A CAT(0) ALTERNATIVE FOR AMENABLE GROUPS AND A KAZHDAN-TYPE RIGIDITY PRINCIPLE

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ABSTRACT. We prove that finitely generated amenable groups acting on CAT(0) spaces satisfy the following alternative: either every action on a geodesically complete CAT(0) space with bounded geometry (or finite dimension) has a global fixed point, or the group admits a fixed-point-free action on $\mathbb{R}^n$. As a consequence, finitely generated amenable torsion groups and finitely generated virtually simple amenable groups cannot act nontrivially on geodesically complete CAT(0) spaces with bounded geometry or on finite-dimensional complete CAT(0) spaces.

The proof relies on a Kazhdan-type rigidity theorem for groups with the Euclidean fixed point property: if such a group acts on a geodesically complete CAT(0) space of bounded geometry with almost fixed points, then it has a genuine fixed point. This yields several further corollaries, including a rigidity dichotomy for drift and that any finitely generated torsion group acting on a geodesically complete visibility CAT(0) space with bounded geometry must have a global fixed point. These results make substantial progress on the longstanding problem of understanding actions of torsion groups on CAT(0) spaces.

1. Introduction

The study of which finitely generated groups can act on various classes of complete CAT(0) spaces in a non-degenerate way is a central topic for Riemannian geometry and geometric group theory. For example, the classical theorems of Preissmann and Myers demonstrated how curvature strongly constrains the algebraic structure of fundamental groups. Together with Cartan's fixed point theorem, this suggests that torsion phenomena typically appear in positive curvature, but the extent to which torsion groups can act on CAT(0) spaces has remained an important open problem. The seminal rigidity theory of Mostow-Margulis concerns groups of isometries of symmetric spaces of nonpositive curvature. The representation theoretic property (T) of Kazhdan plays an important role in that theory as well as for many other topics, for example expander graphs. It can in turn be reformulated as a fixed point property on Hilbert spaces.

In this paper, we establish a rigidity principle for CAT(0) spaces that can be viewed as a nonlinear analogue of Kazhdan's property (T). It imposes strong constraints on isometric actions of groups that cannot act on Euclidean spaces without a fixed point. Our methods do not require the action to be proper or discrete, nor do they rely on the presence of a large isometry group of the space. In particular, they provide a different approach from arguments based on generalized versions of the Margulis lemma, which typically require properness. The two main theorems, and most of their consequences

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below, are new even in the setting of proper actions. Our first main theorem is the following alternative for amenable groups:

Theorem A. (CAT(0) alternative for amenable groups) *Let Γ be a finitely generated amenable group. Then exactly one of the following holds:*

(1) *every isometric action of Γ on a geodesically complete* CAT(0) *space with bounded geometry has a global fixed point;*
(2) Γ *admits an isometric action on some $\mathbb{R}^n$ without a global fixed point.*

Moreover, the minimal dimension of a complete CAT(0) *space admitting a fixed-point-free action by Γ equals the minimal dimension of a Euclidean space admitting such an action.*

The assumption of bounded geometry in the first part of the theorem is necessary, see Remark 5.1. In infinite dimensions, every infinite finitely generated amenable group acts on a Hilbert space without a global fixed point [5].

Our theorem significantly generalizes [22] and should be compared with the foundational theorems by Burger-Schroeder [12], Adams-Ballmann [2] and Caprace-Lytchak [14], which state that actions by amenable groups either stabilize a flat or fix a boundary point at infinity. Theorem A strengthens this by identifying Euclidean actions as the unique obstruction to global fixed points for amenable groups acting on CAT(0) spaces. Amenable group actions on CAT(0) spaces have been studied extensively by Caprace and Monod, often under the assumption that the full isometry group acts cocompactly, see [13, 15, 16]. Farb [19] and Bridson [10] investigated the largest dimension $\mathrm{FixDim}_{\mathrm{CAT}(0)}(\Gamma)$ of complete CAT(0) spaces for which Γ-actions always has a global fixed point. Theorem A asserts that

$$\mathrm{FixDim}_{\mathrm{CAT}(0)}(\Gamma) = \mathrm{FixDim}_{\mathrm{Eucl}}(\Gamma),$$

for every finitely generated amenable group. Here $\mathrm{FixDim}_{\mathrm{Eucl}}(\Gamma)$ denotes the largest integer n for which Γ has the *fixed-point property for* $\mathbb{R}^n$, that is, any isometric action of Γ on $\mathbb{R}^n$ has a fixed point.

We now describe the new rigidity principle that underlies most of our results. Let Γ be a finitely generated group acting on a CAT(0) space Y. We say that the action has *almost fixed points* if

$$\inf_{x\in Y} \max_{s\in S} d(x, sx) = 0$$

for some (hence any) finite generating set S of Γ. The expression on the left is called the *joint minimal displacement* of S in [9].

Theorem B. (CAT(0) property (T)) *Let Γ be a finitely generated group with the fixed-point property for all finite-dimensional Euclidean spaces, and let Y be a geodesically complete* CAT(0) *space with bounded geometry. Suppose that Γ acts isometrically on Y with almost fixed points. Then there is a point in Y fixed by Γ.*

This applies to groups with Kazhdan's property (T) and groups with only finite image, finite-dimensional linear representations over the reals, since the isometry group of a Euclidean space is linear. The latter class includes torsion groups and nonlinear just infinite groups, and there are many such examples within the family of weakly branch groups [1, 4]. Moreover, since linear groups are residually finite, groups with no nontrivial finite quotients, such as simple groups, cannot act on Euclidean spaces without a fixed point. Theorem B is used in the proof of Theorem A as well as in most of the results below.

We now turn to describing the corollaries. Note first that Theorem A implies that if a finitely generated amenable group admits a fixed-point free action on a finite-dimensional complete CAT(0) space, then it must possess an infinite-image, finite-dimensional linear representation. This responds partially to a general question raised by Margulis in [30, §2, Problem 4] about the linear nature of groups acting on Cartan-Hadamard manifolds. The amenable alternative therefore has two immediate corollaries since such infinite-image linear representations can be ruled out: for torsion groups by a theorem of Schur (in response to the Burnside problem), and for simple groups since linear groups are residually finite (see [22] for details).

Corollary C. *Finitely generated, amenable, torsion groups cannot act on a geodesically complete* CAT(0) *space of bounded geometry, nor on a complete finite-dimensional* CAT(0) *space, without a global fixed point.*

It is remarkable that infinite such groups exist. Indeed, infinite, finitely generated torsion groups were first discovered by Golod in the 1960s. The first amenable such examples are the Grigorchuk groups, see [31] for further examples and references. Corollary C answers, in the case of amenable groups, an old and well-known question in geometric group theory, see [32, 20] for the history of this problem. For this class of group, the answer is more complete than perhaps was anticipated, since there are no other assumptions on the action or the space. Large classes of nonamenable groups contain free subgroups, such as linear groups and hyperbolic groups, and hence they are of course not torsion groups, on the other hand, some Burnside groups are nonamenable. Corollary C significantly improves on the recent result in [22]. Serre showed that finitely generated torsion groups cannot act on 1-dimensional complete CAT(0) spaces without a fixed point (the proof in [33] written for simplicial trees works in general). In two dimensions, a recent theorem of Norin, Osajda, and Przytycki [32] answers the question for torsion groups acting on 2-dimensional CAT(0) complexes (with some small extra conditions).

Corollary D. *Finitely generated, amenable, virtually simple groups and finitely generated, amenable, just infinite, nonlinear groups cannot act on a geodesically complete* CAT(0) *space of bounded geometry, nor on a complete finite-dimensional* CAT(0) *space, without a global fixed point.*

Examples of such groups can be found within the important class of branch groups [4], in particular in the work of Grigorchuk. Juchenko-Monod [25] proved the existence of finitely generated simple amenable groups confirming a conjecture of Grigorchuk-Medynets. Further examples appear in the work of Matte-Bon and Nekrashevych, see [31]. Corollary D significantly extends a result in [22].

