Bosons and Gaussian fields

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I review the classical connection between bosons and Gaussian fields. I describe Wick ordering in its various guises and discuss its relationship to normal ordering of field operators. As an application, I point out a completely elementary combinatorial proof of hypercontractive moment bounds, which, in the regime relevant for most applications, are marginally sharper than their classical form.

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1. Introduction

The purpose of this note is to review the connection between bosons and Gaussian fields, including the relationship between Wick ordering and normal ordering. This material is classical, going back at least to the inception of constructive quantum field theory in the 1960s. These notes are an attempt at an elementary yet comprehensive account, which I had trouble finding in the literature when I tried to clarify these notions to myself. They are essentially self-contained, assuming only basic functional analysis and some probability theory. For the benefit of readers less familiar with quantum mechanics, I have also included some tangential remarks on the connection to quantum mechanics, as well as an appendix on tensor products of Hilbert spaces.

As an application of the main material, I also point out an almost trivial combinatorial proof of hypercontractive moment bounds, which, in the regime relevant for most applications, are marginally sharper than their classical form.

The main sources that I used are the classic book [8, Chapter 1] as well as [6, Chapter 1], [1, Section 5] and [2, Appendix A]. Further standard references are [3, 5].

2. Bosons on Fock space

Let $\mathcal{H}$ be a separable (real or complex) Hilbert space with scalar product $\langle \cdot \,, \cdot \rangle$. If $\mathcal{H}$ is complex, we assume $\langle \cdot \,, \cdot \rangle$ to be linear in the second argument[1]. The space $\mathcal{H}$ is known as the *one-particle space*.

The tensor product space[2] $\mathcal{H}^{\otimes n}$ for $n \in \mathbb{N}^*$ is spanned by vectors of the form $f_1 \otimes \cdots \otimes f_n$, where $f_1, \dots, f_n \in \mathcal{H}$. It is again a Hilbert space with scalar product $\langle f_1 \otimes \cdots \otimes f_n \,, g_1 \otimes \cdots \otimes g_n \rangle := \langle f_1 \,, g_1 \rangle \cdots \langle f_n \,, g_n \rangle$, extended to $\mathcal{H}^{\otimes n}$ by linearity. By convention, $\mathcal{H}^{\otimes 0}$ is $\mathbb{R}$ or $\mathbb{C}$, depending on whether $\mathcal{H}$ is real or complex.

On $\mathcal{H}^{\otimes n}$ we define the symmetrisation[3] P_n through

$$P_n f_1 \otimes \cdots \otimes f_n := \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \,,$$

where the sum ranges over all permutations σ of n elements. Clearly, P_n is an orthogonal projection on $\mathcal{H}^{\otimes n}$. We define the *bosonic Fock space*

$$\mathcal{F} := \bigoplus_{n \in \mathbb{N}} \mathcal{F}_n \,, \qquad \mathcal{F}_n := P_n \mathcal{H}^{\otimes n} \,.$$

We use the notation $\Psi = (\Psi_n)_{n \in \mathbb{N}} \in \mathcal{F}$ for the elements of $\mathcal{F}$, where $\Psi_n \in \mathcal{F}_n$. The space $\mathcal{F}_n$ is known as the *n-particle space*. By definition, the *vacuum* of $\mathcal{F}$ is

$$\mathcal{V} := (1, 0, 0, \dots) \in \mathcal{F} \,.$$

The Fock space is a Hilbert space with scalar product

$$\langle \Psi \,, \Psi' \rangle := \sum_{n \in \mathbb{N}} \langle \Psi_n \,, \Psi'_n \rangle \,.$$

We denote by $\mathcal{F}_*$ the subspace of $\mathcal{F}$ consisting of sequences $\Psi = (\Psi_n)_{n \in \mathbb{N}}$ with only finitely many nonzero components.

[1] This convention seems more common in the physics literature, while the opposite convention is more prevalent in the mathematics literature. I would like to argue that, as it unfortunately so often happens, it is the physicists who got things right. Indeed, it is natural to regard vectors $f \in \mathcal{H}$ as column vectors, so that the action of an operator H on a vector f can be regarded as matrix multiplication Hf. With the column vector $f \in \mathcal{H}$ one can associate its *dual* vector f^*, which, in matrix notation, is the transpose conjugate of f (a row vector). Then the scalar product is naturally written as $\langle f \,, g \rangle = f^* g$, where the right-hand side can again be naturally interpreted as matrix multiplication.

[2] See Appendix A for a brief review of tensor products of Hilbert spaces.

[3] An analogous construction can be done for *fermions* instead of bosons, where in the definition of P_n one has to weight the terms of the sum over σ with $\operatorname{sgn} \sigma$.

Definition 2.1 (creation and annihilation operators)**.** Let $f \in \mathcal{H}$. We define the *creation operator* $a^*(f) : \mathcal{F}_n \to \mathcal{F}_{n+1}$ and the *annihilation operator* $a(f) : \mathcal{F}_n \to \mathcal{F}_{n-1}$ as follows. For $n \in \mathbb{N}$ and $\Psi_n \in \mathcal{F}_n$, we set

$$a^*(f)\Psi_n := \sqrt{n+1}\, P_{n+1}\, f \otimes \Psi_n\,,$$

and for $n \in \mathbb{N}^*$ and $P_n\, f_1 \otimes \cdots \otimes f_n \in \mathcal{F}_n$ with $f_1, \dots, f_n \in \mathcal{H}$, we set[4]

$$a(f)\, P_n\, f_1 \otimes \cdots \otimes f_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \langle f\,, f_i \rangle\, P_{n-1}\, f_1 \otimes \overset{\not{i}}{\cdots} \otimes f_n\,,$$

By definition, $a(f)\mathcal{V} := 0$.

Remark 2.2. By linearity and density of product tensors, the creation and annihilation operators extend to linear operators on $\mathcal{F}_*$, on which they are each other's adjoints (exercise), $a(f)^* = a^*(f)$. They are unbounded, but it is not hard to show (exercise) that $a^*(f)$ and $a(f)$ are closable on $\mathcal{F}$ and extend to closed linear operators, which are each other's adjoints. We denote the latter operators again by $a^*(f)$ and $a(f)$.

Remark 2.3. The operator $a^*(f)$ is linear in f, while $a(f)$ is conjugate linear in f. Sometimes they are also written as $a^*(f) = \langle a\,, f\rangle$ and $a(f) = \langle f\,, a\rangle$.

Remark 2.4. For the reader unfamiliar with quantum mechanics, it might be helpful to give some context to these definitions; I emphasise that this remark is entirely tangential and can be ignored without fear of dire consequences. Suppose we are studying quantum particles in some region $D \subset \mathbb{R}^3$ of physical space. The one-particle space is then $\mathcal{H} = L^2(D, \mathbb{C})$ (with respect to Lebesgue measure). The state of a single particle in D is given by the *one-particle wave function* $f \in \mathcal{H}$. Famously, the quantity $\frac{1}{\|f\|^2}|f(x)|^2\,\mathrm{d}x$ describes the probability of finding the particle in the region $\mathrm{d}x$ given that it is in the state f. (The wave function determines values of other observables as well, such as momentum or energy – something we do not discuss here.)

The one-particle state f can be obtained by *creating* a particle in state f from the vacuum: $f = a^*(f)\mathcal{V}$. Similarly, we can obtain the vacuum from the state f by *annihilating* a particle in state f: $a(f)f = \mathcal{V}$.

