ON THE LIMIT SOLUTION OF REGULARIZED PIECEWISE-SMOOTH DYNAMICAL SYSTEMS*

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Abstract. This work deals with piecewise-smooth dynamical systems and with regularizations, where the jump discontinuities in the vector field are smoothed out in an ε-neighbourhood by using a continuous transition function. It addresses the questions whether the solution of the regularization converges to a Filippov solution of the discontinuous problem, and under which condition the limit for $\varepsilon \to 0$ of the regularized solution is independent of the transition function. The results are complemented by numerical simulations.

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1. Introduction. Piecewise smooth dynamical systems arise in many applications and they are an active field of recent research. Historically, one of the first examples is Coulomb friction in mechanical systems, where the force of friction is proportional to the sign of velocity (see [2]). Many interesting applications can be found in the monograph [3]: relay control systems, where the control variable admits jump discontinuities; converter circuits, where switching devices lead to a non-smooth dynamics; models in the social and financial sciences, where continuous change can trigger discrete actions. Discontinuity points are also created by the activation/deactivation of inequality constraints in mixed constrained optimization problems. See [17] for a particular application arising in the modelling of atmospheric particles.

For a mathematical formulation of the problem we consider discontinuity hyper-surfaces

$$\Sigma_j = \{ y \in \mathbb{R}^n \mid \alpha_j(y) = 0 \}, \qquad j = 1, \ldots, d, \tag{1.1}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}^d$ (with $d < n$) is assumed to be sufficiently differentiable and such that these hyper-surfaces intersect transversally. We denote the discontinuity set by $\Sigma = \bigcup_{j=1}^d \Sigma_j$. The hyper-surfaces Σ_j divide the phase space $\mathbb{R}^d \setminus \Sigma$ into 2^d open regions

$$\mathcal{R}^{\mathbf{k}} = \big\{ y \in \mathbb{R}^n \,\big|\, k_j \alpha_j(y) > 0 \text{ for } j = 1, \ldots, d \big\}, \tag{1.2}$$

where $\mathbf{k} = (k_1, \ldots, k_d)$ is a multi-index with $k_j \in \{-1, 1\}$. The discontinuous dynamical system is then given by

$$\dot y = f^{\mathbf{k}}(y) \qquad \text{for} \qquad y \in \mathcal{R}^{\mathbf{k}}. \tag{1.3}$$

We assume that the functions $f^{\mathbf{k}}(y)$ are defined in a neighbourhood of the closure of $\mathcal{R}^{\mathbf{k}}$ and that they are sufficiently differentiable. In the discontinuity set Σ the right-hand side of (1.3) is considered to be multi-valued with values from the neighbouring domains. We are thus concerned with a differential inclusion and we adopt the approach of Filippov [9, 10] for the concept of solutions. Besides classical solutions, which cross the discontinuity surfaces, there are also sliding modes evolving in the discontinuity set Σ.

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For a co-dimension d surface $\bigcap_{j=1}^{d}\Sigma_j$, Filippov's proposal consists in considering the differential inclusion

$$\dot{y} \in F(y)=\sum_{\mathbf{k}}\mu_{\mathbf{k}}(y)f^{\mathbf{k}}(y), \qquad \text{where} \quad \mu_{\mathbf{k}}(y)\geq 0, \text{ and } \quad \sum_{\mathbf{k}}\mu_{\mathbf{k}}(y)=1, \tag{1.4}$$

subject to the constraint that $F(y)$ lies in the tangent space $T_y\big(\bigcap_{j=1}^{d}\Sigma_j\big)$ at y:

$$\alpha_j'(y)F(y)=0 \quad \text{for all} \quad j=1,\dots,d. \tag{1.5}$$

The case of co-dimension $d=1$ is well understood. There are two regions, $\mathcal{R}^1,\mathcal{R}^{-1}$, with vector fields f^1 and f^{-1}, and $\Sigma=\{y\in\mathbb{R}^n : \alpha_1(y)=0\}$ with $\alpha_1:\mathbb{R}^n\to\mathbb{R}$. We assume that in $\mathcal{R}^{-1}$ we have $\alpha_1(y)<0$, and in $\mathcal{R}^1$ we have $\alpha_1(y)>0$. If $\alpha_1'(y)f^{-1}(y)>0$ and $\alpha_1'(y)f^1(y)<0$ at some point $y\in\Sigma$ (a condition called nodal attractivity), (1.4)-(1.5) provide a unique sliding vector on Σ. Indeed, we can write $\mu^1=\gamma$, $\mu^{-1}=1-\gamma$, and imposing (1.5) we immediately get the differential system on Σ:

$$\dot{y}=(1-\gamma)f^{-1}(y)+\gamma f^1(y), \qquad \gamma=\frac{\alpha_1'(y)f^{-1}(y)}{\alpha_1'(y)f^{-1}(y)-\alpha_1'(y)f^1(y)}\,. \tag{1.6}$$

However, already for co-dimension $d=2$, the construction based on (1.4)-(1.5) in general does not select a unique sliding vector field on $\bigcap_{j=1}^{d}\Sigma_j$, since it gives a system of 3 equations in 4 unknowns. This ambiguity (which naturally appears in higher codimension) prevents from having a well defined differential equation (of Filippov type) governing the evolution of the system (we refer to [15] for a more extended discussion).

Closely connected to a discontinuous dynamical system is a regularization, where the jump discontinuities are replaced in an ε-neighbourhood by a continuous transition. In this way the differential inclusion is transferred to an ordinary differential equation. It is natural to consider regularizations because, as mentioned in [3, page 1], "... there is strictly speaking no such thing as a piecewise-smooth dynamical system and that in reality all physical systems are smooth". This is precisely what happens in the analysis of gene regulatory networks [8, 19], where steep sigmoid-type nonlinearities are approximated by step functions.

But this is not the only reason for considering regularizations. It is known that the discontinuous problem (1.3) (with given initial value) can have more than one solution, whereas the solution of the regularized ordinary differential equation is unique. From a theoretical viewpoint, the limit (for $\varepsilon\to 0$) of the solution of the regularized differential equation (if it exists) gives us an interesting possibility for selecting a physically meaningful solution. From a practical viewpoint, the numerical solution of the regularized problem permits us to apply standard software for differential equations, which does not require switching decisions.

The present article is mainly concerned with regularizations of the discontinuous problem (1.3). It addresses the following questions:

- For a fixed transition function, does the solution of the regularized differential equation converge, for $\varepsilon\to 0$, to a Filippov solution of the discontinuous problem (also in the case of non-uniqueness)?
- Is the limit solution, if it exists, independent of the transition function of the regularization?

Since the definition of Filippov solutions is ambiguous in the intersection of discontinuity surfaces, we specify in Section 2 the meaning of solutions for (1.3) and we present the class of considered regularizations. The main results are given in Section 3. For many important situations it is shown that the solution of the regularized differential equation converges, for $\varepsilon \to 0$, to a Filippov solution of the discontinuous problem. Sufficient conditions on the vector fields are presented (including cases with multiple Filippov solutions) for which the limit solution is independent of the transition function. We present counter-examples, one with a solution entering the intersection of two discontinuity surfaces and one exiting a codimension-2 sliding mode, for which the limit solution can depend on the transition function. The proofs of the main results are given in Sections 4 and 5. More insight into the second counter-example is given in Section 6. A conclusion terminates the present work.

2. Solution concept and regularization.

The definition of Filippov solutions for a discontinuous dynamical system (1.3) is ambiguous, because in the intersection of discontinuity hyper-surfaces a convex combination of the adjacent vector fields has too many degrees of freedom. We restrict our study to special convex combinations having m parameters in the intersection of m hyper-surfaces Σ_j. Such convex combinations (for $m = 2$) are called "blending" in [1] and "bilinear interpolation" in [5, 4], see also [7, 18]. For arbitrary m they are called "convex canopy" in [15]. We consider regularizations that are closely connected to such convex combinations, and we call them "multi-linear interpolation".

2.1. Solution concept – classical solutions and sliding modes.

For a fixed multi-index $\mathbf{k} = (k_1, \ldots, k_d)$ with $k_j \in \{-1, 1\}$ the equation (1.3) is a regular ordinary differential equations on the open domain $\mathcal{R}^{\mathbf{k}}$, and the standard theory on existence, uniqueness, and continuous dependence on parameters and initial values applies. In this case the solution of (1.3) is called *classical*.

