Path-regularization of linear neutral delay differential equations with several delays

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Abstract

For differential equations with discontinuous right-hand side and, in particular, for neutral delay equations it may happen that classical solutions do no exist beyond a certain time instant. In this situation, it is common to consider weak solutions of Utkin (Filippov) type. This article extends the concept of weak solutions and proposes a new regularization which eliminates the discontinuities. Codimension-1 and codimension-2 weak solutions are considered. Numerical experiments show the advantages of the new regularization.

Key words: Neutral delay equations, weak solutions, discontinuous differential equations, path-regularization, stability
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1. Introduction

Delay differential equations arise when phenomena with memory are modelled. This article considers systems of linear neutral delay differential equations

$$\begin{aligned} \dot y(t) &= c(y(t)) + \sum_{j=1}^{m} A_j(y(t))\dot y(\alpha_j(y(t))) \qquad \text{for} \quad t > 0 \\ y(t) &= \varphi(t) \qquad \text{for} \quad t \le 0, \end{aligned} \tag{1}$$

where $y \in \mathbb{R}^n$ and the functions $c(y)$, $A_j(y)$, $\varphi(t)$ and $\alpha_j(y)$ are sufficiently differentiable. We consider time intervals where the solution satisfies $\alpha_j(y(t)) < t$ for all j. The right derivative of the solution of (1) at $t = 0$ is

$$\dot y_0^+ = c(\varphi(0)) + \sum_{j=1}^{m} A_j(\varphi(0))\dot\varphi(\alpha_j(\varphi(0))), \tag{2}$$

and in general it is different from its left derivative $\dot y_0^- = \dot\varphi(0)$. This produces a jump discontinuity of the derivative $\dot y(t)$ at $t = 0$. Since *neutral* delay equations are considered, the vector field of (1) has a discontinuity when the solution $y(t)$ crosses one of the manifolds given by $\alpha_j(y) = 0$, or evolves in one of them.

Neutral delay differential equations are closely connected to piecewise smooth dynamical systems (PWS) and in general to ordinary differential equations with discontinuous right-hand side. Excellent monographs on these subjects are those by Filippov [6], Utkin [14] and more recently by Budd, Di Bernardo, Champneys and Kowalczyk [1]. It is well-known that in general ordinary differential equations with discontinuous right-hand-side f do not have a classical solution, and a weak solution concept becomes necessary. A quite popular definition of weak solution is due to Filippov, see [6], who suggested to replace - at a discontinuity point - the right-hand-side by a certain differential inclusion. The resulting method is known as Filippov convexification (a methodology which treats all components of the solution in the same way, as we will see). Similarly when the vector field f is discontinuous due to a discontinuity of one of its arguments, Utkin convexification consists in replacing the discontinuous argument by a convex combination of its left and right limits. Since the source of discontinuity in a neutral delay differential equation of the form $\dot{y}(t) = f(t, y(t), \dot{y}(\alpha(y(t))))$ is a jump of the solution derivative at a past instant, it would seem natural to follow Utkin's approach and replace the discontinuous argument $\dot{y}(\alpha(y(t)))$ by a convex combination of its left and right-hand limits. In the present work, due to the fact the we consider linear problems, Filippov and Utkin convexifications coincide.

However, in presence of a discontinuity hypersurface of codimension bigger than 1, such a convexification becomes ambiguous and a bunch of possible weak solutions arises, so that the contemporary literature has focused its attention on this situation. Recent papers by Dieci, Elia and Lopez have discussed possible ways to retain a unique weak solution. In [4] and [5] and more recently in [2] the authors have provided a systematic way for defining vector fields on the intersection of several surfaces. Their model passes through the use of a multivalued sign function and is restricted to cases where the sliding manifold is *attractive*.

The possibility of choosing several weak solutions is a main issue of the present paper. An alternative to an a-priori motivated choice for selecting a sliding vector-field, is that of considering regularizations. This idea has also been partially explored in the literature (see e.g. [13], [7], [8], [3], [12]) where singularly perturbed smooth systems are proposed to replace the original PWS and analyzed.

For neutral delay differential equations the discontinuity is due to a jump at breaking points of the derivative of the solution, which appears on the right hand side of (1) and is determined by the fact that the derivative of the initial function at $t = 0$ is different from the derivative of the solution. Here we consider a so-called path-regularization, which replaces the derivative of the initial datum by a continuous function. Such a path-regularization mainly aims to achieve a few peculiar properties for the regularized solution, as the *closeness* to a classical solution, when it exists, and the absence of high frequency small oscillations, which is a typical drawback (also known as chattering) of introducing singular perturbations in non-smooth systems.

An interesting by-product of our analysis is the appearence of a so-called hidden (or dummy) dynamics (see also [11]), which models the instantaneous behaviour of the discontinuous system at a discontinuity. This is the key point to answer a fundamental question, which involves the (hidden) instantaneous dynamics of a PWS at a discontinuity, which is apparently not described by the system of ODEs.

The present article starts with a summary of previous results (Section 2) and continues with a numerical experiment (Section 3), where the effect of the new path-regularization is illustrated. Section 4 discusses the concept of weak solutions for neutral delay equations. It extends the approach of Filippov [6] and Utkin [14], by allowing more flexibility, which turns out to be essential for the space regularisations introduced in Section 5. Based on singular perturbation techniques [9] the characterisation of [10] for the kind of solution, which is approximated by the regularization, is extended to arbitrary paths and to codimension-2 weak solutions. The stability of codimension-2 weak solutions is studied in Section 6 with technical proofs postponed to Section 7. In particular, it is shown that the new path-regularization can avoid highly oscillatory approximations that can be present in the standard space regularization.

