Polygon spaces and their cohomology rings

Cornell University, November 2007

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joint works with

A. Knutson (1998)

E. Rodriguez (2004)

M. Farber and D. Schütz (2007)

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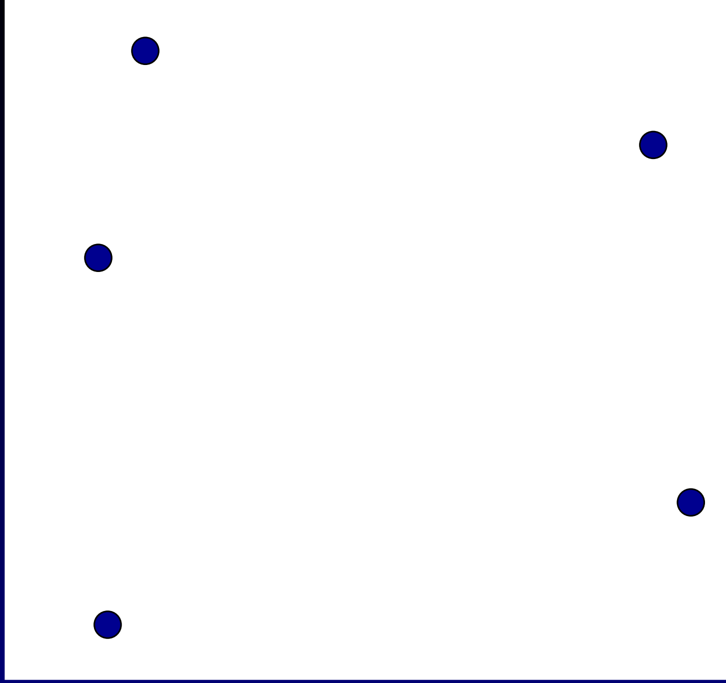

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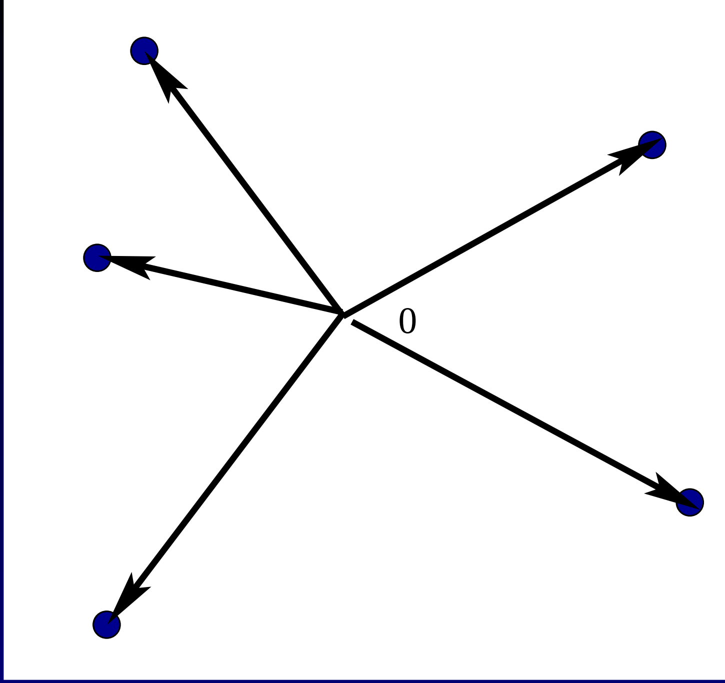

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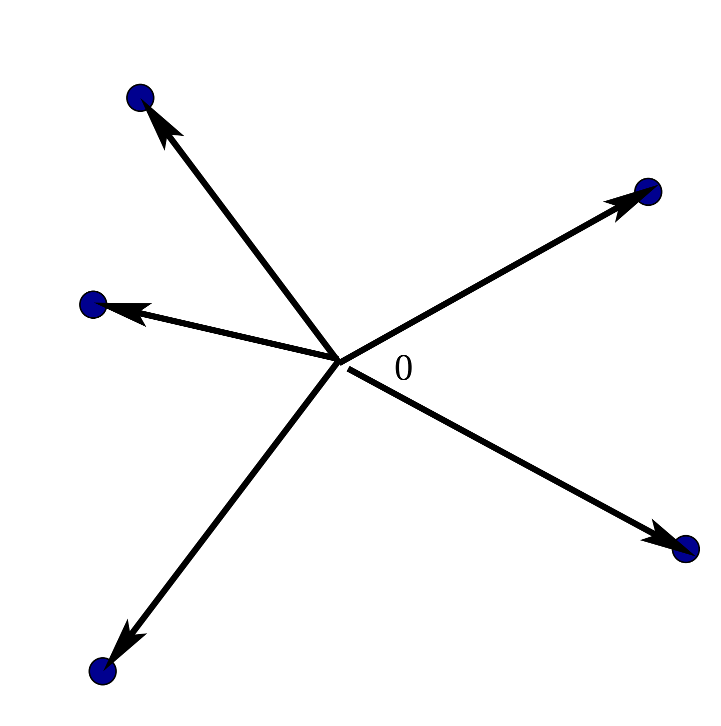

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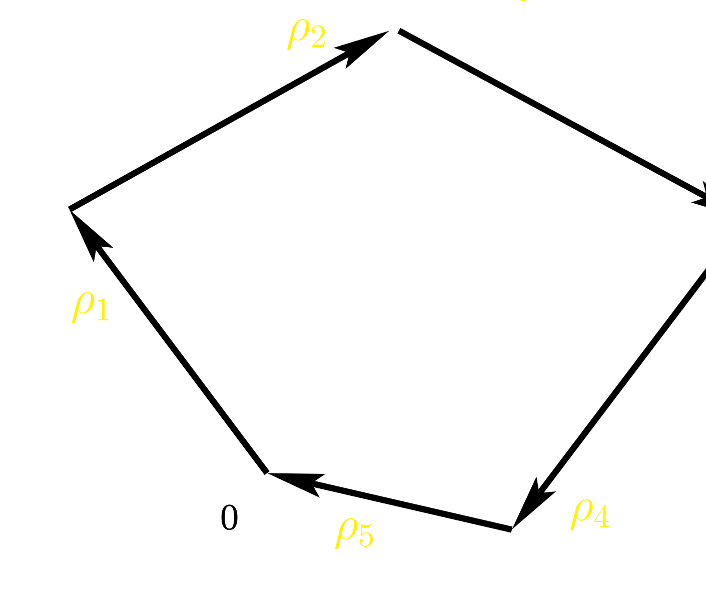

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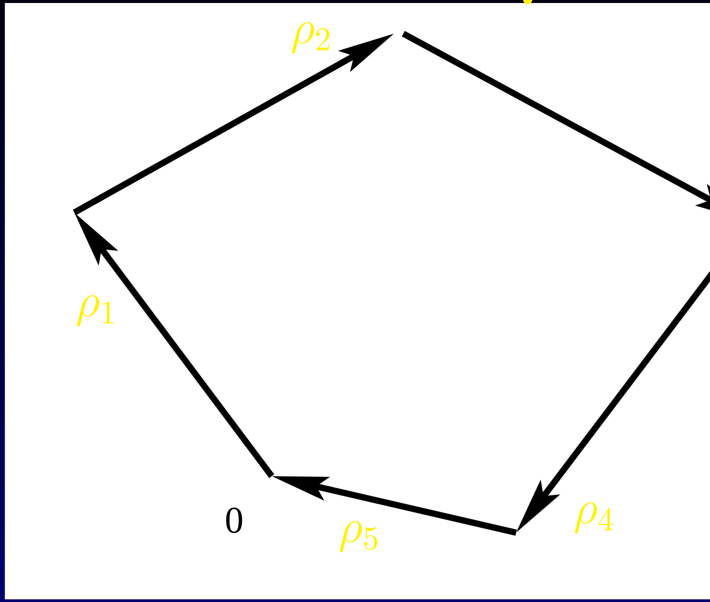

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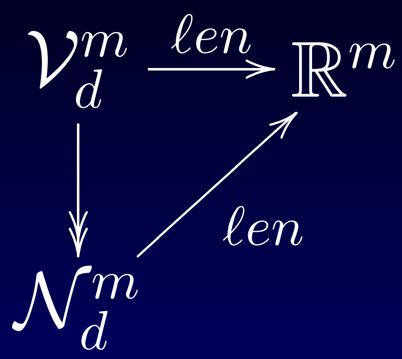

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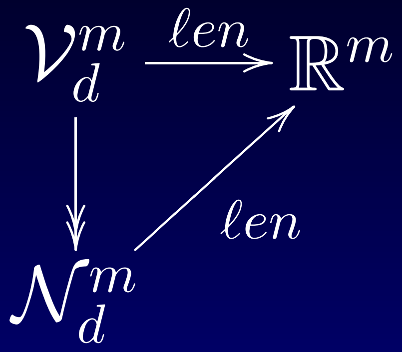

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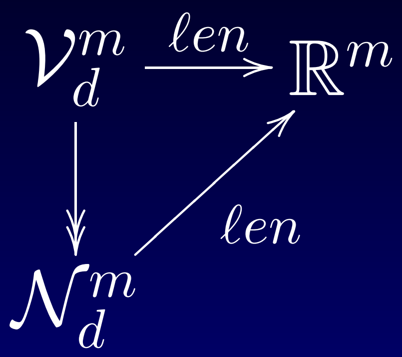

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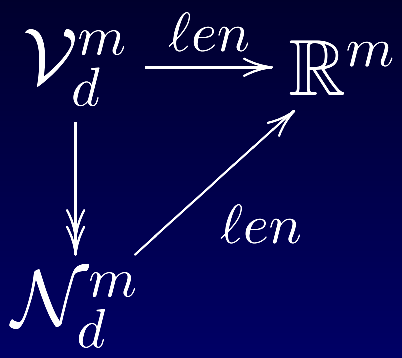

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Examples:

ℓ	$\mathcal{N}_3^m(\ell)$	$\bar{\mathcal{N}}_2^m(\ell)$	$\mathcal{N}_2^m(\ell)$
$(1, \cdots, 1, m-2)$	$\mathbb{C}P^{m-3}$	$\mathbb{R}P^{m-3}$	S^{m-3}
$(\varepsilon, \cdots, \varepsilon, 1, 1, 1)$	$(S^2)^{m-3}$	T^{m-3}	$T^{m-3} \dot{\cup} T^{m-3}$

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Moreover, there is a ring isomorphism

$$H^{2*}(\mathcal{N}_3^m(\ell);\mathbb{Z}_2) \xrightarrow{\approx} H^*(\bar{\mathcal{N}}_2^m(\ell);\mathbb{Z}_2)$$

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In fact, $\mathcal{N}_3^m(\ell)$ is a conjugation space. *(T. Holm – V. Puppe – JCH, 2005)*

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PROPOSITION

$[\omega_\ell]$ is integral $\iff$ $\ell_i \in \mathbb{Z}[1/2]$ and $\sum \ell_i \in \mathbb{Z}$.