We continue with a result for torsion groups that complements [24, Theorem 1.5] since we do not assume that the action is properly discontinuous, on the other hand, we insist on geodesic completeness, which is not assumed in [24]. Recall that a CAT(0) space Y is called a *visibility space* if for every two distinct points in the geometric boundary ∂Y of Y, there is a geodesic line whose endpoints are exactly these two points. Typical examples include CAT(-1) spaces, and more generally Gromov hyperbolic CAT(0) spaces. One may also see [23] for visibility manifolds with finite-volume quotients of bounded curvatures that are not Gromov hyperbolic. We prove:

Theorem E. *Let Y be a geodesically complete visibility space with bounded geometry, and let Γ be a finitely generated torsion group. Whenever Γ acts isometrically on Y, there is a global fixed point in Y.*

From Theorem B we can also deduce:

Theorem F. (Drift dichotomy) *Let Γ be a finitely generated group with fixed-point property for all finite-dimensional Euclidean spaces. Let $\rho : \Gamma \to \mathrm{Isom}(Y)$ be an isometric action on a geodesically complete* CAT(0) *space Y with bounded geometry. Then exactly one of the following holds:*

1. *for every symmetric probability measure with finite second moment whose support generates Γ, the drift $l_\rho(\Gamma) > 0$;*
2. *Γ has a global fixed point in Y.*

Note the rigidity implication: if the orbit has sublinear growth, $d(y, \gamma(y)) = o(||\gamma||)$, then in fact it is bounded, $d(y, \gamma(y)) < C$. When the action comes from a real finite-dimensional linear representation (π, V), and Y is the corresponding symmetric space $\mathrm{GL}(V)/\mathrm{O}(V)$, positive drift means positive top Lyapunov exponent and fixing a point means that $\pi(\Gamma)$ lies in a compact subgroup. This is an important topic in rigidity theory and dynamical systems, see [30, §3], [6, Sect. 4.7], and [7].

Symmetric spaces of non-compact type and Gromov hyperbolic spaces satisfy a geometric Berger-Wang identity, see the paper by Breuillard-Fujiwara [9] and section 5 below. For such spaces in particular, the following applies:

Corollary G. *Let Y be a geodesically complete* CAT(0) *space with bounded geometry; Then any isometric action by a finitely generated torsion group Γ on Y, which satisfies the geometric Berger-Wang identity for some finite generating set of Γ, must fix a point in Y.*

Corollaries C and G, as well as, Theorems E and F taken together provide substantial progress on a question partly originating with Swenson's work in the 1990s. This well-known open problem, concerning to what extent torsion groups can act on CAT(0) spaces, has since appeared in several problem lists in geometric group theory; see [32, p. 1379] for a compilation of references to such lists. In particular, our results confirm parts of [32, Conjecture 1.5], even without the assumption that the group acts on a CAT(0) *complex*.

Methods. The novel methods developed in this paper may be useful for other problems. In particular, taking the ultralimit of spaces is for us a very useful device. It does not increase the dimension and respects well bounded geometry. Moreover, under some conditions we get compactness of the Tits boundary in the limit. This is then combined with the main theorem in Karlsson-Margulis [28] in the proof of Theorem B.

Further ingredients in the proof of Theorem B are Busemann functions and harmonic maps. The latter tool enters via Theorem 5.1 below, taken from [21]. The proof of Theorem E also uses random walks and Busemann functions.

For Theorem A, concerning the case of amenable groups, it should be said that the theorems of Adams-Ballmann [2] and Caprace-Lytchak [14] are crucial for the induction.

As a final remark, although we make extensive use of the visual boundary at infinity, as is standard in the subject, all of our main results concern fixed points in the space itself rather than on the boundary. This contrasts with some parts of the existing literature.

2. A resolution of boundary fixed points

Let ω be a non-principal ultrafilter on $\mathbb{N}$. Given a sequence of complete CAT(0) spaces Y_n with basepoints p_n, the associated ultralimit (Y_∞, p_∞) is a complete CAT(0) space [11, Chapter II.3 Theorem 3.9]. If the dimensions of the spaces Y_n are at most D, then the dimension of Y_∞ is also at most D [22, Proposition 6].

The following ultralimit space Z, associated with an isometric group action Γ on a complete CAT(0) space Y, is fundamental and useful in this work. One may consult [29, Proposition 2.8] and [15, Proposition 3.3] for constructions without using ultralimits.

Lemma 2.1. *Let Y be a complete* CAT(0) *space, and let Γ be a finitely generated group acting on Y via a homomorphism $\rho : \Gamma \to \mathrm{Isom}(Y)$. Suppose that $\rho(\Gamma)$ fixes a point ξ in ∂Y. Let $c : [0, \infty] \to Y$ be a geodesic ray with $c(\infty) = \xi$. Then the ultralimit*

$$(Z, o') = \omega\text{-}\lim(Y, c(n))$$

is a complete CAT(0) *space on which Γ acts via a homomorphism $\rho' : \Gamma \to \mathrm{Isom}(Z)$ satisfying*

$$\inf_{q\in Z} \max_{s\in S} d(q, \rho'(s)q) \geq \inf_{p\in Y} \max_{s\in S} d(p, \rho(s)p)$$

for any finite generating set $S \subset \Gamma$. Moreover, Z contains a $\rho'(\Gamma)$-invariant convex subset Z' that splits isometrically as $Z_0 \times \mathbb{R}$, on which the action of $\rho'(\Gamma)$ preserves the splitting.

Proof. Without loss of generality, we may assume that c has unit speed. Take $o = c(0)$, and denote by c_γ the unit speed geodesic ray starting from $\rho(\gamma)o$ and terminating at ξ, $\gamma \in \Gamma$. Note that $c_e = c$, where e is the identity element of Γ.

Since $d(c(n), \rho(\gamma)c(n)) \leq d(c(0), c_\gamma(0))$, we have a well-defined homomorphism $\rho' \colon \Gamma \to \mathrm{Isom}(Z)$. For any $q = \omega\text{-}\lim p_n \in Z$ with $p_n \in Y$, we have

$$d(q, \rho'(\gamma)q) = \omega\text{-}\lim d(p_n, \rho(\gamma)p_n),$$

this implies

$$\inf_{q\in Z} \max_{s\in S} d(q, \rho'(s)q) \geq \inf_{p\in Y} \max_{s\in S} d(p, \rho(s)p).$$

Let us define $c_\gamma^{(n)} \colon \mathbb{R} \to Y$ by

$$c_\gamma^{(n)}(t) = \begin{cases} c_\gamma(0) & t \leq -n \\ c_\gamma(t+n) & t > -n. \end{cases}$$

Note that we have, for any $t \in \mathbb{R}$, and $\gamma, \gamma' \in \Gamma$,

$$(\star) \qquad d(c_\gamma^{(n)}(t), c_{\gamma'}^{(n)}(t)) \leq d(c_\gamma(0), c_{\gamma'}(0)),$$

and $c_\gamma^{(n)}$ restricted to $[-n, \infty)$ is a geodesic ray. Then

$$\omega\text{-}\lim c_\gamma^{(n)}(0) = \omega\text{-}\lim c_\gamma(n) = \omega\text{-}\lim \rho(\gamma) c_e(n) = \rho'(\gamma) o',$$

and therefore $c'_\gamma \colon t \mapsto \omega\text{-}\lim c_\gamma^{(n)}(t)$ is a geodesic defined on whole $\mathbb{R}$ passing through $\rho'(\gamma)o'$. Since $\rho(\gamma')c_\gamma^{(n)}(t) = c_{\gamma'\gamma}^{(n)}(t)$, we get

$$\rho'(\gamma')c'_\gamma(t) = \omega\text{-}\lim \rho(\gamma')c_\gamma^{(n)}(t) = \omega\text{-}\lim c_{\gamma'\gamma}^{(n)}(t) = c'_{\gamma'\gamma}(t).$$

Recalling $(\star)$, we see that $d(c'_\gamma(t), c'_{\gamma'}(t)) \leq d(c_\gamma(0), c_{\gamma'}(0))$ holds for any $t \in \mathbb{R}$, that is, c'_γ and $c'_{\gamma'}$ are parallel to each other. We denote by ξ' the point in ∂Z given by $c'_\gamma(\infty)$ for any $\gamma \in \Gamma$; $\rho'(\Gamma)$ fixes ξ'.

