique solution on $[0,T]$ which admits the asymptotic expansions

$$\begin{aligned}y(t) &= y_0(t) + \varepsilon y_1(t) + \ldots + \varepsilon^N y_N(t) + \mathcal{O}(\varepsilon^{N+1}),\\ u(t) &= u_0(t) + \varepsilon u_1(t) + \ldots + \varepsilon^N u_N(t) + \mathcal{O}(\varepsilon^{N+1}),\\ v(t) &= v_0(t) + \varepsilon v_1(t) + \ldots + \varepsilon^N v_N(t) + \mathcal{O}(\varepsilon^{N+1}),\end{aligned}$$

where $y_j(t), u_j(t), v_j(t)$ are the functions constructed in Section 2.2.

Proof The present proof follows closely the ideas of the proof of Theorem 3.2 in [16, Chapter VI.3]. We consider the truncated series (13). The construction of the coefficient functions $y_j(t), u_j(t), v_j(t)$ and the fact that $\varepsilon^{-1}\alpha\big(\hat y + \varepsilon^{N+1}y_{N+1} + \mathcal{O}(\varepsilon^{N+2})\big) = \hat u(t) + \mathcal{O}(\varepsilon^{N+1})$ yields

$$\begin{aligned} \dot{\hat y} &= f(\hat y, \hat u, \hat v) + \mathcal{O}(\varepsilon^{N+1}) \\ \varepsilon\, \dot{\hat u} &= \alpha'(\hat y) f(\hat y, \hat u, \hat v) + \mathcal{O}(\varepsilon^{N+1}) \\ \varepsilon\, \dot{\hat v} &= \beta'(\hat y) f(\hat y, \hat u, \hat v) + \mathcal{O}(\varepsilon^{N+1}). \end{aligned} \tag{16}$$

In the following we write $z = (u,v)^{\mathsf T}$, so that the second and third lines of (14) and (16) become $\varepsilon \dot z = g(y,z)$ and $\varepsilon \dot{\hat z} = g(\hat y, \hat z) + \mathcal{O}(\varepsilon^{N+1})$, respectively.

a) We first consider the situation, where there exists an inner product norm such that $\langle \Delta z, G(y,z)\Delta z\rangle \le \mu \|\Delta z\|^2$ for all Δz and for all $(y,z) \in \mathcal{U}$. Subtracting (14) from (16) and exploiting Lipschitz conditions for f and g we obtain

$$\begin{aligned} D_+\|\hat y(t) - y(t)\| &\le L_1\|\hat y(t) - y(t)\| + L_2\|\hat z(t) - z(t)\| + C_1\varepsilon^{N+1} \\ \varepsilon\, D_+\|\hat z(t) - z(t)\| &\le L_3\|\hat y(t) - y(t)\| + \mu\|\hat z(t) - z(t)\| + C_2\varepsilon^{N+1}, \end{aligned}$$

where D_+ denotes the Dini derivative. To solve this inequality we replace $\le$ by $=$ and so obtain

$$\begin{aligned} \dot\eta &= L_1\eta + L_2\zeta + C_1\varepsilon^{N+1}, \quad & \eta(0) &= \|\hat y(0) - y(0)\| = \mathcal{O}(\varepsilon^{N+1}), \\ \varepsilon\, \dot\zeta &= L_3\eta + \mu\zeta + C_2\varepsilon^{N+1}, \quad & \zeta(0) &= \|\hat z(0) - z(0)\| = \mathcal{O}(\varepsilon^{N+1}). \end{aligned}$$

Since the system is quasi-monotone, we get the estimates

$$\|\hat y(t) - y(t)\| \le \eta(t), \qquad \|\hat y(t) - y(t)\| \le \zeta(t).$$

Solving the linear differential equation for (η,ζ) and using $\mu < 0$, one verifies that $\eta(t) = \mathcal{O}(\varepsilon^{N+1})$ and $\zeta(t) = \mathcal{O}(\varepsilon^{N+1})$ on compact intervals. This proves the statement of the theorem.

b) In the general case we fix $\delta > 0$ such that $\mu + \delta < 0$. For a given $t \in [0,T]$ we construct an inner product $\langle\cdot,\cdot\rangle_t$ for which

$$\langle \Delta z, G(y,z)\Delta z\rangle_t \le (\mu+\delta)\|\Delta z\|_t^2 \quad \text{for} \quad (y,z) = (y,u,v) \in \mathcal{U}_t,$$

where $\mathcal{U}_t$ is a neighbourhood of the point $\big(y_0(t), u_0(t), v_0(t)\big)$. This can be done by transforming the matrix $G\big(y_0(t), u_0(t), v_0(t)\big)$ to diagonal form (or to Jordan canonical form with a sufficiently small off-diagonal element), and by using the continuity of $G(y,u,v)$.

The collection $\{\mathcal{U}_t\}_{t\in[0,T]}$ forms an open covering of the compact set $K = \big\{\big(y_0(t), u_0(t), v_0(t)\big)\,;\, t \in [0,T]\big\}$. By compactness of K, finitely many sets among $\{\mathcal{U}_t\}_{t\in[0,T]}$ cover K. As a consequence there exists a partition $0 = t_0 < t_1 < \cdots < t_m = T$, such that the set $K_j = \big\{\big(y_0(t), u_0(t), v_0(t)\big)\,;\, t \in [t_j, t_{j+1}]\big\}$ lies entirely in one of the $\mathcal{U}_t$, say, in $\mathcal{U}_{\tau_j}$. On the interval $[t_j, t_{j+1}]$ we thus get the estimate of part (a) in the norm $\|\cdot\|_{\tau_j}$. The estimate over the whole interval $[0, T]$ is then obtained by patching together the estimates over $[t_j, t_{j+1}]$ (for $j = 0, \ldots, m-1$), and by using the equivalence of norms on a finite dimensional vector space. $\square$

Remark 1 Theorem 1 gives a sufficient condition for the robustness of codimension-2 sliding modes. It is perfectly conceivable that at some time instant t (with $t \in (0, T)$) a classical solution or a codimension-1 sliding mode starts to exist. As long as the assumptions of the previous theorem hold, the solution of the regularized problem will nevertheless remain close to the codimension-2 sliding mode.