2. A summary of previous results

In this section we summarize the main results obtained in the recent articles [10] and [9], which are concerned with the same kind of problem addressed here. In [10] we have considered neutral state dependent delay differential equations with a *single state-dependent delay* $\alpha(y)$, i.e.

$$\begin{aligned} \dot{y}(t) &= f\Big(y(t), \dot{y}(\alpha(y(t)))\Big) \\ y(t) &= \varphi(t) \qquad \text{for} \quad t \le 0 \end{aligned} \tag{3}$$

where $y \in \mathbb{R}^n$ and $f(y,z)$, $\varphi(t)$ and $\alpha(y)$ are smooth functions, and we assume $\dot{\varphi}(0) \neq \dot{y}(0^+)$. The possible discontinuities in the right-hand-side, due to jumps in the solution derivative at breaking points, may determine existence termination for the solution. Since such a situation can occur only when the solution meets the manifold $\alpha(y) = 0$ (or $\alpha(y) \in \{\text{breaking points}\}$), we have to weaken the concept of solution to so-called sliding modes along such a manifold. This is a codimension-1 sliding and is therefore well-described by Filippov and Utkin methodologies. Our goal is not that of applying such methodologies directly but to study suitable regularizations and analyze their limit behaviour (with respect to the regularization parameter ε). In [10] we have considered two regularizations.
First regularization. By rewriting (3) in the equivalent form

$$\begin{aligned} \dot{y}(t) &= z(t) \\ 0 &= f(y(t), z(\alpha(y(t)))) - z(t), \end{aligned} \tag{4}$$

where $y(t) = \varphi(t)$ and $z(t) = \dot{\varphi}(t)$ for $t \le 0$, the regularization is based on the replacement of the initial function $\dot{\varphi}(t)$ on the interval $-\varepsilon \le t \le 0$ by the function $\pi(t/\varepsilon)$, where $\pi : [-1, 0] \to \mathbb{R}^n$ is a smooth monotonic function satisfying $\pi(-1) = \dot{y}_0^-$ and $\pi(0) = \dot{y}_0^+$.
Second regularization. Next we have considered the non-neutral delay equation which is obtained by associating to (4) the singularly perturbed delay differential equation (with $0 < \varepsilon \ll 1$)

$$\begin{aligned} \dot{y}(t) &= z(t) \\ \varepsilon \dot{z}(t) &= f(y(t), z(\alpha(y(t)))) - z(t). \end{aligned} \tag{5}$$

To study the solution of the regularized problems for small ε we have constructed an asymptotic expansion after the first breaking point (i.e. $t_0(\varepsilon)$ such that $\alpha(y(t_0(\varepsilon))) = 0$). In [10] we have proved that the solutions of both regularizations remain close to the solution of (3) both when this is a classical one and when it is a weak one (in the sense of Utkin). This means that in presence of a unique solution, either weak or classical, the regularized solutions converge to it when $\varepsilon \to 0$. The main differences between the two regularizations are relevant to the case when the solution is not unique. In those cases when weak and classical solutions coexist, the first regularization always approaches a weak solution. Instead, the second regularization can approach a weak or a classical solution depending on the behaviour of an underlying dynamical system.

In [9] we have considered a natural extension of the second regularization to neutral equations with *several state-dependent delays*. We have in fact associated to the neutral equation

$$\begin{aligned} \dot y(t) &= f\big(y(t), \dot y(\alpha_1(y(t))), \dot y(\alpha_2(y(t)))\big) \qquad \text{for} \quad t > 0 \\ y(t) &= \varphi(t) \qquad \text{for} \quad t \le 0, \end{aligned} \tag{6}$$

the singularly perturbed non-neutral delay equation

$$\begin{aligned} \dot y(t) &= z(t) \\ \varepsilon\, \dot z(t) &= f\big(y(t), z(\alpha_1(y(t))), z(\alpha_2(y(t)))\big) - z(t). \end{aligned} \tag{7}$$

The main interest in [9] is an analysis of the approximation of weak codimension-2 solutions. This can be done by studying a 4-dimensional dynamical system. The main phenomenon we have underlined is the occurrence of rapidly oscillating solutions of (7) (with frequency proportional to $1/\varepsilon$), which are associated to limit cycles of the underlying 4-dimensional system. In some cases a modified regularization allows to damp these oscillations, but there are also cases where such damping is not possible. This phenomenon makes a numerical integration quite challenging.

The goal of the present article is that of generalizing the first regularization based on the modification of the derivative of the initial function to the case of two delays, that means to codimension-2 discontinuity manifolds.

3. Numerical experiment

For a simple but instructive example, we let

$$\dot y(t) = c + A\dot y(\alpha(y(t))) + B\dot y(\beta(y(t))) \tag{8}$$

with $y(t) = \varphi$ for $t \le 0$, $\alpha(y) = y_1 - 1, \beta(y) = y_2 - 1$, and

$$c = \begin{pmatrix} 2 \\ 1/4 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 1/4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -24 \\ 0 & -4 \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 \\ 9/10 \end{pmatrix}$$

(these data are taken from problem (P3) in [9]).

Since $\dot y(t) = \dot\varphi(t) = 0$ for $t < 0$, the solution of this problem is given by $y_1(t) = c_1 t$, $y_2(t) = 0.9 + c_2 t$ until the first breaking point (at $t_0 = 0.4$), which is determined by $y_2(t_0) = 1$. The solution cannot continue into the region $y_2 > 1$, because there the derivative of $y_2(t)$ would be negative: $c_2 - 4c_2 = -3/4 < 0$. Beyond $t_0 = 0.4$ we thus have a codimension-1 sliding in the manifold $y_2(t) = 1$. The second component of equation (8), which reads $0 = c_2 - 4\dot y_2(0)$ fixes the value $\dot y_2(0) = 1/16$. Inserted into the first component of (8) this yields $y_1(t) = 0.8 + 0.5(t - 0.4)$ for $0.4 \le t \le 0.8$. Since $y_1(t)$ cannot enter the region $y_1 > 1$, we have a codimension-2 sliding for $t \ge 0.8$ which gives $y_1(t) = 1$, $y_2(t) = 1$ for $t \ge 0.8$.