Let Z' be the union of the images of geodesics parallel to c'_e. Z' is a closed convex subset of Z and hence a complete CAT(0) space. By [11, p. 183, 2.14 A Product Decomposition Theorem], we have a splitting $Z' = Z_0 \times \mathbb{R}$, and $\{z\} \times \mathbb{R}$ corresponds to a geodesic parallel to c'_e for each $z \in Z_0$. This completes the proof. $\square$

Lemma 2.2. *Let Y, Γ, ξ be as in Lemma 2.1. Assume further that*

$$\delta := \inf_{p \in Y} \max_{s \in S} d(p, \rho(s)p) > 0$$

for some finite generating set $S \subset \Gamma$, that Γ admits no nontrivial homomorphism into $\mathbb{R}$, and that Y has finite dimension. Then there exists a complete CAT(0) *space Z_0 and a homomorphism $\rho_0 : \Gamma \to \mathrm{Isom}(Z_0)$ such that*

(1) $\dim Z_0 < \dim Y$*;*

(2) $\rho_0(\Gamma)$ *does not fix any point in $Z_0 \cup \partial Z_0$.*

Proof. Take $o \in Y$ and let c be the geodesic ray starting from o and terminating at ξ. Consider $(Z, o') = \omega\text{-}\lim(Y, c(n))$. By Lemma 2.1, there exists a complete CAT(0) space $Z' = Z_1 \times \mathbb{R} \subset Z$, on which Γ acts via a homomorphism $\rho' : \Gamma \to \mathrm{Isom}(Z')$. We have $\dim Z_1 < \dim Z' \leq \dim Y$.

Let $\pi : Z' = Z_1 \times \mathbb{R} \to \mathbb{R}$ be the projection onto the second factor. Since the action of $\rho'(\Gamma)$ on $Z_1 \times \mathbb{R}$ preserves the splitting, the map

$$\gamma \mapsto \pi(\rho'(\gamma)o') - \pi(o')$$

gives a homomorphism from Γ into $\mathbb{R}$. By our assumption on Γ, this homomorphism must be trivial. It follows that $\rho'(\Gamma)$ induces an action on Z_1 (still denoted by $\rho'(\Gamma)$) that satisfies

$$\inf_{q \in Z_1} \max_{s \in S} d(q, \rho'(s)q) \geq \delta.$$

In particular, $\rho'(\Gamma)$ does not have fixed points in Z_1.

Suppose that $\rho'(\Gamma)$ fixes a point in ∂Z_1. We can then repeat the procedure described above to obtain a complete CAT(0)-space Z_2 with $\dim Z_2 < \dim Z_1$, together with a homomorphism $\rho'' : \Gamma \to \mathrm{Isom}(Z_2)$ satisfying

$$\inf_{q \in Z_2} \max_{s \in S} d(q, \rho''(s)q) \geq \delta,$$

which implies that the action does not fix any point in Z_2. Iterating this process, in a finite number of steps, say k, we get a CAT(0) space Z_k on which Γ acts without any fixed points in $Z_k \cup \partial Z_k$. Otherwise, eventually we reach a one-dimensional complete CAT(0) space $\tilde{Z}$ on which Γ acts without fixed points in $\tilde{Z}$ but fixes a point in $\partial\tilde{Z}$. Then again by taking an ultralimit $\tilde{Z}'$ of $\tilde{Z}$, we obtain a subset $\tilde{Z}'' \subset \tilde{Z}'$ consisting of the images of parallel geodesic lines. Since $\dim \tilde{Z}'' \leq 1$, we have $\tilde{Z}'' = \mathbb{R}$. Therefore, the action of Γ on $\tilde{Z}''$, either has a fixed point, or induces a nontrivial homomorphism from Γ to $\mathbb{R}$, both of which contradict our assumptions on Γ. □

Remark 2.1. We cannot remove the assumption that Γ does not admit a nontrivial homomorphism into $\mathbb{R}$. Indeed, let $g \in \mathrm{Isom}(Y)$ be a hyperbolic element and $\Gamma = \langle g \rangle \cong \mathbb{Z}$. Then Γ translates every axis of g. By the Caprace-Lytchak theorem ([14]) we cannot find a finite-dimensional CAT(0) space Z and an action of $\Gamma \cong \mathbb{Z}$ on Z that does not fix any point in ∂Z.

3. An alternative for amenable groups acting on CAT(0) spaces

Note that if we have a fixed-point free action of a finitely generated group Γ on a finite-dimensional complete CAT(0) space, then we can always find a finite-dimensional complete CAT(0) space Y and a homomorphism $\rho\colon \Gamma \to \mathrm{Isom}(Y)$ satisfying

$$\inf_{q\in Y} \max_{s\in S} d(q, \rho(s)q) \geq \delta > 0$$

by the important argument in [8, Proposition 3.1], see also [22]. In particular, $\rho(\Gamma)$ has no fixed point in Y. Together with Lemma 2.2, we get the following.

Proposition 3.1. *Let Γ be a finitely generated group that admits no nontrivial homomorphism into $\mathbb{R}$. Suppose that Γ acts on a finite-dimensional complete* CAT(0) *space Y via a homomorphism $\rho : \Gamma \to \mathrm{Isom}(Y)$, and that $\rho(\Gamma)$ fixes no points in Y. Then there exists a complete* CAT(0)*-space Z, with $\dim Z \leq \dim Y$, on which Γ acts without fixed points in $Z \cup \partial Z$.*

Now assume that the group Γ considered in Proposition 3.1 is amenable. We prove the following alternative when the space is of finite dimension.

Theorem 3.2. (Dimension alternative) *Let Γ be a finitely generated amenable group and n a positive integer. Then either*

(1) *every isometric action of Γ on an n-dimensional complete* CAT(0) *space has a global fixed point, or*
(2) *Γ admits an isometric action on $\mathbb{R}^n$ without a global fixed point.*

Proof. Suppose (1) does not hold. That is, there exists an n-dimensional complete CAT(0) space Y such that an isometric action of Γ on Y does not have any fixed points.

If Γ admits a nontrivial homomorphism into $\mathbb{R}$, then Γ acts on $\mathbb{R}$ without a global fixed point. This clearly implies (2).

If Γ admits no nontrivial homomorphism into $\mathbb{R}$, it follows from Proposition 3.1 that there exists a complete CAT(0) space Z with $\dim Z \leq n = \dim Y$, on which Γ acts without fixed points in $Z \cup \partial Z$. Since Γ is amenable, according to [2, Main Theorem] (or [14, Theorem 1.7]), the action of $\rho(\Gamma)$ on Z must leave a nontrivial flat subspace invariant, and this action has no fixed point in the flat. This also implies (2). □

We now study the largest dimension $\mathrm{FixDim}_{\mathrm{CAT}(0)}(\Gamma)$ of complete CAT(0) spaces, or of Euclidean spaces, on which the group Γ must have a fixed point when acting by isometries. If no fixed-point-free action exists, we define this dimension to be infinite. For example, $\mathrm{FixDim}_{\mathrm{CAT}(0)}(G) = \infty$ for any finite group G.

Now we are ready to prove the second part of Theorem A.

Theorem 3.3 (Theorem A, equal dimension)**.** *For any finitely generated amenable group* Γ*,*

$$\mathrm{FixDim}_{\mathrm{CAT}(0)}(\Gamma) = \mathrm{FixDim}_{\mathrm{Eucl}}(\Gamma).$$

Proof. The inequality, $\mathrm{FixDim}_{\mathrm{CAT}(0)}(\Gamma) \leq \mathrm{FixDim}_{\mathrm{Eucl}}(\Gamma)$, is trivial since Euclidean spaces are special cases of complete CAT(0) spaces.

For the other direction, if $\mathrm{FixDim}_{\mathrm{CAT}(0)}(\Gamma) = \infty$, then we are done by the above trivial inequality. Now we suppose that $\mathrm{FixDim}_{\mathrm{CAT}(0)}(\Gamma) = n < \infty$. Then there exists a fixed-point free action by Γ on an $(n+1)$-dimensional complete CAT(0) space. This means that the first alternative in Theorem 3.2 is ruled out for the dimension $n+1$ and thus by the second alternative, $\mathrm{FixDim}_{\mathrm{Eucl}}(\Gamma) \leq n = \mathrm{FixDim}_{\mathrm{CAT}(0)}(\Gamma)$. □

4. Bounded geometry and scaling ultralimits

Let $X = (X, d)$ be a metric space and let $\varepsilon > 0$. A subset N of X is called ε-*sparse* if $d(x, x') \geq \varepsilon$ for all distinct $x, x' \in N$. For $A \subset X$, we denote by $n_\varepsilon(A)$ the maximal cardinality of an ε-sparse subset of A. Note that if $N \subset A$ is an ε-sparse subset of maximal cardinality, then $A \subset \bigcup_{x \in N} B(x, \varepsilon)$, where $B(x, \varepsilon)$ denotes the open ball centered at x with radius ε. For $r > 0$ and $\varepsilon > 0$, we set

$$n_{r,\varepsilon}(X) = \sup_{x \in X} n_\varepsilon(B(x, r)).$$

Definition 4.1 ([13, §4.1],[17])**.** A metric space $X = (X, d)$ is said to have *bounded geometry* if, for all $r > \varepsilon > 0$, $n_{r,\varepsilon}(X)$ is finite. In this case, we call $n_{r,\varepsilon}(X)$ the (r, ε)-*capacity* of X.