Remark 2 It is possible to use different regularization parameters for the two discontinuity surfaces in (4). This is equivalent to replacing $\alpha(y)$ and $\beta(y)$ by $\kappa_\alpha\alpha(y)$ and $\kappa_\beta\beta(y)$, where $\kappa_\alpha > 0$ and $\kappa_\beta > 0$. In this case the rows of the matrix G are multiplied by κ_α and κ_β, respectively. This leaves the sign of $\det G$ invariant, but it can influence the sign of the trace of G, and hence also the stability of G.

The main interest of the present article is the study of solutions of the regularized differential equation when they leave a codimension-2 sliding mode. According to Theorem 1 this can happen in the following two situations:

A: the solution $u_0(t), v_0(t)$ of the reduced problem (10) leaves the unit square $-1 \le u, v \le 1$, and the eigenvalues of $G\big(y_0(t), u_0(t), v_0(t)\big)$ stay in the negative half-plane;

B: the solution $u_0(t), v_0(t)$ stays in the unit square, but the real part of at least one of the eigenvalues of the matrix $G\big(y_0(t), u_0(t), v_0(t)\big)$ tends to zero. Generically, either the real part of one eigenvalue changes sign, or the solution $u_0(t), v_0(t)$ ceases to exist (meaning that the equilibrium ceases to exist).

We shall discuss both situations separately in the next two sections. In the present work we do not consider the situation, where the approximation of a codimension-2 sliding mode is highly oscillatory around a smooth solution. This corresponds to the existence of a limit cycle in the hidden dynamics (for the frozen parameter t). Note that such limit cycles can often be avoided by suitably choosing the ratio between the parameters κ_α and κ_β of Remark 2 above (see the section on ‘Stabilization’ in [15]).

3 Leaving codimension-2 sliding mode - case A

In this section we study the case A of an exit from a codimension-2 sliding mode for the solution of the regularized differential equation. More precisely, we assume that the solution $u_0(t), v_0(t)$ of the reduced problem (10) leaves the unit square $-1 \le u, v \le 1$ at $t = 0$, and that both eigenvalues of the matrix $G(t) = G\big(y_0(t), u_0(t), v_0(t)\big)$ of (15) have negative real part on an interval $-\delta \le t \le 0$. These assumptions guarantee the existence of an asymptotic expansion by Theorem 1. Since we are mainly interested in generic situations, we exclude an exit at one of the corners and we assume (without loss of generality) that the solution $u_0(t), v_0(t)$ leaves the square transversally at the right side, i.e., $u_0(0) = 1$, $\dot u_0(0) > 0$, $v_0(0) = v_0^* \in (-1, 1)$.

The component $\big(u(t), v(t)\big)$ of the ε-dependent solution of the regularized differential equation (14) leaves the square $-1 \le u, v \le 1$ at a time $t^* = t^*(\varepsilon) = \varepsilon t_1^* + \varepsilon^2 t_2^* + \ldots$. This follows from the Implicit Function Theorem applied to $u(t, \varepsilon) = 1$ (indicating explicitly the ε-dependence), because $u(0, 0) = 1$ and $\partial_t u(0, 0) > 0$. By Theorem 1 the solution of (14) at $t^*(\varepsilon)$ has an expansion

$$y\big(t^*(\varepsilon)\big) = y_0^* + \varepsilon y_1^* + \varepsilon^2 y_2^* + \mathcal{O}(\varepsilon^3), \tag{17}$$

for which

$$\frac{\beta\big(y\big(t^*(\varepsilon)\big)\big)}{\varepsilon} = v_0^* + \varepsilon v_1^* + \varepsilon^2 v_2^* + \mathcal{O}(\varepsilon^3).$$

These expansions serve as initial value for the leaving solution.

3.1 Asymptotic expansion after leaving the codimension-2 sliding

The assumption $\dot u_0(0) > 0$ and the fact that the vector field (4) is continuous, imply that $u(t) > 1$ (we let $u(t) = \alpha(y(t))/\varepsilon$ also outside the unit square) on an interval $(t^*, t^* + \delta^*]$ with $\delta^* > 0$. By continuity we also have $-1 \le v(t) \le 1$ on this interval. In the region $u \ge 1$ and $v \in [-1, 1]$ the regularized differential equation becomes

$$\dot y = f(y, 1, v), \qquad v = \beta(y)/\varepsilon. \tag{18}$$

Initial values at $t^* = t^*(\varepsilon)$ are given by continuity of the solution. For $t \le t^*$ the solution has an asymptotic expansion (Theorem 1). Do we also have an asymptotic expansion of the solution for $t > t^*$? Assuming its existence, we can find the coefficient functions as in Section 2.2. The functions $y_0(t)$, $v_0(t)$ have to satisfy (for $t > t^*$)

