Maximal Hamiltonian tori for polygon spaces

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June 21 2002

Abstract

We study the poset of Hamiltonian tori for polygon spaces. We determine some maximal elements and give examples where maximal Hamiltonian tori are not all of the same dimension.

1 Introduction

Let M be a symplectic manifold and let $\mathcal{S}(M)$ be the group of symplectomorphisms of M. A sub-torus of $\mathcal{S}(M)$ is called a *symplectic torus*; these tori are partially ordered by inclusions. In this paper, we study the maximal symplectic tori of polygon spaces with a particular emphasis on bending tori (see the definitions below). Since polygon spaces are simply connected, symplectic tori act on M in a Hamiltonian fashion so we refer to them as *Hamiltonian tori.*

Let E be a finite set together with a function $\lambda : E \to \mathbf{R}_+$. Define the space $\widetilde{\mathrm{Pol}}\,(E,\lambda)$ by

$$\widetilde{\mathrm{Pol}}\,(E,\lambda) := \left\{ \rho : E \to \mathbf{R}^3 \;\middle|\; \sum_{e\in E} \rho(e) = 0 \text{ and } |\rho(e)| = \lambda(e) \;\forall e \in E \right\}.$$

The *polygon space* $\mathrm{Pol}\,(E,\lambda)$ is the quotient $\mathrm{Pol}\,(E,\lambda) := \widetilde{\mathrm{Pol}}\,(E,\lambda) \,/\, SO_3$. By choosing a bijection between E and $\{1,\dots,m\}$, the space $\mathrm{Pol}\,(E,\lambda)$ is regarded as the space of configurations in $\mathbf{R}^3$ of a polygon with m edges of length $\lambda_1,\dots,\lambda_m$, modulo rotation, whence the name "polygon space". Also, we call an element of E an *edge* and λ the *length function.*

*Both authors thank the Swiss National Fund for Scientific Research for its support. The second author is partially supported by a Sloan Fellowship and a National Science Foundation Grant.

A length function λ is called *generic* if there is no map $\varepsilon : E \to \{\pm 1\}$ so that $\sum_{e\in E} \varepsilon(e)\lambda(e) = 0$. This guarantees that the polygon cannot collapse to a line. In this paper, we always assume that λ is generic and that $\mathrm{Pol}\,(E,\lambda)$ is not empty. In this case, $\mathrm{Pol}\,(E,\lambda)$ is a closed smooth symplectic manifold of dimension $2(|E|-3) \geq 0$. The polygon spaces are better known as the moduli spaces of (weighted) ordered points on $\mathbf{P}^1$, and also arise via other symplectic reductions (see [Kl], [KM], [HK1] and the proof of Proposition 2.4 below).

A subset I of E is called *lopsided* if there exists $e_0 \in I$ such that $\lambda(e_0) > \sum_{e\in I-\{e_0\}} \lambda(e)$. The empty set is not lopsided, while a singleton $\{e\}$ is always lopsided since the length function takes strictly positive values. The total set E is not lopsided since $\mathrm{Pol}\,(E,\lambda)$ is assumed to be non-empty.

For $I \subset E$ define $\rho_I : \widetilde{\mathrm{Pol}}\,(E,\lambda) \to \mathbf{R}^3$ by $\rho_I := \sum_{e\in I} \rho(e)$. The continuous function and $f_I : \widetilde{\mathrm{Pol}}\,(E,\lambda) \to \mathbf{R}$ by $f_I(\rho) := |\sum_{i\in I} \rho_i|$ descends to a function on $\mathrm{Pol}\,(E,\lambda)$, still called f_I. When I is lopsided, this function does not vanish and is therefore smooth. Its Hamiltonian flow Φ_I^t is called the *bending flow* associated to I. Bending flows have been introduced in [Kl] and [KM]. They are periodic (see [Kl, § 2.1] or [KM, Corollary 3.9]): Φ_I^t rotates at constant speed the set of vectors $\{\rho(e) \mid e \in I\}$ around the axis ρ_I.

A *bending torus* is a Hamiltonian torus in $\mathcal{S}(\mathrm{Pol}\,(E,\lambda))$ generated by bending flows. Since the dimension of $\mathrm{Pol}\,(E,\lambda)$ is $2(|E|-3)$, the dimension of any Hamiltonian torus is at most $|E|-3$.

In this paper, we study the poset of bending tori and compare it with that of Hamiltonian ones. For instance, the following result is proved in Section 3 (see Corollary 3.2):

Theorem A *Let $N(\lambda)$ be the minimal number of lopsided subsets which are necessary for a partition of E. Then the maximal dimension of a bending torus for* $\mathrm{Pol}\,(E,\lambda)$ *is* $|E| - \max\{3, N(\lambda)\}$.

We also give a more general statement that allows us to characterize maximal bending tori. In some cases, these coincide with maximal Hamiltonian tori:

Theorem B *Let T be a bending torus of* $\mathrm{Pol}\,(E,\lambda)$ *of dimension* $\geq |E|-5$. *Then T is a maximal Hamiltonian torus if and only if it is a maximal bending torus.*

In Section 5, we give several examples where maximal Hamiltonian tori are not all of the same dimension. Using the work of Y. Karshon [Ka], we show the existence of Hamiltonian tori which are not conjugate to a bending torus (Proposition 5.5). Finally, the relationship with maximal tori in the contactomorphism group of pre-quantum circle bundles, due to E. Lerman [Le], is mentioned in 5.6.