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Generic planar polyon spaces are determined by their integral cohomology ring:

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Theorem not true for real cohomology:

$\ell = (\varepsilon, \varepsilon, 1, 1, 1)$ $\qquad$ $\ell' = (\varepsilon, 1, 1, 1, 2)$

$\mathcal{N}_3^5(\ell) = S^2 \times S^2$ $\qquad$ $\mathcal{N}_3^5(\ell') = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$

$$H^*(\mathcal{N}_3^5(\ell);\mathbb{R}) \approx H^*(\mathcal{N}_3^5(\ell');\mathbb{R})$$

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PROPOSITION *Let ℓ, ℓ' be generic m-length-vectors. Then*

ℓ, ℓ' in the same chamber $\Longrightarrow \mathcal{N}^m_d(\ell) \approx_{diff} \mathcal{N}^m_d(\ell')$

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Number of chambers up to permutations (JCH-Rodriguez, 2002):

m	3	4	5	6	7	8	9
	2	3	7	21	135	2470	175428

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STRATEGY: *for ℓ generic, show that $H^*(\mathrm{Pol}\,(\ell); \mathbb{Z}_2)$ determines $\mathcal{S}_m(\ell)$.*

Cohomology classes

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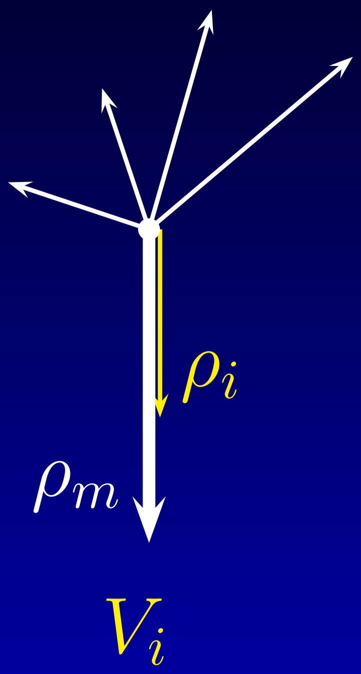
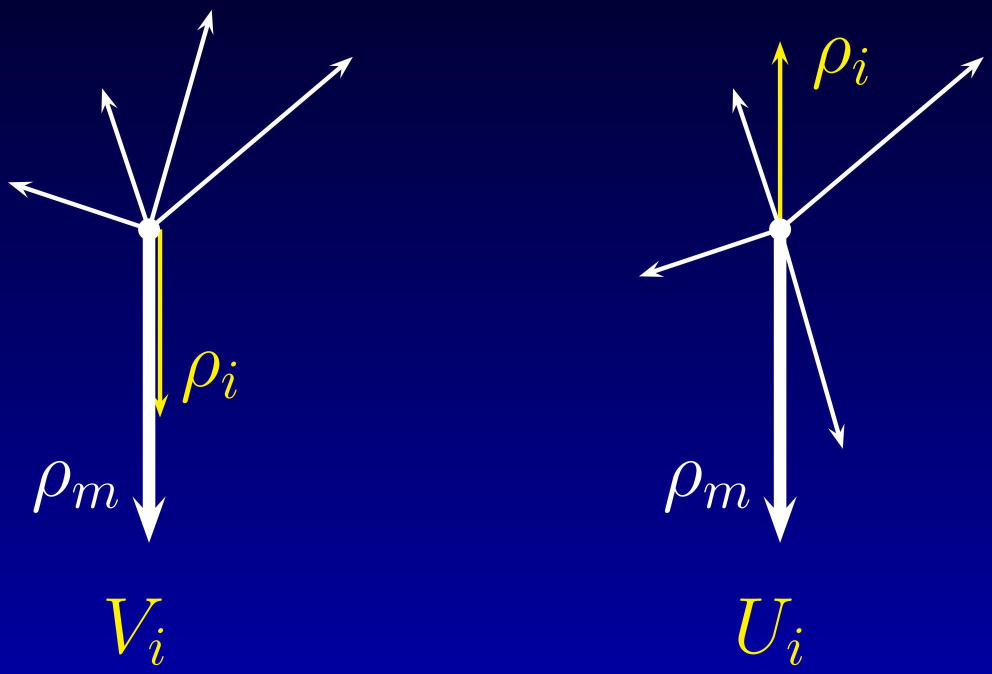


Cohomology classes



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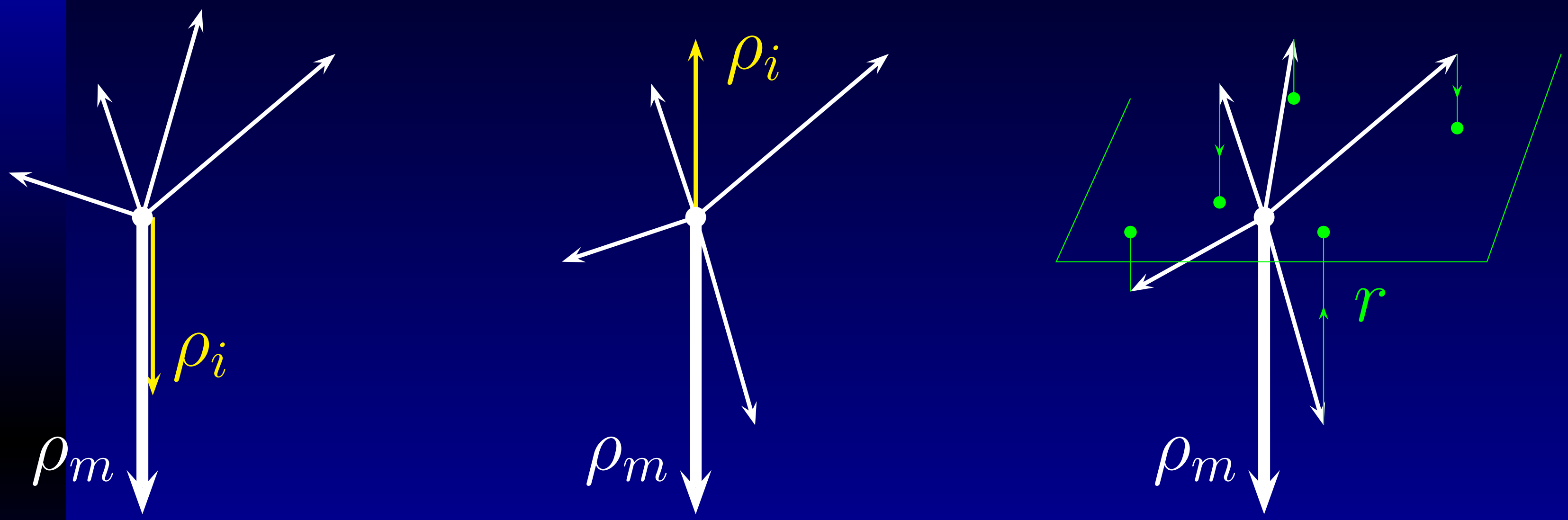
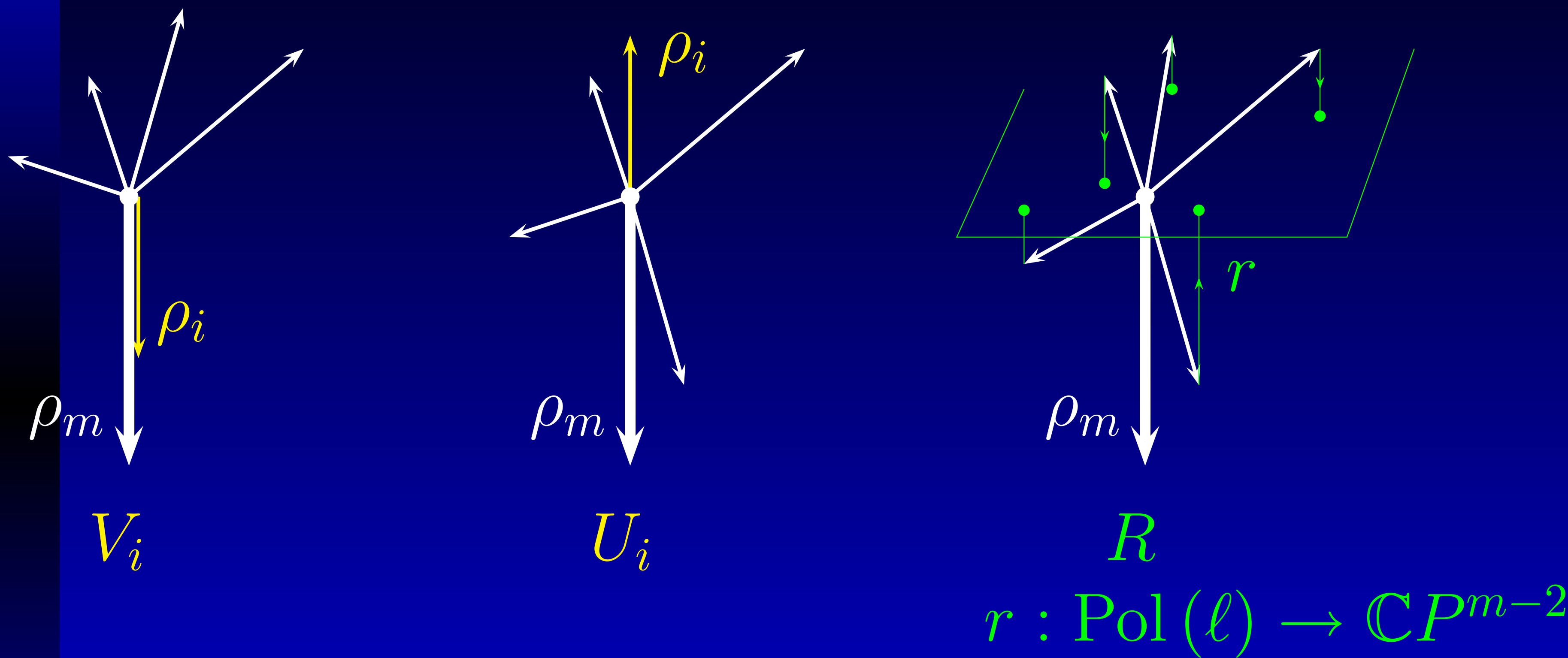