$$\dot y_0 = f(y_0, 1, v_0), \qquad 0 = \beta(y_0). \tag{19}$$

Consistent initial values at $t^*(0) = 0$ are given by y_0^*, v_0^*, because $\beta(y_0^*) = 0$, $\beta'(y_0^*) f(y_0^*, 1, v_0^*) = 0$ (which follows from (11) and from $u_0^* = u_0(0) = 1$). The function $v_0(t)$ can be computed for $t > 0$ from

$$0 = \beta'(y_0) f(y_0, 1, v_0). \tag{20}$$

provided that $\beta'(y_0^*) \partial_v f(y_0^*, 1, v_0^*) \neq 0$ (generic assumption). The functions $y_1(t)$ and $v_1(t)$ will be determined, for $t > 0$, by

$$\dot y_1 = \partial_y f(y_0, 1, v_0) y_1 + \partial_v f(y_0, 1, v_0) v_1, \qquad v_0 = \beta'(y_0) y_1. \tag{21}$$

The algebraic relation is satisfied by y_0^*, v_0^*, y_1^* at $t = 0$. Hence, the left- and right-side limits for $t \to 0$ of the functions $y_0(t)$, $v_0(t)$, $\dot y_0(t)$, and $y_1(t)$ coincide. That this is not the case for $\dot v_0(t)$ follows from differentiation of (11) and (20):

$$\begin{aligned} 0 &= \beta''(y_0)(\dot y_0, \dot y_0) + \beta'(y_0) \big(\partial_y f(y_0, u_0, v_0)\, \dot y_0 + \partial_u f(y_0, u_0, v_0)\, \dot u_0 \\ &\qquad\qquad + \partial_v f(y_0, u_0, v_0)\, \dot v_0 \big) \qquad \text{for} \quad t < 0 \\ 0 &= \beta''(y_0)(\dot y_0, \dot y_0) + \beta'(y_0) \big(\partial_y f(y_0, 1, v_0)\, \dot y_0 + \partial_v f(y_0, 1, v_0)\, \dot v_0 \big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t > 0. \end{aligned}$$

Since $\dot u_0(0) > 0$, there is a jump discontinuity of $\dot v_0(t)$ at $t = 0$. It is given by

$$\beta'(y_0^*) \partial_v f(y_0^*, 1, v_0^*) \left(\dot v_0(0^+) - \dot v_0(0^-) \right) = \beta'(y_0^*) \partial_u f(y_0^*, 1, v_0^*)\, \dot u_0(0).$$

This discontinuity entails jump discontinuities at $t = 0$ in $v_1(t)$, in $y_2(t)$, and in further coefficient functions of an assumed asymptotic expansion.

This motivates to consider the 2-scale ansatz (with $\tau = t/\varepsilon$)

$$\begin{aligned} y\big(t\big) &= y_0(t) + \varepsilon y_1(t) + \varepsilon^2 \big(y_2(t) + \eta_1(\tau) \big) + \mathcal{O}(\varepsilon^3) \\ v\big(t\big) &= v_0(t) + \varepsilon \big(v_1(t) + \nu_1(\tau) \big) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{22}$$

for $t \geq t^*$. The functions $y_0(t)$ and $v_0(t)$ are the solution of the reduced problem (19). Inserting (22) into (18) and comparing the coefficients of ε yields (for conciseness we omit the argument t in $y_0(t), v_0(t), y_1(t), v_1(t)$, but explicitly write the argument τ in $\nu_1(\tau)$)

$$\begin{aligned} \dot y_1 + \eta_1'(\tau) &= \partial_y f(y_0, 1, v_0)\, y_1 + \partial_v f(y_0, 1, v_0) \big(v_1 + \nu_1(\tau) \big) \\ v_1 + \nu_1(\tau) &= \beta'(y_0) \big(y_2 + \eta_1(\tau) \big) + \tfrac{1}{2} \beta''(y_0) \big(y_1, y_1 \big). \end{aligned} \tag{23}$$

We let $y_1(t), v_1(t)$ be the solution of (21), so that the first equation reduces to $\eta_1'(\tau) = \partial_y f\big(y_0(\varepsilon\tau), 1, v_0(\varepsilon\tau) \big)\, \nu_1(\tau)$ which, for $\varepsilon = 0$, gives

$$\eta_1'(\tau) = \partial_v f(y_0^*, 1, v_0^*)\, \nu_1(\tau). \tag{24}$$

The terms with positive powers of ε will be incorporated in higher order transient terms. Since $\nu_1(\tau) = \beta'(y_0^*)\,\eta_1(\tau)$, which follows from (23), a multiplication of (24) with $\beta'(y_0^*)$ yields the scalar equation

$$\nu_1'(\tau) = \gamma\,\nu_1(\tau), \qquad \gamma = \beta'(y_0^*)\,\partial_v f(y_0^*, 1, v_0^*). \tag{25}$$

Differentiating $v_0 = \beta'(y_0)\,y_1$ with respect to t relates v_1 to $\dot v_0$. Together with the above relation between the jump discontinuity of $\dot v_0$ at $t = 0$ and the value $\dot u_0(0)$ this then gives

$$\nu_1(0) = v_1(0^-) - v_1(0^+) = -\frac{\beta'(y_0^*)\partial_u f(y_0^*, 1, v_0^*)}{\big(\beta'(y_0^*)\partial_v f(y_0^*, 1, v_0^*)\big)^2}\,\dot u_0(0).$$

We have to distinguish the cases $\gamma < 0$ and $\gamma > 0$.

*3.2 Exit as codimension-*1 *sliding mode*

If the expression γ of (25) is negative, the transient term $\nu_1(\tau)$, and hence also $\eta_1(\tau)$, converge exponentially fast to zero. Therefore, after a time interval of length $\mathcal{O}(\varepsilon)$ only the smooth part of the expansion (22) is visible.