2 Preliminaries - Bending sets

Lemma 2.1 *Let $\mathcal{I}$ be a family of lopsided subsets of E. The following conditions are equivalent:*

a) The bending flows $\{\Phi^t_I \mid I \in \mathcal{I}\}$ generate a bending torus.

b) For each pair $A, B \subseteq \mathcal{I}$, either $A \cap B = \emptyset$ or one is contained into the other.

PROOF: By [Kl, § 2.1] or [KM, Corollary 3.9], the bending flows are periodic. Therefore, a) is equivalent to the fact that $\{f_A, f_B\} = 0$ for all $A, B \in \mathcal{I}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket. Proposition 2.1.2 of [Kl] shows that $\{f_A^2, f_B^2\} = 0$ if and only if the pair A, B satisfies Condition b). Since f_A and f_B never vanish, the formula

$$\{f_A^2, f_B^2\} = 4\, f_A f_B\, \{f_A, f_B\}$$

implies that $\{f_A^2, f_B^2\} = 0$ if and only if $\{f_A, f_B\} = 0$. □

A set $\mathcal{I}$ of lopsided subsets of E is called a *bending set* if it contains every singleton $\{e\}$ and satisfies the following "absorption condition": *for each pair $A, B \subseteq \mathcal{I}$, either $A \cap B = \emptyset$ or one is contained in the other.*

Bending sets are technically convenient to parametrize bending tori. Indeed, let $\mathcal{I}$ be a bending set. By 2.1, the bending flows $\{\Phi^t_I \mid I \in \mathcal{I}\}$ generate a bending torus $T_{\mathcal{I}}$. Conversely, if T is a bending torus, there is at least one set $\mathcal{I}$ of lopsided subsets satisfying the absorption condition such that $T = T_{\mathcal{I}}$, and one can add singletons to $\mathcal{I}$ to make it a bending set.

The elements of $\mathcal{I}$ are partially ordered by inclusions, so one can associate to $\mathcal{I}$ the family $\mathcal{M}_{\mathcal{I}}$ of its maximal elements. A direct consequence of the definition is that $\mathcal{M}_{\mathcal{I}}$ is a partition of E.

A bending set $\mathcal{I}$ is called *full* if, for each $I \in \mathcal{I}$ which is not a singleton, there exist $I', I'' \in \mathcal{I}$ so that I is the disjoint union of I' and I''. It is easy to check that this condition is equivalent to either of the following.

a) Given I and I' in $\mathcal{I}$ such that $I' \subset I$, the union $\mathcal{I} \cup \{I'\}$ is not a bending set. This justifies the term "full": one can no longer add elements to $\mathcal{I}$ and keep the latter a bending set.

b) For all $I \in \mathcal{I}$ the set $\{I' \in \mathcal{I} : I' \subseteq I\}$ contains $2\,|I| - 1$ elements.

Remark Let $\mathcal{I}$ be a bending set. The reader might find it helpful to consider the graph of this poset. It is a union of disjoint trees, each of which contains a unique maximal element. The bending set $\mathcal{I}$ is full iff these trees are binary: each vertex has one edge leaving it (except the maximal ones which have none) and 2 edges pointing into it (except the singletons which have none).

Lemma 2.2 *Let $\mathcal{I}$ be a bending set. Then there exists a (non-unique) bending set $\hat{\mathcal{I}}$ such that the following conditions hold*

1) $\mathcal{I} \subset \hat{\mathcal{I}}$ (therefore $T_{\mathcal{I}} \subset T_{\hat{\mathcal{I}}}$).

2) $\hat{\mathcal{I}}$ is full.

3) $\mathcal{M}_{\hat{\mathcal{I}}} = \mathcal{M}_{\mathcal{I}}$.

PROOF: If $\mathcal{I}$ is full we are done. Otherwise, we proceed by induction on the number of "non-full" elements of $\mathcal{I}$: those $I \in \mathcal{I}$ which are not singletons and are not the disjoint union of 2 elements of $\mathcal{I}$. Let $I \in \mathcal{I}$ be a minimal "non-full" element.

Let $I_1, \dots, I_r$ be the maximal proper subsets of I which are elements of $\mathcal{I}$. One of them, say I_1, contains the longest edge of I. For $i = 2, \dots, r-1$, define $R_i := I_1 \cup \dots \cup I_i$ and let $\check{\mathcal{I}} := \mathcal{I} \cup \{R_2\} \cup \dots \cup \{R_{r-1}\}$. One has $I = R_{r-1} \sqcup I_r$, $R_{r-1} = R_{r-2} \sqcup I_{r-1}$ etc. As I was minimal, it is no longer non-full in $\hat{\mathcal{I}}$. This gives the inductive step. □

We shall now compute the dimension of a bending tori. We need some knowledge about the critical points of the maps f_I and its symplectic reduction. The following lemma comes from [Ha, Theorem 3.2].