V_i U_i R

$r : \mathrm{Pol}\,(\ell) \to \mathbb{C}P^{m-2}$

Cohomology classes



$$U_i, V_i, R \in H^2(\mathrm{Pol}\,(\ell); \mathbb{Z})$$

$$U_i V_i = 0 \qquad\qquad R = \pm U_i \pm V_i$$

Symplectic toric manifolds

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Let $T = (S^1)^n$ acting Hamiltonianly on a symplectic manifold M^{2n}, with moment map $\Phi : M \to \mathfrak{t}^* = \mathbb{R}^n$.

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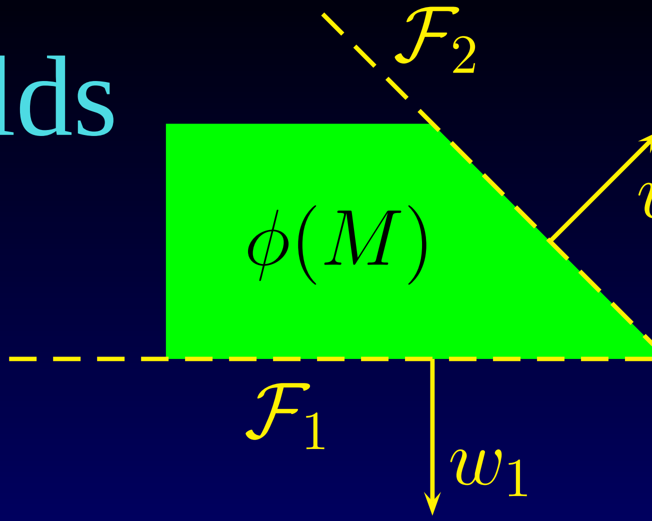
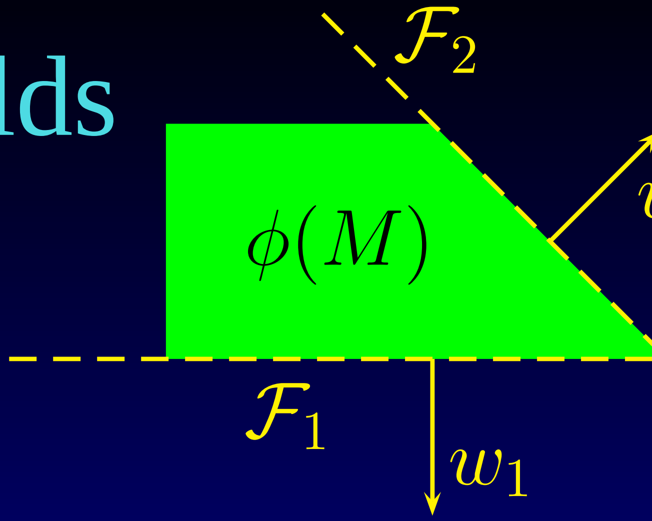


Symplectic toric manifolds



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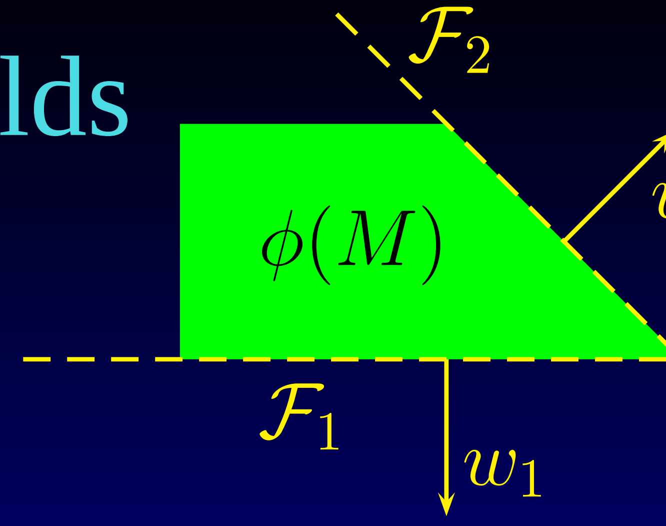
Write

$$\Phi(M) = \{x \in \mathbb{R}^n \mid \langle x, w_j \rangle \leq \lambda_j,\ j = 1 \dots, p\}$$

(convex polytope limited by the hyperplanes $\mathcal{F}_j = \{x \in \mathbb{R}^n \mid \langle x, w_j \rangle = \lambda_j\}$).

Symplectic toric manifolds



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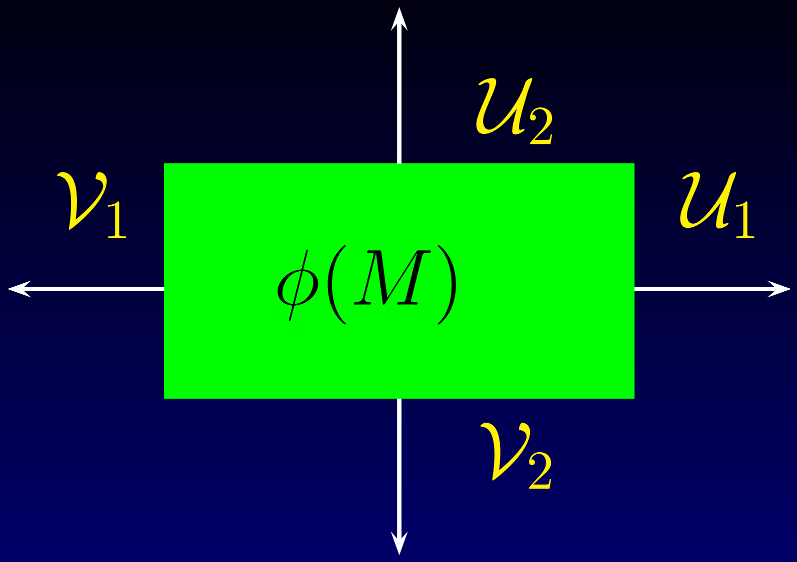
THEOREM (Danilov) $\quad H^*(M; \Lambda) = \Lambda[F_1, \ldots, F_p]/\mathcal{I}$

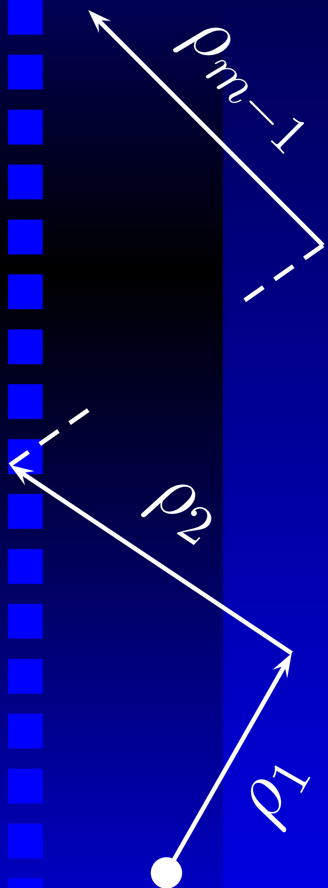
where $\mathcal{I}$ is the ideal generated by the families

- $\sum_{j=1}^{p} \langle e_i, w_j \rangle F_j \qquad (i = 1, \ldots, n)$
- $\prod_{j \in \mathcal{B}} F_j \qquad$ *if* $\bigcap_{j \in \mathcal{B}} \mathcal{F}_j = \emptyset$.

$$M = (S^2)^{m-1}$$

$$\Phi(M) = \prod_{i=1}^{m-1}[-\ell_i, \ell_i]$$





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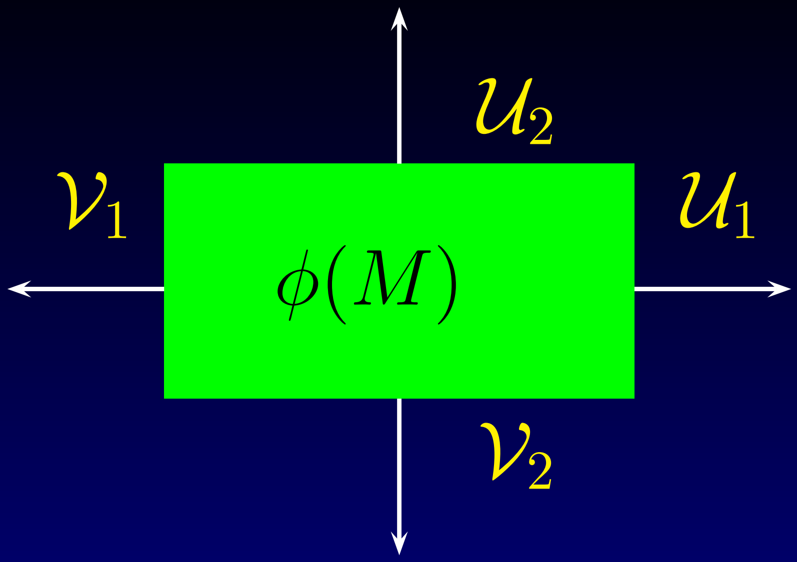