Theorem 2 *Suppose that the solution $\big(u_0(t), v_0(t)\big)$ of (10) crosses the boundary of the unit square for $t = 0$ at $u_0^* = 1$, $v_0^* \in (-1, 1)$ and satisfies $u_0(0) = 1$, $\dot u_0(0) > 0$.*

If $\beta'(y_0^*)\,\partial_v f(y_0^*, 1, v_0^*) < 0$, then the solution of (18) with initial value (17) approximates for $t \ge t^*$ a codimension-1 sliding mode along Σ_β in the region $\alpha(y) > 0$.

Proof We let $\alpha_0(t) = \alpha\big(y_0(t)\big)$, which vanishes identically for $t \le 0$. Since $\dot y_0(t)$ is continuous at $t = 0$, we have $\alpha(0) = \dot\alpha_0(0) = 0$. Similar to the computations of the beginning of this section, we obtain

$$\begin{aligned}\ddot\alpha_0(0^+) &= \alpha'(y_0^*)\partial_v f(y_0^*, 1, v_0^*)\,\big(\dot v_0(0^+) - \dot v_0(0^-)\big)\\ &\quad - \alpha'(y_0^*)\partial_u f(y_0^*, 1, v_0^*)\,\dot u_0(0).\end{aligned}$$

Expressing the difference $\dot v_0(0^+) - \dot v_0(0^-)$ in terms of $\dot u_0(0)$ yields

$$\ddot\alpha_0(0^+) = -\frac{\det G(y_0^*, 1, v_0^*)}{\gamma}\,\dot u_0(0),$$

which implies that $\ddot\alpha_0(0^+) > 0$. Consequently, $\alpha_0(t)$ leaves the origin like a positive parabola.

For small values of $t \ge t^*(\varepsilon)$, further terms in the asymptotic expansion could either be comparable or dominate $\alpha\big(y_0(t)\big)$, so that the solution of the regularized differential equation re-enters the region $\alpha\big(y(t)\big) < \varepsilon$. Since the asymptotically stable equilibrium of the hidden dynamics is still close to the unit square (but outside of it), the solution will rapidly return to the region $\alpha(y) > \varepsilon$. The statement of the theorem now follows from the expansion (22). The remainder term can be estimated as in the proof of Theorem 1. $\square$

3.3 Exit as classical solution

If the expression γ of (25) is positive, the transient term $\nu_1(\tau)$ in the expansion (22) tends to $\pm\infty$. Consequently, the solution of (18) leaves the stripe $\{y \,|\, |\beta(y)| \le \varepsilon\}$ after a time interval of length $\mathcal{O}(\varepsilon)$, and it turns into a classical solution.

Theorem 3 *Assume that we are in the situation of Theorem 2. If $\beta'(y_0^*)\,\partial_v f(y_0^*, 1, v_0^*) > 0$, then the solution of (18) with initial value (17) approximates, after a transient of time $\mathcal{O}(\varepsilon)$, a classical solution:*

- *if $\beta'(y_0^*)\partial_u f(y_0^*, 1, v_0^*) > 0$ this approximation lies in the region $\alpha(y) > 0$, $\beta(y) < 0$,*
- *if $\beta'(y_0^*)\partial_u f(y_0^*, 1, v_0^*) < 0$ this approximation lies in the region $\alpha(y) > 0$, $\beta(y) > 0$.*

Proof As for the proof of Theorem 2 this follows from the expansion (22) and from an estimation of the remainder. $\square$

3.4 An example with explicit solution

For an illustration of Theorems 2 and 3 we consider the discontinuous differential equation with vector fields

$$f^{-+} = \begin{pmatrix} -2a + b - at \\ -2c + d - ct \end{pmatrix} \qquad f^{++} = \begin{pmatrix} b - at \\ d - ct \end{pmatrix}$$
$$f^{--} = \begin{pmatrix} -2a - b - at \\ -2c - d - ct \end{pmatrix} \qquad f^{+-} = \begin{pmatrix} -b - at \\ -d - ct \end{pmatrix}$$

and discontinuity surfaces $\alpha(y^1, y^2) = y^1$ and $\beta(y^1, y^2) = y^2$. For $-\varepsilon \le y^1, y^2 \le \varepsilon$ the regularized differential equation takes the form

$$\begin{aligned} \varepsilon\,\dot y^1 &= a(y^1 - \varepsilon) + by^2 - a\varepsilon t \\ \varepsilon\,\dot y^2 &= c(y^1 - \varepsilon) + dy^2 - c\varepsilon t. \end{aligned} \tag{26}$$

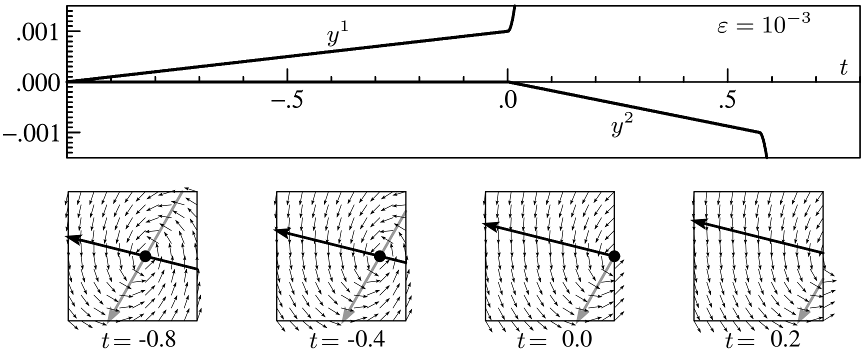

Fig. 1. Upper picture: Solution components y^1, y^2 of the regularized equation (26) with $a = -0.5$, $b = -2$, $c = 3.5$, $d = -2$, and $\varepsilon = 10^{-3}$ as a function of time t. Initial values are $y^1(-1) = y^2(-1) = 0$.
Lower pictures: For some fixed values of t the direction of the vector field of (26) is plotted on a grid in the square $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$ (with y^1 on the horizontal axis and y^2 on the vertical axis). The big black arrow indicates a vanishing first component of the vector field with positive values to the left, the big grey arrow indicates a vanishing second component.