Lemma 2.3 *Let I be a lopsided subset of E. An element $\rho \in \mathrm{Pol}\,(E,\lambda)$ is a critical point for f_I if and only if either the set $\{\rho(e) \mid e \in I\}$ or the set $\{\rho(e) \mid e \notin I\}$ lies in a line.* □

Proposition 2.4 *Let $A \subset E$. Define $\bar{A} := A \cup \{A\}$ and $\lambda^{A,t} : \bar{A} \to \mathbf{R}$ by $\lambda^{A,t}(e) := \lambda(e)$ for $e \in A$ and $\lambda^{A,t}(A) := t$. Then, if A is lopsided, the symplectic reduction of $\mathrm{Pol}\,(E,\lambda)$ at t, for the action of the bending circle T_A, is symplectomorphic to the product of the two polygon spaces*

$$\mathrm{Pol}\,(E,\lambda) \mathbin{/\!/_{t}} T_A \;\cong\; \mathrm{Pol}\,(\bar{A},\lambda^{A,t}) \times \mathrm{Pol}\,(\overline{E-A},\lambda^{E-A,t}).$$

Remark 2.5 a) Proposition 2.4 holds true even if t is not a regular value. If it is, the two right hand polygon spaces of the formula are generic by Lemma 2.3.

b) The following is clear from the proof below: if $T_{\mathcal{I}}$ is a bending torus and $A \in \mathcal{I}$, then the action of $T_{\mathcal{I}}$ descends to the reduced space, giving rise to a product of two bending tori: one for the bending set $\{I \in \mathcal{I} \mid I \subset A\}$ and the other for $\{I \in \mathcal{I} \mid I \not\subset A\}$

c) In this paper, Proposition 2.4 is used only for $|A| = 2$. In this case, the reduction of $\mathrm{Pol}\,(E, \lambda)$ at t is symplectomorphic to a polygon space with $|E| - 1$ edges, since $\mathrm{Pol}\,(\bar{A}, \lambda^{A,t})$ is a point. However, the hypothesis $|A| = 2$ does not simplify the proof.

Proof of Proposition 2.4 : First recall the precise definition for the symplectic structure on $\mathrm{Pol}\,(E, \lambda)$ (for details, see [HK1, § 1]). For $s \in \mathbf{R}$, let $\mathcal{O}(s)$ the coadjoint orbit of $SO(3)$ with symplectic volume $2s$. With the usual identification of $so(3)^*$ with $\mathbf{R}^3$, $\mathcal{O}(s)$ is the 2-sphere centered in 0 of radius r. For $A \subset E$, let $\mu_A : \prod_{e \in E} \mathcal{O}(\lambda(e)) \to \mathbf{R}^3$ be the partial sum $\mu_A((z_e)) := \sum_{e \in A} z_e$. This is the moment map for the diagonal action of $SO(3)$ on the component indexed by $e \in A$. The space $\mathrm{Pol}\,(E, \lambda) = \mu_E^{-1}(0)/SO(3)$ is then the symplectic reduction

$$\mathrm{Pol}\,(E, \lambda) = \prod_{e \in E} \mathcal{O}(\lambda(e)) \underset{0}{/\!\!/} SO(3)$$

for the diagonal action of $SO(3)$. This determines the symplectic structure on $\mathrm{Pol}\,(E, \lambda)$.

The codimension 2-embedding

$$V_t := \mu_A^{-1}(\mathcal{O}(t)) \cap \mu_E^{-1}(0) \hookrightarrow \mu_A^{-1}(\mathcal{O}(t)) \times \mu_{E-A}^{-1}(\mathcal{O}(t)) \tag{1}$$

gives rise to a diffeomorphism

$$\begin{array}{ccc} [V_t/SO(3))]/T_A & \cong & \mu_A^{-1}(\mathcal{O}(t))/SO(3) \times \mu_{E-A}^{-1}(\mathcal{O}(t))/SO(3) \\ \| & & \| \\ \mathrm{Pol}\,(E, \lambda) \underset{t}{/\!\!/} T_A & & \mathrm{Pol}\,(\bar{A}, \lambda^{A,t}) \times \mathrm{Pol}\,(\overline{E-A}, \lambda^{E-A,t}). \end{array} \tag{2}$$

As the embedding (1) is the restriction of the obvious symplectomorphism

$$\prod_{e \in E} \mathcal{O}(\lambda(e)) \cong \prod_{e \in A} \mathcal{O}(\lambda(e)) \times \prod_{e \in E-A} \mathcal{O}(\lambda(e)). \tag{3}$$

and as all group actions preserve the symplectic forms, the diffeomorphism (2) is a symplectomorphism. □

Proposition 2.6 *Let $\mathcal{I}$ be a bending set for* Pol (E,λ). *Then*

$$\dim T_{\mathcal{I}} \leq |E| - \max\{3, |\mathcal{M}_{\mathcal{I}}|\}$$

with equality if and only if $\mathcal{I}$ is full.

Proof: By Lemma 2.2, it is enough to prove the formula when $\mathcal{I}$ is full. We proceed by induction on the number of elements of $\mathcal{I}$ which are not singletons. If there are none, then $\dim T_{\mathcal{I}} = 0 = |E| - |E|$ and the formula holds true (recall that $|E| \geq 3$ since we suppose that Pol $(E,\lambda) \neq \emptyset$). Otherwise, as $\mathcal{I}$ is full, there is $A \in \mathcal{I}$ with $|A| = 2$.

If $|E| = 3$, the formula holds true (the 0-torus, being a quotient of $\mathbf{R}^0$, is of dimension 0). We may then assume that $|E| \geq 4$.