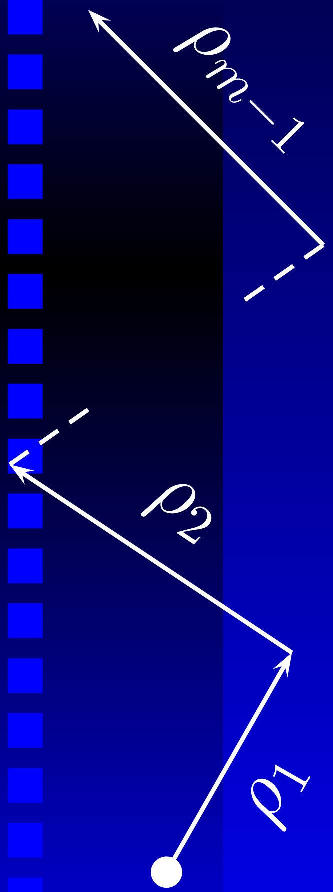

$H^*((S^2)^m; \Lambda) = \Lambda[V_i, U_i]$ quotiented by

- $U_i - V_i \qquad i = 1, \ldots, m-1$
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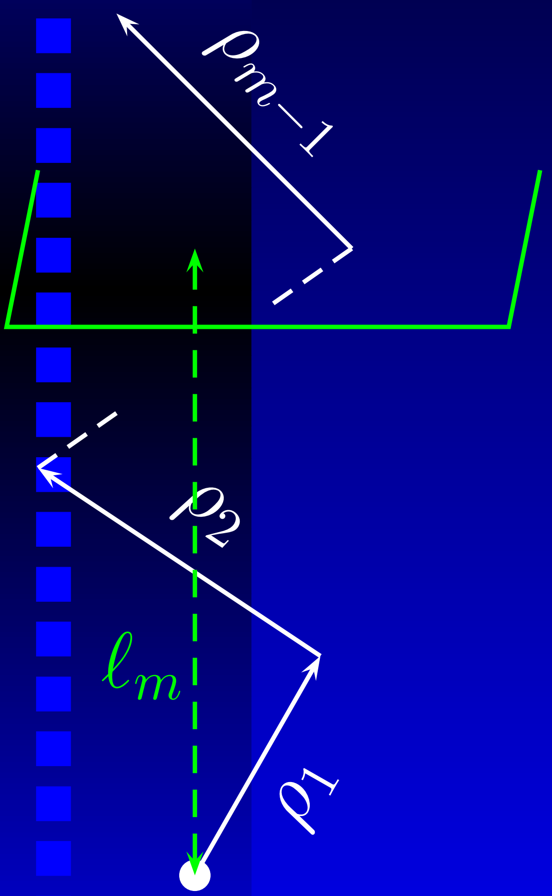
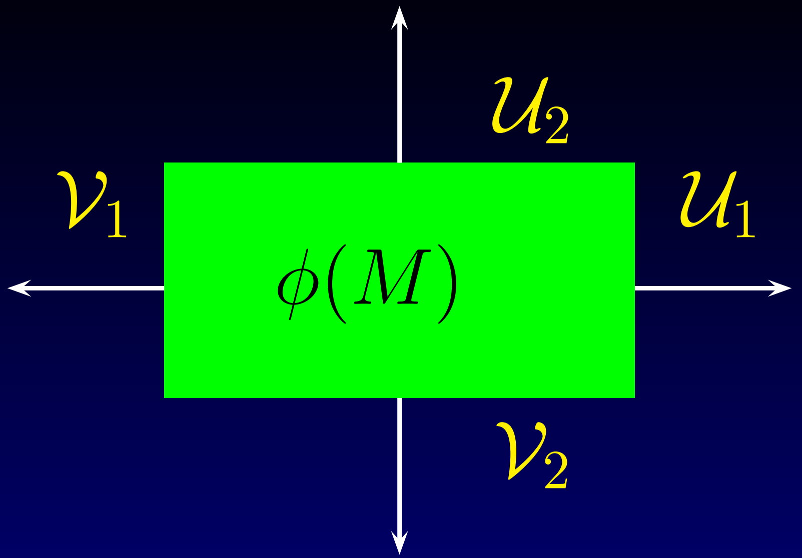
$M = \mathrm{UPol}\,(\ell)$

Upper polygon space





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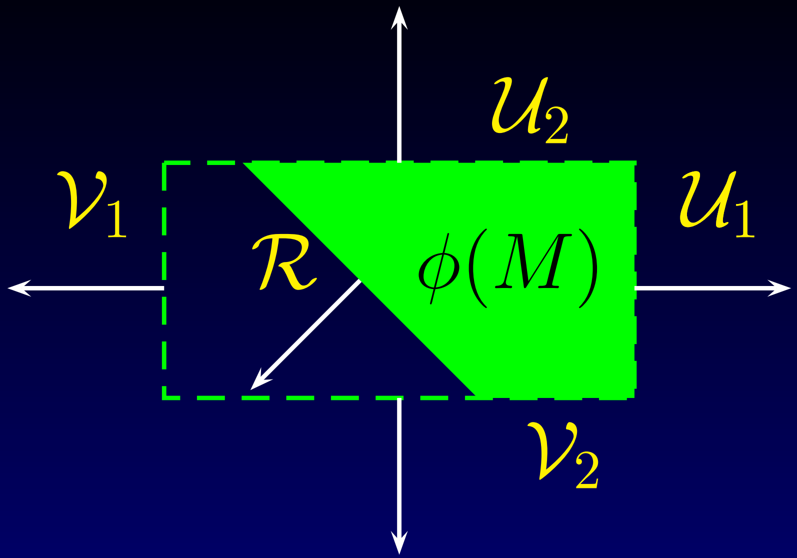

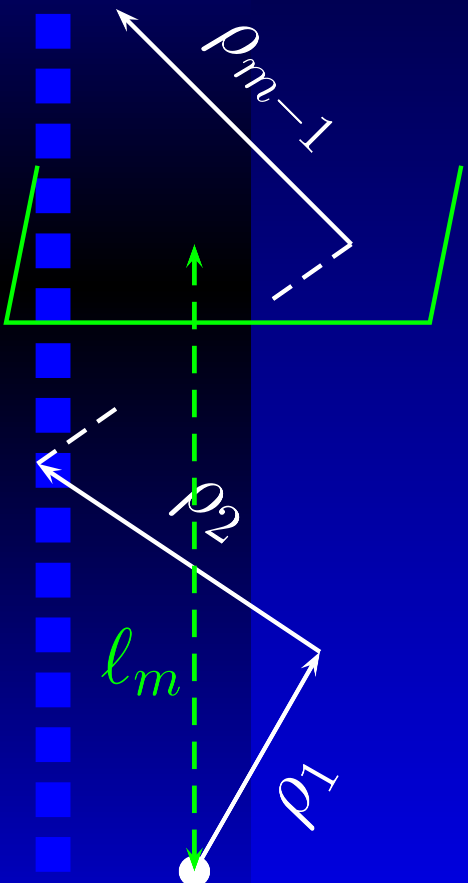

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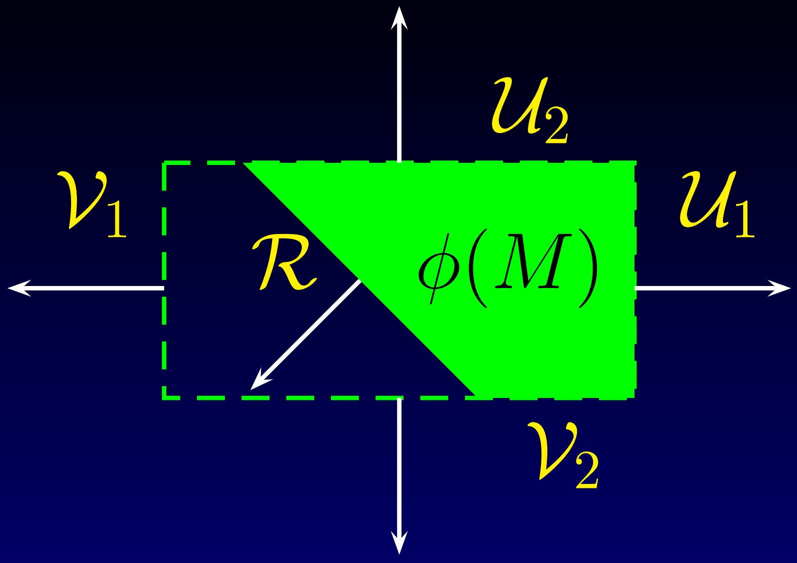

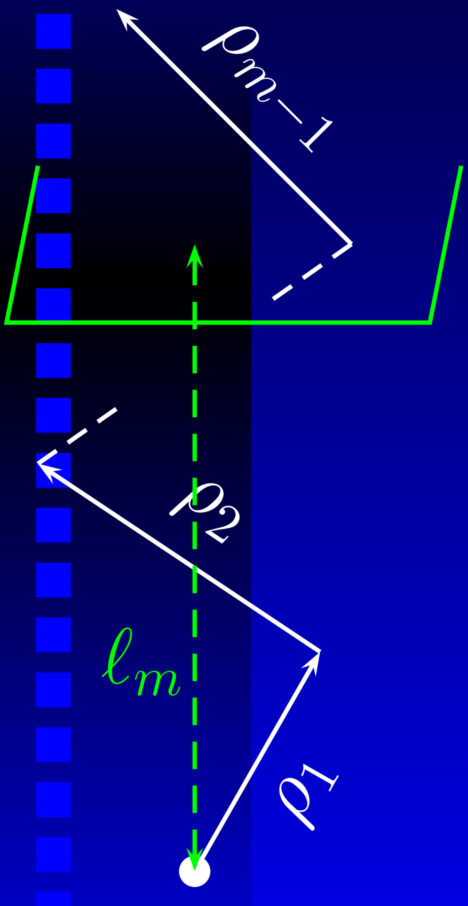

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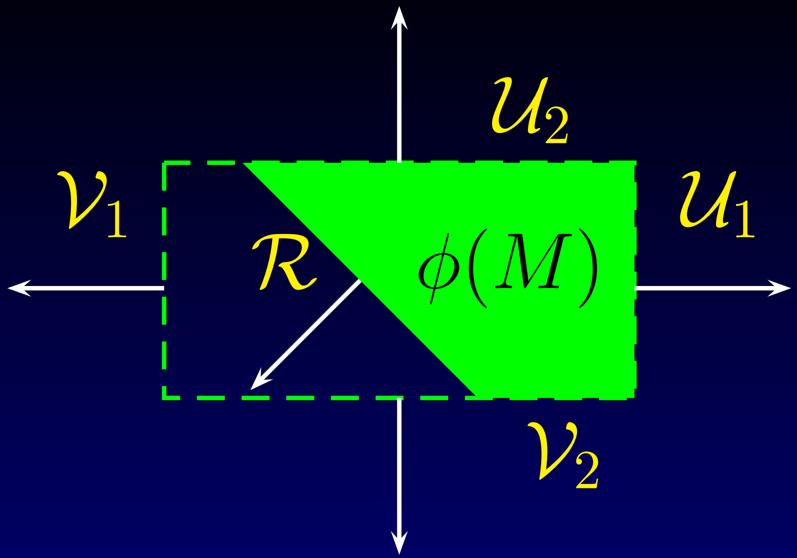