Let us now consider several bosons. The state of n identical bosons in the physical region D is characterised by its n-particle wave function $\Psi_n \in \mathcal{F}_n$. The quantity $\frac{1}{\|\Psi_n\|^2}|\Psi_n(x_1, \dots, x_n)|^2\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$ describes the probability of finding exactly one particle in each of the regions $\mathrm{d}x_1, \dots, \mathrm{d}x_n$. For example, suppose that $f_1, \dots, f_n$ are functions in $\mathcal{H}$ that are compactly supported with disjoint supports. In particular, they are orthogonal. Having n bosons in D in the individual one-particle states $f_1, \dots, f_n$ respectively corresponds to the n-particle wave function $\Psi_n = P_n f_1 \otimes \cdots \otimes f_n$. If we annihilate a particle in state f_1, we obtain the $(n-1)$-particle state $a(f_1)\Psi_n = \frac{1}{\sqrt{n}} P_{n-1} f_2 \otimes \cdots \otimes f_n$. Creating a particle in state f_1 back again results in the state $a^*(f_1)a(f_1)\Psi_n = \Psi_n$. Similarly, we obtain $\sum_{i=1}^n a^*(f_i)a(f_i)\Psi_n = n\Psi_n$. Generally, one easily finds that the operator

$$\mathcal{N} := \sum_{i \in \mathbb{N}^*} a^*(e_i)a(e_i)\,, \tag{2.1}$$

where $(e_i)_{i\in\mathbb{N}^*}$ is an orthonormal basis of $\mathcal{H}$, is equal to n times the identity on $\mathcal{F}_n$. For this reason, $\mathcal{N}$ is called the *particle number operator*.

[4] We use the notation $\overset{\not{i}}{\cdots}$ to indicate that the ith term is excluded from the product.

We denote by $[A, B] := AB - BA$ the *commutator* of two linear operators A, B on $\mathcal{F}$. From Definition 2.1 it is left as an exercise to deduce the following result.

Lemma 2.5 (Canonical commutation relations)**.** *For $f, g \in \mathcal{F}$ we have*

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0\,, \qquad [a(f), a^*(g)] = \langle f\,, g\rangle I\,,$$

where I is the identity on $\mathcal{F}$.

Remark 2.6. The entire Fock space $\mathcal{F}$ is spanned by applying creation operators to the vacuum, since

$$P_n\, f_1 \otimes \cdots \otimes f_n = \frac{1}{\sqrt{n!}}\, a^*(f_1)\cdots a^*(f_n)\mathcal{V}\,,$$

as follows from a simple computation using the definition of a^*.

Definition 2.7 (Field operator)**.** For $f \in \mathcal{H}$ we define the *field operator*

$$\Phi(f) := a(f) + a^*(f)\,.$$

As above, $\Phi(f)$ is defined on $\mathcal{F}_*$, on which it is essentially self-adjoint[5]. We denote its self-adjoint extension again by $\Phi(f)$.

Remark 2.8. If $\mathcal{H}$ is real then Φ is linear in f and the family of self-adjoint operators $\{\Phi(f) : f \in \mathcal{H}\}$ commutes, by Lemma 2.5.

Remark 2.9. We emphasise the importance of the assumption that $\mathcal{H}$ be real for the validity of Remark 2.8. Generally, for complex $\mathcal{H}$ we can restrict ourselves to a real subspace of $\mathcal{H}$. To that end, let $f \mapsto \overline{f}$ be a complex conjugation on $\mathcal{H}$, i.e. an antiunitary involution on $\mathcal{H}$. Defining the real subspace $\mathcal{H}_\mathbb{R} := \{f \in \mathcal{H} : f = \overline{f}\}$, which is a real Hilbert space, we find that Remark 2.8 holds for all $f \in \mathcal{H}_\mathbb{R}$.

Remark 2.10. Analogously to Remark 2.6, the vectors of the form $\Phi(f_1)\cdots\Phi(f_n)\mathcal{V}$ span $\mathcal{F}$. Indeed, using Lemma 2.5 we find

$$\Phi(f_1)\cdots\Phi(f_n)\mathcal{V} - a^*(f_1)\cdots a^*(f_n)\mathcal{V} \in \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{n-1}\,,$$

and the claim follows from Remark 2.6.

Remark 2.11. For $\mathcal{H} = \mathbb{C}^d$, the bosonic Fock space $\mathcal{F}$ is the phase space of the d-dimensional quantum harmonic oscillator. This connection is given by $\mathcal{F} \simeq L^2(\mathbb{R}^d)$, where the implicit isomorphism is defined as follows. Let $e_1, \ldots, e_d$ be the canonical basis of $\mathbb{C}^d$ and abbreviate $a_i := a(e_i)$ and $a_i^* := a^*(e_i)$. On $L^2(\mathbb{R}^d)$ we define the differential operators, for $i = 1, \ldots, d$,

$$A_i := \frac{1}{\sqrt{2}}\left(x_i + \frac{\partial}{\partial x_i}\right) \qquad A_i^* := \frac{1}{\sqrt{2}}\left(x_i - \frac{\partial}{\partial x_i}\right).$$

It is easy to check that A_i and A_i^* satisfy all of the algebraic properties of a_i and a_i^*: they are each other's adjoints and they satisfy the canonical commutation relations

$$[A_i, A_j] = 0\,, \qquad [A_i^*, A_j^*] = 0\,, \qquad [A_i, A_j^*] = \delta_{ij}\,. \tag{2.2}$$

[5] See for instance [8, Section I.3]

The vector $\mathcal{W}$ corresponding to the vacuum in $L^2(\mathbb{R}^d)$ is determined by $A_i \mathcal{W} = 0$ for all i. The unique normalised solution of this equation is the Gaussian

$$\mathcal{W}(x) := \pi^{-d/4} \mathrm{e}^{-\frac{1}{2}|x|^2} .$$

Then we define the isomorphism U through its action

$$U : a^*(f_1) \cdots a^*(f_n)\mathcal{V} \to A^*(f_1) \cdots A^*(f_n)\mathcal{W} ,$$

and extend it by linearity. That U is an isometry follows from the canonical commutation relations Lemma 2.5 and (2.2), as well as from $a_i \mathcal{V} = 0$ and $A_i \mathcal{W} = 0$ for all $i = 1, \dots, d$. To show that U is a unitary map $U : \mathcal{F} \to L^2(\mathbb{R}^d)$, by Remark 2.6, it suffices to show that vectors of the form $A^*_{i_1} \cdots A^*_{i_n} \mathcal{W}$ span $L^2(\mathbb{R}^d)$. The latter follows from the observation that $A^*_{i_1} \cdots A^*_{i_n} \mathcal{W}$ is clearly a polynomial of degree n multiplied by $\mathcal{W}$, and such functions are dense in $L^2(\mathbb{R}^d)$; this standard fact is also worked out in detail after (3.4) below.

The d-dimensional quantum harmonic oscillator is defined by its Hamiltonian

$$H := -\frac{1}{2}\Delta + \frac{1}{2}|x|^2$$

acting $L^2(\mathbb{R}^d)$. It can be written as

$$H = \sum_{i=1}^d \biggl(A_i^* A_i + \frac{1}{2} \biggr) .$$

(Note that in the Fock space representation associated with the isomorphism U, the operator H is, up to a constant, equal to the particle number operator (2.1).) Since $A_i^* A_i (A_i^*)^n \mathcal{W} = n (A_i^*)^n \mathcal{W}$ by (2.2), we conclude that the eigenbasis of H consists of the normalised vectors

$$\Psi_{n_1 \dots n_d} := \prod_{i=1}^d \frac{(A_i^*)^{n_i}}{\sqrt{n_i!}} \mathcal{W} , \tag{2.3}$$

with corresponding eigenvalues $\sum_{i=1}^d (n_i + \frac{1}{2})$. That $\Psi_{n_1 \dots n_d}$ is normalised follows easily from (2.2). Explicitly, we have

$$\Psi_{n_1 \dots n_d}(x_1, \dots, x_d) = \prod_{i=1}^d \frac{1}{\pi^{1/4} \sqrt{n_i!}} \mathrm{e}^{-\frac{1}{2}x_i^2} 2^{n_i/2} H_{n_i}(\sqrt{2} x_i) , \tag{2.4}$$

where

$$H_n(x) := (-1)^n \mathrm{e}^{\frac{1}{2}x^2} \biggl(\frac{\mathrm{d}}{\mathrm{d}x} \biggr)^n \mathrm{e}^{-\frac{1}{2}x^2} \tag{2.5}$$

is the nth Hermite polynomial[6]. The form (2.4) follows from (2.3) and

$$\biggl(x - \frac{\mathrm{d}}{\mathrm{d}x} \biggr)^n \mathrm{e}^{-\frac{1}{2}x^2} = \biggl(x - \frac{\mathrm{d}}{\mathrm{d}x} \biggr)^n \mathrm{e}^{\frac{1}{2}x^2} \mathrm{e}^{-x^2} = \mathrm{e}^{\frac{1}{2}x^2} \biggl(- \frac{\mathrm{d}}{\mathrm{d}x} \biggr)^n \mathrm{e}^{-x^2} = \mathrm{e}^{-\frac{1}{2}x^2} 2^{n/2} H_n(\sqrt{2}x) .$$

[6] In (2.5) we use the "probabilist's convention", where the Hermite polynomials are orthogonal with respect to the standard Gaussian (3.1). In the "physicist's convention", the nth Hermite polynomial is $(-1)^n \mathrm{e}^{x^2} (\frac{\mathrm{d}}{\mathrm{d}x})^n \mathrm{e}^{-x^2} = 2^{n/2} H_n(\sqrt{2}x)$, which is also the polynomial naturally appearing on the right-hand side of (2.4).