We next extend the concept of solution to the discontinuity set Σ. For an index vector $\mathbf{k} = (k_1, \ldots, k_d)$ with $k_j \in \{-1, 0, 1\}$ (note that now k_j can also be zero) we consider the set

$$\mathcal{R}^{\mathbf{k}} = \Big\{ y \in \mathbb{R}^n \,\Big|\, \alpha_j(y) = 0 \text{ if } k_j = 0,\ k_j\alpha_j(y) > 0 \text{ if } k_j \neq 0 \Big\}, \tag{2.1}$$

and if at least one component $k_j = 0$, then $\mathcal{R}^{\mathbf{k}} \subset \bigcap_{\{j \mid k_j = 0\}} \Sigma_j \subset \Sigma$. We assume that $\alpha(y)$ is such that $\mathcal{R}^{\mathbf{k}}$ is a submanifold of $\mathbb{R}^d$ of codimension m, where m counts the number of elements k_j being equal to zero. For $\mathbf{k} = (k_1, \ldots, k_d)$ we define $\mathcal{I}^{\mathbf{k}} = \{j \mid k_j = 0\}$, and we let

$$\mathcal{N}^{\mathbf{k}} = \Big\{ \boldsymbol{\ell} \in \{-1, 1\}^d \,\Big|\, \ell_j \in \{-1, 1\} \text{ if } k_j = 0,\ \ell_j = k_j \text{ if } k_j \neq 0 \Big\}$$

which collects the index vectors $\boldsymbol{\ell}$ such that $\mathcal{R}^{\boldsymbol{\ell}}$ touches $\mathcal{R}^{\mathbf{k}}$. With this notation we consider the differential-algebraic equation (DAE)

$$\begin{aligned} \dot{y} &= \sum_{\boldsymbol{\ell} \in \mathcal{N}^{\mathbf{k}}} \Big(\prod_{j \in \mathcal{I}^{\mathbf{k}}} \frac{(1 + \ell_j \lambda_j)}{2} \Big) f^{\boldsymbol{\ell}}(y) \\ 0 &= \alpha_j(y), \qquad j \in \mathcal{I}^{\mathbf{k}} \end{aligned} \tag{2.2}$$

with algebraic variables $\lambda_j, j \in \mathcal{I}^{\mathbf{k}}$. In the following we denote the right-hand side of the differential equation in (2.2) by $f^{\mathbf{k}}(y, \lambda^{\mathbf{k}})$, where $\lambda^{\mathbf{k}}$ is the vector that collects

$\lambda_j, j \in \mathcal{I}^{\mathbf{k}}$. Differentiating the algebraic constraint of (2.2) with respect to time yields

$$0 = \alpha_j'(y) f^{\mathbf{k}}(y, \lambda^{\mathbf{k}}), \qquad j \in \mathcal{I}^{\mathbf{k}}. \tag{2.3}$$

We assume that the Implicit Function Theorem can be applied to guarantee that locally $\lambda^{\mathbf{k}}$ can be expressed as function of y. This implies that the DAE has index 2. The special case $\mathcal{I}^{\mathbf{k}} = \emptyset$ includes classical solutions of (1.3), because in this case $\mathcal{N}^{\mathbf{k}} = \{\mathbf{k}\}$ consists of only one element and the empty product in (2.2) is interpreted as 1.

For $\lambda_j \in [-1, 1]$ the vector field in (2.2) is a convex combination of the vector fields $f^{\boldsymbol{\ell}}(y)$ (with $\boldsymbol{\ell} \in \mathcal{N}^k$) which are defined on the open domains touching $\mathcal{R}^{\mathbf{k}}$. The solution of (2.2) is therefore a Filippov solution.

DEFINITION 2.1. *Consider an index vector* $\mathbf{k}$ *with* $\mathcal{I}^{\mathbf{k}} \neq \emptyset$ *and let* $m = |\mathcal{I}^{\mathbf{k}}|$ *be the cardinality of* $\mathcal{I}^{\mathbf{k}}$. *Then, a solution* $(y, \lambda^{\mathbf{k}})$ *of the differential-algebraic equation (2.2) is called a codimension-m sliding mode in the set* $\mathcal{R}^{\mathbf{k}}$ *as long as* $\lambda_j \in [-1, 1]$ *for* $j \in \mathcal{I}^{\mathbf{k}}$.

The emphasis of the present work is the study of the limit for $\varepsilon \to 0$ of a solution of the regularized differential equation (2.4). Our assumptions will be such that this limit, if it exists, is a solution in the sense of Definition 2.1 including classical solutions.

DEFINITION 2.2. *A piecewise-smooth, continuous function* $y : [0, T] \to \mathbb{R}^n$ *is called a solution of the discontinuous dynamical system (1.3), if there exists a finite partition* $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$, *such that the following is true: for every subinterval* $[t_i, t_{i+1}]$ *there exists* $\mathbf{k}_i \in \{-1, 0, 1\}^d$ *with* $m_i = |\mathcal{I}^{\mathbf{k}_i}|$ *such that the restriction of* $y(t)$ *to this interval is a codimension-m_i sliding mode in the set* $\mathcal{R}^{\mathbf{k}_i}$ *(a classical solution if* $\mathcal{I}^{\mathbf{k}_i} = \emptyset$*).*

Although the situation, where time instants t_i with a change in the type of solution have a finite accumulation point, is an interesting and important phenomenon (chattering), we do not consider it in the present work.

2.2. Regularization. The main topic of this work is the study of regularizations, where jump discontinuities in the vector field of (1.3) are smoothed out. For this we consider a transition function $\pi(u)$, which is assumed to be continuous, piecewise-smooth, and satisfies $\pi(u) = -1$ for $u \leq 1$ and $\pi(u) = 1$ for $u \geq 1$. We also assume that $\pi'(u) > 0$ for $u \in (-1, 1)$, and that $\pi(u)$ is centrally symmetric (see Figure 1).

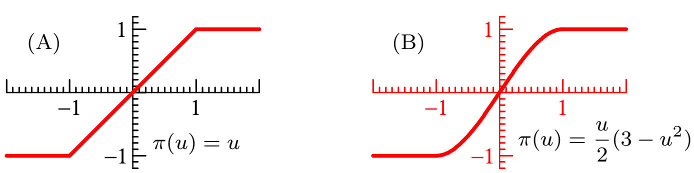

FIG. 1. *Two examples for transition functions.*

For a discontinuous dynamical system (1.3) we consider the regularization

$$\dot{y} = \sum_{\boldsymbol{\ell} \in \{-1,1\}^d} \left(\prod_{j=1}^{d} \frac{(1 + \ell_j \pi(u_j))}{2} \right) f^{\boldsymbol{\ell}}(y) \tag{2.4}$$

where $u_j = \alpha_j(y)/\varepsilon$. We denote the right-hand side of this regularized differential equation by $f\big(y, \pi(u_1), \ldots, \pi(u_d)\big)$. The complete phase space (including the discontinuity set Σ) is the union of 3^d sets

$$\mathcal{R}_\varepsilon^{\mathbf{k}} = \Big\{ y \in \mathbb{R}^n \,\Big|\, |\alpha_j(y)| \le \varepsilon \text{ if } k_j = 0,\; k_j \alpha_j(y) > \varepsilon \text{ if } k_j \ne 0 \Big\}, \tag{2.5}$$

where $\mathbf{k} = (k_1, \ldots, k_d)$ with $k_j \in \{-1, 0, 1\}$. For the case that all $k_j \ne 0$, we have that $\mathcal{R}_\varepsilon^{\mathbf{k}} \subset \mathcal{R}^{\mathbf{k}}$, and $\boldsymbol{\ell} = \mathbf{k}$ is the only vector for which the product in (2.4) is non-zero. Therefore, on the set $\mathcal{R}_\varepsilon^{\mathbf{k}}$ the regularization coincides with the differential equation $\dot y = f^{\mathbf{k}}(y)$ of the un-regularized problem.

For $\mathbf{k}$ with $\mathcal{I}^{\mathbf{k}} \ne \emptyset$ the set $\mathcal{R}_\varepsilon^{\mathbf{k}}$ approximates $\mathcal{R}^{\mathbf{k}}$. On the set $\mathcal{R}_\varepsilon^{\mathbf{k}}$ only the vectors $\boldsymbol{\ell} \in \mathcal{N}^{\mathbf{k}}$ give rise to a non-vanishing product in (2.4). Since $\ell_j \pi(u_j) = k_j \pi(u_j) = 1$ for $\ell \in \mathcal{N}^{\mathbf{k}}$ and $j \notin \mathcal{I}^{\mathbf{k}}$, the regularized differential equation (2.4) becomes

$$\dot y = \sum_{\boldsymbol{\ell} \in \mathcal{N}^{\mathbf{k}}} \bigg(\prod_{j \in \mathcal{I}^{\mathbf{k}}} \frac{(1 + \ell_j \pi(u_j))}{2} \bigg) f^{\boldsymbol{\ell}}(y) \qquad \text{for} \quad y \in \mathcal{R}_\varepsilon^{\mathbf{k}}, \tag{2.6}$$

which is in complete analogy to (2.2). If m denotes the cardinality of $\mathcal{I}^{\mathbf{k}}$, then for $m = 1$ the sum in (2.6) consists of two terms (linear interpolation), for $m = 2$ it consists of four terms (bilinear interpolation), and in general it consists of 2^m terms.

