Cohomology rings of symplectic cuts *

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Abstract

The "symplectic cut" construction [Le] produces two symplectic orbifolds C_- and C_+ from a symplectic manifold M with a Hamiltonian circle action. We compute the rational cohomology ring of C_+ in terms of those of M and C_-.

1 Statement of the results

Let M be a symplectic n-manifold endowed with an Hamiltonian S^1-action with moment map $f : M \to \mathbf{R}$. Suppose that 0 is a regular value of f and consider the manifolds-with-boundary

$$M_- := f^{-1}(\mathbf{R}_{\leq 0}) \quad , \quad M_+ := f^{-1}(\mathbf{R}_{\geq 0}) \quad , \quad M_0 := f^{-1}(0) = M_- \cap M_+ .$$

The symplectic cutting of (M, f) at 0, introduced by E. Lerman [Le], can be described as follows. The S^1-action restricted to M_0 gives rise to the equivalence relation on M:

$$x \sim y \iff \begin{cases} x \in M_0 \quad \text{and} \quad y = \theta \cdot x \text{ for some } \theta \in S^1 \\ \quad\quad \text{or} \\ x = y \quad \text{if } x \notin M_0. \end{cases}$$

As 0 is a regular value of f, the stabilizers of points in M_0 are finite cyclic subgroups of S^1. Therefore, the following quotients are orbifolds:

$$C_- := M_-/\sim \quad , \quad C_+ := M_+/\sim \quad , \quad C_0 := M_0/S^1$$

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Lerman proved that there are unique symplectic forms on these orbifolds so that C_0 is the symplectic reduction of M at 0, the inclusions $C_0 \subset C_\pm$ are symplectic and the diffeomorphisms $M_\pm - M_0 \to C_\pm - C_0$ are symplectomorphisms. The symplectic orbifolds C_+ and C_- are called the *symplectic cuts* of M at 0.

In this paper, we compute the rational cohomology rings $H^*(C_+; \mathbf{Q})$ and $H^*(C_0; \mathbf{Q})$ in terms of those of M and C_-. In what follows, the notation $H^*(-)$ always means $H^*(-; \mathbf{Q})$.

We shall consider the kernel of the group homomorphism

$$H^*(C_-) \oplus H^*(M) \stackrel{p^*-i^*}{\longrightarrow} H^*(M_-)$$

(we denote by $p : M_\pm \to C_\pm$ the various projections and by i any inclusion between spaces). Note that $\ker(p^* - i^*)$ is a subring of $H^*(C_-) \oplus H^*(M)$. Consider the composed homomorphsim:

$$\begin{array}{rcll} \delta : H^*(M_-, M_0) & \stackrel{\text{diag}}{\longrightarrow} & H^*(M_-, M_0) \oplus H^*(M_-, M_0) & \stackrel{\cong}{\longrightarrow} \\ & \stackrel{\cong}{\longrightarrow} & H^*(C_-, C_0) \oplus H^*(M, M_+) & \longrightarrow \\ & \longrightarrow & H^*(C_-) \oplus H^*(M). & \end{array}$$

Observe that the image of δ is an ideal in $\ker(p^* - i^*)$. Our main result is

1.1 Main Theorem *The cohomology ring $H^*(C_+) = H^*(C_+; \mathbf{Q})$ is isomorphic, as a graded ring, to $\ker(p^* - i^*)/Image\,(\delta)$.*

We now mention a few side results. First, the cohomology ring of the symplectic reduction C_0 may be then obtained from that of C_- in the following way:

Proposition 1.2 *The cohomology ring $H^*(C_0)$ is the quotient of $H^*(C_-)$ by the annihilator of the Poincaré dual of the suborbifold C_0. One has a short exact sequence ($n = \dim M$):*

$$0 \to H_{n-*}(M_-; \mathbf{Q}) \to H^*(C_-) \to H^*(C_0) \to 0.$$

It is also worth noting that the cohomology ring of M_- may be obtained from those of C_- and C_0. Let $i_! : H^{*-2}(C_0) \to H^*(C_-)$ be the push-forward homomorphism.