The vector fields are chosen to make disappear the products $y^1 y^2$, so that the regularized differential equation becomes linear which facilitates the computation. A particular solution of (26) is

$$y^1(t) = \varepsilon + \frac{\varepsilon^2 d}{ad - bc} + \varepsilon t, \qquad y^2(t) = -\frac{\varepsilon^2 c}{ad - bc}. \tag{27}$$

We have $u_0(t) = 1 + t$, $v_0(t) = 0$, and the eigenvalues of the matrix $G\big(y_0(t), u_0(t), v_0(t)\big)$ have negative real part provided that $\mathrm{trace}(G) = a + d < 0$ and $\det G = ad - bc > 0$. Under these conditions the matrix is stable for all t (positive and negative). A solution with initial values at $t_0 < 0$ in the square $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$ converges exponentially fast to (27), see Figure 1. It exits the square at $t^* = -\varepsilon d/(ad - bc)$, for which $y^1(t^*) = \varepsilon$. Beyond this point the regularized differential equation becomes

$$\begin{aligned} \varepsilon\, \dot y^1 &= b y^2 - a \varepsilon t \\ \varepsilon\, \dot y^2 &= d y^2 - c \varepsilon t. \end{aligned} \tag{28}$$

With initial values given by (27) at $t = t^*$, its solution is

$$\begin{aligned} y^1(t) &= \varepsilon + \varepsilon(t - t^*) - \frac{ad - bc}{2d}(t - t^*)^2 - \frac{\varepsilon^2 bc}{d^3}\Big(\mathrm{e}^{\tau d} - 1 - \tau d\Big) \\ y^2(t) &= \frac{\varepsilon c}{d} t - \frac{\varepsilon^2 c}{d^2}\Big(\mathrm{e}^{\tau d} - 1\Big), \qquad \tau = (t - t^*)/\varepsilon. \end{aligned}$$

The statements of the previous theorems are confirmed. For $d < 0$ (Theorem 2) the exponential terms are rapidly damped out and the solution is seen to satisfy $-\varepsilon \le y^2(t) \le \varepsilon$ and $y^1(t) \ge \varepsilon$ on an interval (t_0^*, δ) with $\delta \approx |d/c|$. For $d > 0$ (Theorem 3) the exponential terms are soon dominant. By the stability of the matrix G, $d > 0$ implies $a < 0$ and $bc < 0$. Consequently, the sign of c determines whether the classical solution continues in $\beta(y) > 0$ or in $\beta(y) < 0$.

4 Leaving codimension-2 sliding mode - case B

In this section we consider the situation where the stationary point $\big(u_0(t), v_0(t)\big)$ of the hidden dynamics (with t considered as a fixed parameter) either changes from asymptotically stable to unstable or where it disappears. In both situations Theorem 1 can no longer be applied. The study of the case B is more difficult and leads to unexpected results. We present two typical examples which give much insight into the behaviour of the regularized differential equation. We expect the same behaviours for the general situation.

4.1 Equilibrium becomes unstable

We consider the discontinuous differential equation with vector fields

$$f^{++} = \begin{pmatrix} 1 \\ 2t-1 \end{pmatrix} \quad f^{+-} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad f^{-+} = \begin{pmatrix} 1 \\ 2t+1 \end{pmatrix} \quad f^{--} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and discontinuity surfaces $\alpha(y^1, y^2) = y^1$ and $\beta(y^1, y^2) = y^2$. For $-1/2 < t < 1/2$ the solution spirals around the origin. In the square $-\varepsilon \le y^1, y^2 \le \varepsilon$ the regularized differential equation is

$$\begin{aligned} \varepsilon\, \dot y^1 &= y^2 \\ \varepsilon\, \dot y^2 &= -y^1 + t\, y^2 + \varepsilon t. \end{aligned} \tag{29}$$

Again, the vector fields are chosen to obtain a linear differential equation. A particular solution of (29) is given by

$$y^1(t) = a(\varepsilon) t, \qquad y^2(t) = \varepsilon a(\varepsilon), \qquad a(\varepsilon) = \frac{\varepsilon}{1-\varepsilon}. \tag{30}$$

Expanding $(1-\varepsilon)^{-1}$ into a series, we obtain the asymptotic expansion of Theorem 1 in explicit form. In this particular case the series expansion is convergent for $|\varepsilon| < 1$. With $u_0(t) = t$, $v_0(t) = 0$ the eigenvalues of the matrix $G\big(y_0(t), u_0(t), v_0(t)\big)$ satisfy $\lambda_{1,2} = \pm \mathrm{i} + t/2 + \mathcal{O}(t^2)$, which shows that the matrix is stable for $t < 0$ and unstable for $t > 0$.