3. Main results. An essential requirement for a meaningful regularization is that its solution converges, for $\varepsilon \to 0$, to a Filippov solution of the discontinuous problem. With this term we mean that the unique solution y_ε of (2.6) approaches a Filippov solution y of (1.3) as $\varepsilon \to 0$, that is $\lim_{\varepsilon \to 0} \|y_\varepsilon - y\| = 0$ for some norm $\|\cdot\|$.

This question is already addressed in § 8 of the classical monograph [10, Theorem 2 on page 90] which is on the "Dependence of solution on initial data and on the right-hand side of the equation", and in [20]. It is shown (using the Arzelà–Ascoli theorem) that for every $\varepsilon > 0$ the solution of the regularized differential equation is close to *some* solution of the discontinuous problem, and convergence is established if the Filippov solution of the discontinuous problem is unique. These results are obtained in the setting of differential inclusions, and nothing is stated, whether the Filippov solution is of the form considered in Definition 2.1.

In the present work we use the approach of [11] and [12] which is based on asymptotic expansions in powers of ε. For regularizations of the form (2.4) this approach shows that they appoximate a Filippov solution of the form (2.2). This is not surprising due to the similarity of the formulas. The study of asymptotic expansions is independent of whether there is a unique Filippov solution or not. It typically requires certain expressions to be non-zero, which is generically satisfied.

3.1. Desired results. Before we present rigorous statements we formulate the kind of theorem that we would like to prove.

Desired Theorem 3.1. *Consider a discontinuous system (1.3) with initial value $y_0 \in \mathbb{R}^n$, and assume that for every solution (in the sense of Definition 2.2) the matrix*

$$\Big(\alpha_j'(y) \frac{\partial f^{\mathbf{k}_i}}{\partial \lambda_p} \big(y, \lambda^{\mathbf{k}_i}\big) \Big)_{j,p \in \mathcal{I}^{\mathbf{k}_i}} \tag{3.1}$$

is invertible for all $\big(y(t), \lambda^{\mathbf{k}_i}(t)\big)$, $t \in [t_i, t_{i+1}]$.

Then, the solution of the regularized differential equation (with fixed transition function $\pi(u)$) converges for $\varepsilon \to 0$ uniformly to some solution of the discontinuous initial value problem.

The invertibility of the matrix (3.1) is a natural, generic assumption, which permits us to apply the Implicit Function Theorem to the system (2.3), so that the solutions in the sense of Definition 2.2 are well-defined.

Desired Theorem 3.2. *In the situation of Desired Theorem 3.1 the limit solution, for $\varepsilon \to 0$, is independent of the transition function $\pi(u)$.*

Such a result is important for answering the question, which solution of the discontinuous dynamical system (in the case of non-uniqueness of solutions) is the most meaningful. It will turn out that the Desired Theorem 3.2 holds true for many situations, but there are counter-examples to its general validity.

3.2. Rigorous result for the codimension-1 case. Let us start with the situation, where every solution of the discontinuous problem has only classical and codimension-1 solutions. This corresponds to the case $d = 1$ in the system (1.3), so that there are only two vector fields $f^{-1}(y)$ and $f^{1}(y)$. The equation (2.3) becomes

$$\alpha'(y)\Big(\frac{(1+\lambda)}{2}\, f^{1}(y) + \frac{(1-\lambda)}{2}\, f^{-1}(y)\Big) = 0,$$

and condition (3.1) is

$$\alpha'(y)\big(f^{1}(y) - f^{-1}(y)\big) \neq 0. \tag{3.2}$$

Along a codimension-1 sliding mode the projections $\alpha'(y)f^{1}(y)$ and $\alpha'(y)f^{-1}(y)$ cannot have the same sign. Therefore, generically, condition (3.2) is satisfied. In this case the statements of the Desired Theorems 3.1 and 3.2 are true.

Theorem 3.3. *Assume that a solution of (1.3) enters a codimension-1sliding mode through a classical solution, and that (3.2) is satisfied along this sliding. Then the solution of the regularized differential equation (2.6) converges uniformly to the (unique) Filippov solution of (1.3). It is independent of the transition function $\pi(u)$.*

This result is well-known and can be rigorously proved with the technique of asymptotic expansions.

The following example shows that, if at some point of a codimension-1 sliding mode it holds $\alpha'(y)f^{1}(y) = \alpha'(y)f^{-1}(y) = 0$, the limit solution of the regularized differential equation can depend on the point, where the solution enters the codimension-1 sliding mode.

3.3. An illustrative example in dimension 1. We consider the time-dependent problem, where

$$f^{-1}(t,y) = -a(t-1), \qquad f^{1}(t,y) = b(t-c),$$

and the discontinuity surface is given by $\alpha(t,y) = y$. The constants a, b, c are positive. We consider initial values $y(0) = y_0 < 0$.

In the domains $\{(t,y)\,|\,y<0\}$ and $\{(t,y)\,|\,y>0\}$ the solutions are

$$y^{-}(t) = -a\frac{t^2}{2} + at + C^{-} \quad \text{and} \quad y^{+}(t) = b\frac{t^2}{2} - bct + C^{+},$$

respectively. For $y_0 < 0$ the solution of the discontinuous problem follows $y^{-}(t)$ until it hits the discontinuity surface at time $t = t^{-}$. We then distinguish the following situations (see Figure 2):

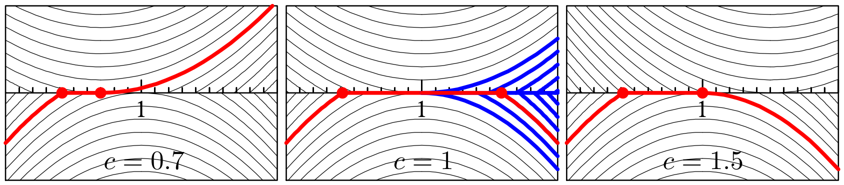

FIG. 2. *Solutions of the discontinuous problem of Section 3.3 with $a = 1.4$, $b = 1$, and three different values of c. For one particular solution (red) the beginning and the end of the sliding mode is indicated by bubbles. For $c = 1$, further solutions with the same initial value are plotted in blue.*

$c < 1$: If $c \le t^- < 1$ the solution crosses the surface at t^- and continues along $y^+(t)$. If $t^- < c$, we have sliding until $t = c$, where it leaves tangentially the surface along $y^+(t)$ into $\{(t,y)\,|\,y > 0\}$.

$c > 1$: If $t^- < 1$ we have sliding until $t = 1$, where it leaves tangentially the surface along $y^-(t)$ into $\{(t,y)\,|\,y < 0\}$.

$c = 1$: If $t^- < 1$ we have sliding until $t = 1$. There we have non-uniqueness: the solution can leave the surface tangentially along $y^-(t)$ into $\{(t,y)\,|\,y < 0\}$ or along $y^+(t)$ into $\{(t,y)\,|\,y > 0\}$ or it can continue as sliding until some time $t = t^+ > 1$, where it leaves the surface transversally along $y^-(t)$ or along $y^+(t)$.

The regularized differential equation is given by

$$\dot y = \frac{1}{2}\pi\Big(\frac{y}{\varepsilon}\Big)\Big((a+b)t - bc - a\Big) + \frac{1}{2}\Big((b-a)t - bc + a\Big), \tag{3.3}$$

which, for the case $c = 1$, becomes

$$\dot y = \frac{1}{2}\pi\Big(\frac{y}{\varepsilon}\Big)(a+b)(t-1) + \frac{1}{2}(b-a)(t-1). \tag{3.4}$$

The function $y^*(t) = \varepsilon u^*$, where u^* solves $\pi(u^*)(a+b) = (a-b)$, is a particular solution of (3.4), and the difference $z(t) = y(t) - \varepsilon u^*$ satisfies

$$\dot z = \frac{1}{2}\Big(\pi\Big(\frac{z}{\varepsilon} + u^*\Big) - \pi(u^*)\Big)(a+b)(t-1).$$

Separation of the variables yields

$$\int_{z_1}^{z_2} \frac{2\,\mathrm{d}z}{\pi\big(\frac{z}{\varepsilon} + u^*\big) - \pi(u^*)} = \int_{t_1}^{t_2} (a+b)(t-1)\,\mathrm{d}t. \tag{3.5}$$

If the solution enters the ε-band around the discontinuity surface from below at $t^-(\varepsilon) = t^- + \mathcal{O}(\varepsilon)$ and exits it at $t^+(\varepsilon)$, then we have $z\big(t^-(\varepsilon)\big) = z\big(t^+(\varepsilon)\big) = -\varepsilon - \varepsilon u^*$, and the relation (3.5) shows that

$$\int_{t^-(\varepsilon)}^{t^+(\varepsilon)} (a+b)(t-1)\,\mathrm{d}t = 0.$$

This implies that, independent of the transition function $\pi(u)$, the exit point is given by $t^+(\varepsilon) = 2 - t^-(\varepsilon)$ (red solution in the central picture of Figure 2). The limit $\varepsilon \to 0$ for the regularization selects this special solution out of all solutions of the discontinuous problem.