Proposition 1.3 *There is a short exact sequence*

$$0 \to H^{*-2}(C_0) \stackrel{i_!}{\longrightarrow} H^*(C_-) \stackrel{p^*}{\longrightarrow} H^*(M_-) \to 0.$$

As in [HK, Proposition 3.2] one can use Propositions 1.2 and 1.3 to get the Poincaré polynomials of C_- and C_0 in terms of that of M_-:

Corollary 1.4 *The Poincaré polynomials for the rational cohomology of C_- and C_0 are given by the equations*

$$\begin{aligned}(1-t^2)P_{C_0}(t) &= P_{M_-}(t) - t^n P_{M_-}(1/t)\\ (1-t^2)P_{C_-}(t) &= P_{M_-}(t) - t^{n+2} P_{M_-}(1/t).\end{aligned}$$

Remarks :

1.5 It seems likely that the results of this paper are true for the cohomology with coefficients any ring in which the orders of the finite stabilizer groups are invertible. The literature on cohomology of orbifolds being still to be developed, we have followed, in order to keep the paper focused on the problem at hand, the lazy trend of using rational coefficients. (See also example 3.1).

1.6 If M admits a Hamiltonian action of a compact Lie group G which commutes with that of S^1 (equivalently: f is G-invariant), the the main theorem is also true for the equivariant cohomology $H^*_G(C_+;\mathbf{Q})$. The proof is the same, using the G-equivariant perfection of the Morse-Bott function f (well known by experts). Note that the circle S^1 itself has this property.

1.7 While we were pursuing this result, the paper [IP] appeared, which calculates some of the *quantum* cohomology of M in terms of that of M_+, M_-, C_0, and C_0's normal bundle in C_+. Since the quantum cohomology subsumes the ordinary cohomology, it would be interesting to see how our result is a consequence of those of [IP].

1.8 Since a symplectic cut can be defined as a symplectic reduction, and there are now general techniques for computing the cohomology of symplectic quotients (such as [JK] for cohomology pairings or [TW] for generators and relations), the question posed in this paper has other answers. However, those answers are in terms of other ingredients (like fixed point data in the other cut space, rather than its cohomology), so we believe that our result remains useful; in addition, the proofs are short and elementary (without e.g. heavy use of equivariant cohomology).

2 Proofs

Proof of the main theorem. Let C denote the auxiliary space $C := M/\sim = C_- \cup C_+$. Its cohomology will be calculated below. We will then make use of the maps $M\dot{\cup}C_- \to C$ and $C_+ \to C$ whose induced maps in cohomology will turn out to be respectively injective and surjective.

The long exact sequences of the pairs (C, C_-) and (M, M_-) are linked by the commutative diagram

$$\begin{array}{ccccccccc} \to & H^*(C,C_-) & \to & H^*(C) & \stackrel{i^*_-}{\longrightarrow} & H^*(C_-) & \to & H^{*+1}(C,C_-) \to \\ & \downarrow\cong & & \downarrow p^* & & \downarrow i^*_- & & \downarrow\cong \\ \to & H^*(M,M_-) & \to & H^*(M) & \stackrel{i^*_-}{\longrightarrow} & H^*(M_-) & \to & H^{*+1}(M,M_-) \to \end{array} \tag{1}$$

This produces a long exact sequence

$$\to H^{*-1}(M_-) \stackrel{\partial}{\longrightarrow} H^*(C) \stackrel{(i^*_-,p^*)}{\longrightarrow} H^*(C_-) \oplus H^*(M) \stackrel{p^*_- - i^*_-}{\longrightarrow} H^*(M_-) \to \tag{2}$$

where the connecting homomorphism ∂ is obtained using the vertical isomorphisms

$$\partial : H^{*-1}(M_-) \to H^*(M, M_-) \stackrel{\cong}{\longrightarrow} H^*(C, C_-) \to H^*(C).$$

The moment map $f : M \to \mathbf{R}$ is a *perfect* Morse-Bott function (See [Fr] when M is Kähler; as observed by Atiyah (*Bull. London Math. Soc.* 14 (1982) p.7) Frankel's proof works as well when M symplectic). This implies that $i^* : H^*(M) \to H^*(M_-)$ is surjective. Therefore $\partial \equiv 0$ and exact sequence (2) breaks:

$$0 \to H^*(C) \stackrel{(i^*_-,p^*)}{\longrightarrow} H^*(C_-) \oplus H^*(M) \stackrel{p^*_- - i^*_+}{\longrightarrow} H^*(M_-) \to 0. \tag{3}$$

This first map is a ring homomorphism, induced from the map $M\dot{\cup}C_- \to C$, locating $H^*(C)$ as a subring inside the known ring $H^*(M) \oplus H^*(C_-)$.

