Lattice Yang-Mills Theory and the Confinement Problem

Diploma Thesis

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Abstract

We discuss lattice Yang-Mills at both finite and zero temperature and study the confinement problem. The basic concepts of lattice gauge theories are introduced and criteria for confinement discussed.
Confinement at finite temperature is shown for static quarks in the following cases: a gauge group with continuous centre, $2+1$ space-time dimensions, any coupling; any gauge group, any space-time dimension, strong coupling. The existence of a deconfining transition is shown in $3+1$ or higher dimensions.
At zero temperature, deconfinement of monopoles in 3 and 4 space-time dimensions is shown for the following cases: Abelian gauge group, any coupling; any gauge group, strong coupling.

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1 Introduction

Gauge theories are central to particle physics, where they provide the theoretical framework for the description of weak, electromagnetic and strong forces. They are essentially a generalisation of Maxwell's theory of electromagnetism. One of the outstanding problems in the context of gauge theories is the *confinement* of quarks: Despite considerable indirect experimental evidence of the existence of quarks as building blocks of hadrons, the observation of a free quark is still missing. Quarks are said to be confined. From a theoretical point of view, the confinement problem may be seen as the question why the particle content (asymptotic states) of quantum chromodynamics (QCD) consists only of hadrons but not of anything like quarks. Since this is, at least for the moment, a practically intractable problem, one normally considers some simplified version of it.

As gauge theories such as QCD tend to be highly singular, one fruitful approach is to discretise the space-time continuum (or at the very least the spatial dimensions) and replace it with a lattice, thus introducing an ultraviolet cutoff. When combined with the Euclidean (imaginary time) version of the Feynman path integral formulation of a quantum field theory, the gauge theory becomes a problem of classical statistical mechanics on a lattice. One may then use the well developed machinery of statistical mechanics. Taking this analogy further, lattice gauge theories have been used as models to describe the statistical mechanics of defects in ordered media [1]. However, the main reason for the study of lattice gauge theories lies in the hope of gaining some insight into the properties of the continuum theory.

A further simplification consists in eliminating the quarks from the theory and replacing them with "static", or infinitely heavy quarks that have no dynamics of their own. They are test charges used to probe the properties of the "glue", described by the gauge field, that is supposed to keep quarks together. One may then define a "static quark potential" which describes the potential between an infinitely heavy quark-antiquark pair, and provides a concrete criterion for confinement: quarks are confined if the potential energy required to separate them to infinity grows indefinitely with their separation. This approach was introduced by Wilson [2]. In another view of confinement by 't Hooft [3], the gluons exchanged by quarks arrange themselves in a thin tube of chromo-electric flux. Confinement may then be studied by considering whether the energy per unit length of a chromo-electric tube tends to zero or some nonzero value, provided that the tube is allowed infinite room in the transverse directions.

The subject of this work is to study a pure gauge field theory on a lattice with dynamics determined by the *Yang-Mills* action [4], which is essentially a generalisation of the Maxwell action for the electromagnetic field and may also be used to describe the gauge field of QCD (the "glue"). Our aim is to derive some results concerning confinement at both zero temperature (relevant for particle physics) and finite temperature (relevant for astrophysics of the early universe and superdense stars).

Section 2 contains the background necessary for the subsequent parts. We start with a short review of Euclidean quantum field theory in the path integral formulation, as well as introduce gauge fields and the Yang-Mills action. We then move on to describing the lattice approximation. The last part of Section 2 is devoted to a discussion of some confinement criteria.

In Section 3 we discuss confinement at a finite temperature. We show that, if the gauge group has a continuous centre, we always have confinement in $2+1$ space-time dimensions. We also show, using a high-temperature expansion, that if the coupling is sufficiently strong we have confinement for any gauge group and dimension of space-time. The final result is that in $3+1$ space-time dimensions and higher we have deconfinement if the coupling is sufficiently weak and the temperature sufficiently high: quarks are "freed".

In the final Section 4 we investigate confinement at zero temperature, and show that for the gauge group U(1) (i.e. electromagnetism) monopoles are always deconfined. We also prove this for any gauge group providing that the coupling is strong.

2 The setting

2.1 Scalar fields and functional integrals

We begin with a short discussion of the underlying physics. See also [5]; a rigorous treatment may be found in [6].

2.1.1 Quantum field theory of a scalar field

The main ingredients of a quantum field theory of a scalar field in $d+1$ space-time dimensions are:

- There is a Hilbert space $\mathscr{H}$ of physical states containing a vacuum $|0\rangle$.
- On $\mathscr{H}$ there is a unitary representation $U(a,\Lambda)$ of the Poincaré group, where $a \in \mathbb{R}^{d+1}$ is a translation vector and $\Lambda \in \mathrm{O}(1,d)$ is a Lorentz transformation. The vacuum $|0\rangle$ is invariant under the representation U. It is the only vector (up to scalar multiples) with this property.
- (Spectrum condition) The (self-adjoint) generators $P = (P_\mu)$ of the translation subgroup,
$$U(a,\mathbb{1}) = e^{ia^\mu P_\mu}, \tag{2.1}$$
have a spectrum in the forward light cone:
$$\sigma(P) \subset \overline{V}_+ := \{(p_0,\mathbf{p}) \in \mathbb{R}^{d+1} \ : \ p_0 \geq 0,\ p_0^2 - \mathbf{p}^2 \geq 0\}. \tag{2.2}$$
$H := P_0$ is called the Hamiltonian.
- (Observables) There are self-adjoint field operators $\phi(x) = \phi(x)^*$ acting on $\mathscr{H}$. More precisely, $\phi(x)$ is an operator-valued distribution: for a test function $f \in C_0^\infty(\mathbb{R}^{d+1})$,
$$\int dx\ f(x)\phi(x) := \phi(f) \tag{2.3}$$
is a self-adjoint operator on $\mathscr{H}$.
- (Covariance) The fields transform under the representation U according to
$$U(a,\Lambda)\,\phi(x)\,U^{-1}(a,\Lambda) = \phi(\Lambda x + a), \tag{2.4}$$
i.e.
$$U(a,\Lambda)\,\phi(f)\,U^{-1}(a,\Lambda) = \phi(\tilde{f}), \qquad \tilde{f}(x) := f\big(\Lambda^{-1}(x-a)\big). \tag{2.5}$$
- (Locality) The fields commute (we assume we are dealing with bosons) for space-like separations:
$$[\phi(x),\phi(y)] = 0 \tag{2.6}$$
if $x-y$ is space-like (x is space-like if $x_\mu x^\mu < 0$). More precisely, if the supports of f and g are space-like separated, then
$$[\phi(f),\phi(g)] = 0. \tag{2.7}$$

The above list is all that we need; it is a stripped down version of the Wightman and Haag-Kastler axioms (for a complete and rigorous treatment see for instance [6]).

We are interested in the moments (also called Wightman n-point functions)

$$W(x_1, \dots, x_n) := \langle 0 \,|\, \phi(x_1) \dots \phi(x_n) \,|\, 0 \rangle. \tag{2.8}$$

Note that $W(x_1, \dots, x_n) \in \mathscr{D}'(\mathbb{R}^{(d+1)n})$, the space of distributions on $\mathbb{R}^{(d+1)n}$. The Wightman reconstruction theorem [7] states that the Wightman functions contain all the important information of the theory in the sense that from them the Hilbert space and the field operators can be reconstructed.

2.1.2 Euclidean field theory

Our aim is to analytically continue the Wightman functions to the complex plane. Writing $xP := x^\mu P_\mu$ and using

$$\phi(x) = e^{ixP} \phi(0) e^{-ixP} \tag{2.9}$$

we may rewrite

$$W(x_1, \dots, x_n) = \langle 0 \,|\, \phi(0) e^{-i(x_1 - x_2)P} \cdots \phi(0) e^{-i(x_{n-1} - x_n)P} \,|\, 0 \rangle. \tag{2.10}$$

Let now $x_j \in \mathbb{C}^{d+1}$ and write

$$x_j = a_j - ib_j, \qquad a_j, b_j \in \mathbb{R}^{d+1}, \tag{2.11}$$

so that each exponential in (2.10) becomes (the subgroup of translations is Abelian)

$$e^{-i(a_j - a_{j+1})P} \, e^{-(b_j - b_{j+1})P}. \tag{2.12}$$

This is well defined for $b_j - b_{j+1} \in \overline{V}_+$ because of the assumption on the spectrum of P. In particular we may choose the *Euclidean points*

$$x_j = (-ix_j^{d+1}, \mathbf{x}_j). \tag{2.13}$$

If $x_1^{d+1} > \dots > x_n^{d+1}$ we may therefore analytically continue the Wightman function

$$S\big(\{\mathbf{x}_j, x_j^{d+1}\}_{j=1}^n\big) := W\big(\{-ix_j^{d+1}, \mathbf{x}_j\}_{j=1}^n\big). \tag{2.14}$$

S is called a *Schwinger function*; it is the Wightman function evaluated at the imaginary time $x^0 = -ix^{d+1}$. Using $H = P_0$ we may write

$$S(x_1, \dots, x_n) = \langle 0 \,|\, \phi(0, \mathbf{x}_1) e^{-(x_1^{d+1} - x_2^{d+1})H} \phi(0, \mathbf{x}_2) \dots \,|\, 0 \rangle. \tag{2.15}$$