Our conclusion is that, even for the case $c = 1$, where the discontinuous problem has a continuum of solutions, the limit (for $\varepsilon \to 0$) of the solution of the regularized differential equation exists and is independent of the transition function $\pi(u)$.

3.4. Entering the codimension-2 manifold. We are next interested in the case, where codimension-d sliding modes exist with $d \leq 2$. We thus assume $d = 2$. Classical solutions and codimension-1 sliding modes are covered by the theorem of Section 3.2. For codimension-2 sliding modes we have to consider the equation (2.3), i.e., $\alpha_j(y) f^{0,0}(y, \lambda_1, \lambda_2) = 0$, which (when multiplied by 4) becomes

$$\alpha_j'(y)\Big((1+\lambda_1)(1+\lambda_2)\, f^{1,1}(y) + (1+\lambda_1)(1-\lambda_2)\, f^{1,-1}(y) + (1-\lambda_1)(1+\lambda_2)\, f^{-1,1}(y) + (1-\lambda_1)(1-\lambda_2)\, f^{-1,-1}(y)\Big) = 0 \tag{3.6}$$

for $j \in \{1, 2\}$. For the existence of a locally unique solution (λ_1, λ_2) of this system, we assume that the Implicit Function Theorem can be applied, which means that the 2×2 matrix (3.1), i.e.,

$$G(y, \lambda_1, \lambda_2) = \Big(\alpha_j'(y) \frac{\partial}{\partial \lambda_p} f^{0,0}(y, \lambda_1, \lambda_2)\Big)_{j,p=1}^{2} \tag{3.7}$$

is invertible.

THEOREM 3.4. *Assume that a solution of* (1.3) *enters the intersection at* $y^* \in \Sigma_1 \cap \Sigma_2$. *Generically, we then have in a neighbourhood of* y^* *uniform convergence of the solution of the regularized differential equation either to a classical solution, or to a codimension-*1 *sliding mode, or to a codimension-*2 *sliding mode.*

Note that this theorem and the other theorems of this subsection cover the situation, where there exist several solutions (in the sense of Definition 2.2) starting at $y^* \in \Sigma_1 \cap \Sigma_2$. The proof of the theorems will be given in Section 4. It is based on the classification of limit solutions of the regularized system (2.4) given in [11]. The meaning of "generically" in the statement will be explained in Section 4.4.

In the following theorem we use the notation $f_j^{\mathbf{k}} = \alpha_j'(y) f^{\mathbf{k}}(y)$ for $j \in \{1, 2\}$ and $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$ and $k_2 \neq 0$.

THEOREM 3.5. *Assume that a solution of* (1.3) *enters the intersection at* $y^* \in \Sigma_1 \cap \Sigma_2$ *through a codimension-*1 *sliding along* $\Sigma_1 \cap \{\alpha_2(y) < 0\}$. *If at least one of the following conditions is satisfied at* y^*:

- $f_1^{-1,1} f_1^{1,-1} - f_1^{-1,-1} f_1^{1,1} > 0$,
- $f_1^{-1,1} f_1^{1,-1} - f_1^{-1,-1} f_1^{1,1} < 0$ *and* $f_2^{1,-1} > 0$,
- *the system* (3.6) *does not have a solution in* $(-1, 1) \times (-1, 1)$,

then, in a neighbourhood of y, *the limit solution of the regularization is independent of the transition function* $\pi(u)$.

If the solution enters the intersection generically through spiraling, then, in a neighbourhood of the entry-point, the limit of the regularization is independent of the transition function $\pi(u)$.

In the nodally attractive case [7], where $f_2^{1,-1} > 0$, we have generically convergence of the regularization and independence of the transition function.

3.5. Example with limit solution depending on the transition function. The aim of this example is to give evidence that the limit solution, for $\varepsilon \to 0$, can depend on the transition function $\pi(u)$. We consider the case $d = 2$ with discontinuity surfaces given by $\alpha_1(y) = y_1$ and $\alpha_2(y) = y_2$. The (constant) vector fields on the four

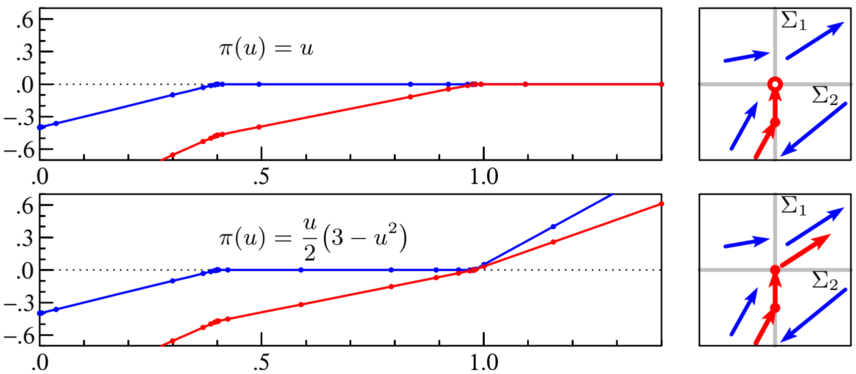

FIG. 3. *Regularized solutions of the discontinuous problem of Section 3.5 corresponding to two different transition functions (regularization parameter $\varepsilon = 10^{-5}$).*

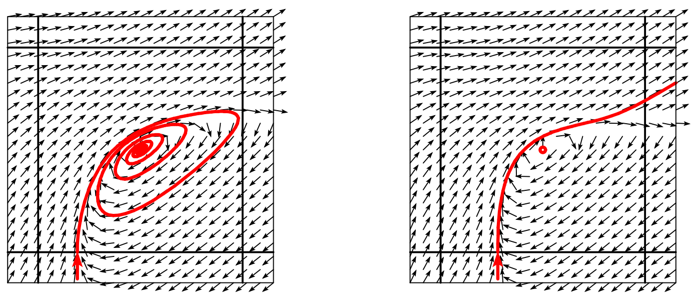

FIG. 4. *Vector field and solution of the discontinuous problem of Section 3.5 on the square $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$ for the transition function $\pi(u) = u$ (left picture) and the transition function $\pi(u) = \frac{u}{2}(3 - u^2)$ (right picture).*

regions are

$$f^{-1,1} = \begin{pmatrix} 1 \\ 0.18 \end{pmatrix} \qquad f^{1,1} = \begin{pmatrix} 2.25 \\ 1.45 \end{pmatrix}$$
$$f^{-1,-1} = \begin{pmatrix} 1 \\ 1.82 \end{pmatrix} \qquad f^{1,-1} = \begin{pmatrix} -4.25 \\ -3.45 \end{pmatrix}.$$

These vector fields are such that starting with initial values $y_1(0) = -0.4$ and $y_2(0) = -1.2$ the solution enters Σ_1 at time $t = 0.4$, and after a codimension-1 sliding it enters the intersection $\Sigma_1 \cap \Sigma_2$ at $t = 1.0$, see Figure 3. Beyond $t = 1.0$ the discontinuous problem admits two solutions: a classical solution in $\mathcal{R}^{1,1}$ and a codimension-2 sliding mode.

For the transition function $\pi(u) = u$ (for $u \in [-1, 1]$) the limit solution of the regularized differential equation is the codimension-2 sliding mode, whereas for the transition function $\pi(u) = \frac{u}{2}(3 - u^2)$ the limit solution is the classical solution. This is verified numerically and illustrated in Figure 3, where both solution components are plotted as a function of time for the regularization parameter $\varepsilon = 10^{-5}$.

This behaviour can be explained by the study of asymptotic expansions where, close to the entry point into the intersection $\Sigma_1 \cap \Sigma_1$, the solution is split into a smooth part and into a fast transient (see [11, Section 4]). The fast transient (hidden dynamics) determines whether a classical solution or a codimension-2 sliding mode is approximated by the regularization. Figure 4 shows the vector field and the solution of the regularized differential equation in the phase space (y_1, y_2) close to the origin (codimension-2 discontinuity manifold). Since the four vector fields are constant and of dimension 2, the pictures are - up to a scaling - independent of ε, and show the transient part.

3.6. During a codimension-2 sliding. Assume that, after entering a codimension-2 manifold at $y \in \mathbb{R}^n$, the solution of the regularized differential equations approximates a codimension-2 sliding mode. It follows from singular perturbation theory that the solution stays close to it as long as the eigenvalues of

$$G_\pi(y, u_1, u_2) = \Big(\alpha_j'(y) \frac{\partial}{\partial u_p} f^{0,0}\big(y, \pi(u_1), \pi(u_2)\big) \Big)_{j,p=1}^{2}, \tag{3.8}$$

with u_1, u_2 given by $\lambda_i = \pi(u_i)$, have negative real part (see Section 4.2 for its connection to the stability of the stationary point of the hidden dynamics). Note that for $\pi(u) = u$, the matrix G_π coincides with the matrix G of (3.7). The eigenvalues of G_π have negative real part iff $\det G_\pi > 0$ and $\operatorname{trace} G_\pi < 0$. Since $\pi'(u_i) > 0$, the sign of $\det G_\pi$ does not depend on the transition function. However, the sign of $\operatorname{trace} G_\pi$ may depend on it. The condition

$$\det G_\pi > 0 \qquad \text{and} \qquad \text{diagonal elements of } G_\pi \text{ are negative} \tag{3.9}$$

is independent of the transition function and provides a sufficient condition for the stability of G_π. Consequently, we have that under the assumption (3.9) the solution of the regularized differential equation converges to a codimension-2 sliding mode and the limit is independent of the transition function.