We conclude this section with Wick's theorem for the field operators. We shall use the following definitions throughout this note.

Definition 2.12. Let I be a finite set.

(i) We denote by $\mathfrak{P}_I$ the set of complete pairings of I. That is, $\mathfrak{P}_I$ consists of all sets of disjoint pairs of elements in I whose union is I. In particular, $\mathfrak{P}_I = \emptyset$ if $|I|$ is odd.

(ii) We denote by $\mathfrak{M}_I$ the set of partial pairings of I. That is, $\mathfrak{M}_I$ consists of all sets of disjoint pairs of elements in I.

(iii) For a partial pairing $\Pi \in \mathfrak{M}_I$ we denote by $\langle \Pi \rangle := \bigcup_{\pi \in \Pi} \pi$ the subset of paired indices.

Moreover, for $n \in \mathbb{N}^*$ we abbreviate $[n] := \{1, \dots, n\}$.

Lemma 2.13 (Wick's theorem)**.** *For $f_1, \dots, f_n \in \mathcal{H}$ we have*

$$\langle \mathcal{V}, \Phi(f_1) \cdots \Phi(f_n) \mathcal{V} \rangle = \sum_{\Pi \in \mathfrak{P}_{[n]}} \prod_{\{i,j\} \in \Pi} \langle f_i \, , f_j \rangle \, ,$$

where for each pair $\{i, j\} \in \Pi$ we use the convention[7] $i < j$.

Proof. Since $a^*(f)$ is the adjoint of $a(f)$ and since $[a(f), \Phi(g)] = \langle f \, , g \rangle I$ by Lemma 2.5, we find

$$\begin{aligned}
\langle \mathcal{V}, \Phi(f_1) \cdots \Phi(f_n) \mathcal{V} \rangle &= \langle \mathcal{V}, a(f_1) \Phi(f_2) \cdots \Phi(f_n) \mathcal{V} \rangle \\
&= \langle \mathcal{V}, \Phi(f_2) a(f_1) \Phi(f_3) \cdots \Phi(f_n) \mathcal{V} \rangle + \langle f_1 \, , f_2 \rangle \langle \mathcal{V}, \Phi(f_3) \cdots \Phi(f_n) \mathcal{V} \rangle \\
&= \dots \\
&= \sum_{i=2}^{n} \langle f_1 \, , f_i \rangle \langle \mathcal{V}, \Phi(f_2) \overset{\not i}{\cdots} \Phi(f_n) \mathcal{V} \rangle \, ,
\end{aligned}$$

and the claim follows by induction. □

3. Gaussian field and the isomorphism

Throughout this section, we suppose that $\mathcal{H}$ is real. Let ϕ be the Gaussian field associated with the Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot \, , \cdot \rangle$. That is, $(\phi(f))_{f \in \mathcal{H}}$ is a Gaussian process with mean zero and covariance

$$\mathbb{E}[\phi(f)\phi(g)] = \langle f \, , g \rangle \, .$$

An explicit construction goes as follows. Let $\omega = (\omega_n)_{n \in \mathbb{N}}$ be independent identically distributed standard normal $\mathcal{N}(0, 1)$ random variables. The underlying probability space is $\Omega = \mathbb{R}^{\mathbb{N}}$, equipped with the Borel cylinder σ-algebra $\mathcal{F}$; it carries the product probability measure $\mathbb{P} = \mu^{\otimes \mathbb{N}}$, where

$$\mu(\mathrm{d}x) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2} \, \mathrm{d}x \tag{3.1}$$

is the law of the standard Gaussian. Let $(e_n)_{n \in \mathbb{N}^*}$ be an orthonormal basis of $\mathcal{H}$, and define the random variable

$$\phi(f) := \sum_{n \in \mathbb{N}^*} \omega_n \langle e_n \, , f \rangle$$

[7] If $\mathcal{H}$ is real than this order does not matter.

on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is easy to check that the sum converges in $L^2(\Omega) \equiv L^2(\Omega, \mathcal{F}, \mathbb{P})$.

The Gaussian field ϕ satisfies the following version of Wick's lemma.

Lemma 3.1 (Wick lemma for Gaussian field). *For $f_1, \dots, f_n$ we have*

$$\mathbb{E}[\phi(f_1) \cdots \phi(f_n)] = \sum_{\Pi \in \mathfrak{P}_{[n]}} \prod_{\{i,j\} \in \Pi} \langle f_i \,, f_j \rangle \,.$$

In fact, Lemma 3.1 is a special case of Lemma 4.5 whose proof is given below.

Lemmas 2.13 and 3.1 suggest an equivalence between $\Phi(f)$ and $\phi(f)$, which is the content of the following lemma.

Lemma 3.2 (Isomorphism). *There is an isomorphism (i.e. a unitary map) $D : L^2(\Omega) \to \mathcal{F}$ satisfying $D1 = \mathcal{V}$ and $D\phi(f)D^{-1} = \Phi(f)$ for all $f \in \mathcal{H}$.*

Proof. Such an operator clearly has to satisfy

$$D\phi(f_1) \cdots \phi(f_n) = \Phi(f_1) \cdots \Phi(f_n)\mathcal{V} \,,$$

and we use this formula to define D on $L^2(\Omega)$-random variables of the form $\phi(f_1) \cdots \phi(f_n)$. By Lemmas 2.13 and 3.1, this linear map is an isometry. Its surjectivity on $L^2(\Omega)$ follows from Remark 2.10 and Lemma 3.3 below. ☐

Lemma 3.3. *The span of random variables of the form $\phi(f_1) \cdots \phi(f_n)$, where $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathcal{H}$, is dense in $L^2(\Omega)$.*

Proof. We approximate $\mathcal{H}$ with finite-dimensional subspaces: for $N \in \mathbb{N}$ let $\mathcal{H}_N := \operatorname{span}\{e_1, \dots, e_N\}$, where $(e_n)_{n \in \mathbb{N}^*}$ is an orthonormal basis of $\mathcal{H}$. Denote by $\mathcal{F}_N$ the σ-algebra generated by the random variables $\{\phi(f) : f \in \mathcal{H}_N\}$. We shall prove that, for any $N \in \mathbb{N}$,

$$L^2(\Omega, \mathcal{F}_N, \mathbb{P}) = \overline{\operatorname{span}\{\phi(f_1) \cdots \phi(f_n) : n \in \mathbb{N}, f_1, \dots, f_n \in \mathcal{H}_N\}} \,. \tag{3.2}$$

Supposing (3.2) for the moment, we show how to conclude the argument. Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then $X_N := \mathbb{E}[X | \mathcal{F}_N]$ is a (Doob) martingale and, by Doob's martingale convergence theorem (see e.g. [9, Chapter 12]) we have $X_N \to X$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Since, by (3.2), X_N can be approximated by finite linear combinations of random variables of the form $\phi(f_1) \cdots \phi(f_n)$, the claim follows.