In order to define the Schwinger function for any values of its parameters, consider first the two-point function $W(x_1, x_2) = \tilde{W}(x)$, $x := x_1 - x_2$ [see (2.10)]. As a function of complex x^0 $\tilde{W}$ is analytic on the complex lower-half plane. Define

$$\tilde{W}_\pi(x) := \tilde{W}(-x), \tag{2.16}$$

which is analytic on the complex upper-half plane. Assume that $\mathbf{x} \neq 0$. Then, for $x^0 \leq |\mathbf{x}|$, x is space-like so that $[\phi(x_1), \phi(x_2)] = 0$ and therefore

$$\tilde{W}_\pi(x) = \tilde{W}(x), \qquad x^0 \in [-|\mathbf{x}|, |\mathbf{x}|]. \tag{2.17}$$

Thus the two analytic functions are equal on a real interval and consequently

$$\tilde{W}_\pi(x) = \tilde{W}(x) \tag{2.18}$$

for any $x^0 \in \mathbb{C}$ excluding the section $\{\xi \in \mathbb{R} \ : \ |\xi| \geq |\mathbf{x}|\}$ of the real line. $\tilde{W}$ may be analytically continued to $\mathbb{C}$ excluding the above strip and in particular to the imaginary axis. Moreover it is even. Similarly, we may show that the Schwinger function $S(x_1, \dots, x_n)$ is well defined for disjoint $\{x_1, \dots, x_n\}$, as well as symmetric under any transposition and consequently under all permutations.

Knowing the Schwinger functions we may recover the Wightman functions by analytic continuation:

$$W\big(\{x_j^0, \mathbf{x}_j\}_{j=1}^n\big) = \lim_{\substack{\varepsilon_j \to 0:\\ \varepsilon_j > \varepsilon_{j+1}}} S\big(\{\mathbf{x}_j, ix_j^0 + \varepsilon_j\}_{j=1}^n\big), \tag{2.19}$$

and consequently reconstruct the whole quantum field theory. In practice the Schwinger functions are easier to work with and also allow a formulation in terms of functional integrals which we discuss in the next section.

Note that since

$$W\big(\Lambda x_1 + a, \dots, \Lambda x_n + a\big) = W\big(x_1, \dots, x_n\big), \tag{2.20}$$

where $\Lambda \in \mathrm{O}(1, d)$ is a Lorentz transformation, we get

$$S\big(Ox_1 + a, \dots, Ox_n + a\big) = S\big(x_1, \dots, x_n\big), \tag{2.21}$$

where $O \in \mathrm{O}(d+1)$ is an orthogonal transformation, so that the Schwinger functions are invariant under Euclidean transformations.

2.1.3 Functional integral formulation

Instead of describing a quantum field theory using the properties introduced above, we turn the argumentation upside down and define our theory using the Schwinger functions. As a motivation, we start with the formal expression for the Schwinger functions

$$S(x_1, \dots, x_n) = \frac{\int \mathcal{D}\phi \ \phi(x_1) \dots \phi(x_n) e^{-S(\phi)}}{\int \mathcal{D}\phi \ e^{-S(\phi)}}, \tag{2.22}$$

where $S(\phi)$ is the classical Euclidean action and $\mathcal{D}\phi$ is a (nonexisting) uniform integration measure on $\mathscr{D}'(\mathbb{R}^{d+1})$. It was found in a heuristic way by Feynman [8, 9] and Kac related it, at least in one space dimension, to the rigorous Work of Wiener and Ornstein-Uhlenbeck on path integrals [10, 11]. Typically, for a complex scalar field ϕ, the Euclidean action is of the form

$$S(\phi) = \int dx \ \Big[\frac{1}{2}\,\overline{\phi(x)}(\Box + m^2)\phi(x) + U\big(\overline{\phi(x)}\phi(x)\big)\Big], \tag{2.23}$$

where $\Box = -\Delta$ is the negative Laplacean, m is a "mass", and U is an "interaction potential".

One possibility to give some meaning to the above expression is to consider only a bounded region of discretised space-time, and therefore a well defined integral, and to hope that the continuum limit exists. Another approach is to identify formally

$$d\mu(\phi) = e^{-S(\phi)}\mathcal{D}\phi \tag{2.24}$$

as a measure on $\mathscr{D}'(\mathbb{R}^{d+1})$. This works for instance if we set $m > 0$ and $U = 0$ in the action above, so that we have a Gaussian measure with covariance $(\Box + m^2)^{-1}$. We therefore follow the idea of Symanzik [12] and consider the Euclidean fields $\phi \in \mathscr{D}'(\mathbb{R}^{d+1})$ as random variables with probability measure $d\mu$. We then define the Schwinger functions as the moments of $d\mu$:

$$S(x_1, \ldots, x_2) = \int d\mu(\phi)\ \phi(x_1)\ldots\phi(x_n). \tag{2.25}$$

In order to recover a sensible quantum field theory as described above, $d\mu$ needs to satisfy certain criteria. The Osterwalder-Schrader axioms [13] are a common choice. The conditions are usually stated for the Schwinger functions S and include regularity (a purely technical property), symmetry (obvious if S is expressed as a moment of a measure), Euclidean covariance (2.21), as well as *reflection positivity*, which we discuss later in the context of lattice gauge theories.

Assuming the Schwinger functions satisfy the required conditions it is possible to return to a quantum field theory formulated in terms of operators on a Hilbert space that has all the properties listed in Section 2.1.1 (see [6]).

2.2 Gauge fields and the Yang-Mills action

We work in the space-time $M := \mathbb{R}^{d+1}$ with Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$; from now on we relabel the imaginary time component x^{d+1} as x^0.

2.2.1 Local gauge invariance and gauge fields

Take n complex "matter" fields $\phi^i(x)$, $x \in M$. These may be considered as the components of a complex vector field $\phi : M \longmapsto \mathbb{C}^n$. In analogy to (2.23) we take an action of the form

$$S(\phi) = \int dx\ \left[\overline{\phi(x)} \cdot (\Box + m^2)\phi(x) + U\left(\overline{\phi(x)} \cdot \phi(x)\right)\right]. \tag{2.26}$$

S is obviously invariant under *global gauge transformations*, i.e. transformations of the form

$$\phi \longmapsto \Lambda^{-1}\,\phi, \qquad \Lambda \in \mathrm{U}(n), \tag{2.27}$$

but not under *local gauge transformations*, i.e. transformations of the form

$$\phi \longmapsto \Lambda^{-1}(x)\,\phi, \qquad \Lambda(x) \in \mathrm{U}(n). \tag{2.28}$$

Motivated by the "Nahewirkungsprinzip" which forbids parallelism at a distance, we require a physically reasonable theory to be invariant under any local choice of basis, i.e. invariant under local gauge transformations. Any terms of the form $\overline{\phi(x)} \cdot \phi(x)$ are already locally gauge invariant, but terms involving derivatives are not. This is not surprising, since the derivative involves comparison of two neighbouring points in space-time with different gauge transformation properties. In order to define a sensible derivative, we need to be able to compare neighbouring points in space-time in a way compatible with local gauge transformations. This is accomplished by defining a parallel transporter U, i.e. a mapping that associates to every curve γ in M a matrix $U(\gamma) \in \mathrm{U}(n)$ with the following properties:

1. $U(\gamma_2 \circ \gamma_1) = U(\gamma_2)\,U(\gamma_1)$, where $\gamma_2 \circ \gamma_1$ denotes the path obtained by joining γ_1 and γ_2 (in that order). Note that the end point of γ_1 must agree with the beginning point of γ_2.

2. $U(\gamma) = \mathbb{1}$ if γ is trivial (i.e. $|\gamma| = 0$).

3. $U(\gamma^{-1}) = U(\gamma)^{-1}$, where γ^{-1} denotes the path obtained from γ by reversing the orientation.

Now it is clear how to define the transformation property of the parallel transporter under a local gauge transformation. Let γ be a path joining the two points x and y. If the matter fields transform as

$$\phi'(x) = \Lambda^{-1}(x)\,\phi(x), \tag{2.29}$$

then the parallel transporter must transform as

$$U'(\gamma) := \Lambda^{-1}(y)\,U(\gamma)\,\Lambda(x), \tag{2.30}$$

to ensure that

$$\big(U(\gamma)\phi(x)\big)' = U'(\gamma)\phi'(x). \tag{2.31}$$

Denote by $U(y,x)$ the parallel transporter along the path $[x,y]$. Then we may expand the parallel transporter for infinitesimal distances:

$$U(x+\delta x, x) = \mathbb{1} - A_\mu(x)\,\delta x^\mu + \mathcal{O}(\delta x^2), \tag{2.32}$$

where A_μ are generators of $\mathrm{U}(n)$, i.e. antihermitean matrices. A_μ is called a *gauge field*. The covariant derivative of ϕ in the direction $\xi \in \mathbb{R}^{d+1}$ is defined as

$$\begin{aligned} D_\xi\,\phi(x) &:= \lim_{\varepsilon\to 0} \frac{U(x, x+\varepsilon\xi)\,\phi(x+\varepsilon\xi) - \phi(x)}{\varepsilon} \\ &= D\phi(x)\;\xi, \end{aligned} \tag{2.33}$$

where

$$D\phi(x) = D_\mu\phi(x)\;dx^\mu, \qquad D_\mu\phi(x) = \big(\partial_\mu + A_\mu(x)\big)\,\phi(x) \tag{2.34}$$

is the covariant differential.