3.7. Exiting the codimension-2 manifold. We turn our attention to the situation of sliding motions exiting a codimension-2 manifold.

THEOREM 3.6. *Assume that condition* (3.9) *holds along a codimension-*2 *sliding mode until (and including) the point* $y = y(t^*)$*, and that either* $\lambda_1(t)$ *or* $\lambda_2(t)$ *leaves the interval* $(-1, 1)$ *at* $t = t^*$.

Generically, we then have uniform convergence of the solution of the regularized differential equation to a solution of (1.3) *(in the sense of Definition 2.2) until time* t^* *and on a non-empty interval beyond it. This convergence is independent of the transition function* $\pi(u)$.

The behaviour of solutions exiting a codimension-2 manifold is discussed in [12]. The assumptions of Theorem 3.6 correspond to the case (A) of that publication. For the case (B), which covers the cases, where the eigenvalues of (3.7) cross the imaginary axis away from the origin at some time t^* or the quadratic system (3.6) does no longer have a solution in $[-1, 1] \times [-1, 1]$ for $t > t^*$, we do not know of a rigorous proof for the behaviour of the solution beyond t^*. Numerical experiments indicate that the solution of the regularized differential equation still converges (for $\varepsilon \to 0$) to a solution od (1.3) (Definition 2.2), although convergence may depend on the transition function.

The example of the next section illustrates the behaviour of the solution of the regularized system in the case, where the eigenvalues of (3.7) cross the imaginary axis away from the origin.

3.8. Example with outward spiraling limit solution depending on the transition function. The example of this section shows that in the case of exiting a codimension-2 sliding mode, the limit solution, for $\varepsilon \to 0$, can depend on the transition function $\pi(u)$. As we have seen in Theoren 3.6, in case of inward spiraling the limit solution does not depend on $\pi(u)$. We remark that in general the outward spiraling behavior does not necessarily follow an inward spiraling into the discontinuity manifold.

We consider $d = 2$, discontinuity surfaces $\alpha_1(y) = y_1$ and $\alpha_2(y) = y_2$, and the vector fields

$$
\begin{aligned}
f^{-1,1} &= \begin{pmatrix} 3(1-t-t^2) \\ 1+5t \end{pmatrix} \qquad & f^{1,1} &= \begin{pmatrix} 3(1+t-t^2) \\ -5(1-t) \end{pmatrix} \\
f^{-1,-1} &= \begin{pmatrix} -3(1+t+t^2) \\ 5+t \end{pmatrix} \qquad & f^{1,-1} &= \begin{pmatrix} -3(1-t+t^2) \\ -1+t \end{pmatrix}
\end{aligned}
$$

Similar examples with spiraling dynamics are considered in [6, Example 4.3] and [16, Section IV.C]. Our example has the peculiarity that the quadratic terms in the regularized differential equation (2.4) are not present, which simplifies an analytic treatment. It is given by

$$
\begin{aligned}
\dot{y}_1 &= 3t\pi(u_1) + 3\pi(u_2) - 3t^2 \qquad & u_1 &= y_1/\varepsilon \\
\dot{y}_2 &= -3\pi(u_1) + 2(t-1)\pi(u_2) + 3t \qquad & u_2 &= y_2/\varepsilon.
\end{aligned} \tag{3.10}
$$

We substitute εu_j for y_j and then put $\varepsilon = 0$. The resulting system

$$
\begin{aligned}
0 &= 3t\pi\big(u_{1,0}(t)\big) + 3\pi\big(u_{2,0}(t)\big) - 3t^2 \\
0 &= -3\pi\big(u_{1,0}(t)\big) + 2(t-1)\pi\big(u_{2,0}(t)\big) + 3t
\end{aligned} \tag{3.11}
$$

yields functions satisfying $\pi\big(u_{1,0}(t)\big) = t$ and $\pi\big(u_{2,0}(t)\big) = 0$. We consider the following two cases (see Section 2.2):

(A) $\pi(u) = u$: in this case we have $u_{1,0}(t) = t$ and $u_{2,0}(t) = 0$;

(B) $\pi(u) = \frac{u}{2}\big(3 - u^2\big)$: in this case we have $u_{2,0}(t) = 0$, but $u_{1,0}(t)$ is solution of $u_{1,0}(t)\big(3 - u_{1,0}(t)^2\big) = 2t$.

Figure 5 shows both components y_1 and y_2 of the solution of (3.10) for different values of ε (red for the transition function (A), and blue for (B)). Initial values are $y_1(0) = y_2(0) = 0.075$. An accurate solution is computed numerically with an 8th order explicit Runge–Kutta method (code DOP853 of [13, Appendix]). In the beginning, until approximately $t \approx 0.1$, the solution spirals around the origin before entering a codimension-2 sliding. The second component $y_2(t)$ is $\mathcal{O}(\varepsilon^2)$-close to the horizontal axis, whereas $y_1(t)$ increases (linearly) with time. The remarkable fact is that, for relatively large ε, the solution leaves the codimension-2 sliding at about $t \approx 0.7$ for the transition function (A), and at about $t \approx 0.8$ for (B). Then it again turns into a spiraling motion around the origin. But what can be said for the limit $\varepsilon \to 0$? Can we trust the numerical solution (obtained with double precision arithmetics) with the smaller value $\varepsilon = 0.003$?

We repeated the same experiment in higher precision arithmetics (quadruple instead of double precision, and a more stringent accuracy requirement of the time integrator). For the parameters $\varepsilon = 0.01$ and $\varepsilon = 0.0065$ we get identical results. However, for $\varepsilon = 0.003$ (see Figure 6) we get a solution that leaves the codimension-2 sliding at about $t \approx 0.8$ (for the transition function (B)), as it was the case for larger values of ε with computations in double precision. This shows that the third picture of Figure 5 is affected by round-off errors. In order to get more insight into the limit behaviour for $\varepsilon \to 0$, we refer to Section 6.

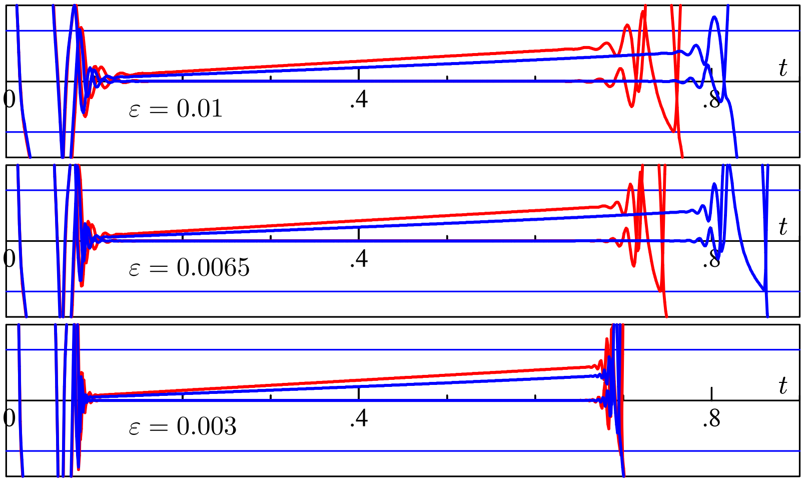

FIG. 5. *Solution of (3.10) with initial values $y_1(0) = y_2(0) = 0.075$: in red with transition function (A) and in blue with (B). Horizontal blue lines are at $y = -\varepsilon$ and at $y = \varepsilon$, and indicate the region of codimension-2 sliding.*

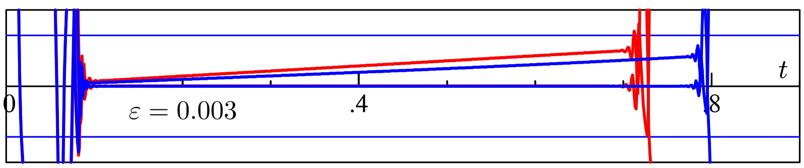

FIG. 6. *Same experiment as in the third picture of Figure 5, with the difference that the computation is done in quadruple precision.*

4. Proof of Theorems 3.4 and 3.5. The proof of Theorems 3.4 and 3.5 relies on the classification of the hidden dynamics [11], when a solution of the regularized differential equation approaches the intersection $\Sigma_1 \cap \Sigma_2$. For the solution of (1.3) there are three possibilities of entering $\Sigma_1 \cap \Sigma_2$: it can enter as classical solution, or through a codimension-1 sliding, or by spiralling around it. We discard the first possibility, because it is not generic.