With the aim of applying standard ODE/DDE software, we regularize the discontinuous functions arising in our problem. More precisely, we replace the expressions $\dot y_1(y_1 - 1)$ and $\dot y_2(y_2 - 1)$ of the righthand side of (8) by $\pi_1(y_1 - 1)$ and $\pi_2(y_2 - 1)$, respectively, where

$$\pi_j(x) = \begin{cases} 0 & \text{for } x \le -\varepsilon/\kappa_j \\ c_j\,(1 + \kappa_j x/\varepsilon) & \text{for } -\varepsilon/\kappa_j \le x \le 0 \\ c_j & \text{for } x \ge 0 \end{cases}$$

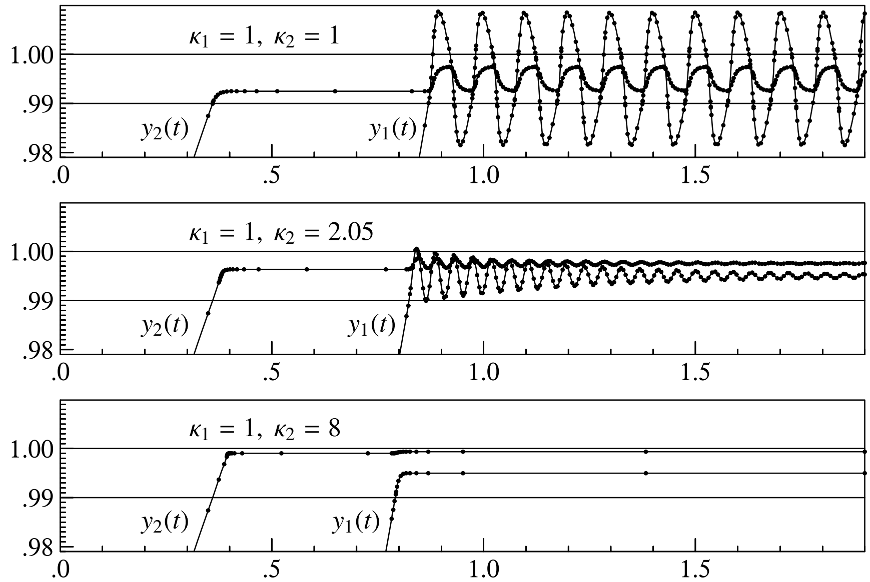

Figure 1: Solution of the regularized problem of Section 3 with $\varepsilon = 10^{-2}$ and different choices of κ_j. The horizontal lines indicate the interval $[1-\varepsilon, \varepsilon]$. The small points on the solution indicate natural output of the numerical integrator.

The standard choice is $\kappa_1 = \kappa_2 = 1$. We introduce some additional freedom by allowing for $\kappa_j \geq 1$ satisfying $\min_j \kappa_j = 1$. The first picture of Figure 1 shows the solution of the regularized differential equation for the standard choice with $\varepsilon = 10^{-2}$. One notices oscillations of amplitude $O(\varepsilon)$ and of frequency $O(\varepsilon^{-1})$, which are not present in the exact solution. These high oscillations make the computation with an ODE solver rather inefficient. When looking at the two discontinuous functions, one notices that the jump in $\dot{y}_2(0)$ is by a factor of 8 smaller than that in $\dot{y}_1(0)$. This motivates us to consider $\kappa_2 > 1$, so that $\pi_2(x)$ agrees with $\dot{y}_1(x)$ on a larger interval. The result is shown in Figure 1. Increasing κ_2 from 1 to 2 does not affect the oscillatory behaviour, but reduces the amplitude of the oscillations. As soon as $\kappa_2 > 2$, the oscillations are damped (second picture of Figure 1). For $\kappa_2 = 8$ we even get a monotonic behaviour of the solutions. The choice $\kappa_2 = 8$ is natural in the following sense that the slopes of the two regularizing functions are the same. Notice that the slope is responsible for stiffness of the regularized problem. It is not increased as long as $\kappa_2 \leq 8$.

4. Weak solutions associated to a path

Due to the discontinuity of $\dot{y}(t)$ at $t = 0$, there is an ambiguity in the definition of the right-hand side of (1) when $y(t)$ is in one of the manifolds $\mathcal{M}_j = \{y \,|\, \alpha_j(y) = 0\}$. If there exists a solution in $\mathcal{M}_j^- = \{y \,|\, \alpha_j(y) < 0\}$ and one in $\mathcal{M}_j^+ = \{y \,|\, \alpha_j(y) > 0\}$ which both approach $y_1 \in \mathcal{M}_j$ and have slopes $\dot{y}_1^-$ and $\dot{y}_1^+$, respectively, and if $\alpha_j'(y_1)\dot{y}_1^-$ and $\alpha_j'(y_1)\dot{y}_1^+$ have the same sign, then we have a (classical) solution that transverses $\mathcal{M}_j$ (left picture of Figure 2). If they have opposite sign and, in particular, if the derivative vectors $\dot{y}_1^-$ and $\dot{y}_1^+$ point towards $\mathcal{M}_j$ from both sides (right picture of Figure 2), it is natural to consider "solutions" that satisfy:

- they evolve in the manifold, i.e., $\alpha_j(y(t)) = 0$,

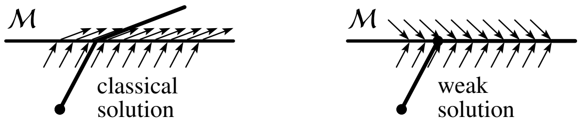

Figure 2: Illustration of the difference between classical and weak solutions.

- the undefined value $\dot{y}(0)$ in the right-hand side of (1) is replaced by an element between the one-sided limits $\dot{y}_0^-$ and y_0^+.

Whereas the first condition is mathematically clear, the second one requires a precision of the meaning of "between". In one dimension there is no ambiguity, but in higher dimension there are several possible definitions. A common approach is to consider the line segment that connects both one-sided limits. This then leads to weak solutions in the sense of Filippov [6] or Utkin [14]. Nothing prevents us to consider elements for which the jth component lies between the jth components of the one-sided limits (rectangle in Figure 3). Since the components of y typically represent different quantities, there is no reason to insist on the same convex combination for all components. To get a well-defined weak solution we propose to consider values for $\dot{y}(0)$ that lie on a path connecting $\dot{y}_0^-$ with y_0^+.

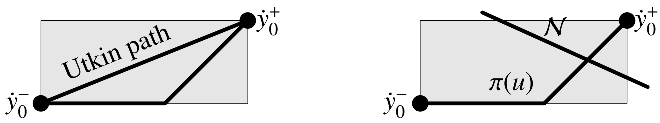

Figure 3: Admissible paths and manifold $\mathcal{N}$ of weak solutions.