We note that if a complete metric space X has bounded geometry, then X is proper ([13, Lemma 4.1(i)]).

Remark 4.1. It is known that a CAT(0) space admitting a geometric group action always has bounded geometry ([13, Lemma 4.1(ii)]). A detailed proof can be found, for example, in [34].

Let μ be a symmetric probability measure on Γ with finite first moment, whose support generates Γ. For any isometric action $\rho : \Gamma \to \mathrm{Isom}(X, d)$, the *drift* $l_\rho(\Gamma)$ of $\rho(\Gamma)$ with respect to μ is defined as

$$l_\rho(\Gamma) := \lim_{n \to \infty} \frac{\int_\Gamma d(x, \rho(\gamma)x) d\mu^{*n}(\gamma)}{n},$$

where μ^{*n} denotes the nth convolution of μ. It is well-known that the limit always exists and is independent of the choice of $x \in X$, see [26] for more information.

The following result is analogous to Lemma 2.2, where the finite-dimensional condition forces the iteration of ultralimits to end. We show that the same conclusion holds for spaces with bounded geometry.

Lemma 4.1. *Let Γ be a finitely generated group that admits no nontrivial homomorphism into $\mathbb{R}$. Suppose that Γ acts on a complete $\mathrm{CAT}(0)$ space Y with bounded geometry via a homomorphism $\rho : \Gamma \to \mathrm{Isom}(Y)$. Then there exists a complete $\mathrm{CAT}(0)$ space Z_0 and a homomorphism $\rho_0 : \Gamma \to \mathrm{Isom}(Z_0)$ such that*

1. *Z_0 has bounded geometry;*
2. *$\rho_0(\Gamma)$ does not fix any point in ∂Z_0.*
3. *ρ_0 satisfies*
$$\inf_{q\in Z_0} \max_{s\in S} d(q, \rho_0(s)q) \geq \inf_{p\in Y} \max_{s\in S} d(p, \rho(s)p),$$
for any finite generating set $S \subset \Gamma$.

Furthermore,

4. *$l_{\rho_0}(\Gamma) \leq l_\rho(\Gamma)$, where both drifts are computed with respect to the same symmetric probability measure with finite first moment.*

Proof. If $\rho(\Gamma)$ does not fix any point in ∂Y, then the lemma is proved by setting $Z_0 = Y$ and $\rho' = \rho$.

It remains to consider the case that $\rho(\Gamma)$ fixes a point $\xi \in \partial Y$. Let $c : [0, \infty] \to Y$ be a geodesic ray with $c(0) = o$ and $c(\infty) = \xi$. Consider the ultralimit $(Z, o') = \omega\text{-}\lim(Y, c(n))$.

By Lemma 2.1, there exists a complete $\mathrm{CAT}(0)$ space $Z' = Z_1 \times \mathbb{R} \subset Z$, on which Γ acts via a homomorphism $\rho' : \Gamma \to \mathrm{Isom}(Z')$ that preserves the splitting. Since Γ admits no nontrivial homomorphism into $\mathbb{R}$, the action of $\rho'(\Gamma)$ on the $\mathbb{R}$-factor is trivial. Therefore, $\rho'(\Gamma)$ induces an isometric action of Γ on Z_1, which we still denote by ρ'. Write $o' = (o_1, 0)$ with $o_1 \in Z_1$.

Repeating the argument in the proof of Lemma 2.1 yields that
$$\inf_{q\in Z_1} \max_{s\in S} d(q, \rho'(s)q) \geq \inf_{p\in Y} \max_{s\in S} d(p, \rho(s)p).$$

Moreover, we have $d(o_1, \rho'(\gamma)o_1) = d(o', \rho'(\gamma)o') \leq d(o, \rho(\gamma)o)$ for every $\gamma \in \Gamma$. It follows that
$$\begin{aligned} l_{\rho'}(\Gamma) &= \lim_{n\to\infty} \frac{\int_\Gamma d(o_1, \rho'(\gamma)o_1) d\mu^{*n}(\gamma)}{n} \\ &\leq \lim_{n\to\infty} \frac{\int_\Gamma d(o, \rho(\gamma)o) d\mu^{*n}(\gamma)}{n} = l_\rho(\Gamma). \end{aligned}$$

For any $r > r' > \varepsilon' > \varepsilon > 0$ and any $q \in Z_1$, let $\{x_1, \cdots, x_N\} \subset B(q, r')$ be an ε'-sparse set in Z_1. Set $z_i = (x_i, 0) \in Z', 1 \leq i \leq N$ and $z_{N+1} = (q, \varepsilon'), z_{N+2} = (q, -\varepsilon') \in Z'$. It is easy to verify that $\{z_1, \cdots, z_{N+2}\}$ is an ε'-sparse set in $B((q, 0), r') \subset Z'$. It follows from the definition of ultralimits that we can find in Y an ε-sparse set of cardinality $N+2$, contained in a ball of radius r. Since Y has bounded geometry, let $n_{r,\varepsilon} = n_{r,\varepsilon}(Y)$ be its (r,ε)-capacity, then we have

$$N+2 \leq n_{r,\varepsilon},$$

which implies that Z_1 has bounded geometry and its (r',ε')-capacity is at most $n_{r,\varepsilon}-2$.

The space Z_1 associated with the action ρ' satisfies all conditions of a complete CAT(0) space required in Lemma 4.1 except possibly condition (2). If $\rho'(\Gamma)$ does not fix any point in ∂Z_1, then we are done by setting $Z_0 = Z_1$; otherwise, we can iterate the preceding process described above. At step $k \leq n_{r,\varepsilon}$, suppose that there exists a $(\varepsilon + \frac{k(\varepsilon'-\varepsilon)}{n_{r,\varepsilon}})$–sparse set of cardinality K contained in a ball of radius $\left(r - \frac{k(r-r')}{n_{r,\varepsilon}}\right)$ in Z_k. Then by the discussion in the preceding paragraph, we can find a $\left(\varepsilon + \frac{(k-1)(\varepsilon'-\varepsilon)}{n_{r,\varepsilon}}\right)$–sparse set of cardinality $K+2$ contained in a ball of radius $\left(r - \frac{(k-1)(r-r')}{n_{r,\varepsilon}}\right)$ in Z_{k-1}. Iterating backward to $k=1$ we obtain

$$n_{r,\varepsilon} \geq K + 2k,$$

which yields

$$k \leq n_{r,\varepsilon}/2,$$

and hence the iteration terminates in finitely many steps. We eventually obtain a complete CAT(0) space Z_0 associated with an isometric action $\rho_0 : \Gamma \to \mathrm{Isom}(Z_0)$ that fulfills all conditions $(1)-(4)$. □

Let Y be a complete CAT(0) space. Recall that for any sequences $\{p_n\} \subset Y$, $\{r_n\} \subset \mathbb{R}$ with $r_n \to \infty$, and any non-principal ultrafilter ω on $\mathbb{N}$, the ultralimit

$$(Y_\infty, p_\infty) = \omega\text{-}\lim(r_n Y, p_n)$$

is a complete CAT(0) space. The following proposition is fundamental and essential in the proofs of Theorems B, E and F.

Proposition 4.2. *Let Y be a geodesically complete* CAT(0) *space with bounded geometry. Given $\{p_n\} \subset Y$, $\{r_n\} \subset \mathbb{R}$ with $r_n \to \infty$, consider the ultralimit $(Y_\infty, p_\infty) = \omega\text{-}\lim(r_n Y, p_n)$, we have the following*

(1) *For any $r > \varepsilon > 0$,*

$$n_{r,\varepsilon}(Y) \geq n_{r,\varepsilon}(Y_\infty).$$

In particular, Y_∞ also has bounded geometry.

(2) *The Tits boundary $\partial_T Y_\infty$ of Y_∞ is compact.*

Before proving Proposition 4.2, we establish two useful lemmas.