The transformation law of A_μ is a direct consequence of the corresponding transformation law for the parallel transporter:

$$\begin{aligned} A'_\mu(x) &= -\frac{\partial}{\partial \xi^\mu}\bigg|_{\xi=0} U'(x+\xi,x) \\ &= -\frac{\partial}{\partial \xi^\mu}\bigg|_{\xi=0} \Lambda^{-1}(x+\xi)\, U(x+\xi,x)\, \Lambda(x) \\ &= -\big(\partial_\mu \Lambda^{-1}(x)\big)\, \Lambda(x) + \Lambda^{-1}(x)\, A_\mu(x)\, \Lambda(x). \end{aligned} \tag{2.35}$$

Using

$$\partial_\mu(\Lambda^{-1}\Lambda) = 0 \quad \Longrightarrow \quad \partial_\mu \Lambda^{-1}\Lambda = -\Lambda^{-1}\,\partial_\mu \Lambda \tag{2.36}$$

we get

$$A'_\mu = \Lambda^{-1}\big(\partial_\mu + A_\mu\big)\Lambda. \tag{2.37}$$

The covariant derivative is indeed covariant:

$$\begin{aligned} D'_\mu \phi' &= (\Lambda^{-1}\,\partial_\mu + \partial_\mu \Lambda^{-1} + \Lambda^{-1}\,\partial_\mu \Lambda\, \Lambda^{-1} + \Lambda^{-1}\, A_\mu)\, \phi \\ &\overset{(2.36)}{=} \Lambda^{-1}\,(\partial_\mu + A_\mu)\phi \\ &= (D_\mu \phi)' \end{aligned} \tag{2.38}$$

2.2.2 Geometric interpretation and generalisations

The above discussion can be generalised in a geometric setting. Let G be a compact Lie group and M a smooth manifold (the "space-time manifold"). A gauge theory lives on a *principal bundle* E of G over M: a manifold that looks locally like $M \times G$, i.e. at each point x of space-time we have an internal symmetry space G.

More precisely, a principal bundle of G over M consists of the following elements:

1. A smooth manifold E.

2. A smooth projection $\pi : E \longmapsto M$.

3. An open covering $\{U_\alpha\}$ of M.

4. A set of diffeomorphisms $\psi_\alpha : U_\alpha \times G \longmapsto \pi^{-1}(U_\alpha)$ such that

$$\pi\big(\psi_\alpha(x,g)\big) = x. \tag{2.39}$$

   The functions ψ_α are called *local trivialisations.*

5. A set of smooth functions $g_{\beta\alpha} : U_\alpha \cap U_\beta \longmapsto G$ such that

$$(\psi_\beta^{-1} \circ \psi_\alpha)\,(x,g) = (x, g_{\beta\alpha}(x)g). \tag{2.40}$$

   The functions $g_{\beta\alpha}$ are called *transition functions.*

Thus the two points represented by $(x, g_\alpha) \in U_\alpha \times G$ and $(x, g_\beta) \in U_\beta \times G$ are equal if $g_\beta = g_{\beta\alpha}(x)\, g_\alpha$. A choice of the local trivialisations can be interpreted as a choice of a gauge. A (local) gauge transformation changes the local trivialisations: ψ_α becomes $\psi'_\alpha(x,g) = \psi_\alpha(x, \Lambda_\alpha^{-1}(x)\, g)$, where $\Lambda_\alpha(x) \in G$. This is obviously the most general transformation on ψ_α allowed by (2.39).

If the space-time manifold M is not simply connected E is in general nontrivial, i.e. not homeomorphic to $M \times G$. This will become relevant in the following since we shall often consider the case when M is finite with periodic boundary conditions, and thus has the topology of a cylinder or a torus.

In this geometrical picture a gauge field is a *connection* on the principal bundle: a prescription telling in which direction to move in the bundle E when a direction is given on the base manifold M. In a local trivialisation, a connection is a 1-form $A = A_\mu(x)\, dx^\mu$ on the base manifold with values in the Lie algebra of G. The connection A may be used to lift curves from M to E, i.e. define a parallel transport. From another point of view, parallel transport is a means of comparing elements of *fibres* $\pi^{-1}(x) \cong G$ over different space-time points x. For simplicity we assume a single local trivialisation for M (or at least the area of M under consideration). Let $\gamma(t)$ be a curve in M. We seek a map $U(t) \in G$ such that the curve $\gamma(t)$ lifts to $(\gamma(t), U(t)g_0)$, where $g_0 \in G$ is given. The prescription at the beginning of the paragraph can now be expressed as

$$\dot{U}(t)\, g_0 = -A\big(\gamma(t)\big)\,\dot{\gamma}(t)\, U(t)\, g_0, \qquad U(0) = \mathbb{1}, \tag{2.41}$$

where the minus sign is conventional; it provides a connection to the previous section, since (2.41) is obviously satisfied by U and A as defined there. The definition (2.41) can be solved:

$$U(t) = \mathcal{P}\; e^{-\int_0^t A[\gamma(t')]\dot{\gamma}(t')\, dt'}, \tag{2.42}$$

where $\mathcal{P}$ is the path ordering operator. Thus the parallel transporter $U(\gamma)$ along the curve γ is given by

$$U(\gamma) = \mathcal{P}\; e^{-\int_\gamma A}. \tag{2.43}$$

Note that for Abelian G the path ordering operator may be omitted.

2.2.3 Curvature and the Yang-Mills action

In general a closed curve in M will not lift to a closed curve in E. In this case the connection has curvature. The curvature can be quantified by measuring how much the parallel transporter around a small closed curve differs from the identity. We choose two coordinate directions, μ and ν, and consider the square along these directions with side length ε (see Figure 2.1).

Using

$$A_\mu(x) = A_\mu(x + \frac{\varepsilon}{2}\hat{\mu}) + \mathcal{O}(\varepsilon) \tag{2.44}$$

the parallel transporter along one side can be expanded as

$$\begin{aligned} U(x + \varepsilon\hat{\mu}, x) &= \mathbb{1} - \varepsilon A_\mu(x + \frac{\varepsilon}{2}\hat{\mu}) + \mathcal{O}(\varepsilon^2) \\ &= e^{-\varepsilon A_\mu(x+\frac{\varepsilon}{2}\hat{\mu}) - \varepsilon^2 B_\mu(x+\frac{\varepsilon}{2}\hat{\mu}) + \mathcal{O}(\varepsilon^3)}, \end{aligned} \tag{2.45}$$

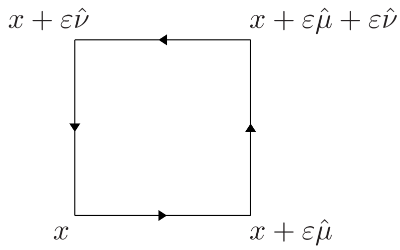

Figure 2.1: *A small loop in the μ-ν-plane.*

for some $B_\mu(x)$. The parallel transporter in the reverse direction is (set $x \mapsto x+\varepsilon$, $\varepsilon \mapsto -\varepsilon$ above)

$$U(x, x+\varepsilon\hat\mu) = e^{\varepsilon A_\mu(x+\frac{\varepsilon}{2}\hat\mu) - \varepsilon^2 B_\mu(x+\frac{\varepsilon}{2}\hat\mu) + \mathcal{O}(\varepsilon^3)}. \tag{2.46}$$

Since we require

$$U(x+\varepsilon\hat\mu, x)^{-1} = U(x, x+\varepsilon\hat\mu) \tag{2.47}$$

we get $B_\mu(x+\frac{\varepsilon}{2}\hat\mu) = 0$. The parallel transporter around the above path is

$$U^\varepsilon_{\mu\nu}(x) := U(x, x+\varepsilon\hat\nu)\, U(x+\varepsilon\hat\nu, x+\varepsilon\hat\mu+\varepsilon\hat\nu)\, U(x+\varepsilon\hat\mu+\varepsilon\hat\nu, x+\varepsilon\hat\mu)\, U(x+\varepsilon\hat\mu, x). \tag{2.48}$$