What remains, therefore, is the proof of (3.2). It suffices to show that if $X \in L^2(\Omega, \mathcal{F}_N, \mathbb{P})$ satisfies the orthogonality condition

$$\mathbb{E}[X\phi(f_1) \cdots \phi(f_n)] = 0 \tag{3.3}$$

for all $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathcal{H}_N$, then $X = 0$. Since X is $\mathcal{F}_N$-measurable, we can write it as $X = h(\phi(e_1), \dots, \phi(e_N))$ for some measurable function $f : \mathbb{R}^N \to \mathbb{R}$ that is square integrable with respect to the standard Gaussian measure $\mu^{\otimes N}$ on $\mathbb{R}^N$. The condition (3.3) hence reads

$$\int_{\mathbb{R}^N} \mu^{\otimes N}(\mathrm{d}x) \, h(x) \, P(x) = 0 \tag{3.4}$$

for any polynomial P of N variables. We then expand the Fourier transform

$$\int_{\mathbb{R}^N} \mu^{\otimes N}(\mathrm{d}x) \, h(x) \, \mathrm{e}^{\mathrm{i}\lambda \cdot x} = \sum_{n \geqslant 0} \frac{\mathrm{i}^n}{n!} \int_{\mathbb{R}^N} \mu^{\otimes N}(\mathrm{d}x) \, h(x) \, (\lambda \cdot x)^n \,, \tag{3.5}$$

where the right-hand side is absolutely convergent by Cauchy-Schwarz:

$$\left|\int_{\mathbb{R}^N} \mu^{\otimes N}(\mathrm{d}x)\, h(x)\, (\lambda\cdot x)^n\right| \leqslant \|X\|_{L^2}\sqrt{\int_{\mathbb{R}^N} \mu^{\otimes N}(\mathrm{d}x)\, |\lambda\cdot x|^{2n}}$$
$$= \|X\|_{L^2}|\lambda|^n\sqrt{\int_{\mathbb{R}} \mu(\mathrm{d}x)\, |x|^{2n}} \leqslant \|X\|_{L^2}(C\sqrt{n}|\lambda|)^n$$

for some universal constant C. This proves the absolute convergence of the sum in (3.5) for any $\lambda \in \mathbb{R}^N$. By the condition (3.4), we therefore conclude that (3.5) vanishes for all $\lambda \in \mathbb{R}^N$. Hence, h is almost surely zero, which concludes the proof. □

4. Wick ordering

Let X be a random variable with finite moments. The nth Wick power of X, denoted by $:X^n:$, is by definition a polynomial in X satisfying

$$:X^0: = 1\,, \qquad \frac{\mathrm{d}}{\mathrm{d}X}:X^n: = n:X^{n-1}:\,, \qquad \mathbb{E}[:X^n:] = 0\,. \tag{4.1}$$

For example, $:X: = X - \mathbb{E}[X]$ and $:X^2: = X^2 - 2\mathbb{E}[X]X + 2\mathbb{E}[X]^2$. The Wick exponential of X is by definition the formal power series

$$:\mathrm{e}^{\lambda X}: = \sum_{n\geqslant 0}\frac{\lambda^n}{n!}:X^n:\,.$$

From (4.1) we obtain that $\frac{\mathrm{d}}{\mathrm{d}X}:\mathrm{e}^{\lambda X}: = \lambda:\mathrm{e}^{\lambda X}:$ and $\mathbb{E}[:\mathrm{e}^{\lambda X}:] = 1$, which implies that

$$:\mathrm{e}^{\lambda X}: = \frac{\mathrm{e}^{\lambda X}}{\mathbb{E}[\mathrm{e}^{\lambda X}]}\,.$$

Suppose now that X is a standard Gaussian random variable. Then the Wick exponential is convergent and

$$:\mathrm{e}^{\lambda X}: = \mathrm{e}^{\lambda X - \frac{1}{2}\lambda^2}\,.$$

Since the right-hand side is the generating function of the Hermite polynomials H_n from (2.5), we find

$$:X^n: = H_n(X)\,. \tag{4.2}$$

More precisely, by completing the square and using that the standard Gaussian is its own Fourier transform, we obtain

$$H_n(X) = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n \mathrm{e}^{\lambda X - \frac{1}{2}\lambda^2}\bigg|_{\lambda=0} = \mathrm{e}^{\frac{1}{2}X^2}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n \mathrm{e}^{-\frac{1}{2}(\lambda - X)^2}\bigg|_{\lambda=0}$$
$$= \mathrm{e}^{\frac{1}{2}X^2}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n \frac{1}{\sqrt{2\pi}}\int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}t(\lambda - X)}\mathrm{e}^{-\frac{1}{2}t^2}\bigg|_{\lambda=0} = (-1)^n \mathrm{e}^{\frac{1}{2}X^2}\left(\frac{\mathrm{d}}{\mathrm{d}X}\right)^n \mathrm{e}^{-\frac{1}{2}X^2}\,, \tag{4.3}$$

in agreement with (2.5).

The above is easily generalised to general Gaussian vectors.

Definition 4.1 (Wick ordering)**.** Let $X = (X_1, \dots, X_n)$ be a real[8] Gaussian vector with mean zero. We define the *Wick ordering of the monomial* $X_1 \cdots X_n$ through

$$:X_1 \cdots X_n: = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \frac{\mathrm{e}^{\lambda \cdot X}}{\mathbb{E}[\mathrm{e}^{\lambda \cdot X}]}\bigg|_{\lambda=0}. \tag{4.4}$$

The expectation in the denominator of (4.4) is equal to

$$\mathbb{E}[\mathrm{e}^{\lambda \cdot X}] = \mathrm{e}^{\frac{1}{2}\lambda \cdot \mathcal{C}\lambda},$$

where $\mathcal{C} = (\mathcal{C}_{ij})_{i,j=1}^n$ with

$$\mathcal{C}_{ij} := \mathbb{E}[X_i X_j]$$

is the covariance matrix of X. Hence (4.4) becomes

$$:X_1 \cdots X_n: = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \mathrm{e}^{\lambda \cdot X - \frac{1}{2}\lambda \cdot \mathcal{C}\lambda}\bigg|_{\lambda=0}. \tag{4.5}$$

There are several equivalent characterisations of Wick order that follow easily from this definition.

Remark 4.2 (Definition by pairings)**.** Computing the derivatives in (4.5) explicitly, we easily find

$$:X_1 \cdots X_n: = \sum_{\Pi \in \mathfrak{M}_{[n]}} \prod_{i \in [n] \setminus \langle \Pi \rangle} X_i \prod_{\{i,j\} \in \Pi} (-\mathbb{E}[X_i X_j]), \tag{4.6}$$

where we recall the notation from Definition 2.12.

Remark 4.3 (Definition by recursion)**.** By splitting the summation in (4.6) over Π satisfying $n \notin \langle \Pi \rangle$ and $n \in \langle \Pi \rangle$, we obtain the recursion

$$:X_1 \cdots X_n: = :X_1 \cdots X_{n-1}: X_n - \sum_{i=1}^{n-1} \mathbb{E}[X_i X_n] :X_1 \overset{\not i}{\cdots} X_{n-1}:. \tag{4.7}$$

Remark 4.4 (Definition by partial derivatives)**.** Denote by ∇ the gradient in the variable X. Using that $\nabla \mathrm{e}^{\lambda \cdot X} = \lambda \mathrm{e}^{\lambda \cdot X}$ and that partial derivatives commute, we get from (4.5)

$$:X_1 \cdots X_n: = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \mathrm{e}^{-\frac{1}{2}\lambda \cdot \mathcal{C}\lambda}\, \mathrm{e}^{\lambda \cdot X}\bigg|_{\lambda=0} = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \mathrm{e}^{-\frac{1}{2}\nabla \cdot \mathcal{C}\nabla}\, \mathrm{e}^{\lambda \cdot X}\bigg|_{\lambda=0} = \mathrm{e}^{-\frac{1}{2}\nabla \cdot \mathcal{C}\nabla} X_1 \cdots X_n, \tag{4.8}$$

where all exponentials are interpreted as formal power series.