Lemma 4.3. *Suppose X is a geodesically complete* CAT(0) *space with bounded geometry, and for every $r > \varepsilon > 0$, denote by $n_{r,\varepsilon}$ its (r,ε)-capacity. Then for any $a > 1$, we have*

$$n_{ar,a\varepsilon} \geq n_{r,\varepsilon}.$$

Proof. For any $p \in X$, let $\{x_1, \cdots, x_N\} \subset B(p,r)$ be an ε-sparse subset. Since X is geodesically complete, every geodesic segment starting from p can be extended (not necessarily uniquely) to a geodesic ray. For every $1 \le i \le N$, let $c_i : [0,\infty) \to X$ be a geodesic ray with $c_i(0) = p$ that passes through x_i. Choose $y_i \in c_i$ so that $d(p,y_i) = ad(p,x_i) < ar$. By standard CAT(0) estimates we obtain

$$d(y_i,y_j) \ge ad(x_i,x_j) \ge a\varepsilon \ (i \ne j).$$

Thus $\{y_1, \cdots, y_N\}$ is a $a\varepsilon$-sparse subset of $B(p,ar)$. The claim then follows from the arbitrariness of p. □

Recall that a family $\mathcal{X}$ of metric spaces is said to be *uniformly compact* ([11, p. 74, 5.39 Definition]) if

(i) There exists $C > 0$ such that $\mathrm{diam}(X) \le C$ for any $X \in \mathcal{X}$, and

(ii) For any $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $N(\varepsilon)$ balls of radius ε cover X for any $X \in \mathcal{X}$; in other words, $n_{2\varepsilon}(X) \le N(\varepsilon)$ for any $X \in \mathcal{X}$.

The second lemma is as follows.

Lemma 4.4. (compare [11, p. 79, 5.52 Exercise]) *Let $\{(X_n,p_n)\}$ be a uniformly compact sequence of pointed metric spaces. For any non-principal ultrafilter ω on $\mathbb{N}$, there exists a subsequence that converges in the Gromov-Hausdorff topology to a pointed metric space (D,x), such that the ultralimit $(X_\infty,p_\infty) = \omega\text{-}\lim(X_n,p_n)$ is isometric to (D,x).*

Proof. We first show that X_∞ has bounded geometry. For any $\varepsilon > 0$, let $\{x_1, \cdots, x_K\} \subset X_\infty$ be an ε-sparse set. By the definition of ultralimits, for some sufficiently large $k \in \mathbb{N}$ we can find points $x_{k,1}, \cdots, x_{k,K} \in X_k$ so that

$$|d_{X_k}(x_{k,i},x_{k,j}) - d_{X_\infty}(x_i,x_j)| < \varepsilon/2 \ \ (1 \le i,j \le K).$$

It follows that $\{x_{k,1}, \cdots, x_{k,K}\}$ is an $\varepsilon/2$-sparse set in X_k, whence $K \le N(\varepsilon/4)$.

Now suppose that the ε-sparse set $\{x_1, \cdots, x_K\} \subset X_\infty$ is chosen to be maximal, then it is ε-dense in X_∞. In this case for every $1 \le i \le K$, let $\{x_{n,i}\}$ be a sequence with $x_{n,i} \in X_n$ whose ultralimit represents x_i. We denote $x_i = (x_{n,i})_{X_\infty}$.

We select a sequence $\{q_n\}$ with $q_n \in X_n$ as follows. If X_n is covered by $\bigcup_{i=1}^{K} B(x_{n,i},2\varepsilon)$, then set $q_n = x_{n,1}$; otherwise choose any $q_n \in X_n \setminus \bigcup_{i=1}^{K} B(x_{n,i},2\varepsilon)$. We claim that $\omega\{n : q_n = x_{n,1}\} = 1$, otherwise we would have $d_{X_\infty}(x_i,(q_n)_{X_\infty}) = \omega\text{-}\lim d_{X_n}(x_{n,i},q_n) \ge 2\varepsilon$ for all $1 \le i \le K$, which contradicts the maximality of $\{x_1, \cdots, x_K\}$ in X_∞. Therefore, for ω-almost every n, the set $\{x_{n,1}, \cdots, x_{n,K}\}$ is 2ε-dense.

Therefore, the Gromov-Hausdorff distance $D_H(X_k,X_\infty) \le 2\varepsilon$ for infinitely many k. Letting $\varepsilon \to 0$ and applying Gromov's precompactness criterion([11, p. 77, 5.45 Theorem]), we obtain a subsequence of $\{(X_n,p_n)\}$ that converges in the Gromov-Hausdorff topology to a pointed metric space (D,x), which is isometric to (X_∞,p_∞). □

Now we are ready to prove Proposition 4.2.

Proof of Proposition 4.2. (1). For any $q \in Y_\infty$, let $\{x_1, \cdots, x_N\}$ be a finite ε-sparse subset of $B(q,r)$. Then $r' := \max_{1 \le i \le N} d_{Y_\infty}(q,x_i) < r$. For every $n \in \mathbb{N}^*$ and $1 \le$

$i \le N$, choose $q_n, x_{n,i} \in r_nY$ so that $q = \omega\text{-}\lim q_n$ and $x_i = \omega\text{-}\lim x_{n,i}$. We have $\omega\text{-}\lim d_{r_nY}(q_n, x_{n,i}) \le r'$ and $\omega\text{-}\lim d_{r_nY}(x_{n,i}, x_{n,j}) \ge \varepsilon$ for all $i \ne j$. By the definition of ultralimit, for some sufficiently large n, we have

(i) $r_n > 1$;
(ii) $d_{r_nY}(q_n, x_{n,i}) < \frac{r+r'}{2}$;
(iii) $d_{r_nY}(x_{n,i}, x_{n,j}) \ge \frac{r'+r}{2r}\varepsilon$.

Since each scaling space r_nY is also geodesically complete, we can extend the unit speed geodesic segment emanating from q_n and terminating at $x_{n,i}$ so that the new endpoint $x'_{n,i}$ satisfies $d_{r_nY}(q_n, x'_{n,i}) = \frac{2r}{r+r'} d_{r_nY}(q_n, x_{n,i}) < r$. We have

$$d_{r_nY}(x'_{n,i}, x'_{n,j}) \ge \frac{2r}{r+r'} d_{r_nY}(x_{n,i}, x_{n,j}) \ge \varepsilon \quad \text{for all } i \ne j.$$

Therefore $\{x'_{n,1}, \cdots, x'_{n,N}\}$ is an ε-sparse subset of $B(q_n, r) \subset r_nY$. Via the natural identification $i_n : r_nY \to Y$, we get immediately that $\{i_n(x'_{n,1}), \cdots, i_n(x'_{n,N})\}$ is a $\frac{\varepsilon}{r_n}$-sparse subset of $B(q_n, \frac{r}{r_n}) \subset Y$, thus

$$N \le n_{\frac{r}{r_n}, \frac{\varepsilon}{r_n}}(Y) \le n_{r,\varepsilon}(Y),$$

where the second inequality follows from Lemma 4.3. This finishes the proof of Part (1).

(2). First note that, since $r_n \to \infty$, the space $(Y_\infty, p) = \omega\text{-}\lim(r_nY, p_n)$ is isometric to $\omega\text{-}\lim(r_n\bar{B}(p_n, r), p_n)$ for any $r > 0$, where $\bar{B}(p_n, r)$ denotes the closed ball in Y centered at p_n with radius r. For any fixed r, consider the ultralimit $\omega\text{-}\lim(\bar{B}(p_n, r), p_n)$. Since Y has bounded geometry, by Lemma 4.4, $\omega\text{-}\lim(\bar{B}(p_n, r), p_n)$ is isometric to the Gromov-Hausdorff limit of a subsequence of $\{(\bar{B}(p_n, r), p_n)\}$, which we denote by (D, x). Note that D, and hence ∂D, is compact.

We will show that ∂D bounds $\partial_T Y_\infty$.

Since Y is geodesically complete, any geodesic can be extended to a geodesic line. Therefore for every $\xi \in \partial Y_\infty$, the geodesic ray in c_p^ξ in Y_∞ with $c_p^\xi(0) = p$, $c_p^\xi(\infty) = \xi$ is a scaling ultralimit of geodesic segments $c_{p_n}^{q_n}$ in $\bar{B}(p_n, r)$ with endpoints p_n and $q_n \in \partial B(p_n, r)$. We denote $c_p^\xi = (c_{p_n}^{q_n})_{Y_\infty}$. On the other hand, the ultralimit (without scaling) of $c_{p_n}^{q_n}$ represents a geodesic segment in D starting from x and terminating at ∂D, which we denote by $(c_{p_n}^{q_n})_D$. Every geodesic in Y_∞ (resp. D) starting from p (resp. x) and terminating at $\xi \in \partial Y_\infty$ (resp. terminating at $q \in \partial D$) is obtained in this way.