Using the Campbell-Baker-Hausdorff formula

$$e^a\, e^b = e^{a+b+\frac{1}{2}[a,b]+\ldots} \tag{2.49}$$

we get

$$\begin{aligned}
U^\varepsilon_{\mu\nu}(x) = {} & e^{\varepsilon A_\nu(x+\frac{\varepsilon}{2}\hat\nu)+\mathcal{O}(\varepsilon^3)}\, e^{\varepsilon A_\mu(x+\varepsilon\hat\nu+\frac{\varepsilon}{2}\hat\mu)+\mathcal{O}(\varepsilon^3)} \\
& e^{-\varepsilon A_\nu(x+\varepsilon\hat\mu+\frac{\varepsilon}{2}\hat\nu)+\mathcal{O}(\varepsilon^3)}\, e^{-\varepsilon A_\mu(x+\frac{\varepsilon}{2}\hat\mu)+\mathcal{O}(\varepsilon^3)} \\
= {} & e^{\varepsilon A_\nu(x+\frac{\varepsilon}{2}\hat\nu)+\varepsilon A_\mu(x+\varepsilon\hat\nu+\frac{\varepsilon}{2}\hat\mu)+\frac{\varepsilon^2}{2}[A_\nu(x+\frac{\varepsilon}{2}\hat\nu),A_\mu(x+\varepsilon\hat\nu+\frac{\varepsilon}{2}\hat\mu)]+\mathcal{O}(\varepsilon^3)} \\
& e^{-\varepsilon A_\nu(x+\varepsilon\hat\mu+\frac{\varepsilon}{2}\hat\nu)-\varepsilon A_\mu(x+\frac{\varepsilon}{2}\hat\mu)+\frac{\varepsilon^2}{2}[A_\nu(x+\varepsilon\hat\mu+\frac{\varepsilon}{2}\hat\nu),A_\mu(x+\frac{\varepsilon}{2}\hat\mu)]+\mathcal{O}(\varepsilon^3)} \\
= {} & e^{\varepsilon A_\nu(x+\frac{\varepsilon}{2}\hat\nu)-\varepsilon A_\nu(x+\varepsilon\hat\mu+\frac{\varepsilon}{2}\hat\nu)+\varepsilon A_\mu(x+\varepsilon\hat\nu+\frac{\varepsilon}{2}\hat\mu)-\varepsilon A_\mu(x+\frac{\varepsilon}{2}\hat\mu)} \\
& {}^{+\frac{\varepsilon^2}{2}[A_\nu(x+\frac{\varepsilon}{2}\hat\nu),A_\mu(x+\varepsilon\hat\nu+\frac{\varepsilon}{2}\hat\mu)]+\frac{\varepsilon^2}{2}[A_\nu(x+\varepsilon\hat\mu+\frac{\varepsilon}{2}\hat\nu),A_\mu(x+\frac{\varepsilon}{2}\hat\mu)]} \\
& {}^{-\frac{\varepsilon^2}{2}[A_\nu(x+\frac{\varepsilon}{2}\hat\nu)+A_\mu(x+\varepsilon\hat\nu+\frac{\varepsilon}{2}\hat\mu),A_\nu(x+\varepsilon\hat\mu+\frac{\varepsilon}{2}\hat\nu)+A_\mu(x+\frac{\varepsilon}{2}\hat\mu)]+\mathcal{O}(\varepsilon^3)}.
\end{aligned}$$

The exponent is equal to

$$\varepsilon^2\, \partial_\nu A_\mu(x) - \varepsilon^2\, \partial_\mu A_\nu(x) + \varepsilon^2 [A_\nu(x), A_\mu(x)] + \mathcal{O}(\varepsilon^3) \tag{2.50}$$

and therefore

$$U^\varepsilon_{\mu\nu}(x) = \mathbb{1} - \varepsilon^2 F_{\mu\nu}(x) + \mathcal{O}(\varepsilon^3), \tag{2.51}$$

where

$$F_{\mu\nu}(x) := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)] \tag{2.52}$$

are the components of an antisymmetric tensor, the curvature tensor. The associated *curvature form* F is a 2-form with values in the Lie algebra of G:

$$F = \sum_{\mu<\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu. \tag{2.53}$$

We may now rewrite (2.52) as

$$F = dA + A \wedge A. \tag{2.54}$$

Note that curvature can also be expressed as a commutator of covariant derivatives:

$$F_{\mu\nu} = [D_\mu, D_\nu]. \tag{2.55}$$

The gauge transformation behaviour of $F_{\mu\nu}$ follows from the fact that it is the leading order term in a parallel transporter around a closed curve, and hence

$$F'_{\mu\nu}(x) = \Lambda^{-1}(x)\, F_{\mu\nu}(x)\, \Lambda(x); \tag{2.56}$$

thus $F_{\mu\nu}$ is not gauge invariant.

In the Abelian case

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \tag{2.57}$$

and F is the familiar electromagnetic field tensor. Moreover, $F_{\mu\nu}$ is gauge invariant.

Although we used the matter field ϕ as a starting point for introducing the gauge field A, we may consider the gauge field on its own without being coupled to other fields. In the following we shall concentrate on the case where no matter field is present. To proceed, we need some action $S(A)$ to determine the dynamics of the gauge field. One such action was proposed by Yang and Mills [4]:

$$S_{\mathrm{YM}}(A) = -\frac{1}{2g^2} \int dx \sum_{\mu<\nu} \mathrm{tr}\, F_{\mu\nu}^2, \tag{2.58}$$

where a coupling constant g has been introduced conventionally. The action S_{YM} is gauge invariant by cyclicity of the trace, as well as real and positive since $F_{\mu\nu}$ is antihermitean. It is a generalisation for non-Abelian fields of the Maxwell action for the classical electromagnetic field without sources

$$S_{\mathrm{ed}}(A) = \frac{1}{4} \int dx\; F_{\mu\nu} F^{\mu\nu} \tag{2.59}$$

to which it reduces in the Abelian case, provided that we substitute A_μ with $-igA_\mu$. (Thus the covariant derivative becomes the familiar $D_\mu = \partial_\mu - igA_\mu$, A_μ being real and g representing the elementary charge e.)

Note that for Abelian fields the action is quadratic in the field and describes therefore a free field. In the non-Abelian case there are higher-order terms arising from the nonzero commutator in (2.52).

2.3 Lattice gauge fields and Wilson's action

Euclidean field theory requires the construction of a measure with density $e^{-S(\phi)}$ on the field space. Such an integral is even more singular for gauge theories than in scalar models, and hence a regularisation is necessary. The lattice approximation proposed by Wilson [2] is a particularly convenient choice, since it is compatible with gauge invariance. Local (ultraviolet) regularity is achieved by discretising space-time and thus introducing a cutoff for small length scales. The infinite volume (infrared) problem can be addressed by considering a finite lattice. Any estimates should remain uniform when the approximations are removed.

2.3.1 Lattice discretisation

We discretise the Euclidean space-time using a finite lattice Λ. For simplicity, we assume the lattice to be of the form

$$\Lambda = \varepsilon\mathbb{Z}_{N_0} \times \varepsilon\mathbb{Z}_{N_1} \times \cdots \times \varepsilon\mathbb{Z}_{N_d}, \tag{2.60}$$

a rectangular lattice of dimensions $N_0 \times \cdots \times N_d$ with lattice spacing ε. Sometimes it will prove useful to consider an independent lattice spacing τ in the time direction $\mu = 0$. On Λ we shall always use the ∞-norm

$$|x| := \max\{x^0, \ldots, x^d\}. \tag{2.61}$$

To return to a physically more meaningful situation, two limits may be taken:

- The continuum limit: $\varepsilon \longrightarrow 0$, $N_\mu \longrightarrow \infty$, while $\varepsilon N_\mu =: L_\mu$ remains constant.
- The thermodynamic limit: ε remains constant, $N_\mu \longrightarrow \infty$.

2.3.2 Link variables, plaquettes and gauge transformations

In the continuum the field variables are $A_\mu(x)$ which can be interpreted as infinitesimal parallel transporters. On a lattice the smallest length scale is a *link* $b = \langle x, y\rangle$, i.e. an ordered set of nearest neighbours. The field variables in a lattice gauge theory are thus the parallel transporters across all links, called *link variables*. We shall denote the link variable of b by $g_b = g_{yx}$. Since $g_{xy} = g_{yx}^{-1}$ a choice of a positive orientation of the links is required to count only independent link variables. We choose the positive orientation to be $y_\mu \geq x_\mu$, $\forall\mu$. Thus the independent link variables can be labelled as $g_{yx} = g_x^\mu$, where $y = x + \varepsilon\hat{\mu}$.

More generally, a k-cell is a k-dimensional oriented hypercube with side length ε. A 0-cell is a lattice site, a 1-cell a link, a 2-cell a *plaquette*, etc. We shall index plaquettes as $p_x^{\mu\nu}$, a square in the μ-ν-plane with base point x. The positive orientation is defined by $\mu < \nu$. To every k-cell q can be associated its boundary ∂q, a set of $(p-1)$-cells.

A k-form on Λ is a function of the positively oriented k-cells of Λ. Thus a 0-form is a function of the lattice sites. A 1-form is a function of the oriented links; the link variables g_x^μ may be interpreted as the components of a 1-form. A 2-form is a function of the oriented plaquettes and so on.

In view of finding a lattice version of the action (2.58) we define the quantity corresponding to the curvature $F_{\mu\nu}$, a 2-form called *plaquette variable*. Let $p = p_x^{\mu\nu}$ be a plaquette. The corresponding plaquette variable g_p is the parallel transporter around ∂p:

$$g_p := g_x^{\mu\nu} := (g_x^{\nu})^{-1} \, (g_{x+\varepsilon\hat{\nu}}^{\mu})^{-1} \, g_{x+\varepsilon\hat{\mu}}^{\nu} \, g_x^{\mu}. \tag{2.62}$$

Generally if γ is a path in Λ, i.e. a sequence of connected links, we define g_γ to be the parallel transporter along γ. Thus g_γ is a (path ordered) product of link variables.

A gauge transformation can be taken directly over from our discussion of the continuum: a function h of the lattice sites into G. The link variables transform as

$$g'_{yx} = h_y^{-1} \, g_{yx} \, h_x. \tag{2.63}$$

Thus for closed paths γ expressions of the form $\operatorname{tr} g_\gamma$ are gauge invariant by cyclicity of the trace.