Since both sides of (4.6), (4.7) and (4.8) are linear in $X_1, \dots, X_n$, using any of Remark 4.2, Remark 4.3 or Remark 4.4 we can directly define $:X_1 \cdots X_n:$ for a complex Gaussian vector $(X_1, \dots, X_n)$.

Moreover, from the definition (4.4) we find that Wick ordering commutes with differentiation, since

$$\frac{\partial}{\partial X_1} :X_1 \cdots X_n: = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \frac{\lambda_1 \mathrm{e}^{\lambda \cdot X}}{\mathbb{E}[\mathrm{e}^{\lambda \cdot X}]}\bigg|_{\lambda=0} = \frac{\partial^{n-1}}{\partial \lambda_2 \cdots \partial \lambda_n} \frac{\mathrm{e}^{\sum_{i=2}^n \lambda_i X_i}}{\mathbb{E}[\mathrm{e}^{\sum_{i=2}^n \lambda_i X_i}]}\bigg|_{\lambda=0} = :X_2 \cdots X_n:. \tag{4.9}$$

[8] When dealing with complex vectors, we split them into their real and imaginary parts.

Note that the variables $X_1, \dots, X_n$ are treated as independent for the differentiation, although they need not be stochastically independent. For instance, (4.9) implies that $\frac{\mathrm{d}}{\mathrm{d}X} :X^n: = n:X^{n-1}:$, in agrement with (4.1). We also caution against regarding $:\cdot:$ as a linear map on the space of polynomials in X; it is merely a formal operation on a monomial (see Remark 5.4 below).

The following is a generalisation of the Wick lemma, Lemma 3.1, to moments of Wick-ordered monomials. It reduces to Lemma 3.1 in the case where Q is the trivial partition with n blocks.

Lemma 4.5 (General Wick lemma). *Let $X = (X_1, \dots, X_n)$ be a complex Gaussian vector with mean zero. Let Q be a partition of $[n]$. Then*

$$\mathbb{E}\Biggl[\prod_{q \in Q} :\prod_{i \in q} X_i:\Biggr] = \sum_{\Pi \in \mathfrak{P}_{[n]}(Q)} \prod_{\{i,j\} \in \Pi} \mathbb{E}[X_i X_j]\,,$$

where $\mathfrak{P}_{[n]}(Q)$ is the set of complete pairings Π of the set $[n]$ such that no pair $\{i,j\} \in \Pi$ satisfies $i, j \in q$ for some $q \in Q$.

Proof. By linearity, we may assume that $(X_1, \dots, X_n)$ is real. From the definition (4.4) we find

$$\prod_{q \in Q} :\prod_{i \in q} X_i: = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \mathrm{e}^{\lambda \cdot X} \prod_{q \in Q} \exp\biggl(-\frac{1}{2} \sum_{i,j \in q} \mathcal{C}_{ij} \lambda_i \lambda_j\biggr)\bigg|_{\lambda = 0}\,,$$

so that taking the expectation yields

$$\mathbb{E}\Biggl[\prod_{q \in Q} :\prod_{i \in q} X_i:\Biggr] = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \exp\biggl(\frac{1}{2} \sum_{i,j \in [n]} \mathcal{C}^Q_{ij} \lambda_i \lambda_j\biggr)\bigg|_{\lambda = 0}\,,$$

where $\mathcal{C}^Q_{ij} := (1 - \sum_{q \in Q} \mathbf{1}_{i,j \in q}) \mathcal{C}_{ij}$. The claim now follows by differentiation. □

Lemma 4.5 provides an interpretation of how Wick ordering renormalises monomials by forbidding pairings in Lemma 3.1: instead of allowing all pairings as in Lemma 3.1, thanks to Wick ordering only pairings between different Wick-ordered monomials are allowed.

5. Polynomial chaos

Let ϕ be the Gaussian field associated with the Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$.

Definition 5.1. Let $n \in \mathbb{N}$. The nth polynomial chaos, denoted by $\mathcal{W}_n$, is the L^2-closure of

$$\operatorname{span}\{:\phi(f_1) \cdots \phi(f_n): : f_1, \dots, f_n \in \mathcal{H}\}.$$

Lemma 5.2. *We have $L^2(\Omega) = \bigoplus_{n \in \mathbb{N}} \mathcal{W}_n$.*

Proof. First, we note that if $n \neq m$ then $\mathcal{W}_n$ and $\mathcal{W}_m$ are orthogonal. Indeed, by Lemma 4.5,

$$\mathbb{E}[:\phi(f_1) \cdots \phi(f_n)::\phi(f_{n+1}) \cdots \phi(f_{n+m}):] = 0$$

for all $f_1, \dots, f_{n+m} \in \mathcal{H}$ because $\mathfrak{P}_{[n+m]}(\{\{1, \dots, n\}, \{n+1, \dots, n+m\}\}) = \emptyset$. The claim now follows from Lemma 3.3 and the observation that

$$:\phi(f_1) \cdots \phi(f_n): - \phi(f_1) \cdots \phi(f_n) \in \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_{n-1}\,. \tag{5.1}$$

□

Remark 5.3. Define $\mathcal{X}_n$ as is the L^2-closure of

$$\operatorname{span}\{\phi(f_1)\cdots\phi(f_n) : f_1,\dots,f_n \in \mathcal{H}\}.$$

Denote by Π_n the orthogonal projection onto the orthogonal complement of $\mathcal{X}_0 \oplus \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n$. Then we claim that

$$:\phi(f_1)\cdots\phi(f_n): = \Pi_{n-1}\phi(f_1)\cdots\phi(f_n)\,. \tag{5.2}$$

Indeed, (5.2) follows from the observations that (i) both sides of (5.2) are in $\mathcal{W}_n$ and (ii) the difference of the left-hand side and right-hand side is in $\mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_{n-1}$. Both of these observations follow from (5.1), which in particular implies $\mathcal{X}_0 \oplus \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n$. As a consequence, we also have

$$\mathcal{W}_n = \Pi_{n-1}\mathcal{X}_n$$

Remark 5.4. In (5.2), the index $n-1$ of Π_{n-1} depends on the degree of the monomial to which it is applied. This illustrates the danger of regarding $:\cdot:$ as a linear operator on the space of polynomials spanned by $\phi(f)$, $f \in \mathcal{H}$. For instance (writing $X = \phi(f)$ for some $f \in \mathcal{H}$), we have

$$\Pi_1(X^2 + X) = :X^2: + 0 \neq :X^2: + :X: = \Pi_1(X^2) + \Pi_0(X)\,.$$

Example 5.5 (Wiener chaos)**.** For $\mathcal{H} := L^2(\mathbb{R}_+)$ the associated Gaussian field ϕ is white noise on the interval $\mathbb{R}_+$, i.e.

$$\mathbb{E}[\phi(f)\phi(g)] = \int_0^\infty \mathrm{d}t\, f(t)g(t)\,.$$

Then the process $B = (B(t))_{t\in\mathbb{R}_+}$ with

$$B(t) := \phi(\mathbf{1}_{[0,t)})$$

is Brownian motion. More precisely, B has the same finite-dimensional distributions as Brownian motion, and hence by Kolmogorov's continuity theorem it has an a.s. continuous modification, which we henceforth call B. Note that for $f \in L^2(\mathbb{R}_+)$ we have

$$\phi(f) = \int_0^\infty \mathrm{d}B(t)\, f(t)\,.$$

Next, for $n \in \mathbb{N}^*$ we define the simplex

$$\Delta_n := \{(t_1,\dots,t_n) : 0 < t_n < \cdots < t_1 < \infty\} \subset \mathbb{R}_+^n\,.$$

For $h \in L^2(\Delta_n)$ we define the iterated Itô integral

$$I(h) := \int_0^\infty \mathrm{d}B(t_1) \int_0^{t_1} \mathrm{d}B(t_2) \cdots \int_0^{t_{n-1}} \mathrm{d}B(t_n)\, h(t_1,\dots,t_n)\,.$$