4.1. Entering the intersection through a codimension-1 sliding. By changing the sign of $\alpha_j(y)$ and/or by exchanging $\alpha_1(y)$ and $\alpha_2(y)$ we can assume without loss of generality that the solution of (1.3) enters $\Sigma_1 \cap \Sigma_2$ through a codimension-1 sliding along $\mathcal{R}^{0,-1}$. We therefore assume (all vector fields are evaluated at the entry point)

$$
\begin{gathered}
f_1^{-1,-1} > 0, \qquad f_1^{1,-1} < 0, \qquad f_2^{-1,-1} > 0, \\
f_1^{-1,-1} f_2^{1,-1} - f_1^{1,-1} f_2^{-1,-1} > 0.
\end{gathered}
\tag{4.1}
$$

We again use the notation $f_j^{\mathbf{k}} = \alpha_j'(y) f^{\mathbf{k}}(y)$ for $j \in \{1,2\}$ and $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$ and $k_2 \neq 0$, and we denote $f_2^{1,0}(\lambda_2) = \alpha_2'(y) f^{1,0}(y, \lambda_2)$. The first two inequalities of (4.1) imply that both vector fields, $f^{-1,-1}$ and $f^{1,-1}$, point towards $\mathcal{R}^{0,-1}$. The remaining two inequalities imply that there is a sliding motion along $\mathcal{R}^{0,-1}$ in direction of the intersection $\Sigma_1 \cap \Sigma_2$. The flowchart of Figure 7 collects the statements of [11, Theorem 6.1] under the assumption (4.1). The flowchart of Figure 8 collects those of

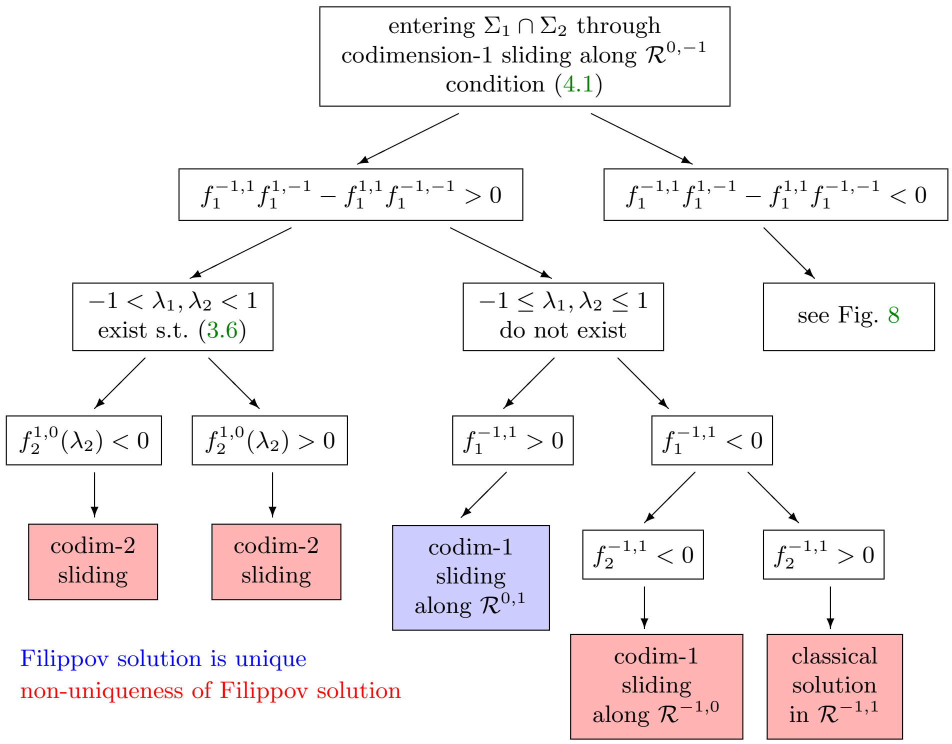

FIG. 7. *Flowchart of Theorem 6.1 of [11]. In the case of multiple solutions of* (3.6), λ_2 *is the value that is closest to* -1. *Here, and in Figures 8 and 9, the term "Filippov solution" has to be interpreted as a solution according to Definition 2.2.*

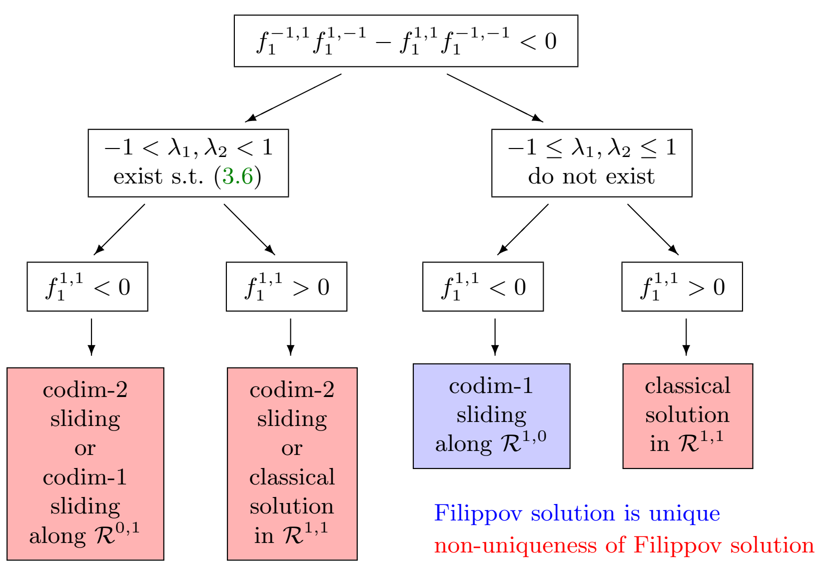

FIG. 8. *Flowchart of Theorem 6.2 of [11]. Condition* (4.2) *is assumed in addition to the assumption* (4.1) *of the flowchart of Figure 7.*

[11, Theorem 6.2], where in addition the condition

$$f_2^{1,-1} < 0 \tag{4.2}$$

is assumed.[1] The condition (4.2) can be considered without loss of generality, because the case $f_2^{1,-1} > 0$ can be reduced to that of Figure 7 by interchanging $\lambda_1 \leftrightarrow -\lambda_1$ and $\alpha_1(y) \leftrightarrow -\alpha_1(y)$.

Following a sequence of arrows in the flowcharts tells us to which kind of solution of (1.3) the solution of the regularized differential equation converges on a non-empty interval after the entry point. We have put the limit solution in a blue box, whenever the solution (in the sense of Definition 2.2) of the discontinuous problem (1.3) is unique. A red box indicates instead a situation of non-uniqueness. For example, the red boxes in Figure 7 correspond to a situation, where an additional classical solution may exist. Nevertheless, it is proved in [11] that the solution of the regularized differential equation does not converge to it for any transition function.

Note that the expressions appearing in the flowcharts of Figures 7 and 8 depend only on the discontinuous problem (1.3), and are independent of the transition function of the regularization. This terminates the proof of Theorem 3.4 for the situation, where the solution enters $\Sigma_1 \cap \Sigma_2$ through a codimension-1 sliding.

For the proof of Theorem 3.5 we note that the dependence on the transition function can happen only in the situations identified by the two red boxes at the left of Figure 8. The counter-example of Section 3.5 illustrates that the limit solution of the regularized problem can indeed depend on the transition function.

We still have to precise the meaning of the term "generically". This will be done in Section 4.4 below.

4.2. More details of the proof. Let us explain in some more detail how the results of [11], which are stated there for $\pi(u) = u$, can to be applied to get the statements illustrated in Figures 7 and 8. The classification of [11] is based on the study of the *hidden dynamics*, which describes the fast transition when the solution enters at y_0 the intersection $\cap_{j=1}^d \Sigma_j$ of the discontinuity surfaces. It is given by the solution $u(\tau)$ (with $\tau = t/\varepsilon$) of the autonomous system

$$u_i' = \sum_{\boldsymbol{\ell} \in \{-1,1\}^d} \Bigg(\prod_{j=1}^d \frac{(1 + \ell_j \pi(u_j))}{2} \Bigg) \alpha_i'(y_0) f^{\boldsymbol{\ell}}(y_0).$$

For $d = 2$, we write $\alpha(y) = \alpha_1(y)$, $\beta(y) = \alpha_2(y)$, $u = u_1$, $v = u_2$, and for the hidden dynamics (as in [11])

$$\begin{aligned} u' &= g_\alpha\big(\pi(u), \pi(v)\big) \\ v' &= g_\beta\big(\pi(u), \pi(v)\big). \end{aligned} \tag{4.3}$$

A typical result of [11], in fact, part (a1) of Theorem 6.1 is the following:

> *Assume that* $g_\alpha(-1,-1) > 0, g_\beta(-1,-1) > 0, g_\alpha(1,-1) < 0$, *and that there exists* $u_0 \in (-1,1)$ *such that* $g_\alpha(u_0,-1) = 0$ *and* $g_\beta(u_0,-1) > 0$. *Assume further that* $\partial_v g_\alpha(u_0,-1) < 0$, *and that the branch of the hyperbola* $g_\alpha(u,v) = 0$ *starting at* $(u_0,-1)$ *intersects transversally the hyperbola* $g_\beta(u,v) = 0$ *inside the unit square at* (u^*,v^*). *If*

[1] The cases (a1) and (a2) of [11, Theorem 6.2] are incorporated into the statements (a3) and (a4), so that the four statements of Figure 8 correspond to (a3), (a4), (b), and (c) of [11, Theorem 6.2].