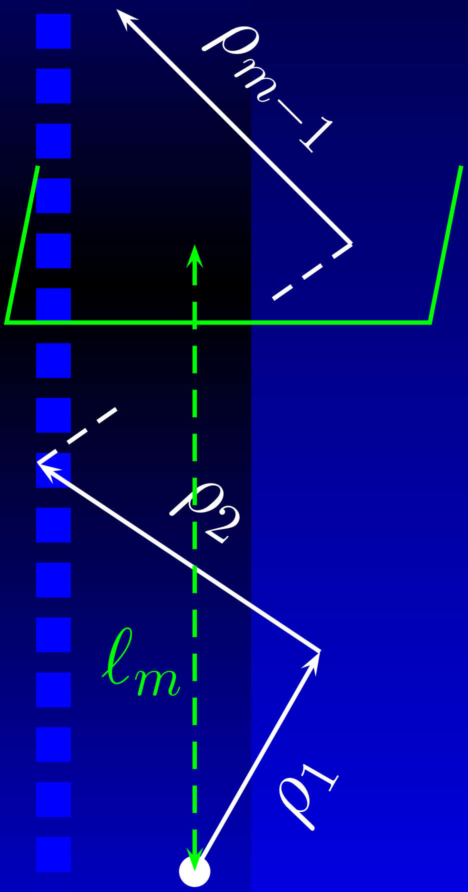

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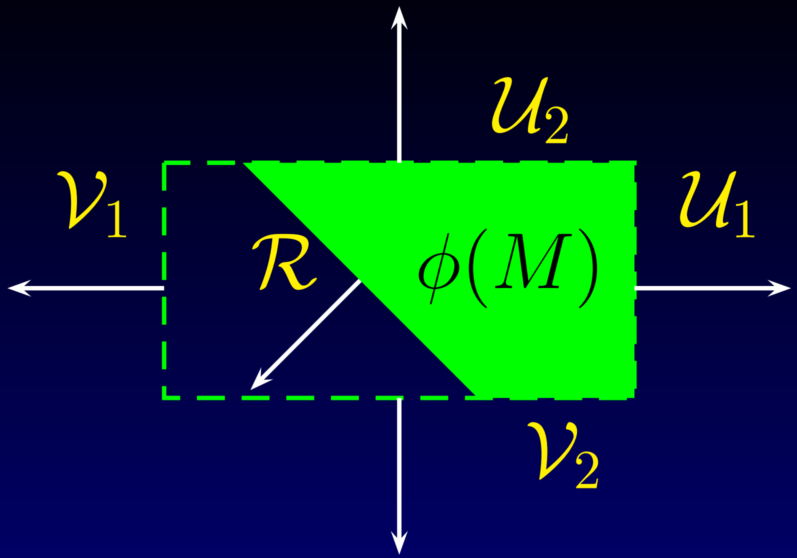

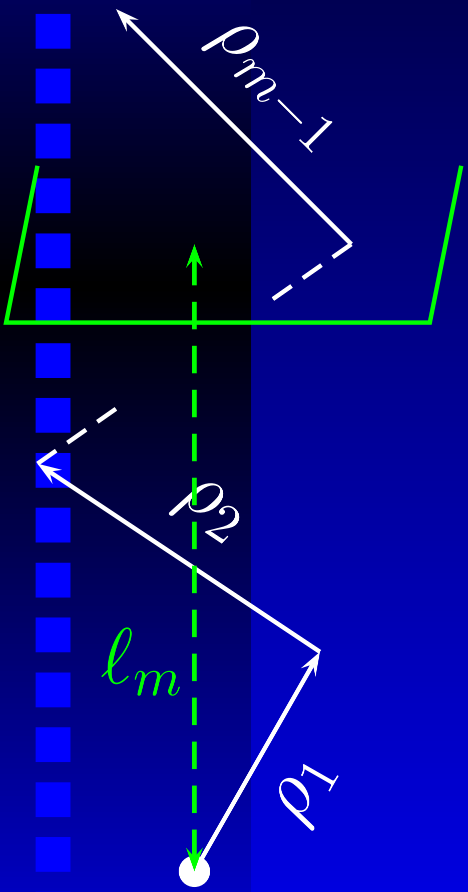

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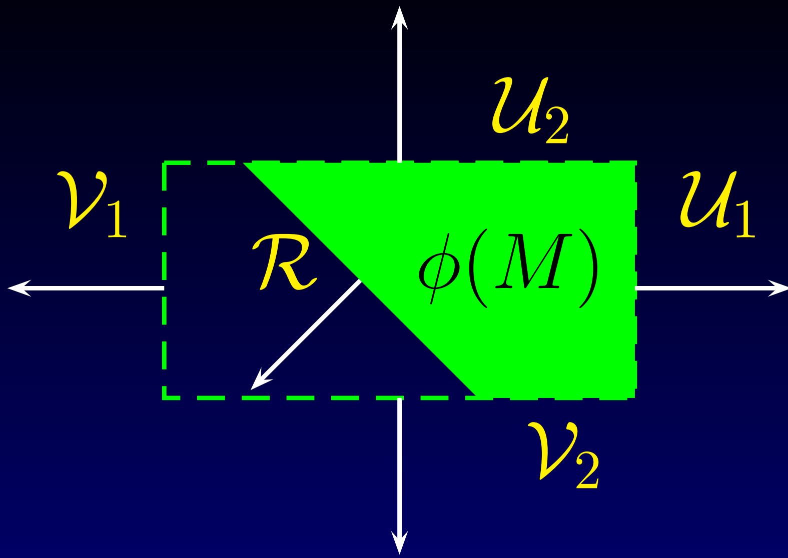

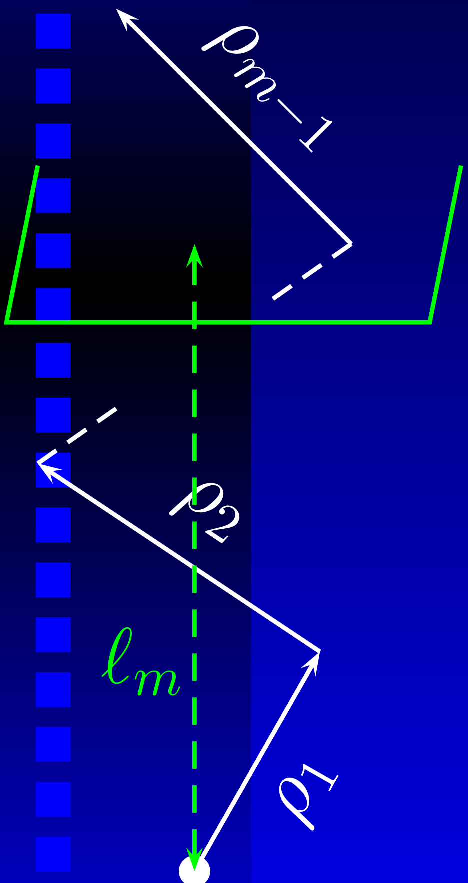

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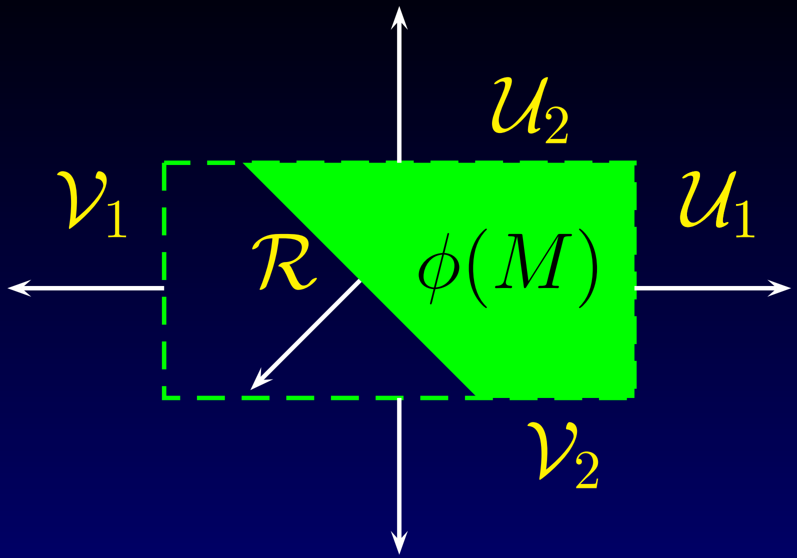