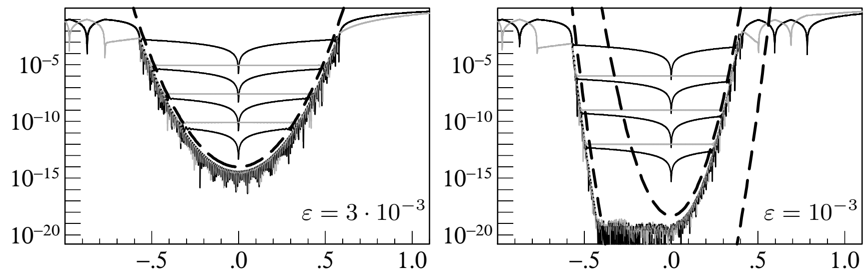

Fig. 2. For two choices of ε, the top two curves are the absolute value of the solution components of the regularized equation of Section 4.1 as a function of time t: $|y^1|$ is in black and $|y^2|$ in grey. The other curves (in descending order) are the absolute value of the remainder $y^j(t) - \sum_{k=1}^N \varepsilon^k y_k^j(t)$ ($j = 1$ in black and $j = 2$ in grey) of the truncated asymptotic expansion for $N = 1, 2, 3, \infty$. The dashed curves represent the theoretical bounds on the remainder for $N = \infty$.

The general solution of (29) is a superposition of the particular solution (30) and the general solution of the homogeneous equation

$$\begin{aligned} \varepsilon\, \dot y^1 &= y^2 \\ \varepsilon\, \dot y^2 &= -y^1 + t\, y^2. \end{aligned} \tag{31}$$

The norm $r(t) = \|y(t)\| = \sqrt{y^1(t)^2 + y^2(t)^2}$ of the solution of (31) satisfies

$$\varepsilon\, r(t)\, \dot r(t) = t\, y^2(t)^2 \le t\, r(t)^2 \qquad \text{for} \qquad t \ge 0.$$

Solving this differential inequality for $r(t)$ yields

$$r(t) \le r(0) \exp\Big(\frac{t^2}{2\varepsilon}\Big). \tag{32}$$

We note that with $\big(y^1(t), y^2(t)\big)$ also $\big(-y^1(-t), y^2(-t)\big)$ is a solution of (31). This implies that the estimate (32) is not only valid for $t \ge 0$, but also for $t \le 0$.

Figure 2 shows both components of the solution of the regularized differential equation corresponding to initial values $y^1(-1) = y^2(-1) = 0.075$. During the codimension-2 sliding the first component is of size $\mathcal{O}(\varepsilon)$ and changes sign at $t = 0$, whereas the second component is of size $\mathcal{O}(\varepsilon^2)$. The pictures also show the remainder of the truncated asymptotic expansion: $y^j(t) - \sum_{k=1}^N \varepsilon^k y_k^j(t)$ for $N = 1, 2, 3$ and for $N = \infty$.

For relatively large values of ε (left picture of Figure 2) the difference between the solution of the regularized differential equation and the (convergent) asymptotic expansion of the smooth solution looks like a parabola in the picture. This corresponds to a behaviour of the type $C\exp\big(t^2/(c\varepsilon)\big)$, and matches very well with estimate (32). The dashed curve in the figure shows this bound for $c = 3.8$ (this constant is fitted to the numerical experiment; it does not rely on theoretical estimates). We conclude that the entry and exit points of the codimension-2 sliding are symmetric with respect to the time instant $t = 0$, where the hidden dynamics changes stability.

For small values of ε (right picture of Figure 2) round-off errors perturb the picture, when the remainder term (for $N = \infty$) is below the accuracy of the machine, which in double precision is about $\|y(t)\| \cdot 10^{-16}$. We have drawn two parabolas. The outer parabola corresponds to the remainder during the phase when the solution enters the codimension-2 sliding mode. In the absence of round-off errors (or in high precision arithmetics) the solution would exit at the time instant symmetric to $t = 0$. However, due to round-off errors, the solution exits earlier and follows closely the parabola that matches the value $\|y(t)\| \cdot 10^{-16}$ at $t = 0$. This explains why the numerically obtained solution of the regularized differential equation exits the codimension-2 sliding at a distance of size $\mathcal{O}(\sqrt{\varepsilon})$ from $t = 0$, independent of the entry point into the codimension-2 sliding.

Remark 3 (Numerical treatment) We note that stiff integrators have to be applied carefully to the regularized differential equation. The eigenvalues of the linearized problem cross the imaginary axis away from the origin. Therefore, one has to be cautious that unstable modes are not damped by the numerical integrator. This “dangerous property” of stiff integrators has first been observed in [21]. As a remedy, the step size should be decreased, so that its product with the unstable eigenvalues lies outside the stability region.

A standard application of stiff integrators (e.g., Radau5 of [16]) would miss the correct exit point and leave the codimension-2 sliding mode only at $t \approx 1$, where the first component of the smooth solution (30) becomes larger than ε.

Numerical experiments with examples that are qualitatively identical to the problem of this section, have recently been carried out in [9] and in [20]. In [9, Example 4.3] it is observed that “stiff integrators behave poorly”, because they miss the correct exit point. Difficulties with numerical computations are reported in [20, Section IV.C], because the “exit point along the intersection is very sensitive to numerical imprecision”. The misbehaviours illustrated in these papers do not appear to be due to an unpredictable behaviour of the regularization, but to a wrong behaviour of the numerical integrators, which produce numerical artefacts.