2.3.3 Wilson's action

Wilson [2] proposed the following action for a lattice gauge theory with gauge group SU(n) or U(n):

$$S_{\mathrm{W}}(\{g_x^{\mu}\}) := -J \sum_p (\operatorname{Re} \operatorname{tr} g_p - \operatorname{tr} \mathbb{1}), \tag{2.64}$$

where the sum ranges over all positively oriented plaquettes, and J is some coupling constant. The factor $\operatorname{tr} \mathbb{1}$ is physically insignificant and we shall sometimes leave it out. Note that (2.64) is gauge invariant, real and positive.

One justification for the choice of (2.64) as our action is that in the continuum limit it reduces to the Yang-Mills action (2.58) whenever the lattice gauge field arises from a smooth continuum gauge field A_μ.

Proof. As shown in Section 2.2.3 the plaquette variable $g_x^{\mu\nu}$ is infinitesimally equal to the curvature in the μ-ν-plane:

$$\begin{aligned} g_x^{\mu\nu} &= \mathbb{1} - \varepsilon^2 F_{\mu\nu}(x) + \mathcal{O}(\varepsilon^3) \\ &= e^{-\varepsilon^2 F_{\mu\nu}(x) + \mathcal{O}(\varepsilon^3)}. \end{aligned} \tag{2.65}$$

Since the exponent is antihermitean, we get

$$\operatorname{Re} \operatorname{tr} g_x^{\mu\nu} = \operatorname{tr} \mathbb{1} + \frac{\varepsilon^4}{2} \operatorname{tr} F_{\mu\nu}^2(x) + \mathcal{O}(\varepsilon^6). \tag{2.66}$$

Consider x in the box with fixed dimensions $L_\mu = \varepsilon N_\mu$. Then

$$\begin{aligned} S_{\mathrm{W}} &= -J \sum_x \sum_{\mu<\nu} (\operatorname{Re} \operatorname{tr} g_x^{\mu\nu} - \operatorname{tr} \mathbb{1}) \\ &= -\frac{J}{\varepsilon^{d+1}} \sum_x \varepsilon^{d+1} \sum_{\mu<\nu} (\frac{\varepsilon^4}{2} \operatorname{tr} F_{\mu\nu}^2(x) + \mathcal{O}(\varepsilon^6)). \end{aligned} \tag{2.67}$$

Thus for $\varepsilon \to 0$ the leading order term becomes

$$-\frac{J}{2\varepsilon^{d-3}} \int dx \sum_{\mu<\nu} \operatorname{tr} F_{\mu\nu}^2, \tag{2.68}$$

i.e. we recover the Yang-Mills action if we set

$$J = \frac{\varepsilon^{d-3}}{g^2}. \tag{2.69}$$

□

In the case that the lattice spacing in the time direction τ is different from ε we separate plaquettes into spatial plaquettes and plaquettes containing links in the time direction. We call the former magnetic plaquettes and the latter electric plaquettes in analogy to their interpretation as the components of the magnetic and electric fields in the Abelian case. In the action (2.64) the magnetic plaquettes have a "magnetic" coupling constant J_{M} and the electric plaquettes an "electric" coupling constant J_{E}. We may repeat the above calculation to find expressions for J_{M} and J_{E} in the continuum limit: In the Riemann sum approximation of the integral we have now $\varepsilon^d\tau$ instead of ε^{d+1}. Moreover, for electric plaquettes, $\varepsilon\tau$ replaces ε^2 as the factor of the curvature tensor in the plaquette variable. Thus we have

$$J_{\mathrm{M}} = \frac{\tau}{\varepsilon} J \tag{2.70a}$$

$$J_{\mathrm{E}} = \frac{\varepsilon}{\tau} J. \tag{2.70b}$$

Thus the lattice constants ε and τ only appear in the coupling constants J_{M} and J_{E}, and in the following we shall often take $\varepsilon = \tau = 1$.

A generalisation of Wilson's action is

$$S(\{g_x^\mu\}) = -J \sum_p \big(\operatorname{Re} \chi(g_p) - \chi(\mathbb{1})\big), \tag{2.71}$$

where χ is some faithful character, i.e. the trace of some injective representation, or, equivalently, a character such that $\operatorname{Re} \chi(g) < \chi(\mathbb{1})$ for $g \neq \mathbb{1}$. This allows us to study other gauge groups than $\mathrm{U}(n)$ that need not be matrix groups. We shall denote the unitary representation belonging to χ by ρ and write $u_{yx} := \rho(g_{yx})$.

Wilson's action is by no means unique, and many other lattice actions have been proposed that reproduce the same continuum limit. For simplicity, however, we shall only consider the Wilson action.

2.3.4 The functional integral

We now return to the original task of defining the functional integral (2.22). The field variables are now a finite number of link variables

$$\underline{g} := \{g_x^\mu \;:\; x \in \Lambda,\ \mu = 0, \dots d\}. \tag{2.72}$$

The natural uniform measure on each g_x^μ is the normalised Haar measure dg_x^μ on the group G, assumed to be compact. We denote by

$$d\underline{g} := \prod_{x,\mu} dg_x^\mu \tag{2.73}$$

the product measure on the compact set

$$\mathscr{G} := \prod_{x,\mu} G. \tag{2.74}$$

This corresponds to the formal uniform measure $\mathcal{D}\phi$ in (2.22). The desired weighted measure is the "Boltzmann" measure on $\mathscr{G}$

$$d\mu_{J,\Lambda}(\underline{g}) := \frac{1}{Z_{J,\Lambda}} e^{-JS(\underline{g})}\, d\underline{g}, \tag{2.75}$$

where

$$Z_{J,\Lambda} := \int_{\mathscr{G}} e^{-JS(\underline{g})}\, d\underline{g} \tag{2.76}$$

is the "partition function" so that $d\mu_{J,\Lambda}(\underline{g})$ is a probability measure. The action $S(\underline{g})$ is

$$S(\underline{g}) = -\sum_p (\operatorname{Re}\chi(g_p) - \chi(\mathbb{1})) \tag{2.77}$$

where we pulled out the constant J that can be interpreted as an "inverse temperature"[1] (In the following we shall sometimes include J in the action S). We shall often omit the index "Λ" and even "J" if no confusion is possible. We have thus transformed the study of a lattice gauge theory into a problem of classical statistical mechanics.

For any function $f = f(\underline{g})$ we define the *expected value of* f as

$$\langle f \rangle_J := \int_{\mathscr{G}} f(\underline{g})\, d\mu_J(\underline{g}). \tag{2.78}$$

In particular,

$$\langle f \rangle_0 = \int_{\mathscr{G}} f(\underline{g})\, d\underline{g}. \tag{2.79}$$

2.3.5 Observables and gauge fixing

Gauge transformations act on the space $\mathscr{G}$ according to (2.63). We denote the gauge transformation by $\underline{h} := \{h_x \,:\, x \in \Lambda\}$ and $\underline{g}'$ by $\underline{g}^{\underline{h}}$.

We define an observable to be a gauge invariant function of the link variables. Note that from any function f we may construct a gauge invariant function $\tilde{f}$ such that

$$\langle f \rangle_J = \langle \tilde{f} \rangle_J \tag{2.80}$$

by "averaging over all gauges":

$$\tilde{f}(\underline{g}) := \int d\underline{h}\; f(\underline{g}^{\underline{h}}), \tag{2.81}$$

[1]Not to be confused with the physical inverse temperature β introduced in Section 2.7.1.

where $d\underline{h} := \prod_{x\in\Lambda} dh_x$ and dh_x is the normalised Haar measure on G. Indeed, by invariance of the Haar measure and gauge invariance of $S(\underline{g})$, we get

$$\begin{aligned} Z_J \langle \tilde{f} \rangle_J &= \int d\underline{h} \int d\underline{g}\; e^{-J\,S(\underline{g})}\; f(\underline{g}^{\underline{h}}) \\ &\overset{\underline{g}'=\underline{g}^{\underline{h}}}{=} \int d\underline{h} \int d\underline{g}'\; e^{-J\,S(\underline{g}')}\; f(\underline{g}'). \\ &= \int d\underline{g}'\; e^{-J\,S(\underline{g}')}\; f(\underline{g}'). \\ &= Z_J \langle f \rangle_J. \end{aligned} \tag{2.82}$$

Furthermore $\tilde{f}$ is gauge invariant by invariance of the Haar measure dh_x.

As previously mentioned, $W(\gamma) := \chi(g_\gamma)$ is gauge invariant for closed paths γ and hence an observable. It is called a *Wilson loop* and plays an important role in studying confinement (see Section 2.8). Durhuus [14] has shown that any continuous observable can be approximated arbitrarily well by linear combinations of observables of the form

$$W(\gamma_1)\dots W(\gamma_n). \tag{2.83}$$

The presence of gauge invariance can be interpreted as the existence of superfluous degrees of freedom in $\underline{g}$. These can be removed by *gauge fixing*. Let $\mathscr{F}$ be some subset of the links in Λ that does not contain any closed loops ($\mathscr{F}$ is called a *forest*). Denote by $\underline{g}_1$ the link variables in $\mathscr{F}$ and by $\underline{g}_2$ the remaining link variables. Thus $\underline{g} = (\underline{g}_1, \underline{g}_2)$. Then we may set the value of $\underline{g}_1$ to any given value $\underline{g}_1'$ using some appropriate gauge transformation $\underline{h}$. More precisely, for any values of $\underline{g}$ and $\underline{g}_1'$ we may find a gauge transformation $\underline{h}$ and some $\underline{g}_2'$ such that $\underline{g}^{\underline{h}} = \underline{g}'$. That this is the case can be seen by considering a connected component of $\mathscr{F}$ (called a *tree*), in which we choose some lattice point (the "root" of the tree). We start at the root and move along one "branch", taking at each lattice point as the gauge transformation a group element that sets the immediately preceding link to the desired value. By recursion all links in the tree may be thus fixed (see Figure 2.2).