*4.1. Weak solutions – codimension-*1

We consider the case $m = 1$ in (1) and we let $A(y) = A_1(y)$ and $\alpha(y) = \alpha_1(y)$. We further fix a path $\pi : [-1, 0] \to \mathbb{R}^n$ satisfying $\pi(-1) = \dot{y}_0^-$ and $\pi(0) = \dot{y}_0^+$. Our main interest is in paths whose jth component $\pi_j(u)$ is constant on $[-1, -\kappa_j^{-1}]$ and linear on $[-\kappa_j^{-1}, 0]$ for some $\kappa_j \geq 1$ (see Figure 3, where $\kappa_1 = 1$ and $\kappa_2 = 5/3$). The choice $\kappa_j = 1$ for all j corresponds to the Utkin path. A weak solution is defined as solution of the differential-algebraic equation

$$\begin{aligned} \dot{y} &= c(y) + A(y)\,\pi(u) \\ 0 &= \alpha(y). \end{aligned} \tag{9}$$

Consistent initial values satisfy

$$\alpha(y_1) = 0 \qquad \text{and} \qquad \alpha'(y_1)\big(c(y_1) + A(y_1)\,\pi(u_1)\big) = 0. \tag{10}$$

This means that at the one hand $y_1 \in \mathcal{M} = \{y \,|\, \alpha(y) = 0\}$, and on the other hand $\pi(u_1)$ lies on the intersection of the chosen path with the manifold $\mathcal{N} = \{w \,|\, \alpha'(y_1)(c(y_1) + A(y_1)\,w) = 0\}$ (see the right picture of Figure 3). If

$$\alpha'(y_1)A(y_1)\,\pi'(u_1) \neq 0, \tag{11}$$

then the differential-algebraic equation (9) has a locally unique solution satisfying $y(t_1) = y_1$ and $u(t_1) = u_1$ (if $\pi(u)$ is not differentiable at u_1, the right derivative has to be considered if $u(t)$

increases, and the left derivative if $u(t)$ decreases at t_1). Note that the solution $y(t)$ is independent of the choice of the parametrization of the path.

We remark that the intersection point $\pi(u_1)$, and hence also the weak solution, depends on the chosen path.

4.2. Weak solutions – codimension-2

For the case $m = 2$ in (1) we make use of the notation $A(y) = A_1(y)$, $B(y) = A_2(y)$ and $\alpha(y) = \alpha_1(y)$. $\beta(y) = \alpha_2(y)$. This time we fix two paths $\pi^\alpha : [-1,0] \to \mathbb{R}^n$ and $\pi^\beta : [-1,0] \to \mathbb{R}^n$ connecting $\dot y_0^-$ with $\dot y_0^+$. A codimension-2 weak solution is then defined as solution of the differential-algebraic equation

$$\begin{aligned} \dot y &= c(y) + A(y)\,\pi^\alpha(u) + B(y)\,\pi^\beta(v) \\ 0 &= \alpha(y) \\ 0 &= \beta(y). \end{aligned} \tag{12}$$

Consistent initial values satisfy $\alpha(y_2) = \beta(y_2) = 0$ and

$$\begin{aligned} \alpha'(y_2)\big(c(y_2) + A(y_2)\,\pi^\alpha(u_2) + B(y_2)\,\pi^\beta(v_2)\big) &= 0 \\ \beta'(y_2)\big(c(y_2) + A(y_2)\,\pi^\alpha(u_2) + B(y_2)\,\pi^\beta(v_2)\big) &= 0. \end{aligned} \tag{13}$$

If the derivative of (13) with respect to (u, v) at (y_2, u_2, v_2) is invertible, then the differential-algebraic equation (12) has a locally unique solution satisfying $y(t_2) = y_2$, $u(t_2) = u_2$, and $v(t_2) = v_2$.

5. Path-regularization

To avoid the discontinuity in the derivative $\dot y(t)$ at $t = 0$ we choose a small regularization parameter ε, we do not touch the initial function $\varphi(t)$, but we replace its derivative $\dot\varphi(t)$ by

$$z(t) = \begin{cases} \dot\varphi(t) & \text{for} \quad t \le -\varepsilon \\ \pi(t/\varepsilon) & \text{for} \quad -\varepsilon \le t \le 0, \end{cases} \tag{14}$$

where $\pi(u)$ defines a path connecting $\dot y_0^-$ with $\dot y_0^+$ as in Section 4. For ease of presentation we assume that $\varphi(t)$ is constant for $t \le 0$. An extension to the general situation is straight-forward. The publication [10] considers codimension-1 weak solutions for the Utkin path. Here, we focus on codimension-2 weak solutions and on paths depending on parameters $\kappa_j \ge 1$ as explained in Section 4.1.

5.1. Regularized weak solution – codimension-1

We briefly recall some results from [10] and thereby extend them to arbitrary paths $\pi(u)$. Since this section is concerned with codimension-1 weak solutions, we can assume $m = 1$ in (1), so that the equation becomes

$$\begin{aligned} \dot y(t) &= c\big(y(t)\big) + A\big(y(t)\big)\,\dot y\big(\alpha(y(t))\big) \qquad \text{for} \quad t > 0 \\ y(t) &= \varphi(t) \qquad \text{for} \quad t \le 0. \end{aligned} \tag{15}$$

The solution of (15) is not affected by the regularization as long as $\alpha(y(t)) \le -\varepsilon$. We let t_1 be the first breaking point of the problem (15), i.e., $a_0 = y(t_1)$ satisfies $\alpha(a_0) = 0$. Assuming that the solution enters transversally the manifold $\alpha(y) = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(y(t))\Big|_{t=t_1} = \alpha'(a_0)\,\big(c(a_0) + A(a_0)\,\dot{y}_0^-\big) > 0, \tag{16}$$

where $\dot{y}_0^- = \dot{\varphi}(0)$. Correspondingly, we let $t_1(\varepsilon)$ be the first time instant such that

$$\alpha(y(t_1(\varepsilon))) = -\varepsilon. \tag{17}$$

Because of (16), the implicit function theorem guarantees that $t_1(\varepsilon)$ can be expanded into a series of powers of ε. Consequently, this is also true for the solution at $t_1(\varepsilon)$ and we have an expansion

$$y(t_1(\varepsilon)) = \sum_{j=0}^{N} a_j\varepsilon^j + O(\varepsilon^{N+1}). \tag{18}$$

Beyond $t_1(\varepsilon)$ and as long as $-\varepsilon \le \alpha(y(t)) \le 0$, the solution of the regularized problem satisfies

$$\dot{y}(t) = c\big(y(t)\big) + A\big(y(t)\big)\,\pi\big(\alpha(y(t))/\varepsilon\big). \tag{19}$$