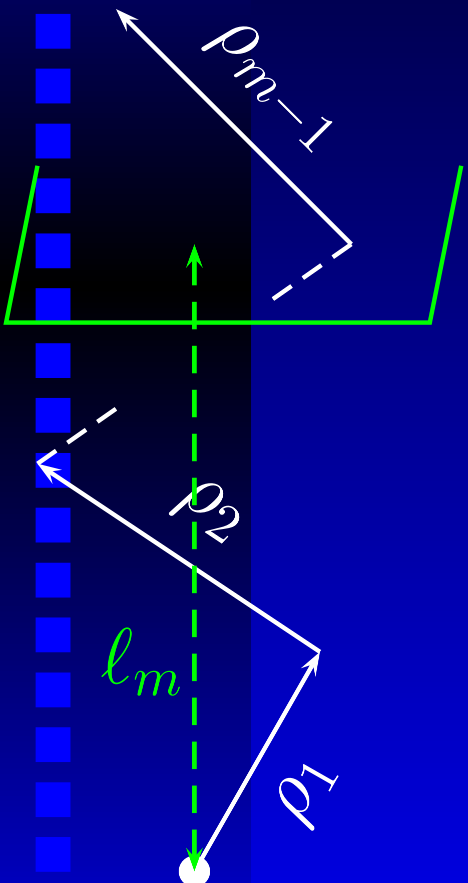

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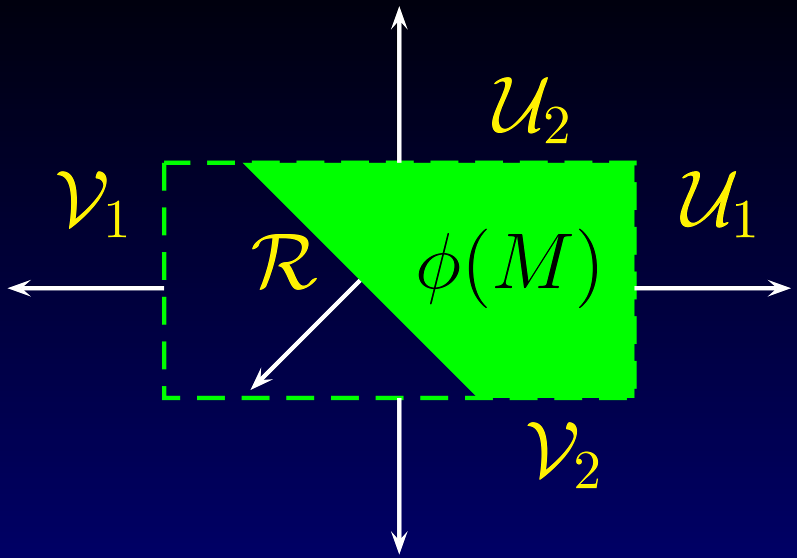

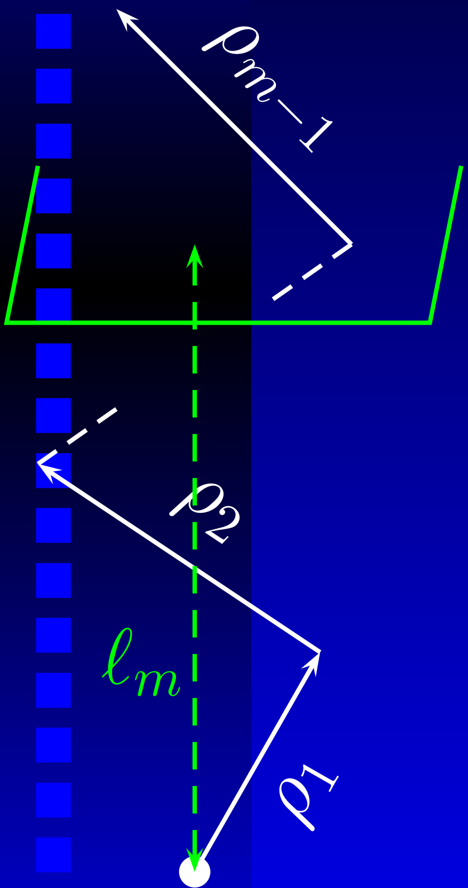

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  L long

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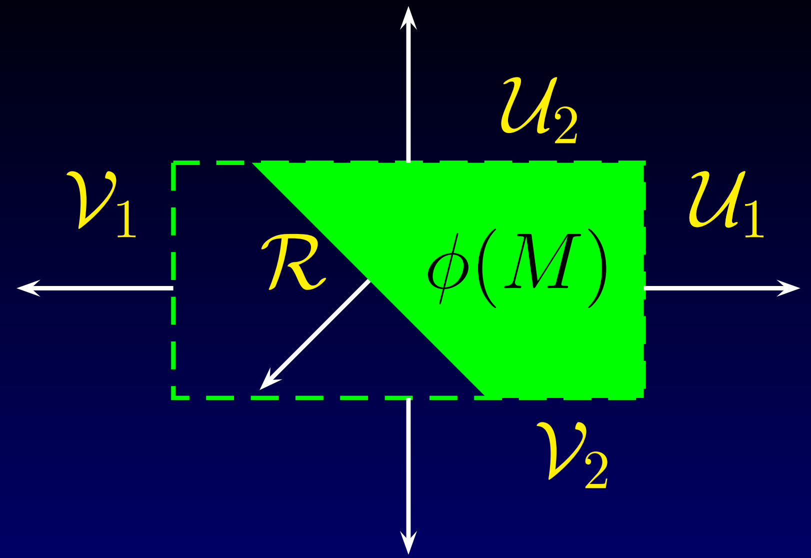

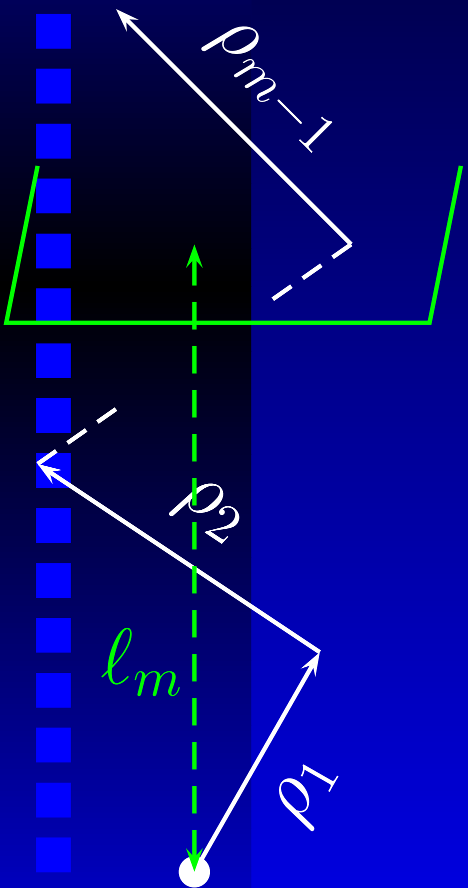

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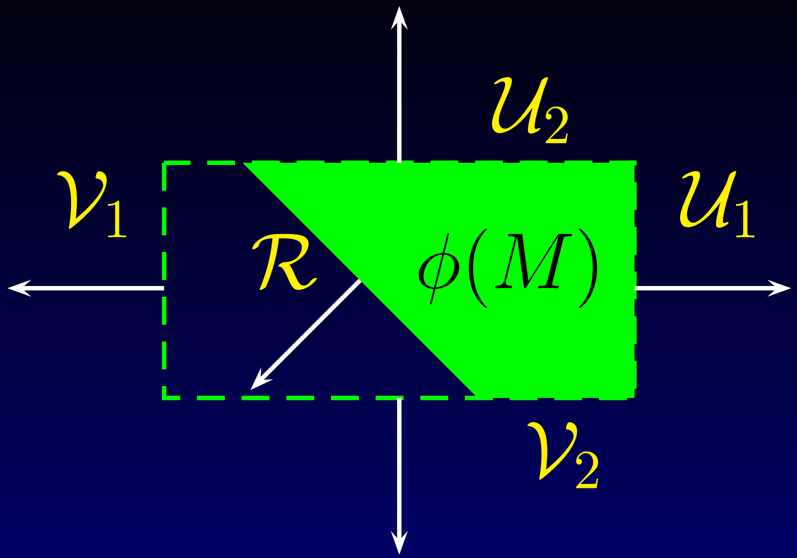

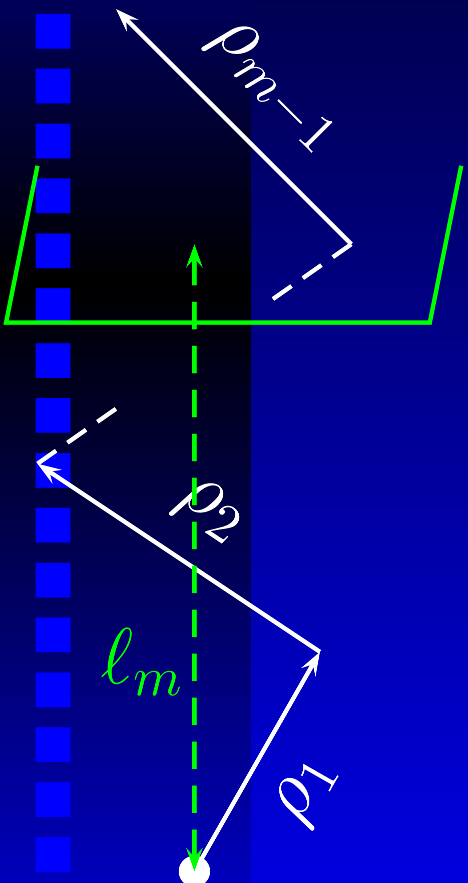

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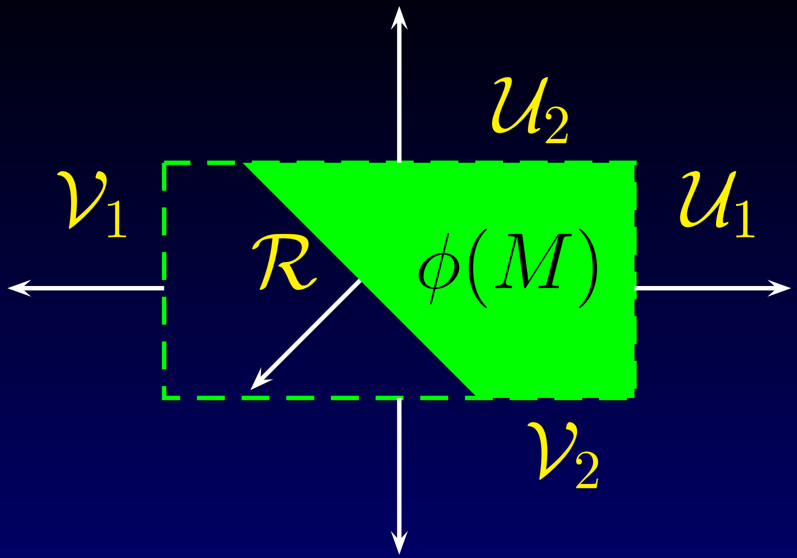