We claim that the nth polynomial chaos (see Definition 5.1) is the space spanned by iterated Itô integrals:

$$\mathcal{W}_n = \{I(h) : h \in L^2(\Delta_n)\}\,. \tag{5.3}$$

To prove (5.3), we first consider functions $h \in L^2(\Delta_n)$ of the form

$$h(t_1,\dots,t_n) = f_1(t_1)\cdots f_n(t_n)\,, \tag{5.4}$$

where

$$f_1, \ldots, f_n \in L^2(\mathbb{R}_+), \qquad \inf \operatorname{supp} f_i > \sup \operatorname{supp} f_{i+1} \quad \forall i = 1, \ldots, n-1.$$

For h as in (5.4), we clearly have

$$I(h) = \int_0^\infty \mathrm{d}B(t_1)\, f_1(t_1) \cdots \int_0^\infty \mathrm{d}B(t_n)\, f_1(t_n) = \phi(f_1) \cdots \phi(f_n) = {:}\phi(f_1) \cdots \phi(f_n){:}\,, \tag{5.5}$$

where the last step follows from Remark 4.2 and the orthogonality of $f_1, \ldots, f_n$. Now (5.3) is an easy consequence of (5.5), the density of functions of the form (5.4) in $L^2(\Delta_n)$, and

$$\mathbb{E}[I(h)^2] = \|h\|^2_{L^2(\Delta_n)}\,, \tag{5.6}$$

which follows from an interated application of the Itô isometry.

6. Normal ordering and Wick ordering

Let us now return to Bosons. In this section we show that, under the isomorphism from Lemma 3.2, *Wick ordering* of Gaussian fields is mapped to *normal ordering* of field operators.

Let $\mathcal{H}$ be a real Hilbert space and recall the creation and annihilation operators from Definition 2.1.

Definition 6.1. Let P be a noncommutative polynomial in the creation and annihilation operators $\{a^*(f), a(f) : f \in \mathcal{H}\}$. Then the *normal ordering of* P, denoted by ${:}P{:}$, is obtained from P by writing all creation operators $a^*(f)$ to the left of all annihilation operators $a(f)$.

For example,

$${:}\Phi(f)^n{:} = {:}(a(f) + a^*(f))^n{:} = \sum_{k=0}^n \binom{n}{k} a^*(f)^k a(f)^{n-k}\,.$$

More generally,

$${:}\Phi(f_1) \cdots \Phi(f_n){:} = \sum_{I \subset [n]} \prod_{i \in I} a^*(f_i) \prod_{i \in [n] \setminus I} a(f_i)\,.$$

Recall the isomorphism D from Lemma 3.2.

Lemma 6.2. *For $f_1, \ldots, f_n \in \mathcal{H}$ we have*

$$D\,{:}\phi(f_1) \cdots \phi(f_n){:}\,D^{-1} = {:}\Phi(f_1) \cdots \Phi(f_n){:}\,.$$

Proof. We proceed by induction on n and verify that normal ordered products of field operators Φ satisfy the same recursion (4.7) from Remark 4.3 as the Gaussian field ϕ, which reads (writing $X_i = \phi(f_i)$)

$$\phi(f_n)\,{:}\phi(f_1) \cdots \phi(f_{n-1}){:} = {:}\phi(f_1) \cdots \phi(f_n){:} + \sum_{j=1}^{n-1} \langle f_j, f_n \rangle\, {:}\phi(f_1) \overset{\not j}{\cdots} \phi(f_{n-1}){:}\,.$$

Indeed, by Lemma 2.5,

$$\begin{aligned}
\Phi(f_n):\Phi(f_1)\cdots\Phi(f_{n-1}): &= (a^*(f_n)+a(f_n)) \sum_{I\subset[n-1]} \prod_{i\in I} a^*(f_i) \prod_{i\in[n-1]\setminus I} a(f_i) \\
&= a^*(f_n) \sum_{I\subset[n-1]} \prod_{i\in I} a^*(f_i) \prod_{i\in[n-1]\setminus I} a(f_i) \\
&\quad + \sum_{I\subset[n-1]} \prod_{i\in I} a^*(f_i) \Bigg(\prod_{i\in[n-1]\setminus I} a(f_i) \Bigg) a(f_n) \\
&\quad + \sum_{j=1}^{n-1} \langle f_j , f_n \rangle \sum_{I\subset[n-1]\setminus\{j\}} \prod_{i\in I} a^*(f_i) \prod_{i\in([n-1]\setminus\{j\})\setminus I} a(f_i) \\
&= :\Phi(f_1)\cdots\Phi(f_n): + \sum_{j=1}^{n-1} \langle f_j , f_n \rangle :\Phi(f_1) \overset{\not j}{\cdots} \Phi(f_{n-1}):,
\end{aligned}$$

which concludes the proof. □

For example, by (4.3) and Lemmas 3.2 and 6.2, for any unit vector $f \in \mathcal{H}$ we have

$$:\Phi(f)^n: = H_n(\Phi(f))\,.$$

7. Hypercontractive moment bounds

We conclude with a remark about hypercontractive moment bounds. Such bounds provide estimates on high moments of a random variable in a polynomial chaos in terms of its variance. They are a classical topic in the field theory literature, and have proven very useful in field theory and the theory of stochastic partial differential equations. They are usually derived as a consequence of the hypercontractive property of the Ornstein-Uhlenbeck semigroup associated with ϕ; see for instance [4, Section 7].

In this section we point out that hypercontractive bounds follow from an almost trivial argument that uses only the (generalised) Wick lemma, Lemma 4.5, and the Cauchy-Schwarz inequality. This material is partly based on [2, Appendix A].

Proposition 7.1. *Let $n \in \mathbb{N}$ and $X \in \mathcal{W}_n$. Then for any $p \in \mathbb{N}^*$ we have*

$$\mathbb{E}[X^{2p}] \leqslant (2p)^{np} \, (\mathbb{E}[X^2])^p\,.$$

Remark 7.2. The proof in fact gives the somewhat stronger estimate

$$\mathbb{E}[X^{2p}] \leqslant (2p)^{np} \, \mathrm{e}^{-h_n p/2} \, (\mathbb{E}[X^2])^p\,, \tag{7.1}$$

where $h_n := \sum_{k=1}^n \frac{1}{k}$. By comparison, the classical form of hypercontractive moment bounds[9] is

$$\mathbb{E}[X^{2p}] \leqslant (2p-1)^{np} \, (\mathbb{E}[X^2])^p\,. \tag{7.2}$$

The estimate (7.1) is sharper than (7.2) in the regime $p \gg n$, which is the case for most applications[10].

[9] See e.g. [4, Eq. (7.2)] or [5, Theorem 5.19].

[10] In practice, however, I am not aware of an application where such differences matter. The sharp constant is given in Remark 7.4 below.