The map f_A : Pol $(E,\lambda) \to \mathbf{R}$ is a moment map for the bending circle T_A. As $|E| \geq 4$, it is not constant. Let s be a regular value of f_A ($s > 0$ since A is lopsided). By Proposition 2.4, the symplectic reduction of Pol (E,λ) at s is a generic polygon space with $|E| - 1$ edges. By Part b) of Remark 2.5, the bending set $\mathcal{I}$ coinduces a bending set $\bar{\mathcal{I}}$ for $\bar{\lambda}$ which is full. The number of non-singletons elements of $\bar{\mathcal{I}}$ is one less than that of $\mathcal{I}$. By induction hypothesis, one has

$$\dim T_{\bar{\mathcal{I}}} = |E| - 1 - \max\{3, |\mathcal{M}_{\bar{\mathcal{I}}}|\} \ .$$

As $\dim T_{\mathcal{I}} = \dim T_{\bar{\mathcal{I}}} + 1$ and $\mathcal{M}_{\bar{\mathcal{I}}} = \mathcal{M}_{\mathcal{I}}$, one gets the required expression for $\dim T_{\mathcal{I}}$. □

3 Maximal bending tori

In this section, we study the poset of bending tori. Let $\mathcal{K}$ and $\mathcal{L}$ be two partitions of E. We say that $\mathcal{L}$ is *coarser* than $\mathcal{K}$ if each element of $\mathcal{L}$ is a union of elements of $\mathcal{K}$.

Theorem 3.1 *Let $\mathcal{I}$ be a bending set for* Pol (E,λ). *Let $N(\lambda,\mathcal{I})$ be the minimal number of lopsided subsets which are necessary for a partition of E which is coarser than $\mathcal{M}_{\mathcal{I}}$. Then, the maximal dimension $n(\lambda,\mathcal{I})$ of a bending torus for* Pol (E,λ) *containing $T_{\mathcal{I}}$ is*

$$n(\lambda,\mathcal{I}) = |E| - \max\{3, N(\lambda,\mathcal{I})\} \ .$$

PROOF: Let T be a bending torus containing $T_{\mathcal{I}}$. By Section 2, $T = T_{\mathcal{J}}$ for a bending set $\mathcal{J}$. By Lemma 2.1, the partition $\mathcal{M}_{\mathcal{J}}$ is coarser than $\mathcal{M}_{\mathcal{I}}$. By 2.6, one has

$$\dim T_{\mathcal{J}} \leq |E| - \max\{3, |\mathcal{M}_{\mathcal{J}}|\} \leq |E| - \max\{3, N(\lambda, \mathcal{I})\}$$

and therefore

$$n(\lambda, \mathcal{I}) \leq |E| - \max\{3, N(\lambda, \mathcal{I})\}.$$

Conversely, let $\mathcal{J}_0$ be a partition of E into lopsided subsets, coarser than $\mathcal{M}_{\mathcal{I}}$, with $N(\lambda, \mathcal{I})$ elements. Let $\mathcal{J} := \mathcal{J}_0 \cup \mathcal{I}$. One check easily that $\mathcal{J}$ is a bending set. Let $\hat{\mathcal{J}}$ be a full bending set associated to $\mathcal{J}$ as in Lemma 2.2. One has $\mathcal{M}_{\hat{\mathcal{J}}} = \mathcal{J}_0$ and, by Proposition 2.6, one has,

$$n(\lambda, \mathcal{I}) \geq \dim T_{\hat{\mathcal{J}}} = |E| - \max\{3, N(\lambda, \mathcal{J})\} \ . \ \square$$

As a corollary, we obtain Theorem A of the introduction:

Theorem 3.2 (Theorem A) *Let $N(\lambda)$ be the minimal number of lopsided subsets which are necessary for a partition of E. Then the maximal dimension of a bending torus for* Pol (E, λ) *is* $|E| - \max\{3, N(\lambda)\}$.

PROOF: Set $\mathcal{I}$ be the sets of singletons of E in the statement of Theorem 3.1. $\square$

We now give a characterization of the maximal bending tori which will be used later. We can restrict our attention to those $T_{\mathcal{I}}$, for $\mathcal{I}$ a full bending set, whose dimension is less than $|E| - 3$ (the maximal possible dimension of a Hamiltonian torus of Pol (E, λ)).

Proposition 3.3 *Let $\mathcal{I}$ be a full bending set so that* $\dim T_{\mathcal{I}} < |E| - 3$. *Then, $T_{\mathcal{I}}$ is a maximal bending torus iff*

$$\bigcap_{J \in \mathcal{M}_{\mathcal{J}}} \mathrm{Image}(f_J) \neq \emptyset$$

PROOF: Observe that $T_{\mathcal{I}}$ is a maximal bending torus if and only if for each pair $I, J \in \mathcal{M}_{\mathcal{I}}$, one has $\mathrm{Image}(f_I) \cap \mathrm{Image}(f_J) \neq \emptyset$ ($I \cup J$ is not lopsided). The condition of Proposition 3.3 is *a priori* stronger than that but in fact equivalent, thanks to the following lemma.

Lemma 3.4 *Let $A_0, \ldots, A_n$ be intervals of the real line. If $A_i \cap A_j \neq \emptyset$ for all i, j, then $A_1 \cap \cdots \cap A_n \neq \emptyset$.*

PROOF: By induction on n, starting with $n = 2$. The condition $A_i \cap A_j \neq \emptyset$ for all i, j implies that $A := A_1 \cup \cdots \cup A_n$ is connected and hence is an interval. The set $\mathcal{A} := \{A_0, \ldots, A_n\}$ is an acyclic covering of A and therefore its nerve $\mathcal{N}(\mathcal{A})$ can be used to compute the cohomology of A: $H^*(A) = H^*(\mathcal{N}(\mathcal{A}))$. By induction hypothesis, the simplicial set $\mathcal{N}(\mathcal{A})$ contains the $n-1$ skeleton of the simplex Δ^n. As $H^{n-1}(A) = 0$, $\mathcal{N}(\mathcal{A})$ must contain Δ^n which is to say $A_1 \cap \cdots \cap A_n \neq \emptyset$. □

4 Maximal Hamiltonian tori

We start with an important special case which illustrate the technique: the almost regular pentagon. A function $\lambda : \{1, \ldots, 5\} \to \mathbf{R}_+$ is called the length function of an *almost regular pentagon* if $\lambda(i) = 1$ for $i = 1, \ldots, 4$ and $1 < \lambda(5) < 2$. In this case, $\dim \mathrm{Pol}\,(E, \lambda) = 4$.