Insight 1 *Suppose that the solution $\big(u_0(t), v_0(t)\big)$ of (10) exists and stays in the interior of the unit square for t in an ε-independent neighbourhood of* 0. *Furthermore, assume that both eigenvalues of the matrix $G\big(y_0(t), u_0(t), v_0(t)\big)$ have negative real part for $t < 0$, and at least one of the eigenvalues has positive real part for $t > 0$.*

Then, the solution of the regularized equation (4) continues to approach a codimension-2 *sliding beyond $t = 0$. The exit point from the codimension-*2 *sliding is independent of ε, but depends on the time instant when the solution enters the sliding. This behaviour is expected to be true in general and not only in the preceding example.*

4.2 Equilibrium disappears

As a final example we consider the discontinuous vector field

$$f^{-+} = \begin{pmatrix} b-1+2a \\ -b+c-t \end{pmatrix} \qquad f^{++} = \begin{pmatrix} -b-1+2a \\ 2+b+c-t \end{pmatrix}$$
$$f^{--} = \begin{pmatrix} b+1+2a \\ -b+c-t \end{pmatrix} \qquad f^{+-} = \begin{pmatrix} -b+1+2a \\ -2+b+c-t \end{pmatrix}$$

and discontinuity surfaces $\alpha(y^1, y^2) = y^1$ and $\beta(y^1, y^2) = y^2$. The parameters a and b are free for the moment and $c = -2a - a^2/b$. The corresponding regularized differential equation is

$$\begin{aligned} \dot y^1 &= -bu - v + 2a \\ \dot y^2 &= u\,v + bu + v + c - t, \end{aligned} \tag{33}$$

where $u = \pi(y^1/\varepsilon)$ and $v = \pi(y^2/\varepsilon)$. We denote the right-hand sides of this system by $g_\alpha(t, u, v)$ and $g_\beta(t, u, v)$, respectively. For $bt < 0$, the equilibrium $\big(u_0(t), v_0(t)\big) = \big((a - \sqrt{-bt})/b, a + \sqrt{-bt}\,\big)$ is the solution of the reduced problem (10). There is no equilibrium solution for $bt > 0$. In view of an application of Theorem 1 we consider the matrix $G(t) = G\big(y_0(t), u_0(t), v_0(t)\big)$. Its determinant is given by $\det G(t) = 2\sqrt{-bt}$ and its trace is $\operatorname{trace} G(t) = -b + 1 + a/b - \sqrt{-bt}$. To have eigenvalues with negative real part for all $t < 0$ we assume

$$b > 0, \qquad a \le b(b-1).$$

By Theorem 1 the solution of the regularized problem (33) stays close to $\big(u_0(t), v_0(t)\big)$ for negative t until very close to $t = 0$. The upper picture of Figure 3 shows both solution components for the parameters $b = 1.2$ and $a = -0.1$. There is indeed a codimension-2 sliding until $t \approx 0$, and after a short transient the solution continues

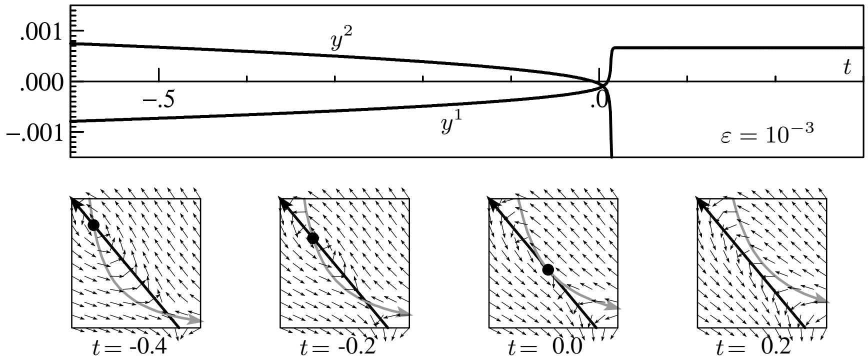

Fig. 3. Upper picture: Solution components y^1, y^2 of the regularized equation (33) with parameters $a = -0.1$, $b = 1.2$, $c = -2a - a^2/b$, and $\varepsilon = 10^{-3}$ as a function of time t. Initial values at $t = -0.6$ are on the equilibrium $\big(u_0(-0.6), v_0(-0.6)\big)$.
Lower pictures: For some fixed values of t the direction of the vector field of (33) is plotted on a grid in the square $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)$ (with y^1 on the horizontal axis and y^2 on the vertical axis). The big black arrow indicates a vanishing first component of the vector field with positive values to the left, the big grey arrow (curved) indicates a vanishing second component.

as codimension-1 sliding mode along Σ_α into the region $\beta(y) < 0$. The lower pictures of Figure 3 show the vector field of (33) for some values of t. The critical point, where the equilibrium disappears, is in the unit square provided that $|a| < \min(1, b)$. The line $g_\alpha(t, u, v) = 0$ crosses the bottom line of the unit square at $(u, v) = \big((1 + 2a)/b, -1\big)$. After $t \approx 0$ the solution continues as codimension-1 sliding if $1 + 2a \le b$, and as classical solution in $\mathcal{R}^{+-}$ if $1 + 2a > b$.