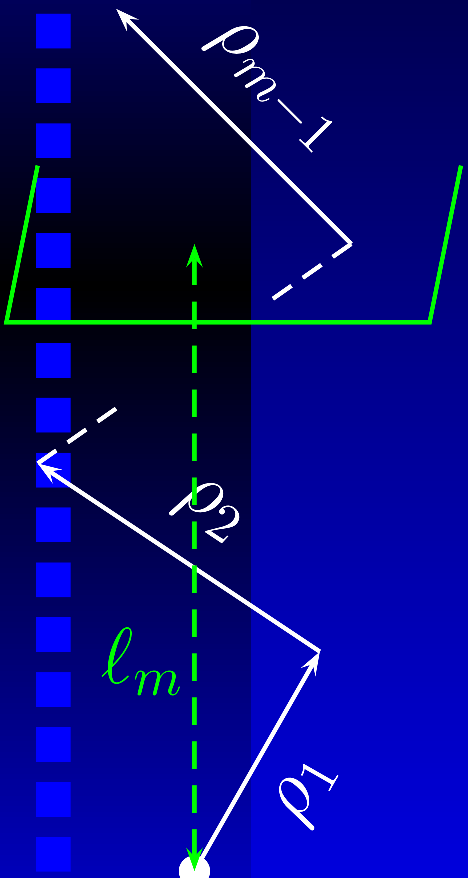

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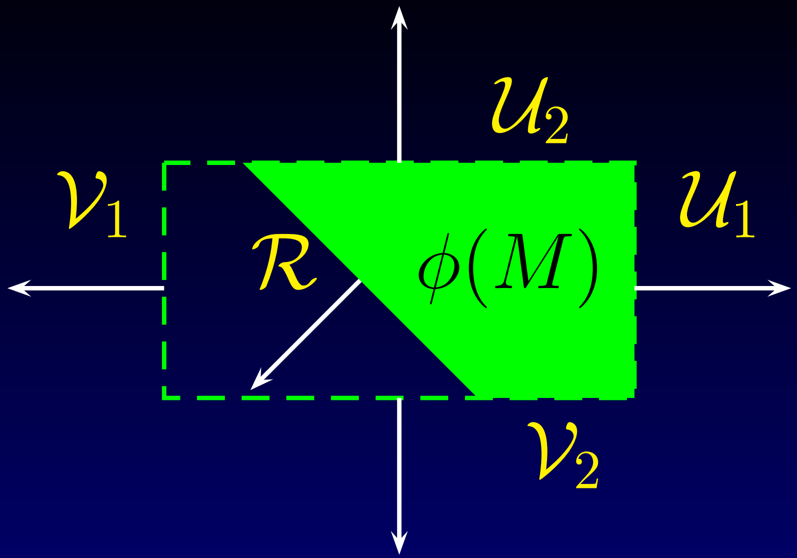

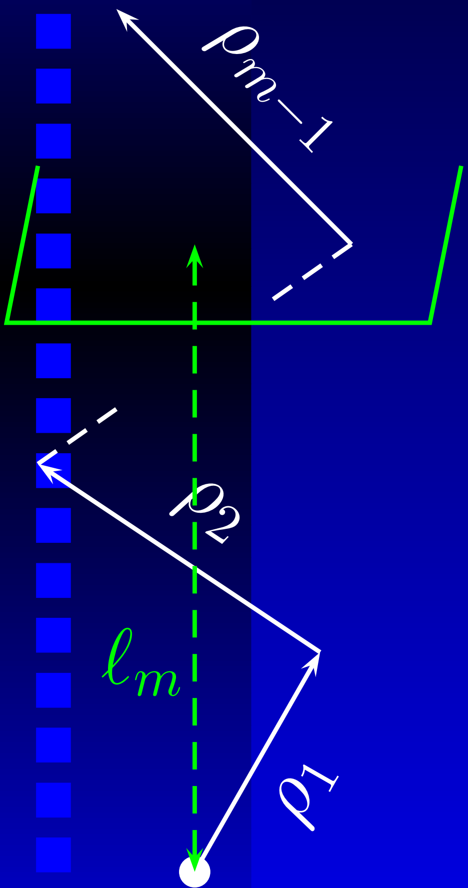

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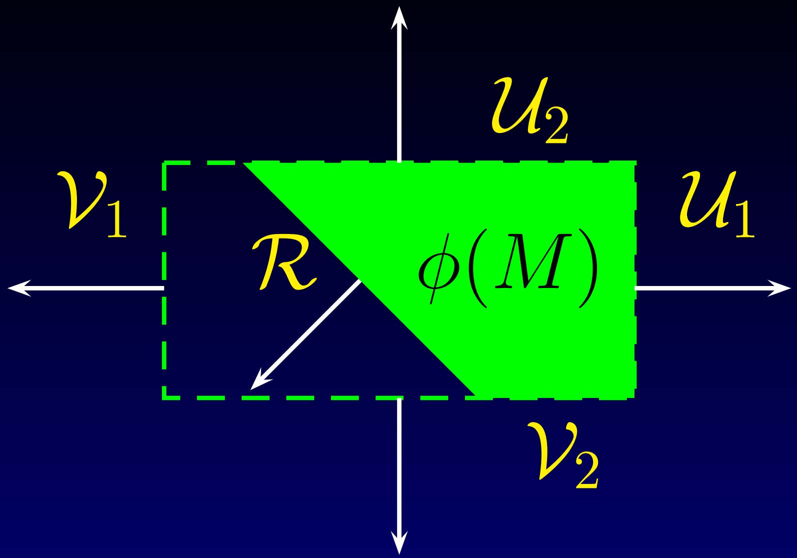

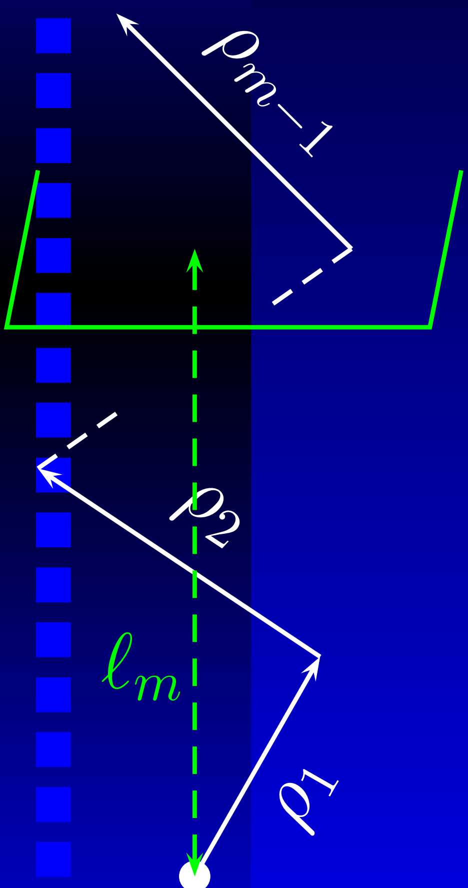

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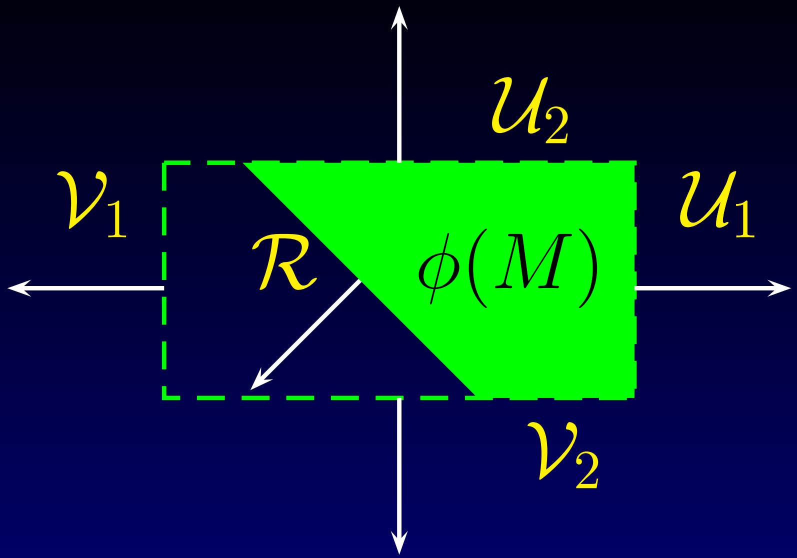