$g_\beta(1,v^*) < 0$, then (u^*,v^*) is an asymptotically stable equilibrium of (4.3), and the solution of the regularized differential equation converges to a codimension-2 sliding mode.

How can we get this statement for an arbitrary transition function? We note that $g_\alpha(-1,-1)$, $g_\beta(-1,-1)$, and $g_\alpha(1,-1)$ are equal to $f_1^{-1,-1}$, $f_2^{-1,-1}$, and $f_1^{1,-1}$, respectively. Together with the next assumption, this is equivalent to (4.1), because with u_0 given by $g_\alpha(u_0,-1)=0$ we have that $g_\beta(u_0,-1)>0$ iff $f_1^{-1,-1}f_2^{1,-1}-f_1^{1,-1}f_2^{-1,-1}>0$. This remains valid, if u_0 is replaced by $\pi(u_0)$. Further, the sign of $\partial_v g_\alpha(u_0,-1)$ is the opposite of that of $f_1^{-1,1}f_1^{1,-1}-f_1^{1,1}f_1^{-1,-1}$ by [11, Lemma 6.3]. The existence of the intersection point (u^*,v^*) is equivalent to the existence of (λ_1,λ_2) satisfying (3.6). Moreover, the sign of $g_\beta(1,v^*)$ is the same as that of $f_2^{1,0}(\lambda_2)$. All these assumption show that we are in the situation, where we follow the arrows to the left in the decision tree of Figure 7.

The study of the hidden dynamics is based on geometric arguments in [11]. The essential idea for transferring the proofs of [11] to the situation, where $\pi(u)$ is different from $\pi(u)=u$, is to argument in the $\big(\pi(u),\pi(v)\big)$ plane rather than in the (u,v) plane. The curves, where the components of the right-hand sides of (4.3) vanish, are again hyperbolas. The flow is vertical on one hyperbola and horizontal on the other. Because of $\pi'(u)>0$ for $u\in(-1,1)$, the sign of $g_\alpha\big(\pi(u),\pi(v)\big)$ to the left, to the right, below, or above a point on the hyperbola $g_\alpha(\tilde u,\tilde v)=0$ is independent of $\pi(u)$. Also the sign of the Jacobian determinant of (4.3) does not depend on $\pi(u)$. Therefore, all the proofs of [11] leading to the statements of Figures 7, 8, and 9 carry over to general transition functions $\pi(u)$, and the convergence of the solution of the regularized differential equation is established. This proves Theorem 3.4.

There are two situations (those to the left of Figure 8) where the limit solution can either be a codimension-2 sliding mode or a codimension-1 sliding (respectively, classical solution). The example of Section 3.5 illustrates that in such a situation the limit solution can very well depend on the transition function. For this reason we have included in Theorem 3.5 assumptions that rule out these two situations.

4.3. Entering the intersection through spiraling. We next consider the situation, where a solution of (1.3) enters the intersection at $y\in\Sigma_1\cap\Sigma_2$ by spiraling inwards. This can be clockwise or counterclockwise. Assuming the second, this is the case, if the vector fields, evaluated at y, satisfy

$$\begin{aligned} &f_1^{-1,-1}>0, \quad f_1^{1,-1}>0, \quad f_1^{1,1}<0, \quad f_1^{-1,1}<0\\ &f_2^{-1,-1}<0, \quad f_2^{1,-1}>0, \quad f_2^{1,1}>0, \quad f_2^{-1,1}<0, \end{aligned} \tag{4.4}$$

and if the contractivity condition

$$0<\gamma<1 \quad \text{with} \quad \gamma=\frac{f_2^{-1,-1}}{f_1^{-1,-1}}\cdot\frac{f_1^{1,-1}}{f_2^{1,-1}}\cdot\frac{f_2^{1,1}}{f_1^{1,1}}\cdot\frac{f_1^{-1,1}}{f_2^{-1,1}} \tag{4.5}$$

holds. Under these two assumptions it follows from [11, Theorem 7.1] that the solution of the regularized differential equation converges uniformly on a non-empty interval after the entry point to a codimension-2 sliding mode of (1.3) for any transition function. Since this codimension-2 sliding mode is unique, the proof for Theorems 3.4 and 3.5 are complete.

4.4. The meaning of "generic" in Theorems 3.4 and 3.5. Let us explain the use of the word "generic" in the formulation of Theorems 3.4 and 3.5.

- We assume that the solution enters the intersection in a "generic way". This is a restriction on the initial values. It excludes the situation, where the solution enters as a classical solution from one region $\mathcal{R}^{\mathbf{k}}$, where $k_j \neq 0$ for all j.
- The word "generically" means that equalities are excluded in the inequality assumptions of (4.1), (4.2), (4.4), (4.5), and in those of the flowcharts in Figures 7 and 8.

The remaining (non-generic) cases, which constitute a set of measure zero, have not been investigated. Note that also the invertibility assumption in Theorem 3.4 is generically fulfilled.

5. Proof of Theorem 3.6. The proof of Theorem 3.6 relies on the results of [12], when a solution of the discontinuous problem leaves a codimension-2 sliding. Along such a sliding the standard theory of asymptotic expansions for singularly perturbed problems [14, Section VI.3] can be applied. Under the assumption of the eigenvalues of the matrix (3.7) we have that the solution of the regularized differential equation converges uniformly and independently of $\pi(u)$, for $\varepsilon \to 0$, to the codimension-2 sliding mode for $t \le t^*$. By assumption of Theorem 3.6 the solution $\big(\lambda_1(t), \lambda_2(t)\big)$ of the system (3.6) leaves the unit square at one side (excluding the corners). Without loss of generality we therefore assume that the following conditions

$$\lambda_1(t^*) = 1, \qquad \dot\lambda_1(t^*) > 0, \qquad -1 < \lambda_2(t^*) < 1 \tag{5.1}$$

hold. The results of [12, Theorems 2 and 3] are illustrated[2] in Figure 9. All vector fields are evaluated at the exit point. According to the decision tree the solution of the regularized differential equation either converges, beyond the exit point, to a codimension-1 sliding mode or to a classical solution. Since the conditions in Figure 9 only depend of the vector fields and not on the transition function, the limit solution is independent of it. This completes the proof of Theorem 3.6.

Note that in all situations of Figure 9 the discontinuous problem can have further solutions (classical or codimension-1). We nevertheless have convergence to the specified solution.

6. More insight into the example of Section 3.8. Much insight into the solution of a singularly perturbed problem, like the equation (3.10), can be obtained by studying an asymptotic expansion in powers of ε. We write $u_j(t)$, for $j = 1, 2$, as

$$u_j(t) = u_{j,0}(t) + \varepsilon u_{j,1}(t) + \ldots + \varepsilon^N u_{j,N}(t) + R_{j,N}(t). \tag{6.1}$$

The dominant functions $u_{j,0}(t)$ are those of (3.11). Further coefficient functions are obtained by inserting the expansion into the differential equation and comparing equal powers of ε. The remainder $R_{j,N}(t)$ can be estimated by studying the stability of the variational equation.

Figure 10 illustrates the asymptotic expansion of (3.10) for two different values of ε and for both transition functions, linear (A) and cubic (B). Plotted are as a function of time t (in logarithmic scale) the modulus of the solution components as well as the modulus of $y_j(t) - \sum_{i=0}^{N} \varepsilon^{i+1} u_{j,i}(t)$ $(N = 0, 1, \ldots)$, which decreases with increasing N. The first component ($j = 1$) is coloured in blue and the second ($j = 2$) in red. Recall

[2] For an application of these theorems we note that $\partial_{u_2} f_2(y, 1, u_2^*) = f_2^{1,1} - f_2^{1,-1}$, where u_2^* is determined by $f_2(y, 1, u_2^*) = 0$. We further note that the sign of $\partial_{u_1} f_2(y, 1, u_2^*)$ is equal to that of $f_2^{-1,1} f_2^{1,-1} - f_2^{1,1} f_2^{-1,-1}$.