Proposition 4.1 *Let $\lambda : \{1, \ldots, 5\} \to \mathbf{R}_+$ be a length function of an almost regular pentagon. Then, the maximal bending tori of $\mathrm{Pol}\,(E, \lambda)$, which are 1-dimensional, are maximal Hamiltonian tori.*

PROOF: The maximal lopsided subset of E are of the form $\{k, 5\}$. Therefore, all maximal bending tori are of dimension 1. Since they are all of the same form, it is enough to prove Proposition 4.1 for one of them, say $T_{\mathcal{I}}$ with $\mathcal{I} := \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$. This gives a Hamiltonian circle action with moment map $f := f_{\{4,5\}} = |\rho(4) + \rho(5)|$. By Lemma 2.3, this map has three critical values:

a) The two extremals $z = \lambda(5) - 1$ and $z = \lambda(5) + 1$ are of course critical values. In both cases, the critical set is a 2-sphere, the configuration spaces of the quadrilateral with side length $(1, 1, 1, z)$.

b) the value 1 for which the critical set consists of three points, namely the configurations $\rho : \{1, \ldots, 5\} \to \mathbf{R}^3$ given by one of the line of equations below

$$\begin{aligned} -\rho(1) &= \rho(2) &&= \rho(3) = -\rho(4) - \rho(5), \\ \rho(1) &= -\rho(2) &&= \rho(3) - \rho(4) - \rho(5) \text{ or} \\ \rho(1) &= \rho(2) &&= -\rho(3) - \rho(4) - \rho(5). \end{aligned}$$

The proof then follows from the lemma below.

Lemma 4.2 *Let $\mu : M \to \mathbf{R}^{m-1}$ be the moment map for a Hamiltonian action of of T^{m-1} on a compact symplectic manifold M^{2m}. Denote by $\mathrm{Crit}\,\mu \subset M$ the set of critical points of μ. Suppose that there is a point*

δ in the interior of the moment polytope $\mu(M)$ such that $\mu^{-1}(\delta) \cap \mathrm{Crit}\,\mu$ has at least 3 connected components. Then the action does not extend to an effective Hamiltonian action of a m-torus.

Proof: Suppose that T extends to a Hamiltonian action of $T \times S^1$ with moment map $\Phi : \mathrm{Pol}\,(E,\lambda) \to \mathbf{R}^n$. Then the moment map f is the composition of Φ with the projection $\mathbf{R}^n \to \mathbf{R}$ onto the last coordinate. Additionally, this action, being effective, would make $Pol(\lambda)$ a symplectic toric manifold. Thus, $\Phi(\rho)$ are distinct points on the boundary of the moment polytope $\phi(\mathrm{Pol}\,(E,\lambda))$ (see [De]), which all project to 1. As at most two points of this boundary can project onto one point of $\mathbf{R}$, we get a contradiction. □

The rest of this section is devoted to the proof of our second main result:

Theorem 4.3 (Theorem B) *Let T be a bending torus of $\mathrm{Pol}\,(E,\lambda)$ of dimension $\geq |E|-5$. Then T is a maximal Hamiltonian torus if and only if it is a maximal bending torus.*

We only need to prove Theorem B in the cases $\dim T = |E|-4$ and $|E|-5$, since it is obvious for $\dim T = |E|-3$.

Proof for $\dim T = |E|-4$ **:** Let $\mathcal{I}$ be a bending set so that $T_{\mathcal{I}}$ is a maximal bending torus of dimension $|E|-4$. We suppose that there is a Hamiltonian circle S^1 commuting with $T_{\mathcal{I}}$; we shall prove that the resulting action of $\widehat{T} := T_{\mathcal{I}} \times S^1$ is not effective.

Let $f_{\mathcal{I}} : \mathrm{Pol}\,(E,\lambda) \to \mathbf{R}^{\mathcal{I}}$ be the product map $f_{\mathcal{I}} := \prod_{A\in\mathcal{I}} f_A$. This is a moment map for the action of $T_{\mathcal{I}}$. Its image Δ is a convex polytope of dimension $|E|-4$. Let μ be the composition of $f_{\mathcal{I}}$ with the projection to the affine space spaned by Δ (the "essential" moment map).

By Proposition 2.6, $\mathcal{I}$ is full and has 4 maximal elements: $\mathcal{M}_{\mathcal{I}} = \{I,J,K,L\}$. By Proposition 3.3, there exists a point c in the intersection of the images of f_I, f_J, f_K and f_L. The proof divides into 3 cases :

Case a) : Suppose that c is in the interior of each image. Then $\vec{c} := (c,c,c,c)$ belongs to the interior of the image of the product map $f := f_I \times f_J \times f_K \times f_L : \mathrm{Pol}\,(E,\lambda) \to \mathbf{R}^4$. This product map is the composition of μ with the projection to $\mathbf{R}^{\mathcal{M}_{\mathcal{I}}}$. Hence, there exists δ in the interior of Δ which projects to $\vec{c}$.