Now let f be some observable. Then the average of f is equal to the "conditional" average of f where the integration variables $\underline{g}_1$ are frozen. In other words

$$\langle f \rangle_J = \langle f \rangle_{J,\,\underline{g}_1}, \tag{2.84}$$

with

$$\langle f \rangle_{J,\,\underline{g}_1} := \frac{1}{Z_J} \int d\underline{g}_2\; e^{-J\,S(\underline{g}_1,\underline{g}_2)}\; f(\underline{g}_1,\, \underline{g}_2). \tag{2.85}$$

PROOF. We need to show that $\langle f \rangle_{J,\,\underline{g}_1}$ is a constant in $\underline{g}_1$. For $\underline{g}_1$, $\underline{g}_1'$ given choose some $\underline{h}$ as above. Then we get, by invariance of the Haar measure, gauge invariance of S and

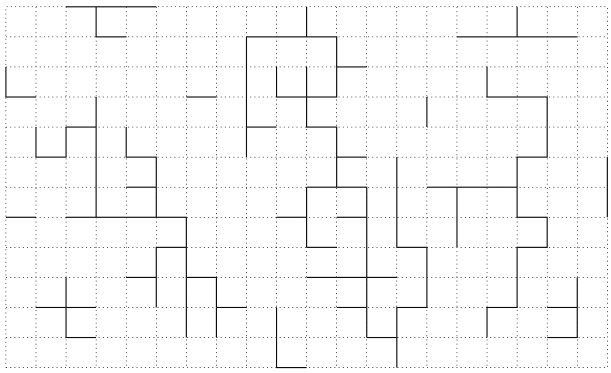

Figure 2.2: *Botany on a lattice. Shown is a forest consisting of trees.*

f, and the substitution $\underline{g}' = \underline{g}^{\underline{h}}$,

$$\begin{aligned} \langle f \rangle_{J,\,\underline{g}_1'} &= \frac{1}{Z_J} \int d\underline{g}_2'\, e^{-J\,S(\underline{g}')}\, f(\underline{g}') \\ &= \frac{1}{Z_J} \int d\underline{g}_2\, e^{-J\,S(\underline{g})}\, f(\underline{g}) \\ &= \langle f \rangle_{J,\,\underline{g}_1}. \end{aligned} \tag{2.86}$$

Since $\langle f \rangle_{J,\,\underline{g}_1}$ is independent of $\underline{g}_1$, integrating over $\underline{g}_1$ has no effect and the statement is proven. □

An example of gauge fixing is the *temporal gauge*, where all links in the time direction are set to unity. This corresponds to the continuum gauge where $A_0 = 0$. Note that some gauge freedom still remains in this case within the spatial link variables. A complete removal of the gauge degrees of freedom can be achieved by fixing the link variables on a *maximal tree*, i.e. a tree to which no link can be added without creating a closed path.

2.4 Reflection positivity

For any reflection hyperplane π in space-time let r denote the map $\Lambda \mapsto \Lambda$ that associates to every lattice site the corresponding site obtained by reflection about π. Then we also have a reflection mapping for the link variables, which we also denote by r, defined by

$$r g_{yx} := g_{ry\,rx}. \tag{2.87}$$

For any observable f we may define the "reflected" observable as

$$\Theta f(\underline{g}) := \overline{f(r\underline{g})}. \tag{2.88}$$

Note that Θ commutes with addition and multiplication; in particular $e^{\Theta f} = \Theta\, e^f$.

Reflection positivity (also called Osterwalder-Schrader positivity) states that for any observable f depending only on the link variables on one side the π

$$\langle f\,\Theta f\rangle_J \geq 0. \tag{2.89}$$

This is not true for any action; in this section we shall prove it for the Wilson action for various plane configurations and boundary conditions. To avoid cluttering the notation, we set $J = 1$.

2.4.1 Free boundary conditions

Consider first the case of free boundary conditions. We take π perpendicular to some coordinate direction, say $\mu = 0$. In order for r to be well defined π must lie either on lattice sites (site-reflection) or half-way between lattice sites (link-reflection). Furthermore, Λ must be closed under the action of r, i.e. the lattice has to be symmetric with respect to π.

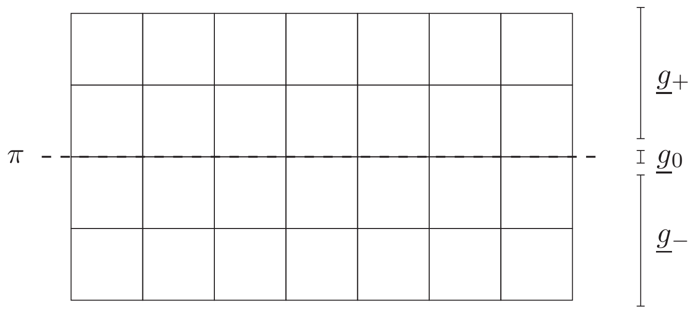

Figure 2.3: *Site-reflection.*

For site-reflection positivity we split the link variables as $\underline{g} = (\underline{g}_-, \underline{g}_0, \underline{g}_+)$, where $\underline{g}_0$ contains all the links in π, $\underline{g}_+$ all the links above π, and $\underline{g}_-$ all the links below π (see Figure 2.3). We now write the action as

$$S(\underline{g}_-, \underline{g}_0, \underline{g}_+) = S_-(\underline{g}_-, \underline{g}_0) + S_+(\underline{g}_+, \underline{g}_0) + S_0(\underline{g}_0), \tag{2.90}$$

where S_0 contains all the plaquettes in π, S_+ the remaining plaquettes above π and S_- the remaining plaquettes below π. The key observation is that

$$S_- = \Theta S_+, \tag{2.91}$$

since the reflection only changes the orientation of the plaquette variables and thus leaves each term in S invariant. Then for any observable $f = f(\underline{g}_0, \underline{g}_+)$ we have

$$\begin{aligned} Z\,\langle f\,\Theta f\rangle &= \int d\underline{g}\; f e^{-S_+}\;\Theta(f e^{-S_+})\; e^{-S_0} \\ &= \int d\underline{g}_0 \Big[\int d\underline{g}_+\; f e^{-S_+}\Big]\,\Big[\int d\underline{g}_-\;\Theta(f e^{-S_+})\Big]\; e^{-S_0} \\ &= \int d\underline{g}_0 \Big[\int d\underline{g}_+\; f e^{-S_+}\Big]\,\overline{\Big[\int d\underline{g}_+\; f e^{-S_+}\Big]}\; e^{-S_0} \\ &\geq 0. \end{aligned} \tag{2.92}$$

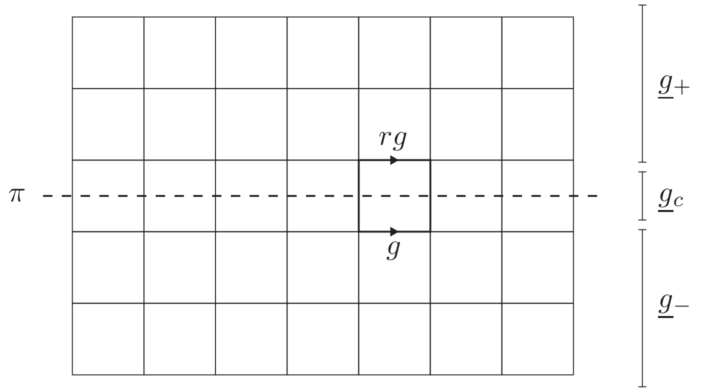

Figure 2.4: *Link-reflection. A plaquette crossing π is drawn.*

Link-reflection positivity is handled with a similar decomposition of the link variables: $\underline{g} = (\underline{g}_-, \underline{g}_c, \underline{g}_+)$, where $\underline{g}_c$ contains all the (temporal) links crossing π, $\underline{g}_+$ the links above π and $\underline{g}_-$ the links below π (see Figure 2.4). We shall work in the temporal gauge, so that all links in $\underline{g}_c$ are set to unity. Note that if f is an observable depending only on $\underline{g}_+$

$$\langle f\,\Theta f\rangle_0 = \Big[\int d\underline{g}_+\; f(\underline{g}_+)\Big]\,\Big[\int d\underline{g}_-\;\Theta f(\underline{g}_-)\Big] = \Big|\int d\underline{g}_+\; f(\underline{g}_+)\Big|^2 \geq 0. \tag{2.93}$$

Define the multiplicative cone $\mathscr{P}$ as the set of linear combinations with positive coefficients of functions of the form $f\,\Theta f$, with f depending only on $\underline{g}_+$. Note that $\mathscr{P}$ is closed under addition, multiplication and multiplication by a positive number, so that $e^F \in \mathscr{P}$ if $F \in \mathscr{P}$. Then (2.93) implies, by linearity,

$$\langle F\rangle_0 \geq 0, \qquad \forall F \in \mathscr{P}. \tag{2.94}$$

Now decompose

$$S(\underline{g}_-, \underline{g}_+) = S_-(\underline{g}_-) + S_+(\underline{g}_+) + S_c(\underline{g}_-, \underline{g}_+), \tag{2.95}$$

with S_c containing the plaquettes crossing π, S_+ the plaquettes above π and S_- the plaquettes below π. As in the case of site-reflection $S_- = \Theta S_+$. Therefore, for any

$f\,\Theta f \in \mathscr{P}$, we have

$$\langle f\,\Theta f\rangle = \frac{1}{Z}\,\langle f\,e^{-S_+}\,\Theta(f\,e^{-S_+})\,e^{-S_c}\rangle_0. \tag{2.96}$$

Thus we only need to prove that $-S_c \in \mathscr{P}$. For a plaquette p in S_c the plaquette variable g_p can be expressed in terms of the link variables as $g_p = g\,(rg)^{-1}$, where g is a spatial link variable (see Figure 2.4). Then

$$\chi(g_p) = \operatorname{tr}\rho\big(g\,(rg)^{-1}\big) = \sum_{i,j}\rho(g)_{ij}\,\overline{\rho(rg)_{ij}} \in \mathscr{P}. \tag{2.97}$$

Therefore $\operatorname{Re}\chi(g_p) \in \mathscr{P}$ and we are done.