Proof of Proposition 7.1. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then we can write

$$X = \sum_{k_1,\dots,k_n\in\mathbb{N}} a_{k_1\cdots k_n} {:}\phi(e_{k_1})\cdots\phi(e_{k_n}){:}\,, \tag{7.3}$$

where $a_{k_1\cdots k_n}$ is symmetric under permutations. By Lemma 2.13 we have

$$\mathbb{E}[X^2] = n! \sum_{k_1,\dots,k_n} a^2_{k_1\cdots k_n}\,. \tag{7.4}$$

To estimate the pth moment, we multiply $2p$ copies of (7.3), take the expectation and apply Lemma 4.5. The latter yields a sum over complete pairings $\Pi \in \mathfrak{P}_{[2p]\times[n]}(Q)$ of the index set $[2p]\times[n]$, where the partition Q consists of $2p$ blocks of size n. Only indices in different blocks can be paired. Because of

$$\mathbb{E}[\phi(e_k)\phi(e_l)] = \delta_{kl}\,,$$

each edge of Π yields a delta function for the indices. Hence, each edge $\pi\in\Pi$ corresponds to a single k-variable. Then we perform the summation over the k-variables by successively summing over each k-variable associated with an edge of Π, in an arbitrary order. This results in the estimate (7.10) below. A representative example of such an estimate, for $p=2$ and $n=3$, is

$$\begin{aligned}
\sum_{k,l,m,u,v,w} a_{klm}\,a_{klu}\,a_{vwu}\,a_{vwm} &\leqslant \sum_{l,m,u,v,w}\Big(\sum_k a^2_{klm}\Big)^{1/2}\Big(\sum_k a^2_{klu}\Big)^{1/2} a_{vwu}\,a_{vwm}\\
&\leqslant \sum_{l,m,u,v}\Big(\sum_k a^2_{klm}\Big)^{1/2}\Big(\sum_k a^2_{klu}\Big)^{1/2}\Big(\sum_w a^2_{vwu}\Big)^{1/2}\Big(\sum_w a^2_{vwm}\Big)^{1/2}\\
&\leqslant \sum_{l,u,v}\Big(\sum_{k,m} a^2_{klm}\Big)^{1/2}\Big(\sum_k a^2_{klu}\Big)^{1/2}\Big(\sum_w a^2_{vwu}\Big)^{1/2}\Big(\sum_{m,w} a^2_{vwm}\Big)^{1/2}\\
&\leqslant \sum_{u,v}\Big(\sum_{k,l,m} a^2_{klm}\Big)^{1/2}\Big(\sum_{k,l} a^2_{klu}\Big)^{1/2}\Big(\sum_w a^2_{vwu}\Big)^{1/2}\Big(\sum_{m,w} a^2_{vwm}\Big)^{1/2}\\
&\leqslant \sum_{v}\Big(\sum_{k,l,m} a^2_{klm}\Big)^{1/2}\Big(\sum_{k,l,u} a^2_{klu}\Big)^{1/2}\Big(\sum_{u,w} a^2_{vwu}\Big)^{1/2}\Big(\sum_{m,w} a^2_{vwm}\Big)^{1/2}\\
&\leqslant \Big(\sum_{k,l,m} a^2_{klm}\Big)^{1/2}\Big(\sum_{k,l,u} a^2_{klu}\Big)^{1/2}\Big(\sum_{u,v,w} a^2_{vwu}\Big)^{1/2}\Big(\sum_{m,v,w} a^2_{vwm}\Big)^{1/2}.
\end{aligned}$$

The estimate has six steps, corresponding to the six edges to be summed out.

The procedure can also be described more formally (though perhaps not necessarily more clearly) as follows. Introduce the index set $I=[2p]\times[n]$ and the blocks $I_i=\{i\}\times[n]$ for $i\in[2p]$. For any $A\subset I$ we use the notation $\mathbf{k}_A=(k_{ij}:(i,j)\in A)\in\mathbb{N}^A$ for the summation variables indexed by the set A. In this notation we can write $X=\sum_{\mathbf{k}_{I_i}} a_{\mathbf{k}_{I_i}} {:}\prod_{j=1}^n \phi(e_{k_{ij}}){:}$ for each $i\in[n]$, and hence we get, using Lemma 4.5,

$$\mathbb{E}[X^{2p}] = \sum_{\mathbf{k}_I}\prod_{i=1}^{2p} a_{\mathbf{k}_{I_i}}\,\mathbb{E}\Bigg[\prod_{i=1}^{2p} {:}\prod_{j=1}^{n}\phi(e_{k_{ij}}){:}\Bigg] = \sum_{\Pi}\sum_{\mathbf{k}_I}\prod_{i=1}^{2p} a_{\mathbf{k}_{I_i}} \prod_{\{(i,j),(i',j')\}\in\Pi} \mathbf{1}_{k_{ij}=k_{i'j'}}\,, \tag{7.5}$$

where the summation ranges over all complete pairings Π of $[2p]\times[n]$ such that for all $\{(i,j),(i',j')\}\in\Pi$ we have $i\neq i'$.

We shall Π and sum over the variables k_{ij} in pairs connected by the delta function on the right-hand side of (7.5). We perform this by an inductive summation of the edges of Π one by one; the order of summation is immaterial. After summing out a number of edges, we obtain a partial pairing $\Sigma\subset\Pi$, whose blocks contain those summation variables that have not yet been summed out. We refer to Figure 7.1 for an illustration of the procedure, which corresponds to the example estimate give n above.

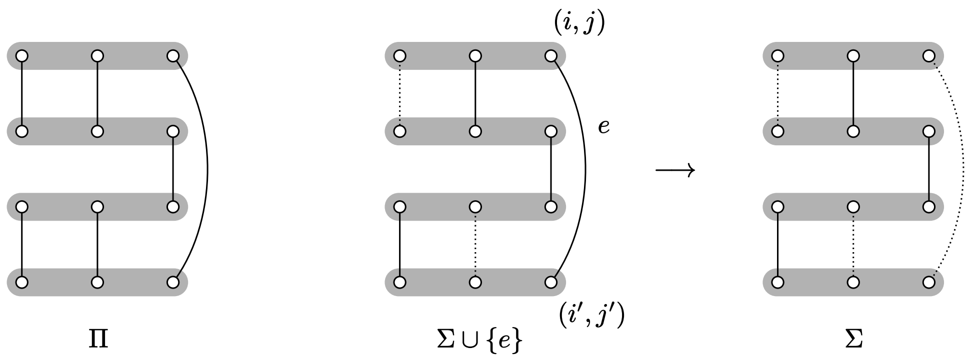

Figure 7.1. An illustration of the inductive algorithm for successively summing out edges of the pairing Π. Here $p=2$ and $n=3$. The pairing Π is illustrated in the figure on the left-hand side. The four grey blocks correspond to the four factors $a_{\mathbf{k}_{I_1}},\dots,a_{\mathbf{k}_{I_4}}$, and each vertex $(i,j)\in[2p]\times[n]$ corresponds to a summation variable k_{ij}. Note that each edge connects vertices from different grey blocks, as is required by Lemma 4.5. A partial pairing $\Sigma\subset\Pi$ is represented in the figure on the right-hand side, where the edges of Σ are drawn using solid lines. The dashed lines, incident to vertices corresponding to the summation variables $\mathbf{k}_{I\setminus\langle\Sigma\rangle}$, have been summed out at this point. This summation contributed a factor $\prod_{i=1}^{2p}\big(\sum_{\mathbf{k}_{I_i\setminus\langle\Sigma\rangle}}a^2_{\mathbf{k}_{I_i}}\big)^{1/2}$, which depends on the remaining summation variables $\mathbf{k}_{\langle\Sigma\rangle}$ that correspond to the vertices incident to the edges of Π. The middle figure corresponds to the partial pairing $\Sigma\cup\{e\}$ with an edge $e\in\Pi\setminus\Sigma$ given by $e=\{(i,j),(i',j')\}$. The induction step underlying the argument, going from $\Sigma\cup\{e\}$ to Σ, is the summation of a single edge e, which amounts to summing over the variables $k_{ij}=k_{i',j'}$, and using Cauchy-Schwarz. For this, it is crucial that $i\neq i'$, i.e. e connects vertices in different grey blocks.