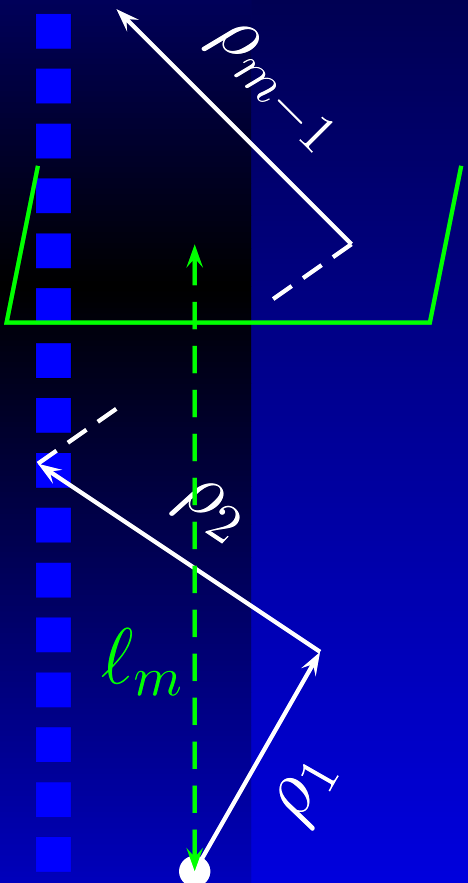

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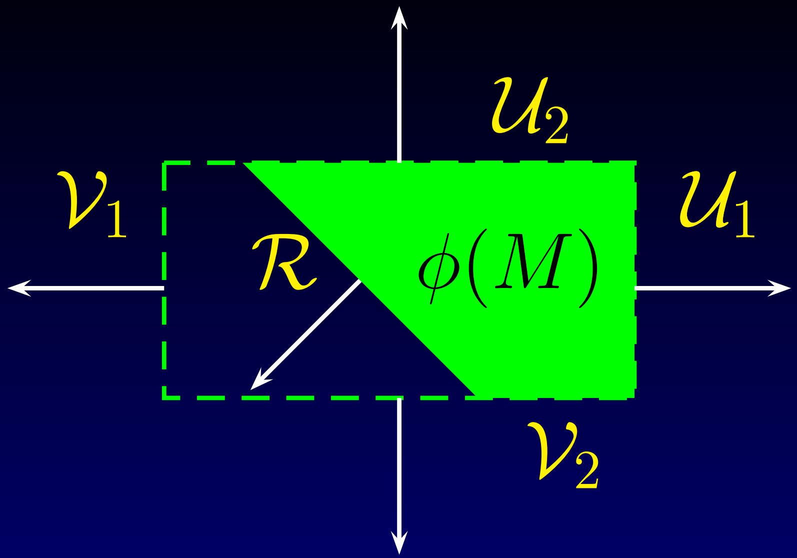
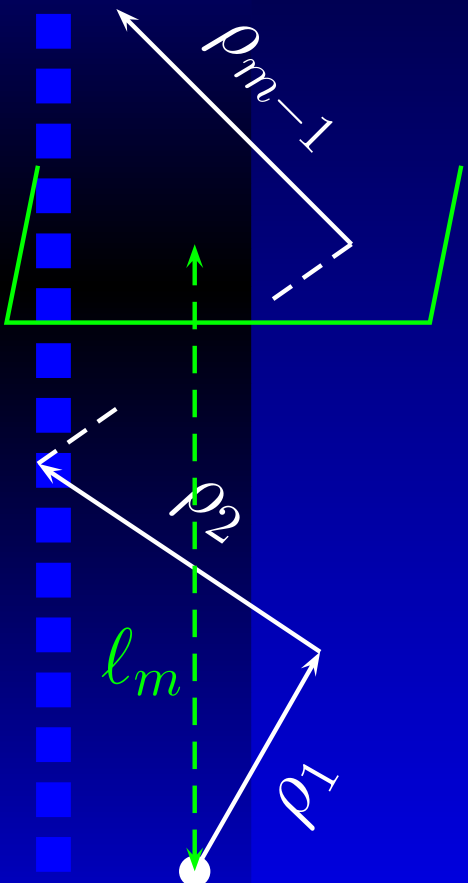
Upper polygon space





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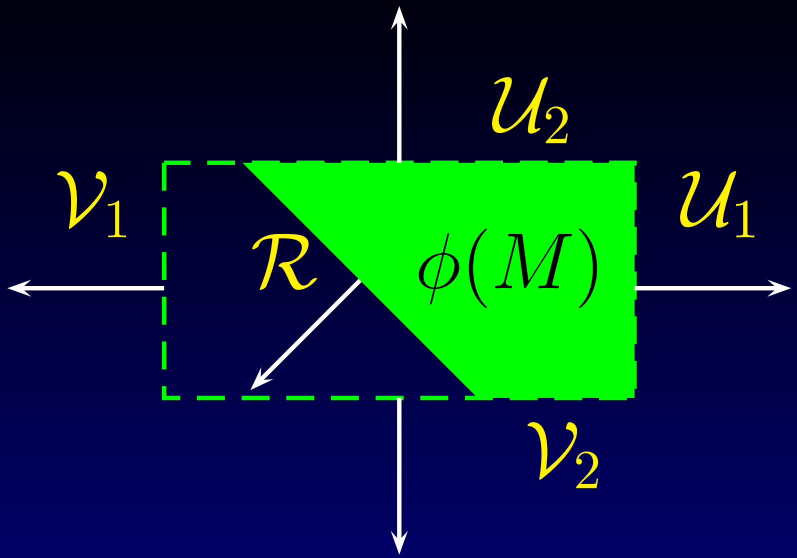

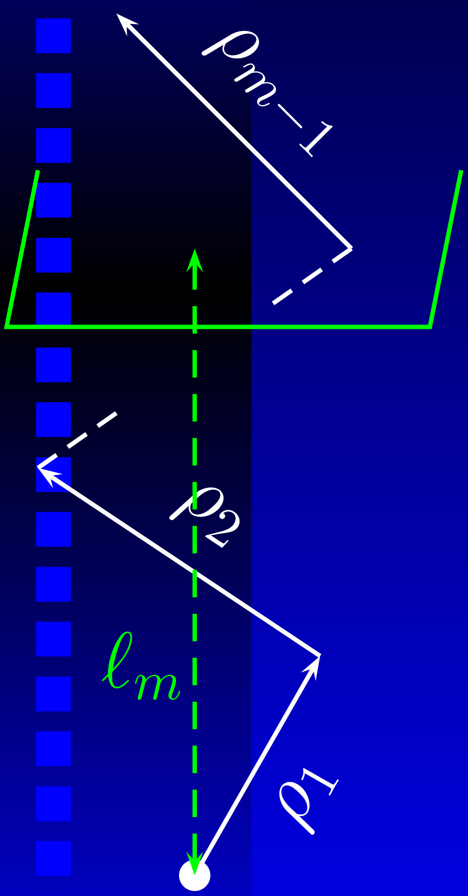

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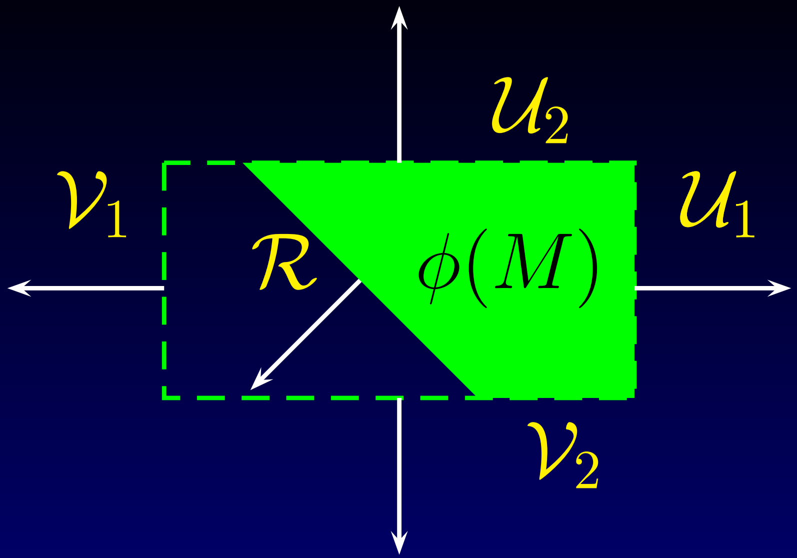
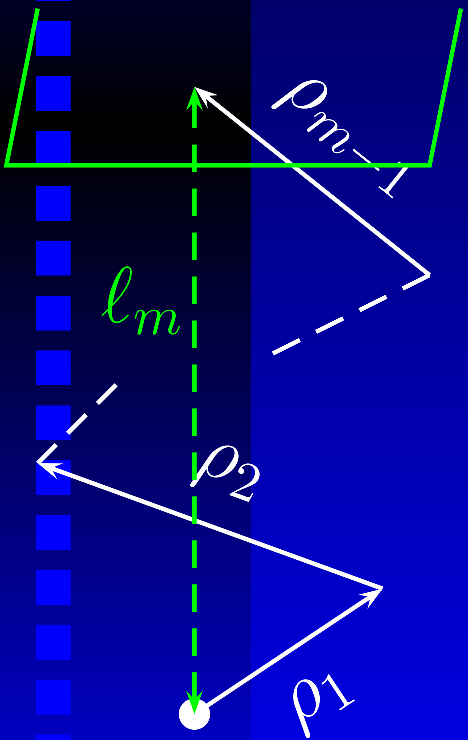
$M = \mathrm{Pol}\,(\ell)$

Polygon space





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THEOREM (JCH-A. Knutson, 1998)

$H^*(\mathrm{Pol}(\ell);\Lambda) = \Lambda[V_i, R]$ *quotiented by*

- $V_i(V_i + R)$ $\quad i = 1, \ldots, m-1$
- $\prod_{j\in B} V_j$ $\quad B \notin \mathcal{S}_m$
- $\displaystyle\sum_{\substack{S\subset L\\ S\in\mathcal{S}_m}} (R^{|L|-|S|-1}\prod_{j\in S} V_j)$ $\quad L \subset \{1, \ldots, m-1\},\ L$ *long*

THEOREM *Assume ℓ is dominated (i.e. $\ell_m = \max \ell_i$).*
$H^*(\mathrm{Pol}\,(\ell); \Lambda) = \Lambda[V_i, R]$ *quotiented by*

- $V_i(V_i + R)$ $\quad i = 1, \ldots, m-1$
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THEOREM *Assume ℓ is dominated (i.e. $\ell_m = \max \ell_i$).*

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$H^*(\mathrm{Pol}\,(\ell); \Lambda) = \Lambda[V_i, R]$ *quotiented by*

- $V_i(V_i + R)$ $\qquad i = 1, \ldots, m-1$
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(Face exterior algebra of $\mathcal{S}_m(\ell)$)

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$$\overset{\text{Gubeladze}}{\iff} \mathcal{S}_m(\ell) \approx \mathcal{S}_m(\ell')$$

$$\implies \mathrm{Pol}\,(\ell) \approx \mathrm{Pol}\,(\ell')$$