For any $\rho \in \widetilde{\mathrm{Pol}}\,(E, \lambda)$ such that $\mu(\rho) = \delta$, there exists $R_I, R_J, R_K, R_L \in SO(3)$ such that

$$R_I(\rho_I) = R_J(\rho_J) = -R_K(\rho_K) = -R_L(\rho_L).$$

Then the configuration ρ' defined by

$$\rho'(e) := R_I(\rho(e)) \text{ if } e \in I\ ,\ \rho'(e) := R_J(\rho(e)) \text{ if } e \in J, \text{ etc.}$$

also satisfies $\mu(\rho') = \delta$ and moreover $\rho'_I = \rho'_J = -\rho'_K = -\rho'_L$. This implies that ρ' is a critical point for the function $h := f_I + f_J - f_K - f_L$ and hence for μ. Indeed, the Hamiltonian flow of h would be a global rotation around the axis ρ_I, and therefore induces the identity on $\mathrm{Pol}\,(E, \lambda)$.

Similarly, one constructs critical configurations in $\mu^{-1}(\delta)$ with $\rho_I = -\rho_J = \rho_K = -\rho_L$ and $\rho_I = -\rho_J = -\rho_K = \rho_L$. By lemma 4.2, this completes the first case.

Case b) : the argument of Case a) works as well if c is in the interior of the image f_A for each $A \in \mathcal{M}_\mathcal{I}$ which is not a singleton (by genericity of λ, there exists at least one such element).

Case c) : in the general case, there may be some set $A \in \mathcal{M}_\mathcal{I}$, such that c is in the boundary of the image of f_A. Let $\mathcal{M}' \subset \mathcal{M}_\mathcal{I}$ be the set of such A's and let $\bar{\mathcal{M}}'$ be the partition of E generated by $\mathcal{M}'$ (formed by the elements of $\mathcal{M}'$ and the singletons). Call $\mathcal{I}'$ the largest sub-poset of $\mathcal{I}$ so that $\mathcal{M}_{\mathcal{I}'} = \bar{\mathcal{M}}'$; this is a full bending set.

In this case, $\bar{P} := f^{-1}(\vec{c})$ is a symplectic submanifold of $\mathrm{Pol}\,(E, \lambda)$ on which $T_{\mathcal{I}'}$ acts trivially. As $\bar{P}$ coincides with the result of successive symplectic reductions at c for the various f_A with $A \in \mathcal{M}'$, it is, by Proposition 2.4, symplectomorphic to the polygon space $\mathrm{Pol}\,(\bar{\mathcal{M}}', \bar{\lambda})$, where

$$\bar{\lambda}(\{e\}) = \lambda(e) \;\text{ and }\; \bar{\lambda}(A) = c \;\text{ if } A \in \mathcal{M}'$$

The bending torus $T_\mathcal{I}$ acts on $\bar{P}$, giving rise to a bending torus $T_{\bar{I}}$ isomorphic to $T_\mathcal{I}/T_{\mathcal{I}'}$. Observe that $\bar{I}$ has 4 maximal elements and that we are in Case b). Therefore, $T_{\bar{I}}$ is a maximal Hamiltonian torus and the induced action of $\widehat{T}$ on $\bar{P}$ has a kernel of dimension strictly larger than that of $T_{\mathcal{I}'}$. Therefore, as

$$\dim \mathrm{Pol}\,(E, \lambda) - \dim \bar{P} = 2\Big(\sum_{A \in \mathcal{M}'} |A| - |\mathcal{M}'|\Big) = 2 \dim T_{\mathcal{I}'},$$

there is a circle in $\widehat{T}$ acting trivially on a tubular neighborhood of $\bar{P}$. Hence, by the generic orbit type theorem [Au, § 2.2], the action of $\widehat{T}$ on $\mathrm{Pol}\,(E, \lambda)$ is not effective. □

Proof for $\dim T = |E| - 5$ **:** Let $\mathcal{I}$ be a bending set so that $T_{\mathcal{I}}$ is a maximal bending torus of dimension $|E| - 5$. We suppose that there is a Hamiltonian circle S^1 commuting with $T_{\mathcal{I}}$ and we shall prove that the resulting action of $\widehat{T} := T_{\mathcal{I}} \times S^1$ is not effective.

Let $\mu : \mathrm{Pol}\,(E, \lambda) \to \mathbf{R}^{|E|-5}$ be the essential moment map, defined as in the proof for $\dim T = |E| - 4$, and let and Δ be the image of μ. Let $\widehat{\mu} : \mathrm{Pol}\,(E, \lambda) \to \Delta \times \mathbf{R}$ be a moment map for the action of $\widehat{T}$ with first component equal to μ and let $\widehat{\Delta}$ be the image of $\widehat{\mu}$.

By Proposition 2.6, $\mathcal{M}_{\mathcal{I}}$ has 5 elements. By Proposition 3.3, there exists a point c in the intersection of the images of f_A for $A \in \mathcal{M}_{\mathcal{I}}$. The proof divides into several cases :

Case 1) : Suppose that $|E| = 5$. Then T_I is of dimension 0 and we have to know that a maximal Hamiltonian torus for a regular pentagon space is also of dimension 0. This is the contents of [HK2, Theorem 3.2].