Following [10], in order to cope with the singularity at $\varepsilon = 0$, we separate the solution by an asymptotic expansion of the form

$$y(t_1(\varepsilon) + t) = \sum_{j=0}^{N} \varepsilon^j y_j(t) + \varepsilon\sum_{j=0}^{N-1}\varepsilon^j\eta_j(t/\varepsilon) + O(\varepsilon^{N+1}). \tag{20}$$

The rationale behind the representation (20) is that of separating the smooth part of the solution from the non-smooth one, which describes the transient. In order to analyze intervals of length independent of ε, we assume (as is done usually) that $\eta_j(\tau)$ converges exponentially fast to zero for $\tau \to \infty$. Observe that (20) has to agree with the expansion (18) for $t = 0$ i.e.,

$$y_0(0) = a_0, \qquad y_j(0) + \eta_{j-1}(0) = a_j \quad \text{for} \quad j \ge 1. \tag{21}$$

It is convenient to write the argument of π in equation (19) as

$$\frac{1}{\varepsilon}\,\alpha(y(t_1(\varepsilon) + t)) = \frac{1}{\varepsilon}\,\alpha(y_0(t)) + \alpha'(y_0(t))(y_1(t) + \eta_0(\tau)) + O(\varepsilon) \tag{22}$$

and assume (to avoid a singularity as $\varepsilon \to 0$)

$$\alpha(y_0(t)) = 0. \tag{23}$$

This allows to analyze the limit, for $\varepsilon \to 0$, of (19) in a neighbourhood of $t_1(\varepsilon)$.

To address the case of a weak solution, we assume the existence of y_1 and $u_1 \in (-1, 0)$ such that (10) holds. Here, $y_1 = a_0$ is the constant term in the Taylor series expansion of $y(t_1(\varepsilon))$ (see (18)). Inserting (20) into (19) with $N = 1$ and expanding into powers of ε, the ε-independent term yields the differential-algebraic equation (9) for the functions $y_0(t)$ and $u_0(t) = \alpha'(y_0(t))y_1(t)$ with consistent initial values $y_0(0) = y_1$ and $u_0(0) = u_1$.

For the transient part we subtract the smooth part from (19), substitute $\varepsilon\tau$ for t, and then put $\varepsilon = 0$. This gives

$$\eta_0'(\tau) = A(y_1)\,(\pi(u(\tau)) - \pi(u_1)), \tag{24}$$

where $u(\tau) = \alpha'(y_1)(y_1(0) + \eta_0(\tau))$. Under the consistency assumption (10) this function is solution of the scalar autonomous equation

$$u'(\tau) = \alpha'(y_1)(c(y_1) + A(y_1)\,\pi(u(\tau))). \tag{25}$$

The initial value $u(0) = -1$ is obtained from (17) and from the fact that $y_1(0) + \eta_0(0) = a_1$ is the coefficient of ε in the Taylor series expansion of $y(t_1(\varepsilon))$. If $\alpha'(y_1)(c(y_1) + A(y_1)\,\pi(-1)) > 0$ (which means that the solution of (15) enters transversally the manifold $\alpha(y) = 0$), the solution $u(\tau)$ is monotonically increasing and approaches either a stationary point or tends to infinity. We thus have (see [10, Theorems 1 and 2]):

Lemma 1. *The scalar differential equation (25) determines which solution of the neutral delay differential equation (15) is approximated by the path regularization:*

- if $u(\tau) \to u_1 \in [-1, 0]$ for $\tau \to \infty$, then it approximates the weak solution corresponding to $\pi(u_1)$;
- if $u(\tau) \approx c\tau$, $c \neq 0$ for $\tau \to \infty$, then it approximates a classical solution.

*5.2. Regularized weak solution – codimension-*2

For the case $m = 2$ in (1) and we write the problem in the form

$$\begin{aligned} \dot y(t) &= c(y(t)) + A(y(t))\,\dot y(\alpha(y(t))) + B(y(t))\,\dot y(\beta(y(t))), \quad t > 0 \\ y(t) &= \varphi(t) \qquad \text{for} \quad t \le 0. \end{aligned} \tag{26}$$

We let $t_2(\varepsilon)$ be the first time instant, such that both delayed arguments, $\alpha(y(t))$ and $\beta(y(t))$, fall into the interval $[-\varepsilon, 0]$. Similar to the previous section we write the solution (for $t \ge 0$) as

$$y(t_2(\varepsilon) + t) = y_0(t) + \varepsilon\Big(y_1(t) + \eta_0(t/\varepsilon)\Big) + O(\varepsilon^2). \tag{27}$$

As long as $\alpha(y(t))$ and $\beta(y(t))$ are in the interval $[-\varepsilon, 0]$, the derivative in the right-hand side of (26) can be replaced by (14). The same analysis as for the codimension-1 case shows that $y_0(t)$ and scalar functions $u_0(t)$, $v_0(t)$ are solution of the differential-algebraic equation (12) with initial values y_2, u_2, and v_2. The study of the transient part leads to the two-dimensional dynamical system

$$\begin{aligned} u'(\tau) &= \alpha'(y_2)(c(y_2) + A(y_2)\,\pi^\alpha(u(\tau)) + B(y_2)\,\pi^\beta(v(\tau))) \\ v'(\tau) &= \beta'(y_2)(c(y_2) + A(y_2)\,\pi^\alpha(u(\tau)) + B(y_2)\,\pi^\beta(v(\tau))) \end{aligned} \tag{28}$$

with initial values $u(0) = u_0 \in (-1, 0)$ and $v(0) = -1$, if the solution approaches the codimension-2 manifold through a sliding along the manifold $\mathcal{M}_\alpha = \{y \,|\, \alpha(y) = 0\}$. The following result, which we state without proof (since it is similar to that of Lemma 1), addresses the three possible situations which may occur.

Lemma 2. *The two-dimensional dynamical system (28) determines which solution of the neutral problem (26) is approximated by the path regularization:*

- *if $(u(\tau), v(\tau)) \to (u_0, v_0)$ for $\tau \to \infty$, then it approximates a codimension-2 weak solution, corresponding to $\pi^\alpha(u_0)$, $\pi^\beta(v_0)$,*
- *if $u(\tau) \to u_0$, $v(\tau) \approx c_\beta \tau$ for $\tau \to \infty$, or if $u(\tau) \approx c_\alpha \tau$, $v(\tau) \to v_0$ for $\tau \to \infty$, then it approximates a codimension-1 weak solution,*
- *if $u(\tau) \approx c_\alpha \tau$, $v(\tau) \approx c_\beta \tau$ for $\tau \to \infty$, then it approximates a classical solution.*

The study of the dynamics of (28) is more difficult than that of (25). The following section is devoted to the asymptotic stability of stationary points. This then explains the approximation of codimension-2 weak solutions by the path-regularization. Note that the system (28) strongly depends on the paths $\pi^\alpha(u)$ and $\pi^\beta(u)$.