Now take any $\xi, \xi' \in \partial Y_\infty$. Then we can find sequences $\{c_{p_n}^{q_n}\}$ and $\{c_{p_n}^{q'_n}\}$ of geodesic segments with $q_n, q'_n \in \partial B(p_n, r)$ so that $c_p^\xi = (c_{p_n}^{q_n})_{Y_\infty}$ and $c_p^{\xi'} = (c_{p_n}^{q'_n})_{Y_\infty}$. Note that, because of our scaling by $\{r_n\}$, we have

$$c_p^\xi(t) = (c_{p_n}^{q_n})_{Y_\infty}(t) = (c_{p_n}^{q_n}(t/r_n))_{Y_\infty}, \quad c_p^{\xi'}(t) = (c_{p_n}^{q'_n})_{Y_\infty}(t) = (c_{p_n}^{q'_n}(t/r_n))_{Y_\infty}.$$

Suppose that $(c_{p_n}^{q_n})_D = (c_{p_n}^{q'_n})_D$ holds. Then we see that $(c_{p_n}^{q_n})_{Y_\infty} = (c_{p_n}^{q'_n})_{Y_\infty}$ holds. Indeed, for any $t > 0$, we have

$$\begin{aligned} d_{Y_\infty}((c_{p_n}^{q_n})_{Y_\infty}(t), (c_{p_n}^{q'_n})_{Y_\infty}(t)) &= \omega\text{-}\lim r_n d_Y(c_{p_n}^{q_n}(t/r_n), c_{p_n}^{q'_n}(t/r_n)) \\ &\leq \omega\text{-}\lim \frac{t}{r} d_Y(c_{p_n}^{q_n}(r), c_{p_n}^{q'_n}(r)) \\ &= \frac{t}{r} d_D((c_{p_n}^{q_n})_D(r), (c_{p_n}^{q'_n})_D(r)) = 0, \end{aligned}$$

where the inequality in the second line follows from the convexity of the distance function (or simply the CAT(0) property) of Y and the fact that $t/r_n < r$ when n is sufficiently large. Thus we have a well-defined surjection

$$\Psi\colon \partial D \to \partial Y_\infty;\ q = (q_n)_D \mapsto \xi = (c_{p_n}^{q_n})_{Y_\infty}(\infty).$$

Now we take a look at the angular metric $\angle(\cdot,\cdot)$ on ∂Y_∞, which can be expressed as

$$2 \sin \frac{\angle(\xi, \xi')}{2} = \lim_{t\to\infty} \frac{1}{t} d_{Y_\infty}(c_p^{\xi}(t), c_p^{\xi'}(t))$$

by [11, p. 281, 9.8 Proposition (4)]. For fixed $t > 0$, we have

$$\frac{1}{t} d_{Y_\infty}(c_p^{\xi}(t), c_p^{\xi'}(t)) = \frac{1}{t}\omega\text{-}\lim r_n d(c_{p_n}^{q_n}(t/r_n), c_{p_n}^{q'_n}(t/r_n)) = \omega\text{-}\lim \frac{r_n}{t} d(c_{p_n}^{q_n}(t/r_n), c_{p_n}^{q'_n}(t/r_n))$$

by definition. Take n large so that $rr_n/t \geq 1$ holds. Then again by the convexity of the distance function, we have

$$\frac{rr_n}{t} d(c_{p_n}^{q_n}(t/r_n), c_{p_n}^{q'_n}(t/r_n)) \leq d(c_{p_n}^{q_n}(r), c_{p_n}^{q'_n}(r)).$$

Thus, for any t, by taking the ultralimit of the both side, we obtain

$$\frac{1}{t} d_{Y_\infty}(c_p^{\xi}(t), c_p^{\xi'}(t)) \leq \frac{1}{r} d_D((q_n)_D, (q'_n)_D),$$

and $q = (q_n)_D$ and $q' = (q'_n)_D$ are points in ∂D. Note that, by the definition of Ψ, we have $\Psi(q) = \xi$ and $\Psi(q') = \xi'$. Thus we obtain

$$2 \sin \frac{\angle(\Psi(q), \Psi(q'))}{2} \leq \frac{1}{r} d_D(q, q').$$

Therefore the surjection $\Psi\colon \partial D \to (\partial Y_\infty, \angle)$ is a continuous map, and we see that $(\partial Y_\infty, \angle)$ is compact. Since $(\partial Y_\infty, \angle)$ is homeomorphic to the Tits boundary $\partial_T Y_\infty$, $\partial_T Y_\infty$ is compact. □

5. A Kazhdan-type rigidity principle

Let Γ be a finitely generated group. We say that Γ has the *fixed-point property for* $\mathbb{R}^n$ if any isometric action of Γ on $\mathbb{R}^n$ has a fixed point. Suppose Γ acts on a metric space $X = (X, d)$ via a homomorphism $\rho\colon \Gamma \to \mathrm{Isom}(X)$. Recall that $\rho(\Gamma)$ has *almost fixed points* if

$$\inf_{x\in X} \max_{s\in S} d(x, \rho(s)x) = 0.$$

for some (hence any) finite generating set S of Γ.

The following formulation of a result of Izeki [21, Theorem A] is essential for the proofs in this section.

Theorem 5.1. *Let Y be a complete* CAT(0) *space which is either proper or of finite dimension, and Γ a finitely generated group equipped with a symmetric generating probability measure μ with finite second moment. Suppose that Γ has the fixed-point property for all finite-dimensional Euclidean spaces, and there exists a homomorphism $\rho : \Gamma \to \mathrm{Isom}(Y)$ satisfying $l_\rho(\Gamma) = 0$, then $\rho(\Gamma)$ fixes a point in $Y \cup \partial Y$.*

If $\rho(\Gamma)$ has a fixed point in Y, then $l_\rho(\Gamma) = 0$ for every probability measure μ on Γ. The following two results relate the vanishing drift to the existence of almost fixed points.

Proposition 5.2. *Let Γ be a finitely generated group with fixed-point property for all finite-dimensional Euclidean spaces, and let Y be a complete* CAT(0) *space with bounded geometry. Suppose that $\rho : \Gamma \to \mathrm{Isom}(Y)$ is a homomorphism such that*

$$l_\rho(\Gamma) = 0$$

for some symmetric generating probability measure μ on Γ with finite second moment. Then $\rho(\Gamma)$ has almost fixed points.

Proof. Recall that a CAT(0) space of bounded geometry is always proper. So by Theorem 5.1 $\rho(\Gamma)$ fixes a point in $Y \cup \partial Y$. It suffices to consider the case when $\rho(\Gamma)$ fixes a point in ∂Y.

Suppose for contradiction that $\rho(\Gamma)$ does not admit almost fixed points:

$$\inf_{x \in Y} \max_{s \in S} d(x, \rho(s)x) \geq \delta > 0.$$

Note that Γ does not admit any nontrivial homomorphisms into $\mathbb{R}$ according to our assumption. Then we can apply Lemma 4.1 to obtain a complete CAT(0) space Z with bounded geometry and a homomorphism $\rho' \colon \Gamma \to \mathrm{Isom}(Z)$ such that $\rho'(\Gamma)$ does not fix a point in ∂Z. We also have

$$\inf_{z \in Z} \max_{s \in S} d(z, \rho'(s)z) \geq \delta > 0.$$

On the other hand, by Lemma 4.1 we have $l_{\rho'}(\Gamma) = 0$. Applying Theorem 5.1 to ρ' it follows that $\rho'(\Gamma)$ has a fixed point in $Z \cup \partial Z$, which yields a contradiction. □

In the finite-dimensional case, the following related statement holds. Although we will not actually use it in the present paper, we think it could be useful to record it here.