2.4.2 Periodic boundary conditions

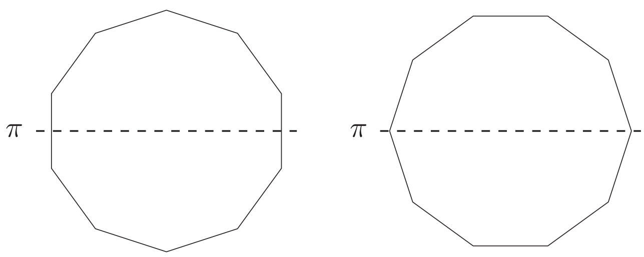

Figure 2.5: *Reflection for an even number of links in the time direction. Shown are link-reflection (left) and site-reflection (right).*

Periodic boundary conditions are treated similarly to free boundary conditions. We again consider reflections in the time direction. Boundary conditions in the spatial directions are irrelevant. The reflection map r is given by reflecting the "space-time cylinder" about a hyperplane that halves it. Because of the periodicity, the hyperplane intersects the lattice in two different locations.

First consider the case when N_0 is even. Then we may either have a site-reflection or a link-reflection, as shown in Figure 2.5. The proofs can be taken over from the previous section. For site-reflection we split as above $\underline{g} = (\underline{g}_-, \underline{g}_0, \underline{g}_+)$ with $\underline{g}_0$ containing all links on π, now consisting of two disjoint sets of link variables. For link-reflection we decompose $\underline{g} = (\underline{g}_-, \underline{g}_c, \underline{g}_+)$, with $\underline{g}_c$ again being all temporal links intersecting π. Due to the periodic boundary conditions we may not choose the temporal gauge. Sufficient for the proof, however, is a gauge in which all links in $\underline{g}_c$ are set to unity.

If N_0 is odd (see Figure 2.6) the proof must be modified slightly. We split $\underline{g} = (\underline{g}_-, \underline{g}_+, \underline{g}_0, \underline{g}_c)$ with $\underline{g}_0$ containing the spatial links in π, $\underline{g}_c$ the temporal links intersecting π, $\underline{g}_+$ the links above π and $\underline{g}_-$ the links below π. We choose a gauge in which all links in $\underline{g}_c$ are set to unity. The action is split as

$$S = S_+(\underline{g}_+, \underline{g}_0) + S_-(\underline{g}_-, \underline{g}_0) + S_0(\underline{g}_0) + S_c(\underline{g}_-, \underline{g}_+) \tag{2.98}$$

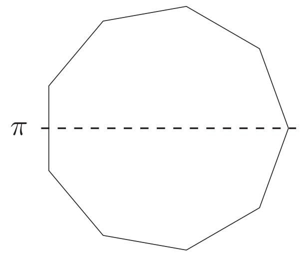

Figure 2.6: *Reflection for an odd number of links in the time direction.*

following the conventions of the previous section. Again $S_- = \Theta S_+$. Then for an observable $f = f(\underline{g}_0, \underline{g}_+)$ we have

$$Z\,\langle f\ \Theta f\rangle = \int d\underline{g}_0\, d\underline{g}_-\, d\underline{g}_+\ \left(f\,e^{-S_+}\right)\,\Theta\,\left(f e^{-S_+}\right)e^{-S_c}\,e^{-S_0}. \qquad (2.99)$$

From above we know that $-S_c \in \mathscr{P}$ and hence reflection positivity follows.

2.4.3 Link splitting

We present an alternate proof of link-reflection positivity that requires no gauge fixing and allows a slight generalisation of the class of functions for which positivity holds. We consider free boundary conditions. The extension to other cases of link-reflection positivity is straightforward.

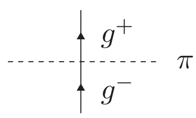

Figure 2.7: *A temporal link crossing π split in two.*

The idea is to split each temporal link in $\underline{g}_c$, labelled by $g_{\mathbf{x}}$, in two:

$$g_{\mathbf{x}} = g_{\mathbf{x}}^+\, g_{\mathbf{x}}^-\,, \qquad (2.100)$$

as shown in Figure 2.7. This induces a splitting of $\underline{g}_c = \underline{g}_c^+\,\underline{g}_c^-$. The reflection r is extended in the natural way:

$$r g_{\mathbf{x}}^+ := (g_{\mathbf{x}}^-)^{-1}. \qquad (2.101)$$

Now we take $\underline{g}_c^+$ and $\underline{g}_c^-$ as independent integration variables. That this does not change the value of the integral follows from the invariance and normalisation of the Haar measure:

$$\int d\underline{g}_+\, d\underline{g}_-\, d\underline{g}_c^+\, d\underline{g}_c^-\ f(\underline{g}_+, \underline{g}_-, \underline{g}_c^+\,\underline{g}_c^-) \overset{\underline{g}_c = \underline{g}_c^+\underline{g}_c^-}{=} \int d\underline{g}_c^- \int d\underline{g}_+\, d\underline{g}_-\, d\underline{g}_c\ f(\underline{g}_+, \underline{g}_-, \underline{g}_c)$$
$$= \int d\underline{g}_+\, d\underline{g}_-\, d\underline{g}_c\ f(\underline{g}_+, \underline{g}_-, \underline{g}_c). \qquad (2.102)$$

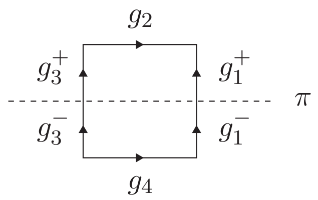

Figure 2.8: *A plaquette crossing* π.

Reflection positivity holds for functions $f = f(\underline{g}_+, \underline{g}_c^+)$. We proceed as above, splitting $S = S_+ + S_- + S_c$ with $S_- = \Theta S_+$. The claim follows from the the fact that each plaquette term in S_c can be written as (see Figure 2.8)

$$\chi\big(g_1^-\, g_4\, (g_3^-)^{-1}\, (g_3^+)^{-1}\, g_2^{-1}\, g_1^+\big) = \sum_{i,j} s_{ij}(\underline{g}_+, \underline{g}_c^+)\; \Theta s_{ij}(\underline{g}_-, \underline{g}_c^-), \tag{2.103}$$

where

$$s_{ij}(\underline{g}_+, \underline{g}_c^+) := \sum_{k,l} \overline{\rho(g_3^+)}_{lj}\, \overline{\rho(g_2)}_{kl}\, \rho(g_1^+)_{ki}. \tag{2.104}$$

2.5 Hilbert space, transfer matrix and Hamiltonian

In the general formalism [15, 16] reflection positivity is used to construct a quantum mechanical Hilbert space and a transfer matrix (and thus a Hamiltonian) in a rather abstract fashion. The fact that the lattice action couples only neighbouring links may be used to construct a simpler and more explicit description of the Hilbert space and the transfer matrix.

Consider a spatial box $\Lambda_0 \subset \mathbb{Z}^d$, a slice at a fixed time of the space-time lattice Λ [see (2.60)]. For convenience we change our notation and write $\underline{g} := \{g_b \;:\; b \subset \Lambda_0\}$. Define the Hilbert space

$$\mathscr{H} := L^2\Bigg(\prod_{b\subset\Lambda_0} G,\; d\underline{g}\Bigg), \tag{2.105}$$

as the space of square-integrable functions Ψ of the link variables in Λ_0.

We now introduce a *transfer matrix* $\mathcal{T} \in \mathscr{J}_1(\mathscr{H})$, a strictly positive trace class[2] operator that corresponds to a translation by one lattice unit in the time direction:

$$\mathcal{T} := MTM. \tag{2.106}$$

Here M is the multiplication operator by

$$m(\underline{g}) := \exp\left[\frac{1}{2}\sum_p J_{\mathrm{M}}\big(\mathrm{Re}\,\chi(g_p) - \chi(\mathbb{1})\big)\right], \tag{2.107}$$

where the sum ranges over all positively oriented plaquettes in Λ_0, and T is the convolution operator with kernel

$$t(\underline{g}) := \exp\left[\sum_b J_{\mathrm{E}}\big(\mathrm{Re}\,\chi(g_b) - \chi(\mathbb{1})\big)\right], \tag{2.108}$$

[2] $\mathscr{J}_1(\mathscr{H})$ stands for the trace class operators on $\mathscr{H}$.

where the sum ranges over all positively oriented links b in Λ_0.