The formal procedure is defined as follows. For any partial pairing $\Sigma\subset\Pi$, we define

$$V_\Pi(\Sigma):=\sum_{\mathbf{k}_{\langle\Sigma\rangle}}\prod_{i=1}^{2p}\Bigg(\sum_{\mathbf{k}_{I_i\setminus\langle\Sigma\rangle}}a^2_{\mathbf{k}_{I_i}}\Bigg)^{1/2}\prod_{\{(i,j),(i',j')\}\in\Sigma}\mathbf{1}_{k_{ij}=k_{i'j'}}\,, \tag{7.6}$$

where we recall the notation from Definition 2.12. This expression has three crucial properties. First, by (7.5) we have

$$\mathbb{E}[X^{2p}]=\sum_\Pi V_\Pi(\Pi) \tag{7.7}$$

Second, by the definition (7.6) we have

$$V_\Pi(\emptyset)=\Bigg(\sum_{k_1,\dots,k_n}a^2_{k_1\cdots k_n}\Bigg)^p. \tag{7.8}$$

Third, for any partial pairing $\Sigma \subset \Pi$ and any edge $e \in \Pi \setminus \Sigma$ we have

$$V_\Pi(\Sigma \cup \{e\}) \leqslant V_\Pi(\Sigma)\,. \tag{7.9}$$

To show (7.9), we suppose that $e = \{(i,j),(i',j')\}$ with $i \neq i'$ (see Figure 7.1), and estimate the sum over $\mathbf{k}_e = (k_{ij}, k_{i'j'})$ in the expression (7.6) for $V_\Pi(\Sigma \cup \{e\})$ as

$$\sum_{\mathbf{k}_e} \Bigg(\sum_{\mathbf{k}_{I_i \setminus [\Sigma \cup \{e\}]}} a^2_{\mathbf{k}_{I_i}}\Bigg)^{1/2} \Bigg(\sum_{\mathbf{k}_{I_{i'}} \setminus [\Sigma \cup \{e\}]} a^2_{\mathbf{k}_{I_{i'}}}\Bigg)^{1/2} \mathbf{1}_{k_{ij} = k_{i'j'}} \leqslant \Bigg(\sum_{\mathbf{k}_{I_i \setminus \langle \Sigma \rangle}} a^2_{\mathbf{k}_{I_i}}\Bigg)^{1/2} \Bigg(\sum_{\mathbf{k}_{I_{i'} \setminus \langle \Sigma \rangle}} a^2_{\mathbf{k}_{I_{i'}}}\Bigg)^{1/2},$$

by Cauchy-Schwarz and the fact that the summation variable $k_{ij} = k_{i'j'}$ appears exactly once in each factor.

From (7.7)–(7.9), we conclude that

$$\mathbb{E}[X^{2p}] \leqslant \sum_{\Pi} \Bigg(\sum_{k_1,\dots,k_n} a^2_{k_1 \cdots k_n}\Bigg)^p\,. \tag{7.10}$$

Using that the number of pairings Π is bounded by the number of complete pairings of the set $[2p] \times [n]$, i.e. $(2pn-1)!!$, we conclude from (7.10) and (7.4) that

$$\mathbb{E}[X^{2p}] \leqslant \frac{(2pn-1)!!}{(n!)^p} (\mathbb{E}[X^2])^p\,.$$

The claim now follows from the combinatorial Lemma 7.3 below. □

Lemma 7.3. *For $p, n \in \mathbb{N}^*$ we have*

$$\frac{(2pn-1)!!}{(n!)^p} \leqslant (2p)^{pn}\, \mathrm{e}^{-h_n p/2}\,,$$

where $h_n := \sum_{k=1}^n \frac{1}{k}$.

Proof. We split

$$\frac{(2pn-1)!!}{(n!)^p} = \prod_{k=1}^n P_k\,, \qquad P_k := \frac{(2pk-1)(2pk-3)\cdots(2pk-2p+1)}{k^p}$$

and estimate, for each k, by concavity of the logarithm,

$$P_k = (2p)^p \prod_{i=1}^p \bigg(1 - \frac{2i-1}{2pk}\bigg) = (2p)^p \exp\Bigg[\sum_{i=1}^p \log\bigg(1 - \frac{2i-1}{2pk}\bigg)\Bigg]$$
$$\leqslant (2p)^p \exp\Bigg[-\frac{1}{2pk}\sum_{i=1}^p (2i-1)\Bigg] = (2p)^p \mathrm{e}^{-\frac{p}{2k}}\,.$$

The claim follows. □

Remark 7.4. The proof of Proposition 7.1 yields the bound

$$\mathbb{E}[X^{2p}] \leqslant \frac{c(p,n)}{(n!)^p} (\mathbb{E}[X^2])^p\,, \tag{7.11}$$

where $c(p,n)$ is the number of complete pairings of the set $[2p] \times [n]$ such that all paired elements have a different second index. The constant in (7.11) is sharp, and it was obtained [5, Corollary 7.36] using more sophisticated stochastic integration techniques. Estimating $c(p,n)$ by $(2p-1)!!$, as in the proof of Proposition 7.1, is slightly wasteful for large n.

A. Tensor products of Hilbert spaces

In this appendix we briefly review the construction of the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two[11] Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$; this material is taken from [7, Section II.5], to which we also refer for further details.

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. For $f_1 \in \mathcal{H}_1$ and $f_2 \in \mathcal{H}_2$ we define the conjugate[12] bilinear form $f_1 \otimes f_2$ on $\mathcal{H}_1 \times \mathcal{H}_2$ through

$$(f_1 \otimes f_2)(h_1, h_2) := \langle h_1 \,, f_1 \rangle \langle h_2 \,, f_2 \rangle \,.$$

Note that the map $(f_1, f_2) \mapsto f_1 \otimes f_2$ is bilinear. Denote by

$$\mathcal{B} := \operatorname{span}\{f_1 \otimes f_2 : f_1 \in \mathcal{H}_1, f_2 \in \mathcal{H}_2\}\,,$$

which is a linear subspace of the conjugate bilinear forms on $\mathcal{H}_1 \times \mathcal{H}_2$. We define the inner product

$$\langle f_1 \otimes f_2 \,, g_1 \otimes g_2 \rangle := \langle f_1 \,, g_1 \rangle \langle f_2 \,, g_2 \rangle \,,$$

and extend it by linearity to $\mathcal{B}$. Then $\langle \cdot \,, \cdot \rangle$ is well-defined (i.e. independent of how the element of $\mathcal{B}$ is represented as a linear combination) and positive definite (see [7, Section II.5]). One then *defines* the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ as the completion of $\mathcal{B}$ under $\langle \cdot \,, \cdot \rangle$.

Lemma A.1. *If $(e_i)_{i \in I}$ and $(f_j)_{j \in J}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then $(e_i \otimes f_j)_{i \in I, j \in J}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

The most important example is $\mathcal{H}_k = L^2(M_k, \mu_k)$ for $k = 1, 2$, in which case we have

$$L^2(M_1, \mu_1) \otimes L^2(M_2, \mu_2) \simeq L^2(M_1 \times M_2, \mu_1 \otimes \mu_2) \,. \tag{A.1}$$

To see (A.1), let $(e_i)_{i \in \mathbb{N}^*}$ and $(f_j)_{j \in \mathbb{N}^*}$ be orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively (we use the index set $\mathbb{N}^*$ because $\mathcal{H}_k$ is separable.). Then the functions $h_{ij}(x, y) := e_i(x) f_j(y)$ form an orthonormal basis of $L^2(M_1 \times M_2, \mu_1 \otimes \mu_2)$. Hence, the map

$$U : e_i \otimes f_j \mapsto h_{ij}$$

maps an orthonormal basis of $L^2(M_1, \mu_1) \otimes L^2(M_2, \mu_2)$ to an orthonormal basis of $L^2(M_1 \times M_2, \mu_1 \otimes \mu_2)$, and therefore extends to a unitary map

$$U : L^2(M_1, \mu_1) \otimes L^2(M_2, \mu_2) \to L^2(M_1 \times M_2, \mu_1 \otimes \mu_2) \,,$$

and thus implies (A.1).

Because of (A.1), one often also writes

$$f(x) g(y) = (f \otimes g)(x, y) \,.$$

For instance, in the finite-dimensional case we have

$$\mathbb{C}^n \otimes \mathbb{C}^m \simeq \mathbb{C}^{n \times m} \,,$$

which can be idenfitied with the space of $n \times m$ matrices. The standard basis of $\mathbb{C}^n \otimes \mathbb{C}^m$ is $(e_i \otimes e_j)_{i \in [n], j \in [m]}$, where $e_1, e_2, \ldots$ denote the standard basis vectors of $\mathbb{C}^n$.

[11] For simplicity of notation, we restrict ourselves to a product of two Hilbert spaces, but the same construction works for a product $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ of n Hilbert spaces.

[12] If $\mathcal{H}$ is real then we can of course omit the word "conjugate".

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