We are interested in the time interval that is needed to change from the codimension-2 sliding to either a codimension-1 sliding or to a classical solution. We denote by $t^*(\varepsilon)$ the time instant, when the solution of (33) leaves the square $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$. To study this value we take an initial value close to $\big(u_0(t), v_0(t)\big)$ for some $t_0 < 0$, and we compute $t^*(\varepsilon)$ for many choices of ε. We expect this exit time to behave like $t^*(\varepsilon) \approx C\varepsilon^\kappa$. Assuming that C changes only slowly with ε, we estimate the exponent $\kappa = \kappa(\varepsilon)$ by

$$\kappa(\varepsilon) = \log\Big(\frac{t^*(\varepsilon + \Delta\varepsilon)}{t^*(\varepsilon)}\Big) \Big/ \log\Big(\frac{\varepsilon + \Delta\varepsilon}{\varepsilon}\Big),$$

using two consecutive approximations for $t^*(\varepsilon)$. The value of κ as a function of ε is plotted in Figure 4 for the parameters $b = 1.2$ and $a = b(b-1) - 0.05 \cdot k$ $(k = 0, 1, \ldots, 8)$. Generically, the function $\kappa(\varepsilon)$ converges to $\kappa \approx 0.67$ for $\varepsilon \to 0$. For the non-generic case $a = b(b-1)$,

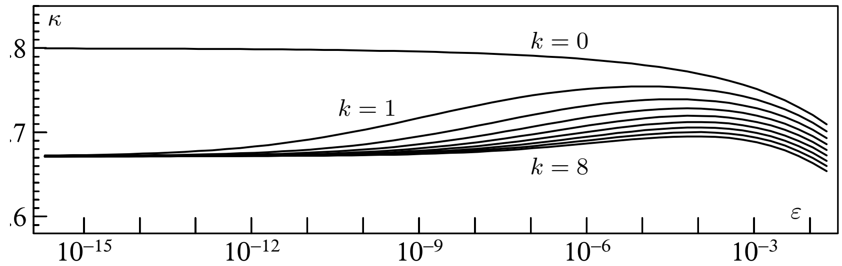

Fig. 4. Exponent κ of the exit time $t^*(\varepsilon) \approx C\varepsilon^\alpha$ as a function ε. The parameters are $b = 1.2$ and $a = b(b-1) - 0.05 \cdot k$, $k = 0, 1, \ldots, 8$.

for which $\det G(t)$ and $\operatorname{trace} G(t)$ converge both to 0 for $t \to 0$, it converges to a value $\kappa \approx 0.80$. For the moment, we are not able to explain this behaviour of $\kappa(\varepsilon)$.

Insight 2 *Suppose that a real solution $\big(u_0(t), v_0(t)\big)$ of (10) exists for $t < 0$ (in the interior of the unit square) and that no real solution exists for $t > 0$. Furthermore, assume that both eigenvalues of the matrix $G\big(y_0(t), u_0(t), v_0(t)\big)$ have negative real part for $t < 0$.*

Then, the solution of the regularized equation (4) leaves the codimension-2 sliding at an exit point that is $\mathcal{O}(\varepsilon^\alpha)$ close to 0 (with some $\alpha < 1$). This behaviour is expected to be true in general.

5 Discussions

The aim of the present work is to study, when and how the solution of the regularized differential equation moves away from a codimension-2 sliding of the corresponding discontinuous system. The essential tool is the solution $\big(u_0(t), v_0(t)\big)$ of the 2-dimensional system (11) and the stability of the matrix $G\big(y_0(t), u_0(t), v_0(t)\big)$, defined in (15). The main results are the following:

– As long as a solution $\big(u_0(t), v_0(t)\big)$ of (11) exists in the unit square $(-1, 1) \times (-1, 1)$ and the matrix $G\big(y_0(t), u_0(t), v_0(t)\big)$ is asymptotically stable (i.e., both eigenvalues are in the negative half-plane), the solution of the regularized problem stays close to a codimension-2 sliding of the discontinuous system (Theorem 1).
– If $\big(u_0(t), v_0(t)\big)$ of (11) leaves the unit square at $t = 0$, then the solution of the regularized problem no longer approaches a codimension-2 sliding beyond $t = 0$, but it continues to approach either a classical solution (Theorem 3) or a codimension-1 sliding mode (Theorem 2). The exit time (i.e., the time that is needed

for the solution to have a distance from the intersection of the discontinuity surfaces that is larger than ε) is of size $\mathcal{O}(\varepsilon)$.

- If $\big(u_0(t), v_0(t)\big)$ of (11) stays in the unit square, but the matrix $G\big(y_0(t), u_0(t), v_0(t)\big)$ becomes unstable at $t = 0$, the solution leaves the codimension-2 sliding after a time that is $\mathcal{O}(1)$ and depends on the entry-time into the codimension-2 sliding (Section 4.1).
- If $\big(u_0(t), v_0(t)\big)$ of (11) stays inside the unit square for $t \leq 0$, but no real solution of (11) exists for $t > 0$, the solution leaves the codimension-2 sliding after a time that is $\mathcal{O}(\varepsilon^\alpha)$ with some $\alpha < 1$ (Section 4.2).

The first two statements are proved rigorously using asymptotic expansions in powers of ε. The last two statements are confirmed by examples and numerical experiments.

6 Conclusion

Differential equations with piece-wise smooth vector fields arise in many important applications. Close to a single discontinuity hypersurface the local dynamics (crossing or sliding) is well understood, analytically as well as numerically. Much more challenging is the study of solutions close to the intersection of two or more discontinuity hypersurfaces. In the situation of non-uniqueness it is commonly adopted to select the solution that can be interpreted as the limit of a suitable regularization. During the last few years much progress has been achieved in the understanding of solutions entering a codimension-2 discontinuity manifold.

The present work yields new insight into the way how solutions can exit a codimension-2 sliding. The effect of a standard regularization of the vector field to the solution is investigated. In this way, not only results for the regularized differential equation are obtained, but it gives also a hint for a numerical treatment of the original discontinuous problem. The study of the behaviour of solutions close to the intersection of more than two discontinuity hypersurfaces is the subject of further research.

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