Case 2) : Suppose that each $A \in \mathcal{M}_{\mathcal{I}}$ contains exactly 2 elements (hence $|E| = 10$) and c is in the interior of the image of f_A. This implies that $\vec{c} := (c, c, c, c, c)$ is a regular value of μ. The reduction Q of $\mathrm{Pol}\,(E, \lambda)$ at $\vec{c}$ is then symplectomorphic to a regular pentagon space (apply Proposition 2.4 five times). The induced Hamiltonian action of $\widehat{T}$ on Q is then trivial by Case 1). This implies that the image of the differential $D\widehat{\mu}$ at any point of $\mu^{-1}(\vec{c})$ is parallel to $\Delta \times \{0\}$. By convexity, we deduce that $\widehat{\Delta}$ and Δ have the same dimension and therefore the action of $\widehat{T}$ is not effective.

Case 3) : The argument of Case 2) works as well if each $A \in \mathcal{M}_{\mathcal{I}}$ has ≤ 2 elements and c is in the interior of the image of f_A when $|A| = 2$. Also, if there are sets $A \in \mathcal{M}_{\mathcal{I}}$ with $|A| = 2$ and c is in the boundary of the image of f_A, one proceeds as in Case c) of the proof for $\dim T_{\mathcal{I}} = |E| - 4$ to deduce that the action of $\widehat{T}$ is not effective. Thus, we are able to prove our result when all the elements of $\mathcal{M}_{\mathcal{I}}$ are either singletons or doubletons.

General case) : For $A \in \mathcal{M}_{\mathcal{I}}$, let $k_A := \max\{0, |A| - 2\}$ and $k := \sum_{A \in \mathcal{M}_{\mathcal{I}}} k_A$. The proof goes by induction on k, the case $k = 0$ being established in Case 3). If $k > 0$, let $A \in \mathcal{M}_{\mathcal{I}}$ such that $|A| \geq 3$. If c lies in the boundary of the image of f_A, one proceeds as in Case c) of the proof for $\dim T_{\mathcal{I}} = |E| - 4$ to deduce that the action of $\widehat{T}$ is not effective (using the induction hypothesis). Otherwise, as $\mathcal{I}$ is full, there exists $B \in \mathcal{I}$ such that $|B| = 2$, $B \subset A$ and $f_B(f_A^{-1}(c))$ is an interval of positive length. It contains an open interval J of regular values of f_B. For $t \in J$, the reduction of $\mathrm{Pol}\,(E, \lambda)$ for the action of the Hamiltonian circle with moment map f_B is, by Proposition

2.4, symplectomorphic to an $(|E|-1)$-gon space $\bar{P}$. The bending torus $T_{\mathcal{I}}$ descends to a bending torus $T_{\bar{\mathcal{I}}}$ for $\bar{P}$. One has $\mathcal{M}_{\bar{\mathcal{I}}} = \mathcal{M}_{\mathcal{I}}$ and $\bar{k} = k-1$. By induction hypothesis, $T_{\bar{\mathcal{I}}}$ is a maximal Hamiltonian torus. This implies that each point of $f_B^{-1}(t)$ has a stabilizer of positive dimension for the action of $\widehat{T}$. This holds true for all $t \in J$, therefore for an open set of $\mathrm{Pol}\,(E,\lambda)$. By the generic orbit type theorem [Au, § 2.2], this implies that the action of $\widehat{T}$ on $\mathrm{Pol}\,(E,\lambda)$ is not effective. □

5 Examples

NOTATIONS : When $E = \{1,\dots,n\}$, we describe $\mathrm{Pol}\,(E,\lambda)$ by writing the values of λ. For instance, $\mathrm{Pol}\,(1,1,1,2)$ stands for $\mathrm{Pol}\,(\{1,2,3,4\},\lambda)$ with $\lambda(1)=\lambda(2)=\lambda(3)=1$ and $\lambda(4)=2$. A bending set is described by listing its elements which are not singletons and labeling the edges by their length.

5.1 *The "two long edge" case :* Suppose that the set of edges E contains two elements a, b such that

$$\lambda(a)+\lambda(b) > \sum_{e\in E-\{a,b\}} \lambda(e)\ .$$

Then E is the disjoint union of E_a and E_b so that E_a is lopsided with longest edge a and E_b is lopsided with longest edge b. One then has $N(\lambda)=2$ and, by Theorem 3.1, $\mathrm{Pol}\,(E,\lambda)$ admits a bending torus of dimension $|E|-3$. In particular, $\mathrm{Pol}\,(E,\lambda)$ is a toric manifold.

5.2 *Almost regular pentagon :* The almost regular pentagon $\mathrm{Pol}\,(1,1,1,1,a)$ with $1<a<2$ (or $0<a<1$) is a very important special case, already used in Proposition 4.1. Notice $\mathrm{Pol}\,(E,\lambda)$ is diffeomorphic to $\mathbf{C}P^2\,\sharp\, 4\,\overline{\mathbf{C}P^2}$ (see [HK1, Example 10.4]).

We used the result of [HK2] that the regular pentagon space admits no non-trivial circle action. This is not known for regular polygon spaces with more edges. Nor it is known whether an almost regular pentagon space is diffeomorphic to a toric manifold.

5.3 *Hamiltonian tori of different dimensions :* Consider a generic pentagon space of the form $P_{a,b} := \mathrm{Pol}\,(1,1,1,a,b)$ with $a \neq 1 \neq b$ and $0 < a-b < 1 < a+b$. The bending circle $\{a,b\}$ is a maximal Hamiltonian torus by Proposition 3.3 and 4.3. However, $\mathrm{Pol}\,(1,1,1,a,b)$ is a toric manifold by the bending tori $T_{\mathcal{I}}$ of the form $\mathcal{I} := \{\{1,a\},\{1,b\}\}$. In this example, one sees that maximal bending tori, as well as maximal Hamiltonian tori, are not all of the same dimension.