Proposition 5.3. *Let Γ be a finitely generated group with the fixed-point property for $\mathbb{R}^n$, and Y an $(n+1)$-dimensional* CAT(0) *space. Suppose that $\rho : \Gamma \to \mathrm{Isom}(Y)$ is a homomorphism satisfying*

(i) *$\rho(\Gamma)$ fixes a point in ∂Y, and*
(ii) *there exists a symmetric generating probability measure μ with finite second moment such that the drift $l_\rho(\Gamma)$ of $\rho(\Gamma)$ with respect to μ is equal to zero.*

Then $\rho(\Gamma)$ has almost fixed points.

Proof. Assume that $\rho(\Gamma)$ does not admit almost fixed points:

$$\inf_{x\in Y}\max_{s\in S} d(x,\rho(s)x)=\delta>0.$$

Note that Γ does not admit any nontrivial homomorphisms into $\mathbb{R}$ according to our assumption. Then we can apply Lemma 2.2 to see that there exists a CAT(0) space Z with $\dim Z<\dim Y$ and a homomorphism $\rho'\colon\Gamma\to\mathrm{Isom}(Z)$ such that $\rho'(\Gamma)$ does not fix a point in ∂Z. We also have

$$(\star\star)\qquad \inf_{z\in Z}\max_{s\in S} d(z,\rho'(s)z)\geq\delta>0.$$

Using an argument similar to that of Lemma 4.1, it is not hard to see that $l_{\rho'}(\Gamma)=0$.

Since Z is finite-dimensional and $\rho'(\Gamma)$ does not fix a point in ∂Z, there exists a ρ'-equivariant μ-harmonic map $f\colon\Gamma\to Z$. Then, by [22, Proposition 11], the convex hull F of $f(\Gamma)$ is isometric to $\mathbb{R}^k$ for some $k\leq\dim Z\leq n$. Since F is invariant under the action of $\rho'(\Gamma)$ and ρ' satisfies $(\star\star)$, $\rho'(\Gamma)$ does not fix a point in F, which contradicts the assumption that Γ has the fixed-point property for $\mathbb{R}^n$. Therefore, $\rho(\Gamma)$ must have almost fixed points. □

Using Proposition 4.2 and Proposition 5.2 above, we prove Theorem B.

Theorem 5.4. (Theorem B) *Let Γ be a finitely generated group with fixed-point property for all finite-dimensional Euclidean spaces, and Y a geodesically complete* CAT(0) *space with bounded geometry. Suppose that Γ acts on Y via a homomorphism $\rho\colon\Gamma\to$* $\mathrm{Isom}(Y)$ *and has almost fixed points. Then there is a point in Y fixed by $\rho(\Gamma)$.*

Remark 5.1. The assumptions "Γ has fixed-point property for $\mathbb{R}^n$" and "Y has bounded geometry" are both essential. Let Z be a CAT(0) space, and consider a warped product $Y=\mathbb{R}\times_f Z$. If we take $f(t)=e^t$, then Y is known to be a CAT(-1) space ([3, Theorem 1.1]), and $\mathbb{R}\times\{z\}$ becomes a geodesic line in Y for each $z\in Z$. Suppose a group Γ acts on Z without fixed point. Then the action of Γ on Z extends naturally to a fixed-point free action on Y by letting Γ act trivially on $\mathbb{R}$, while it strongly fixes a point in ∂Y represented by geodesic rays given by the form of $t\mapsto(-t,z)$, $z\in Z$; in particular, there are almost fixed points along these geodesic rays. If we take Z to be $\mathbb{R}^n$ with a standard $\mathbb{Z}^n$-action by translations, the warped product above gives us a hyperbolic space $\mathbb{H}^{n+1}$, which has bounded geometry, and a $\mathbb{Z}^n$-action with almost fixed points. On the other hand, it is easy to see that unless the boundedness of the geometry of Z is (almost) invariant under scaling (i.e., almost $n_{\alpha r,\alpha\varepsilon}(Z)=n_{r,\varepsilon}(Z)$ holds for $\alpha>0$), like $\mathbb{R}^n$, Y cannot have bounded geometry. In the opposite case, Breuillard-Fujiwara observed [9, Prop 1.10] that for any set of isometries of a Euclidean space that possesses almost fixed-points, there is a common fixed-point.

Proof of Theorem 5.4. Suppose that $\rho(\Gamma)$ does not have a fixed point in Y, while there are almost fixed points. Take q_n so that

$$1/n\geq\delta(q_n):=\max_{s\in S} d(q_n,\rho(s)q_n)$$

for some finite generating set S of Γ. Then, as in [8, Lemma 3.2], we can take $p_n\in Y$ and $\{r_n\}\subset\mathbb{R}$ with $r_n\to\infty$ so that $(Y_\infty,p)=\omega\text{-}\lim(r_nY,p_n)$ and $\rho_\infty\colon\Gamma\to\mathrm{Isom}(Y_\infty)$

satisfies

$$\inf_{q\in Y_\infty} \delta_\infty(q) > 0,$$

where $\delta_\infty(q) = \max_{s\in S} d(q, \rho_\infty(s)q)$.

By Proposition 4.2, the Tits boundary of $\partial_T Y_\infty$ is compact. Note that there is a natural homomorphism $\Phi\colon \mathrm{Isom}(Y_\infty) \to \mathrm{Isom}(\partial_T Y_\infty)$, and $\mathrm{Isom}(\partial_T Y_\infty)$ is compact because of the compactness of $\partial_T Y_\infty$. (We denote $\Phi(\rho_\infty(g))$ by $\rho_\infty(g)$ in what follows.) Note also that $\partial_T Y_\infty$ is homeomorphic to the usual geometric boundary ∂Y_∞, since the identity map $\partial_T Y_\infty \to \partial Y_\infty$ is continuous, $\partial_T Y_\infty$ is compact and ∂Y_∞ is Hausdorff.

Take any symmetric generating probability measure μ with finite second moment on Γ and consider a random walk generated by μ. First suppose that the drift $l_{\rho_\infty}(\Gamma)$ of $\rho_\infty(\Gamma)$ is positive. Then, by [28] of Karlsson-Margulis, almost every sample walk converges to a point in ∂Y_∞, and there is an equivariant map φ from the Poisson boundary $(\partial_P\Gamma, \nu)$ to $\partial_T Y_\infty$. In particular, for almost all sample walk g_n, the sequence of measures $\{(\rho_\infty(g_n))_*(\varphi_*\nu) = \varphi_*((g_n)_*\nu)\}$ converges to a Dirac measure supported on a point in the support of $\varphi_*\nu$. However, since $\mathrm{Isom}(\partial_T Y_\infty)$ is compact, we may assume that $\rho_\infty(g_n)$ converges (up to a subsequence) to some $h \in \mathrm{Isom}(\partial_T Y_\infty)$ (depending on the choice of $\{g_n\}$) and

$$\lim_{n\to\infty} (\rho_\infty(g_n))_*(\varphi_*\nu) = h_*(\varphi_*\nu).$$

Therefore the support of $\varphi_*\nu$, which is the image of the Poisson boundary $\partial_P\Gamma$, must be a single point ξ and $\rho_\infty(\Gamma)$ must fix this ξ.

Now denote by c_p^q the geodesic starting from p and terminating at q. The proof of [28, Theorem 2.1] asserts that, for almost all sample walk g_n of (Γ, μ), the sequence of geodesics $\{c_p^{\rho_\infty(g_n)p}\}$ converges to the geodesic ray c_p^ξ starting from p and terminating to $\xi \in \partial Y_\infty$. Furthermore, since we already assume that the drift $l_{\rho_\infty}(\Gamma) > 0$, we have

$$\lim_{n\to\infty} b_\xi(\rho(g_n)p, p) = -\infty$$

as explained in [26, Proposition 4.2] or [27, §3], where $b_\xi(\cdot, p)$ is the Busemann function associated to the geodesic ray c_p^ξ defined as

$$b_\xi(q, p) = \lim_{t\to\infty} d(q, c_p^\xi(t)) - t.$$

However, since $\rho_\infty(\Gamma)$ fixes ξ and Γ does not admit any nontrivial homomorphism into $\mathbb{R}$, $\rho_\infty(\Gamma)$ leaves each horosphere centered at ξ (which is given as the level set of $b_\xi(\cdot, p)$) invariant, and hence $b_\xi(\rho(g_n)p, p)$ must be constant. This leads to a contradiction. Thus, we conclude that the drift $l_{\rho_\infty}(\Gamma)$ is equal to 0.