Another consequence of the surjectivity of $i^* : H^*(M) \to H^*(M_-)$ is, using diagram (1), that $i^* : H^*(C) \to H^*(C_-)$ is also onto. We use the symmetric statement with C_+ and get the short exact sequence:

$$0 \to H^*(C, C_+) \to H^*(C) \to H^*(C_+) \to 0. \tag{4}$$

Using the diagram

$$
\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \downarrow & & \downarrow & \\
0 \to & H^*(C, C_+) & \stackrel{(i_-^*, p^*)}{\longrightarrow} & H^*(C_-, C_0) \oplus H^*(M, M_+) & \\
 & \downarrow & & \downarrow & \\
0 \to & H^*(C) & \stackrel{(i_-^*, p^*)}{\longrightarrow} & H^*(C_-) \oplus H^*(M) &
\end{array} \qquad (5)
$$

and (4) one sees that

$$H^*(C_+) \cong H^*(C)/H^*(C, C_+) \cong \ker(p^* - i^*)/Image(\delta).$$

which proves our main theorem. □

Proof of Proposition 1.2: It is already established in (4) that the cohomology exact sequence of the pair C_-, C_0 breaks:

$$0 \to H^*(C_-, C_0) \to H^*(C_-) \to H^*(C_0) \to 0.$$

By excision and Poincaré duality,

$$H^*(C_-, C_0) \cong H^*(M_-, M_0) \cong H_{n-*}(M_-).$$

The fact that $\ker(H^*(C_-) \to H^*(C_0))$ is the annihilator of the Poincaré dual of C_0 is proved as in [HK, Proposition 3.3]. The argument there requires that $i_* : H_*(C_0) \to H_*(C_-)$ is injective. This is guaranteed by the perfection of the moment map $\bar{f} : C_- \to \mathbf{R}$ for the induced S^1-action on C_-. □

Proof of Proposition 1.3: Observe that we know from (4) that $p^* : H^*(C_-) \to H^*(M_-)$ is onto. But Proposition 1.3 actually comes from the following exact diagram:

$$
\begin{array}{ccccccc}
 & H^{*-2}(C_0) & \stackrel{i_!}{\longrightarrow} & H^*(C_-) & \stackrel{p^*}{\longrightarrow} & H^*(M_-) & \\
 & & & & & \cong \downarrow \cap[M_-] & \\
 & \cong \downarrow \cap[C_0] & & \cong \downarrow \cap[C_-] & & H_{n-*}(M_-, M_0) & \\
 & & & & & \uparrow \cong & \\
0 \to & H_{n-*}(C_0) & \stackrel{i_*}{\longrightarrow} & H_{n-*}(C_-) & \longrightarrow & H_{n-*}(C_-, C_0) & \to \ 0.
\end{array}
$$

The two zeros on the bottom line come from the perfection of the moment map $\bar{f} : C_- \to \mathbf{R}$ for the induced S^1-action on C_-. □

3 Examples

3.1 Suppose that the only critical point of f in M_- is an isolated minimum u. Then $M_- = D^{2m}$, C_0 and C_- are weighted projective spaces and C_+ is a "weighted blowup" of M (at u).

The relative cohomology $H^*(M, M_-; \mathbf{Z})$ is generated by the fundamental class $v_- \in H^n(M_-, M_0)$ of M_-. Let $\delta(v) = (v_-, v)$. We deduce from Corollary 1.4 that C_- and C_0 have the Poincaré polynomial of $\mathbf{C}P^m$ and $\mathbf{C}P^{m-1}$ respectively. Since (to the top degree) all powers of their symplectic forms are nonzero, the rings must be $H^*(M_-; \mathbf{Q}) = \mathbf{Q}[a]/(a^{m+1})$ and $H^*(M_-; \mathbf{Q}) = \mathbf{Q}[a]/(a^m)$. Passing from $\mathbf{Q}$ to $\mathbf{R}$ for the coefficients, $a \in H^2(C_-; \mathbf{R})$ may be chosen so that $a^m = v_-$. Our main theorem then states that $H^*(C_+; \mathbf{R})$ is the subring of

$$\left(\mathbf{R}[a]/(a^{m+1}) \oplus H^*(M; \mathbf{R})\right) \Big/ (a^m + v)$$

consisting of $\mathbf{R}(1,1)$ in degree 0, and everything in higher degree. The above statement is presumably true with coefficients $\mathbf{Z}[w^{-1/m}]$ where w is the product of the weights of the linear action of S^1 on $T_u M$.

3.2 Let M be the space of Hermitian 3×3 matrices with eigenvalues $-1, 1, 2$. This is a flag manifold, and acquires a symplectic form through identification with a coadjoint orbit of $U(3)$. Let f be the (1,1)-matrix entry. Its Hamiltonian flow is periodic, being the conjugation by $\mathrm{diag}(e^{i\theta}, 1, 1)$. The critical values of f are again $-1, 1, 2$, each with a $\mathbf{C}P^1$ as critical set. Let A be the minimum level set of f.