The moment polytope for $T_{\mathcal{I}}$ shows that $P_{a,b}$ is diffeomorphic to $\mathbf{C}P^2 \sharp 4\,\overline{\mathbf{C}P^2}$ if $a+b<3$ and to $\mathbf{C}P^2 \sharp 3\,\overline{\mathbf{C}P^2}$ if $a+b>3$ (the case $a+b=3$ is not generic). It is known that the other pentagon spaces are 4-manifolds with second Betti number < 3. For them, any Hamiltonian circle action extends to a toric action by [Ka, Th. 1].

An example with maximal Hamiltonian tori of 3 different dimensions is provided by the heptagon spaces Pol$\,(1,1,2,2,3,3,3)$ (it is generic since lengths are integral and the perimeter is odd). The 3 bending sets with maximal (non-singleton) elements of the form

$$\{\{2,1\},\{2,1\}\} \quad , \quad \{\{2,1\},\{3,1\},\{3,2\}\} \quad , \quad \{\{3,1,1\},\{3,2\},\{3,2\}\}$$

determine maximal Hamiltonian tori of dimension respectively $2,3$ and 4. Observe that the bending circle $\{3,2\}$ is contained in two maximal tori of different dimension.

Examples in higher dimension can be constructed by adding "little edges" to the previous one, for instance the $(7+m)$-gon space

$$\mathrm{Pol}\,(1,1,2,2,3,3,3,1/2,1/4,\ldots,1/2^m).$$

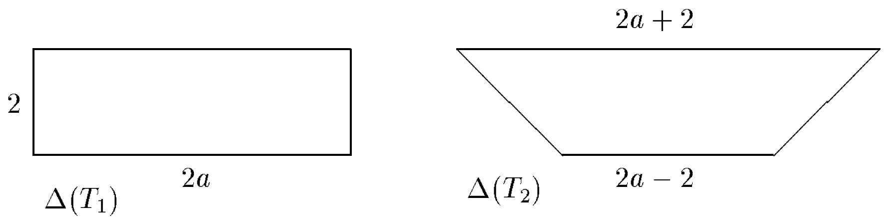
It admit full bending sets with maximal (non-singleton) elements of the form

- $\{\{2,1\},\{2,1\},\{3,1/2,1/4,\ldots,1/2^m\}\}$
- $\{\{2,1\},\{3,1\},\{3,2\},\{3,1/2,1/4,\ldots,1/2^m\}\}$
- $\{\{3,1,1\},\{3,2\},\{3,2\},\{3,1/2,1/4,\ldots,1/2^m\}\}$

which determine maximal Hamiltonian tori of dimension respectively $m+2, m+3$ and $m+4$.

5.4 Let T_1 and T_2 be two Hamiltonian tori of dimension n for a symplectic manifold M^{2n}. Choose isomorphisms $\mathrm{Lie}(T_1)^* \approx \mathbf{R}^n \approx \mathrm{Lie}(T_2)^*$. the moment polytopes Δ_1 and Δ_2 of the two actions are in $\mathbf{R}^n$. By Delzant's theorem, T_1 is conjugate to T_2 in the group $\mathcal{S}(M)$ of sympectomorphism of M if and only if the moment polytopes $\Delta(T_i)$ satisfy $\Delta(T_2) = \psi(\Delta(T_1))$ where ψ is a composition of translations and transformations in $GL(\mathbf{Z}^n)$.

Consider the pentagon space $P := \mathrm{Pol}\,(1, a, c, c, c)$, with $c > a + 1 > 2$. The two bending tori $T_1 = \{\{c, 1\}, \{c, a\}\}$ and $T_2 = \{\{c, 1\}, \{c, a, 1\}\}$ have moment polytopes



Therefore, T_1 and T_2 are not conjugate in in the group $\mathcal{S}(P)$. One can check that any other bending torus is conjugate to either T_1 or T_2.

On the other hand, the polytope $\Delta(T_1)$ shows that P is symplectomorphic to $(S^2 \times S^2, \omega_1 + a\omega_2)$, where ω_1 and ω_2 are the pull back of the standard area form on S^2 via the two projection maps. By [Ka, Th. 2], the number of conjugacy classes of maximal Hamiltonian tori is equal to $\lceil a \rceil$, the smallest integer greater than or equal to a. This proves the following

Proposition 5.5 *If $c > a+1 > 3$, then* $\mathrm{Pol}\,(1, a, c, c, c)$ *admits Hamiltonian tori which are not conjugate to a bending torus.*

5.6 Let (M, ω) be a simply connected symplectic manifold such that $[\omega] \in H^2(M; \mathbf{R})$ is integral. Then there exists a principal circle bundle $S^1 \to Q \to M$ with Euler class $[\omega]$ and Q carries a natural contact distribution by a theorem of Boothby and Wang [BW, Th. 3]. In [Le, Th. 1], E. Lerman recently proved that maximal Hamiltonian tori in M (of dimension k) give rise to maximal tori (of dimension $k + 1$) in the group of diffeomorphism of Q preserving the contact distribution.

By [HK1, Prop. 6.5], the symplectic form on $\mathrm{Pol}\,(E, \lambda)$ is integral when, for example, λ takes integral values. Then, our examples in 5.3 give rise to contact manifolds with maximal tori of different dimensions in their group of contactomorphisms (see [Le, Example 2]).

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