Since Y is a geodesically complete CAT(0) space with bounded geometry, it is known from Proposition 4.2 that Y_∞ also has bounded geometry. Then by Proposition 5.2 this would imply that $\rho_\infty(\Gamma)$ has almost fixed points in Y_∞, which again contradicts that $\inf_{q\in Y_\infty} \delta_\infty(q) > 0$.

The proof is complete. $\square$

Theorem 5.5. (Theorem F) *Let Γ be a finitely generated group with fixed-point property for all finite-dimensional Euclidean spaces. Let $\rho : \Gamma \to \mathrm{Isom}(Y)$ be an isometric* *action on a geodesically complete* CAT(0) *space* Y *with bounded geometry. Then exactly one of the following holds:*

(1) *for every symmetric probability measure with finite second moment whose support generates* Γ*, the drift* $l_\rho(\Gamma) > 0$*;*
(2) Γ *has a global fixed point in* Y*.*

Proof. Suppose (1) does not hold. That is, the drift $l_\rho(\Gamma) = 0$. Then it follows from Proposition 5.2 that Γ has almost fixed points in Y. By Theorem 5.4 we know that Γ fixes some a point in Y, that is, (2) follows. The proof is complete. ☐

We use concepts and notation taken from [9]. Let S be a finite set of isometries of a metric space X. Let

$$L(S) := \inf_{x\in X} \max_{s\in S} d(x, sx),$$

which is a notion that has already played an important role above. Moreover, let

$$L_\infty(S) = \lim_{n\to\infty} L(S^n)/n$$

and in particular $\lambda(g) := L_\infty(\{g\})$. Finally, $\lambda(S) := \max_{s\in S} \lambda(s)$ and $\lambda_\infty(S) = \limsup_{n\to\infty} \lambda(S^n)/n$. Breuillard and Fujiwara say that the geometric Berger-Wang identity is satisfied if

$$\lambda_\infty(S) = L_\infty(S).$$

They show that symmetric spaces of non-compact type, trees and δ-hyperbolic spaces have the Berger-Wang identity for any finite set S. We now turn to the proof of:

Corollary 5.6. (Corollary G) *Let* Y *be a geodesically complete* CAT(0) *space with bounded geometry. Then any isometric action by a finitely generated torsion group* Γ *on* Y*, which satisfies the geometric Berger-Wang identity for some finite generating set of* Γ*, must fix a point in* Y*.*

Proof. Since the group consists entirely of finite order elements, $\lambda(g) = 0$ for every $g \in \Gamma$. Therefore for any finite set S, it holds that $\lambda(S^n) = 0$ for every n, and hence also $\lambda_\infty(S) = 0$. By the Berger-Wang assumption for a specific generating set S, we thus have that $L_\infty(S) = 0$. This implies in particular that the drift of any probability measure supported on S must be 0. The conclusion now follows from Theorem 5.5. ☐

6. Another alternative for amenable groups acting on CAT(0) spaces

In this section, we use Theorem B to prove Theorem A, the bounded geometry version.

First, we make the following preparation.

Proposition 6.1. *Let* Γ *be a finitely generated amenable group with fixed-point property for all finite-dimensional Euclidean spaces, and let* Y *be a complete* CAT(0) *space with bounded geometry. Then any isometric action of* Γ *on* Y *has almost fixed points.*

Proof. Since Γ is amenable, according to [2, Main Theorem] of Adams-Ballmann, Γ fixes either a flat subspace or an infinity boundary point ξ of Y. By our assumption, for the former case, $\rho(\Gamma)$ has a fixed point in Y, in particular, has almost fixed points;

for the latter case, by Lemma 4.1 there exists a CAT(0) space Z_0 of bounded geometry together with an isometric action of Γ such that Γ does not fix any infinity point in ∂Z_0 and

$$\inf_{x\in Y}\max_{s\in S} d(x,s(x)) \le \inf_{x\in Z_0}\max_{s\in S} d(x,s(x)),$$

where S is some finite generating set of Γ. Applying again [2, Main Theorem] on Γ to Z_0, we see that Γ fixes a flat subspace in Z_0. By our assumption, the group Γ must fix a point in Z_0. Thus, we have

$$\inf_{x\in Y}\max_{s\in S} d(x,s(x)) \le \inf_{x\in Z_0}\max_{s\in S} d(x,s(x)) = 0.$$

This also implies that Γ has almost fixed points in Y. □

Now we are ready to prove Theorem A, first part.

Theorem 6.2. (Theorem A, bounded geometry version) *Let Γ be a finitely generated amenable group. Then exactly one of the following holds:*

(1) *every isometric action of Γ on a geodesically complete* CAT(0) *space with bounded geometry has a global fixed point;*
(2) *Γ acts isometrically on some $\mathbb{R}^n$ $(n \ge 1)$ without a global fixed point.*

Proof. Suppose (2) does not hold. That is, the group Γ has fixed-point property for all finite-dimensional Euclidean spaces. Then (1) follows from Proposition 6.1 and Theorem 5.4. The proof is complete. □

7. Torsion groups cannot act on visibility space of bounded geometry without a fixed point

The results in Section 5 are particularly useful when applied to group actions that are not necessarily properly discontinuous. In this section, we examine isometric actions of torsion groups on visibility spaces. The setting differs from earlier works which usually require action to be properly discontinuous; moreover, the argument does not rely on the Margulis Lemma, a tool that applies only to actions with discrete orbits. For further background on this subject, we refer to [11, Chapter II.9], [18] and [23].

Proposition 7.1 ([24, Theorem 3.5])**.** *Let X be a proper visibility* CAT(0) *space and Γ an infinite torsion group. Suppose that Γ acts on X with an unbounded orbit. Then there is a point $\xi \in \partial X$ such that any convergent unbounded sequence $g_n x_0$ in $\bar{X}$ converges to ξ.*

Proposition 7.2. *Let X be a proper visibility* CAT(0) *space and Γ a countable torsion group equipped with a probability measure μ with finite first moment. Suppose that Γ acts on X via a homomorphism $\rho\colon \Gamma \to \mathrm{Isom}(X)$ with an unbounded orbit. Then there is a point $\xi \in \partial X$ fixed by $\rho(\Gamma)$, and the drift $l_\rho(\Gamma)$ with respect to μ is equal to zero.*

Proof. Fix $x_0 \in X$. By Proposition 7.1, every unbounded convergent sequence $\{g_n \circ x_0\}$ in $\bar{X}$ with $g_n \in \rho(\Gamma)$ converges to a unique point $\xi \in \partial X$. Taking any $h \in \rho(\Gamma)$, the sequence $\{(hg_n) \circ x_0\}$ is also an unbounded convergent sequence in $\bar{X}$, and thus converges to ξ. Since $(hg_n) \circ x_0 = h(g_n \circ x_0) \to h(\xi)$, we have $h(\xi) = \xi$; therefore, ξ is fixed by $\rho(\Gamma)$.

Since Γ does not admit nontrivial homomorphism into $\mathbb{R}$, $\rho(\Gamma)$ preserves every horosphere centered at ξ. In particular, $g_n x_0$ approaches ξ along the horosphere centered at ξ passing through x_0 and the Busemann function $b_\xi(g_n x_0, x_0)$ is identically zero. However, as mentioned in the proof of Theorem 5.4 (see [26, Proposition 4.2] and [27, §3]), if $l_\rho(\Gamma) > 0$, it would follow that $b_\xi(g_n x_0, x_0) \to -\infty$ as $n \to \infty$, a contradiction. Therefore, $l_\rho(\Gamma)$ must vanish. □

Theorem 7.3. (Theorem E) *Let Y be a geodesically complete* CAT(0) *visibility space with bounded geometry, and let Γ be a finitely generated torsion group. Suppose that Γ acts on Y via a homomorphism $\rho\colon \Gamma \to \mathrm{Isom}(Y)$. Then $\rho(\Gamma)$ has a fixed point in Y.*

Proof. Since Γ is finitely generated, it admits a symmetric probability measure μ supported on a finite generating set. In particular, μ has finite first and second moments. By Proposition 7.2, we see that $\rho(\Gamma)$ fixes a point in ∂Y and that the drift $l_\rho(\Gamma)$ of $\rho(\Gamma)$ with respect to μ is zero. By Schur's theorem, Γ has the fixed-point property for all finite-dimensional Euclidean spaces. This, combined with Proposition 5.2, implies that $\rho(\Gamma)$ has almost fixed points. Finally, since Y has bounded geometry, we can apply Theorem 5.4, and conclude that $\rho(\Gamma)$ has a fixed point in Y. □

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