When we cut at $f = 0$, we produce $M_- \sim A$ and $C_- \cong A \times \mathbf{C}P^2$. The other cut C_+ is just a blowup of M along A, and is in fact a Bott-Samelson manifold [BS].

The cohomology ring $H^*(M) = H^*(M; \mathbf{Z})$ has as a $\mathbf{Z}$-basis the Schubert classes (indexed by the Weyl group Sym_3) [Fu]:

- degree 0 : $\sigma_{13} = 1$
- degree 2 : σ_{123}, σ_{132}
- degree 4 : σ_{12}, σ_{23}
- degree 6 : σ_1

Schubert calculus gives us the relators:

- $\sigma_{123}\sigma_{132} = \sigma_{12} + \sigma_{23}$,

– $\sigma_{123}^2 = \sigma_{23}$ and $\sigma_{132}^2 = \sigma_{12}$,
– $\sigma_{123}\sigma_{12} = \sigma_{132}\sigma_{23} = \sigma_1$ and $\sigma_{123}\sigma_{23} = \sigma_{132}\sigma_{12} = 0$.

Setting $u := \sigma_{123}$ and $v := \sigma_{132}$, one gets the presentation

$$H^*(M) \cong \mathbf{Z}[u,v]/(uv = u^2 + v^2\,,\ u^3 = v^3 = 0\,,\ u^2v = uv^2\,,\ u^2v^2 = 0)$$

Let $\bar{a}$ be the generator of $H^*(M_-) = \mathbf{Z}[\bar{a}]/(\bar{a}^2 = 0)$. Set also $H^*(C_-) = \mathbf{Z}[a,b]/(a^2 = 0, b^3 = 0)$. The homomorphisms $p^* : H^*(C_-) \to H^*(M_-)$ and $i^* : H^*(M) \to H^*(M_-)$ are given by

$$p^*(a) = \bar{a}\,,\ p^*(b) = 0\,,\ i^*(u) = \bar{a}\,,\ i^*(v) = 0.$$

Setting $x := (u,a)$, $y := (v,0)$ and $z := (0,b)$, the ring $H^*(C)$ is $\mathbf{Z}[x,y,z]$ modulo

– $xy = x^2 + y^2$, $yz = 0$
– $x^2y = xy^2$ and $x^3 = y^3 = z^3 = 0$.

The image of $\delta : H^*(M_-, M_0) \to H^*(C_-) \oplus H^*(M)$ is generated by (v^2, b^2) and (uv^2, ab^2). Therefore, by our main theorem, the ring $H^*(M_+)$ is the quotient of $\mathbf{Z}[x,y,z]$ by the following relations:

– $xy = x^2 + y^2$, $yz = 0$ and $y^2 = -z^2$
– $x^2y = xy^2 = -xz^2$ and $x^3 = y^3 = z^3 = 0$.

In this example, the cohomology ring $H^*(M_+)$ is known [BS, p. 993]:

$$H^*(M_+) = \mathbf{Z}[a,b,c]/(a^2, b(b+a), c(c-a+b)).$$

The isomorphism with our presentation is given by $a \mapsto y + z$, $b \mapsto -z$ and $c \mapsto x + z$.

References

[BS] Bott R. & Samelson H. Applications of the theory of Morse to symmetric spaces. *Amer. J. Math.* 80 (1958) 964–1029.

[Fr] Frankel T. Fixed points and torsion on Kähler manifolds. *Ann. of Math.* 70 (1959) 1–8.

[Fu] Fulton, W. Young tableaux. *London Mathematical Society Student Texts, 35. Cambridge University Press* (1997).

[HK] Hausmann J-C. & Knutson A. The cohomology ring of polygon spaces. *Annales de l'Institut Fourier* 48 (1998) 281–321. PS-file: `http://www.unige.ch/math/biblio/preprint/`

[JK] Jeffrey, L. & Kirwan, F. Localization for nonabelian group actions. *Topology* 34 (1995), no. 2, 291–327.

[IP] Ionel E. & Parker T. Gromov-Witten invariants of symplectic sums. (1998) `http://xxx.lanl.gov/abs/math.SG/9806013`

[Le] Lerman, E. Symplectic cuts. *Math. Res. Lett.* 2 (1995), 247–258.

[TW] Tolman, S. & Weitsman, J. The cohomology rings of abelian symplectic quotients. (preprint 1998) `http://xxx.lanl.gov/abs/math.DG/9807173`

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