In the context of discontinuous ordinary differential equations, and for the special case of an Utkin path, the system (28) is called "dummy system" [11] because its flow corresponds to a motion on a time-interval of length zero.

6. Stability of codimension-2 weak solutions

The aim of this section is the study of asymptotic stability of stationary points of the two-dimensional dynamical system (28). This then explains the behaviour of path-regularisations close to codimension-2 weak solutions.

To be able to work with a more compact notation, we denote the right-hand side of (28) by

$$\begin{aligned} g_\alpha(u,v) &= \alpha'(y_2)(c(y_2) + A(y_2)\,\pi^\alpha(u) + B(y_2)\,\pi^\beta(v)) \\ g_\beta(u,v) &= \beta'(y_2)(c(y_2) + A(y_2)\,\pi^\alpha(u) + B(y_2)\,\pi^\beta(v)) \end{aligned} \tag{29}$$

and its Jacobian matrix by

$$G(u,v) = \begin{pmatrix} \partial_u g_\alpha & \partial_v g_\alpha \\ \partial_u g_\beta & \partial_v g_\beta \end{pmatrix}(u,v). \tag{30}$$

We assume that there exist $u_0, v_0 \in (-1,0)$ such that $g_\alpha(u_0,v_0) = g_\beta(u_0,v_0) = 0$ and that $G(u_0,v_0)$ is invertible. The neutral delay equation (26) then admits a codimension-2 weak solution. Our aim is to investigate whether *all eigenvalues of $G(u_0,v_0)$ have negative real part.* This then implies asymptotic stability of the stationary solution of (28).

Assumption A. We consider a solution of (26), which at time t_2 enters the codimension-2 manifold defined by $\alpha(y) = 0$ and $\beta(y) = 0$. We denote $y_2 = y(t_2)$ and we further assume

- the problem (26) neither has a classical nor a codimension-1 weak solution that continues at y_2;
- there exists a codimension-2 weak solution corresponding to $\pi^\alpha(u_0)$ and $\pi^\beta(v_0)$;

Assumption B. We consider paths as in Section 4.2 and assume

- that the components $\pi^\alpha_j(u)$ are constant on $[-1, -1/\kappa^\alpha_j]$ and affine on $[-1/\kappa^\alpha_j, 0]$ with $\kappa^\alpha_j \geq 1$; similarly, the components $\pi^\beta_j(u)$ are constant on $[-1, -1/\kappa^\beta_j]$ and affine on $[-1/\kappa^\beta_j, 0]$ with $\kappa^\beta_j \geq 1$.

- that each of the curves defined by $g_\alpha(u,v) = 0$ and $g_\beta(u,v) = 0$, respectively, intersects transversally the border of the square $[-1,0]\times[-1,0]$ exactly twice and not at corners.
- that g_α and g_β are differentiable at the intersection point (u_0, v_0).

Assumption A just specifies the situation we are interested to analyse. Assumption B is satisfied for the Utkin path, for which the functions g_α and g_β are affine. In general, these functions are piecewise affine. Assumption B is also satisfied for the path considered in Theorem 4 below.

Theorem 3. *Under the Assumptions A and B we have*

$$\det G(u_0, v_0) > 0.$$

For stability of the codimension-2 weak solution we also have to investigate the trace of G. Whereas the sign of the determinant of G is independent of the values of κ_j^α and κ_j^β (obeying the assumptions of Theorem 3), this is not the case for the trace. For the Utkin path, where $\kappa_j^\alpha = \kappa_j^\beta = 1$ for al j, we distinguish the following situations:

1. $\partial_u g_\alpha < 0$ and $\partial_v g_\beta < 0$ at the point (u_0, v_0);
2. $\partial_u g_\alpha$ and $\partial_v g_\beta$ have opposite sign at the point (u_0, v_0);
3. $\partial_u g_\alpha > 0$ and $\partial_v g_\beta > 0$ at the point (u_0, v_0).

In the first case, the codimension-2 weak solution is stable and there is no need for choosing a different path. In the second case, the trace of G can be positive so that the codimension-2 solution (corresponding to the Utkin path) can be unstable.

Theorem 4. *Under the Assumptions A and B we have: If at least one of the expressions $\partial_u g_\alpha(u_0, v_0)$ and $\partial_v g_\beta(u_0, v_0)$ is negative for the Utkin path, then there exist $\kappa_j^\alpha = \kappa^\alpha \geq 1$ and $\kappa_j^\beta = \kappa^\beta \geq 1$ with $\min(\kappa^\alpha, \kappa^\beta) = 1$ for which the trace of G is negative at the codimension-2 solution. This choice of the path makes the codimension-2 weak solution stable.*

Proof. If $g_\alpha(u,v)$ and $g_\beta(u,v)$ are the functions (29) for the Utkin path, they are $\widetilde{g}_\alpha(u,v) = g_\alpha(\kappa^\alpha u, \kappa^\beta v)$ and $\widetilde{g}_\beta(u,v) = g_\beta(\kappa^\alpha u, \kappa^\beta v)$ for the parametrized path as long as $-1 \leq \kappa^\alpha u, \kappa^\beta v \leq 0$. This path therefore admits a codimension-2 solution corresponding to $(\widetilde{u}_0, \widetilde{v}_0) = (u_0/\kappa^\alpha, v_0/\kappa^\beta)$. The matrices G for the Utkin path and for the parametrized path are thus given by

$$G_{Utkin} = \begin{pmatrix} \partial_u g_\alpha & \partial_v g_\alpha \\ \partial_u g_\beta & \partial_v g_\beta \end{pmatrix}, \qquad G = \begin{pmatrix} \kappa^\alpha \partial_u g_\alpha & \kappa^\beta \partial_v g_\alpha \\ \kappa^\alpha \partial_\alpha g_\beta & \kappa^\beta \partial_v g_\beta \end{pmatrix},$$

respectively. If one of the diagonal elements is negative, say, $\partial_u g_\alpha < 0$, then putting $\kappa^\beta = 1$ and $\kappa^\alpha \geq 1$ sufficiently large makes the trace of G negative. □

Consider the third case, where $\partial_u g_\alpha > 0$ and $\partial_v g_\beta > 0$ at (u_0, v_0), so that the path of the previous proof cannot alter the sign of the trace. In the following theorem we show that this situation is exceptional and cannot occur under Assumption A.