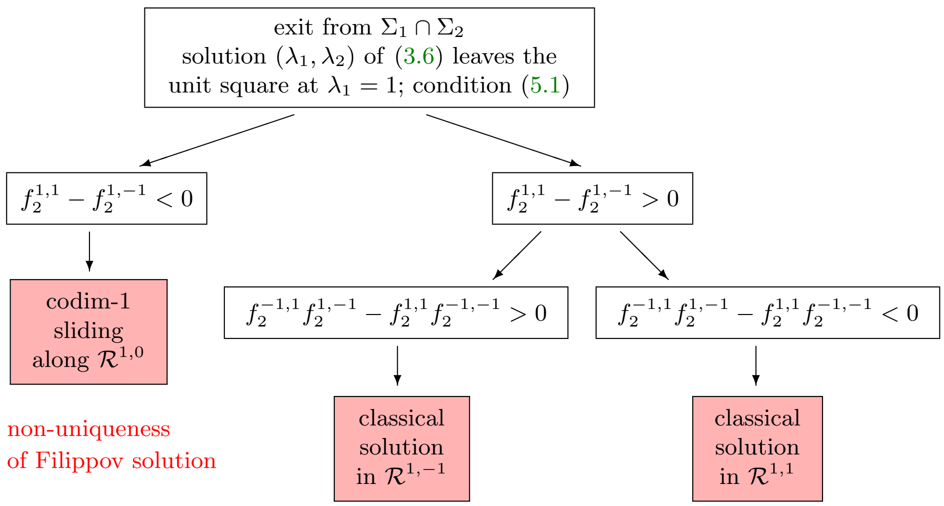

FIG. 9. *Flowchart for case (A) of [12].*

that $u_{2,0}(t) = 0$, so that one red function seems to be missing. Figure 11 repeats the experiment with higher precision arithmetics.

To explain the figures, we note that the stability of the system (3.10) is governed by the matrix

$$G(t) = \begin{pmatrix} 3t\pi'\big(u_{1,0}(t)\big) & 3\pi'\big(u_{2,0}(t)\big) \\ -3\pi'\big(u_{1,0}(t)\big) & 2(t-1)\pi'\big(u_{2,0}(t)\big) \end{pmatrix}.$$

Its determinant

$$\det G(t) = 3\pi'\big(u_{1,0}(t)\big)\pi'\big(u_{2,0}(t)\big)(2t^2 - 2t + 3)$$

is positive for all t, and its trace is given by

$$\text{trace } G(t) = 3t\pi'\big(u_{1,0}(t)\big) + 2(t-1)\pi'\big(u_{2,0}(t)\big).$$

For the transition function (A) of Section 3.8 the trace is equal to $5t - 2$ and changes sign (from negative to positive) at $t = 0.4$. We are in the same situation as in the example of Section 4.1 of [12] and we expect that the remainder of the asymptotic expansion is bounded by $C \exp\big(c\,(t-0.4)^2/\varepsilon\big)$ (which is a parabola, centered at $t = 0.4$, in logarithmic scale). For this reason we have included in the upper pictures of Figures 10 and 11 a parabola representing a bound of the remainder. For $\varepsilon = 0.01$ (and transition function (A)) we observe a perfect agreement. For $\varepsilon = 0.003$ the computation is affected by round-off, and this agreement can be observed only in Figure 11 (quadruple precision). We conclude that for (A) the exit point from the codimension-2 sliding has approximately the same distance to $t = 0.4$ as the entry point.

In the pictures for the transition function (B) we have included the parabolas of the case (A). This permits us to compare the results and to study the effect of the transition functions. In all pictures, with the exception of the lower right picture of Figure 10 (which is affected by round-off) we notice that the exit point from the codimension-2 sliding is significantly farther to the right. A sound explanation is still missing, but we note that for the case (B) the trace of $G(t)$ changes sign later at t, defined implicitly by $t = 0.4 + 0.6\,t\,u_{1,0}(t)^2$.

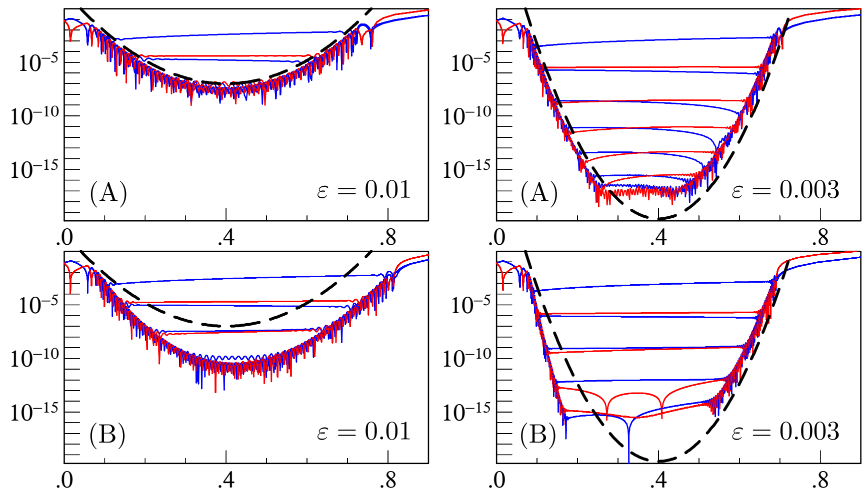

FIG. 10. *Illustration of the asymptotic expansion for the solution of (3.10) for two different ε and for the linear (A) and cubic (B) transition functions.*

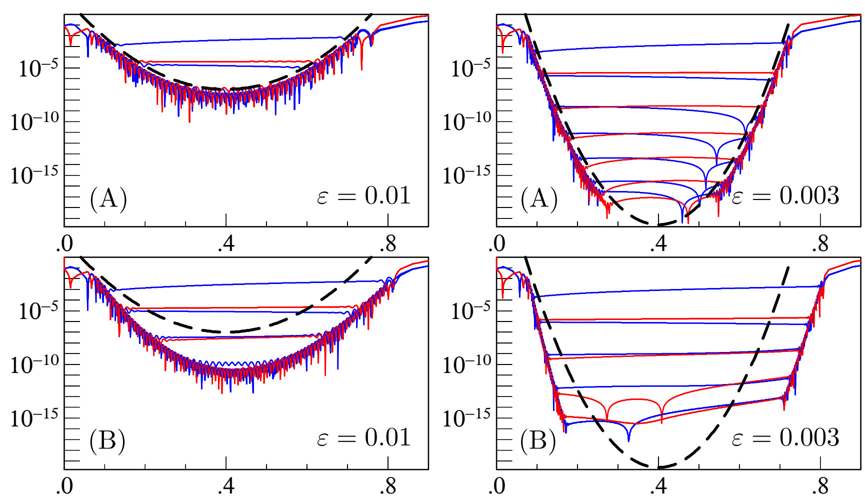

FIG. 11. *Same experiment as in Figure 10, with computation in quadruple precision.*

7. Conclusion. This work is devoted to the study - existence and independence of the transition function - of the limit solution of regularized piece-wise smooth dynamical systems. Solutions evolving in a codimension-1 discontinuity surface are well understood. We therefore focus our study on solutions that enter and/or exit a codimension-2 discontinuity manifold. In Section 3.1 we have formulated two desired results concerning the limit behaviour of regularized solutions.

Concerning the Desired Theorem 3.1 we have shown that during and after entering a codimension-1 or codimension-2 manifold we generically have convergence to a solution of the discontinuous dynamical system (1.3) in the sense of Definition 2.2. The proof, based on asymptotic expansions in powers of ε, uses the classification presented in the publication [11]. The main result is illustrated in Figures 7 and 8, where a checkable decision tree is given that allows to determine whether the solution of the regularized differential equation converges to a classical solution, to a codimension-1 sliding mode, or to a codimension-2 sliding mode.

For solutions exiting a codimension-2 sliding mode, there does not exist a complete classification on the limit of the solution of the regularized differential equation. For the situations that are covered by the study in [12] it is proved (Section 3.7 and Figure 9) that we have convergence of the solution of the regularized problem to a solution of the discontinuous equation (1.3). At present we do not know of an example, for which (under the regularity assumptions of Section 1) the solution of the regularized differential equation does not converge to a solution in the sense of Definition 2.2 for $\varepsilon \to 0$.

Concerning Desired Theorem 3.2 we have characterized situations, where the convergence is independent of the transition function. These are the situations, where in the decision trees of Figures 7, 8, and 9 there is only one possibility of a limit solution. Two exceptions arise when the left arrow is considered in Figure 8. For one of these two situations we have constructed a concrete example (Section 3.5), where different transition functions lead to different limits of the solution of the regularized differential equation.

For solutions exiting a codimension-2 sliding mode the example of Section 3.8 (see also Section 6) shows that the limit solution of the regularized differential equation can depend on the transition function. This insight is obtained by careful numerical experiments.

Since a complete characterization of the behaviour of regularized solutions is still missing at exit points from a codimension-2 sliding mode, the convergence and independence of transition functions is for the moment an open problem. Another challenging open problem is to understand the behaviour of regularized solutions close to a discontinuity manifold of codimension larger than 2.

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