Theorem 5. *Let the Assumptions A and B be fulfilled. If $\partial_u g_\alpha > 0$ and $\partial_v g_\beta > 0$ at (u_0, v_0) and if*

$$\begin{aligned} g_\alpha(0,v) - g_\alpha(-1,v) > 0 \quad &\textit{for} \quad v \in \{-1,0\} \\ g_\beta(u,0) - g_\beta(u,-1) > 0 \quad &\textit{for} \quad u \in \{-1,0\} \end{aligned} \tag{31}$$

then the codimension-2 weak solution is isolated, i.e., there is no classical solution and no codimension-1 weak solution of (26) that enters the codimension-2 manifold.

Note that for the Utkin path, where g_α and g_β are affine functions, condition (31) is a consequence of of the assumption that $\partial_u g_\alpha > 0$ and $\partial_v g_\beta > 0$ at (u_0, v_0). This is also true for the path of Theorem 4 and for all paths, for which the partial functions $g_\alpha(\cdot, v)$ and $g_\beta(u, \cdot)$ are monotonic. The proofs of Theorem 3 and Theorem 5 are postponed to Section 7.

Example. We consider the example of Section 3, for which

$$\begin{aligned} g_\alpha(u,v) &= 2 + 2\pi_1(u) - 6\pi_2(v) \\ g_\beta(u,v) &= 1/4 + \pi_1(u)/2 - \pi_2(v) \end{aligned}$$

The solution of $g_\alpha(u,v) = g_\beta(u,v) = 0$ is given by $\pi_1(u_0) = 1/2, \pi_2(v_0) = 1/2$, which corresponds to $\kappa_1 u_0 = -1/2$ and $\kappa_2 v_0 = -1/2$, so that u_0, v_0 stay in $(-1,0)$ for all $\kappa_j \geq 1$. Assumption A can be verified, and the matrix G becomes

$$G = \begin{pmatrix} 2\kappa_1 & -6\kappa_2 \\ \kappa_1/2 & -\kappa_2 \end{pmatrix}$$

In the Utkin definition of weak solutions we have $\kappa_1 = \kappa_2 = 1$, and the trace of G is positive implying instability of the solution of (28). However, if we choose $\kappa_1 = 1$ and $\kappa_2 > 2$, the codimension-2 solution becomes stable. This choice does not affect the non-existence of classical and codimension-1 solutions, and explains the behaviour of Figure 1 in Section 3.

7. Proofs

Proof (Theorem 3 of Section 6). This proof extends that of Proposition 1 in [9] to the situation where $g_\alpha(u,v)$ and $g_\beta(u,v)$ are not necessarily affine functions.

The signs of $g_\alpha(u,v)$ and $g_\beta(u,v)$ at the four corners of the square $[-1,0] \times [-1,0]$ play an important role. The non-existence of a classical solution in the region $\alpha(y) < 0, \beta(y) < 0$ implies that at least one of the functions g_α and g_β is positive at the corner $(-1,-1)$. Since the involutive transformation

$$\big(g_\alpha(u,v), g_\beta(u,v)\big) \mapsto \big(g_\beta(v,u), g_\alpha(v,u)\big) \tag{32}$$

does not affect $\det G$, we can assume without loss of generality that $g_\alpha(-1,-1) > 0$. Consequently, only the four sign patterns of Figure 4 have to be considered. We treat each of them individually:

(a) $g_\alpha(-1,-1) > 0$, $g_\alpha(0,-1) < 0$, with arbitrary signs in the remaining two corners. In this case there exists a unique $u^* \in (-1,0)$ with $g_\alpha(u^*,-1) = 0$. We have $\partial_u g_\alpha(u^*,-1) < 0$, so that the non-existence of codimension-1 weak solutions along $\alpha(y) = 0$, starting at y_2, implies $g_\beta(u^*,-1) > 0$. Consequently, the arrow for $g_\beta(u,v) = 0$ points into the region $g_\alpha(u,v) > 0$ implying that $\det G > 0$ at the intersection of the curves.

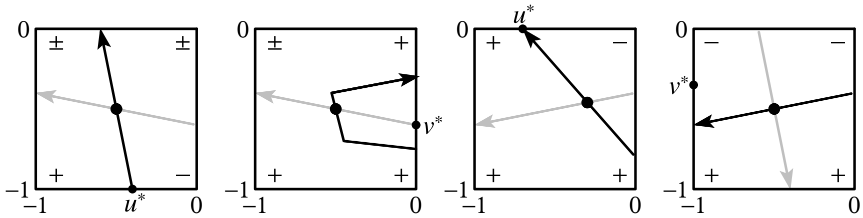

Figure 4: Sign patterns of $g_\alpha(u,v)$ at the corners of the square. The curve in black represents $g_\alpha(u,v) = 0$, the curve in grey $g_\beta(u,v) = 0$. The arrows indicate that the function is positive to the left and negative to the right.

(b) $g_\alpha(-1,-1) > 0, g_\alpha(0,-1) > 0, g_\alpha(0,0) > 0$, with arbitrary sign in the remaining corner. The non-existence of classical solutions implies $g_\beta(0,0) < 0$ and $g_\beta(0,-1) > 0$, so that $v^* \in (-1,0)$ exists with $g_\beta(0,v^*) = 0$ and $\partial_v g_\beta(0,v^*) < 0$. The non-existence of codimension-1 weak solutions along $\beta(y) = 0$ thus implies $g_\alpha(0,v^*) < 0$, so that the curve representing $g_\alpha(u,v) = 0$ has to be as shown in the second picture of Figure 4. Hence, we have $\det G > 0$.

(c) $g_\alpha(-1,-1) > 0, g_\alpha(0,-1) > 0, g_\alpha(0,0) < 0, g_\alpha(-1,0) > 0$. Similar to the case (a) there exists $u^* \in (-1,0)$ with $g_\alpha(u^*,0) = 0$ and $\partial_u g_\alpha(u^*,0) < 0$. This implies $g_\beta(u^*,0) < 0$ by the non-existence of cidimension-1 weak solutions along $\alpha(y) = 0$, and consequently also $\det G > 0$.

(d) $g_\alpha(-1,-1) > 0, g_\alpha(0,-1) > 0, g_\alpha(0,0) < 0, g_\alpha(-1,0) < 0$. The non-existence of classical solutions implies $g_\beta(-1,0) < 0$ and $g_\beta(0,-1) > 0$. If the arrow for $g_\beta(u,v) = 0$ would point into the region $g_\alpha(u,v) < 0$, there would exist $v^* \in (-1,0)$ satisfying $g_\beta(-1,v^*) = 0$, $\partial_v g_\beta(-1,v^*) < 0$, and $g_\alpha(-1,v^*) < 0$. This contradicts the non-existence of stable codimension-1 weak solutions along $\beta(y) = 0$ and proves $\det G > 0$.

The proof of Theorem 3 is thus complete. □

Proof (Theorem 5 of Section 6). Since the transformation (32) does not change the statement of the theorem, it is sufficient to consider the four situations of Figure 4. In the first three situations the value $g_\alpha(u,v)$ is positive to the left of the curve given by $g_\alpha(u,v) = 0$ and negative to its right. This implies $\partial_u g_\alpha(u_0,v_0) < 0$.

It remains to consider the last situation of Figure 4. We study the solution close to the codimension-2 manifold. By definition of the functions g_α and g_β, their signs at the corners of $(-1,0)\times(-1,0)$ tell us whether $\alpha(y(t))$ and $\beta(y(t))$ are increasing or decreasing.

In the region $\{y; \alpha(y) < 0, \beta(y) < 0\}$ the value $\alpha(y(t))$ is increasing because of $g_\alpha(-1,-1) > 0$, and $\beta(y(t))$ is decreasing as a consequence of $g_\beta(-1,-1) < 0$, which can be seen by contradiction as follows: if $g_\beta(-1,-1) > 0$ there would exist $v^* \in (-1,0)$ with $g_\beta(-1,v^*) = 0$ and $\partial_v g_\beta(-1,v^*) < 0$ which is not possible by our assumptions. The situation is illustrated in the lower left quadrant of the left picture in Figure 5. As a consequence, the solution enters and crosses the manifold $\{y; \alpha(y) = 0\}$ to arrive in the region $\{y; \alpha(y) > 0, \beta(y) < 0\}$. There, the values $\alpha(y(t))$ and $\beta(y(t))$ are both increasing, because $g_\alpha(0,-1) > 0$, and $g_\beta(0,-1) > 0$. Similar arguments describe the situation in the remaining two regions (see the left picture of Figure 5. The solution is seen to turn around the codimension-2 manifold. We still have to investigate whether it spirals inwards (stability) or outwards (instability).

Starting in the region $\{y; \alpha(y) < 0, \beta(y) < 0\}$ at a value satisfying $\beta(y) \approx 0$ and $\alpha(y) = -\varepsilon$ with small $\varepsilon > 0$, the solution will satisfy $\alpha(y(t)) \approx -\varepsilon + t g_\alpha(-1,-1)$ and $\beta(y(t)) \approx t g_\beta(-1,-1)$ until $\alpha(y(t))$ vanishes. At that point we have $t \approx \varepsilon / g_\alpha(-1,-1)$ and $\beta(y) \approx \varepsilon g_\beta(-1,-1)/g_\alpha(-1,-1)$. An analogous computation in the other three regions shows that after one complete round the

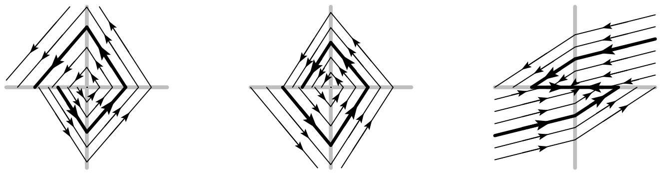

Figure 5: Sketch of solutions close to a codimension-2 manifold. The vertical and horizontal lines indicate the manifolds given by $\alpha(y) = 0$ and $\beta(y) = 0$, respectively. The first two pictures correspond to the situation of the last case in Figure 4: with positive trace of G (left picture) and with negative trace (picture in the middle). The last picture corresponds to the example of Section 3.

solution arrives at a value satisfying $\beta(y) \approx 0$ and $\alpha(y) = -\gamma\varepsilon$ with an amplification factor given by

$$\gamma = \frac{g_\beta(-1,-1)}{g_\alpha(-1,-1)} \cdot \frac{g_\alpha(0,-1)}{g_\beta(0,-1)} \cdot \frac{g_\beta(0,0)}{g_\alpha(0,0)} \cdot \frac{g_\alpha(-1,0)}{g_\beta(-1,0)}.$$

The particular sign pattern and the condition (31) imply that

$$g_\alpha(-1,0) < g_\alpha(0,0) < 0, \qquad 0 < g_\alpha(-1,-1) < g_\alpha(0,-1).$$

This proves the first inequality in

$$\frac{|g_\alpha(-1,0)\, g_\alpha(0,-1)|}{|g_\alpha(0,0)\, g_\alpha(-1,-1)|} > 1, \qquad \frac{|g_\beta(0,0)\, g_\beta(-1,-1)|}{|g_\beta(-1,0)\, g_\beta(0,-1)|} > 1.$$

The second inequality is proved with similar arguments. Consequently, the amplification factor satisfies $|\gamma| > 1$, and the solution cannot approach the codimension-2 manifold. This completes the proof of Theorem 5. □

Conclusions

This article presents a new regularization of neutral delay differential equations which, instead of interpolating all components of the one-sided limits y_0^- and y_0^+ by the same convex combination, permits to use different interpolations for the components (depending on a parameter κ_j). Our theoretical analysis shows that the freedom in choosing the parameters κ_j permits to reduce unphysical oscillations and makes the numerical integration more efficient.

It is still a big challenge to choose suitably these parameters. Numerical experiments have shown that choosing κ_j^{-1} proportional to the size of the jump in the discontinuity of the jth component, usually gives good results. It may also happen that at subsequent breaking points the parameters κ_j have to be modified to get an improved behaviour. Therefore, it is desirable to develop a strategy that permits to choose the parameters adaptively.

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