First note that $\mathcal{T}$ is self-adjoint, since m and t are real and $t(\{g_b^{-1}\}) = t(\{g_b\})$. That $\mathcal{T} \in \mathscr{J}_1(\mathscr{H})$ follows from the fact that M is bounded and $T \in \mathscr{J}_1(\mathscr{H})$, since m and t are continuous functions on the compact set G^{Λ_0}. In particular $\mathcal{T}$ is compact (see for example [17]). Furthermore,

Lemma 2.1. *The transfer matrix $\mathcal{T}$ is strictly positive.*

Proof. Since obviously $M > 0$ we only need to show that $T > 0$. To this end note that $\mathscr{H}$ is a product:

$$\mathscr{H} = \bigotimes_{b \subset \Lambda_0} L^2(G, dg), \tag{2.109}$$

and correspondingly T is a product over links:

$$T = \bigotimes_{b \subset \Lambda_0} T_b, \tag{2.110}$$

where T_b is given (up to an irrelevant constant) by convolution with

$$t(g) := e^{J_{\mathrm{E}} \operatorname{Re} \chi(g)}. \tag{2.111}$$

It suffices to show that $T_b > 0$. Let us find the eigenvalues of T_b. For any $f \in L^2(G, dg)$ we have by the Peter-Weyl theorem (see appendix A for conventions)

$$f(g) = \sum_{\alpha \in \widehat{G}} \sum_{i,j} f_\alpha^{ij} \, D_{ij}^{(\alpha)}(g) \tag{2.112}$$

and

$$t(g) = \sum_{\beta \in \widehat{G}} t_\beta \, \chi_\beta(g) = \sum_{\beta \in \widehat{G}} \sum_k t_\beta \, D_{kk}^{(\beta)}(g). \tag{2.113}$$

Therefore

$$\begin{aligned}
(T_b f)(g) &= \int dh \; t(gh^{-1}) \, f(h) \\
&= \sum_{\alpha,\beta} \sum_{i,j,k} t_b \, f_\alpha^{ij} \int dh \; D_{kk}^{(\beta)}(gh^{-1}) \, D_{ij}^{(\alpha)}(h) \\
&= \sum_{\alpha,\beta} \sum_{i,j,k,l} t_b \, f_\alpha^{ij} \, D_{kl}^{(\beta)}(g) \underbrace{\int dh \; \overline{D_{kl}^{(\beta)}(h)} \, D_{ij}^{(\alpha)}(h)}_{= \frac{1}{d_\alpha} \delta_{\alpha\beta} \delta_{ik} \delta_{jl}} \\
&= \sum_{\alpha} \sum_{i,j} \frac{1}{d_\alpha} t_\alpha \, f_\alpha^{ij} D_{ij}^{(\alpha)}(g).
\end{aligned} \tag{2.114}$$

Thus the eigenvectors of T_b are $\{D_{ij}^{(\alpha)}(g)\}$ and form a basis of $L^2(G, dg)$. The corresponding eigenvalues are

$$\begin{aligned}
m_\alpha := \frac{1}{d_\alpha} t_\alpha &= \frac{1}{d_\alpha} \int dg \; \overline{\chi_\alpha(g)} \, e^{J_{\mathrm{E}} \operatorname{Re} \chi(g)}. \\
&= \frac{1}{d_\alpha} \sum_{m,n} \left(\frac{J_{\mathrm{E}}}{2} \right)^{m+n} \frac{1}{m! n!} \, c_{mn}^\alpha,
\end{aligned} \tag{2.115}$$

where

$$c^{\alpha}_{mn} := \int dg\, \overline{\chi_\alpha(g)}\, \chi(g)^n\, \overline{\chi(g)^m}, \tag{2.116}$$

which is simply the number of times the irreducible representation α occurs in the representation $\rho^{\otimes n} \otimes \overline{\rho}^{\otimes m}$, so that $c^{\alpha}_{mn} \in \mathbb{N}$.

Now since χ was assumed to be faithful, i.e. $\operatorname{Re}\chi(g) < \chi(\mathbb{1})$ for $g \neq \mathbb{1}$, the term $e^{J_{\mathrm{E}} \operatorname{Re}\chi(g)}$ becomes sharply peaked around $\mathbb{1}$ for $J_{\mathrm{E}} \to \infty$ and we thus have

$$\frac{m_\alpha}{m_0} \to 1, \qquad J_{\mathrm{E}} \to \infty, \tag{2.117}$$

where

$$m_0 = \int dg\, e^{J_{\mathrm{E}} \operatorname{Re}\chi(g)} > 0 \tag{2.118}$$

is the eigenvalue corresponding to the trivial representation. Therefore, for each α there must be a $c^{\alpha}_{mn} > 0$, so that $m_\alpha > 0$ for all J_{E}. □

The positivity of $\mathcal{T}$ allows us to define a *Hamiltonian* H_τ:

$$\mathcal{T} =: e^{-\tau H_\tau}, \tag{2.119}$$

in analogy to (2.15). It can be shown [18] that H_τ has a well defined continuum limit $\tau \to 0$, which is the Hamiltonian H of the lattice gauge theory. For our purposes, however, the discretised H_τ is enough.

2.6 External charges: static quarks

A gauge transformation on $\mathscr{H}$ is a function $\underline{h} = \{h_{\mathbf{x}}\}$ of the lattice sites into G. This induces a time-independent gauge transformation on the space-time lattice Λ. As before we write the action of $\underline{h}$ on a field configuration $\underline{g}$ as

$$\underline{g}^{\underline{h}} := \{h_{\mathbf{y}}^{-1}\, g_{\mathbf{yx}}\, h_{\mathbf{x}}\}. \tag{2.120}$$

This defines a unitary representation U of $\mathscr{G} := \prod_{\mathbf{x}\in\Lambda_0} G$ on $\mathscr{H}$:

$$\big(U(\underline{h})\Psi\big)(\underline{g}) := \Psi(\underline{g}^{\underline{h}}). \tag{2.121}$$

In general, $\Psi \in \mathscr{H}$ is not gauge invariant. Since $\mathscr{G}$ is a direct product, its irreducible representations are of the form

$$\rho_{\underline{\alpha}} := \bigotimes_{\mathbf{x}\in\Lambda_0} \rho_{\alpha_{\mathbf{x}}}, \tag{2.122}$$

where $\underline{\alpha} := \{\alpha_{\mathbf{x}} \;:\; \mathbf{x} \in \Lambda_0\}$ and $\alpha_{\mathbf{x}}$ is an irreducible representation of G. We may accordingly decompose $\mathscr{H}$ into representation subspaces

$$\mathscr{H} = \bigoplus_{\underline{\alpha}} \mathscr{H}_{\underline{\alpha}}, \tag{2.123}$$

where $\mathscr{H}_{\underline{\alpha}}$ is the subspace of $\mathscr{H}$ that bears the irreducible representation $\underline{\alpha}$. From appendix A we get the orthogonal projection $\mathcal{P}_{\underline{\alpha}}$ onto $\mathscr{H}_{\underline{\alpha}}$:

$$\mathcal{P}_{\underline{\alpha}} = d_{\underline{\alpha}} \int d\underline{h}\, \overline{\chi_{\underline{\alpha}}(\underline{h})}\, U(\underline{h}). \tag{2.124}$$

This factors over the lattice points:

$$\mathcal{P}_{\underline{\alpha}} = \prod_{\mathbf{x}\in\Lambda_0} \mathcal{P}_{\alpha_{\mathbf{x}}}. \tag{2.125}$$

Here

$$\mathcal{P}_{\alpha_{\mathbf{x}}} := d_{\alpha_{\mathbf{x}}} \int dh\, \overline{\chi_{\alpha_{\mathbf{x}}}(h)}\, U_{\mathbf{x}}(h), \tag{2.126}$$

with $U_{\mathbf{x}}(h) := U(\underline{h})$, where

$$h_{\mathbf{y}} = \begin{cases} h, & \mathbf{y} = \mathbf{x}, \\ \mathbb{1}, & \mathbf{y} \neq \mathbf{x}. \end{cases} \tag{2.127}$$

We say that $\mathscr{H}_{\underline{\alpha}}$ is the subspace having *external charges* $\underline{\alpha}$, so that at each lattice point we have an external charge $\alpha_{\mathbf{x}} \in \widehat{G}$. If $\alpha_{\mathbf{x}} = 0$ is the trivial representation we say there is no charge at $\mathbf{x}$.

Lemma 2.2.

$$[\mathcal{T}, \mathcal{P}_{\underline{\alpha}}] = 0. \tag{2.128}$$

Proof. First note that since $m(\underline{g})$ is gauge invariant $[M, \mathcal{P}_{\underline{\alpha}}] = 0$, so that we only need to show that $[T, \mathcal{P}_{\underline{\alpha}}] = 0$. This is just a calculation. For any $\Psi \in \mathscr{H}$ we have

$$\begin{aligned}(T\,\mathcal{P}_{\underline{\alpha}}\,\Psi)(\underline{g}) &= \int d\underline{l}\; t(\underline{g}\,\underline{l}^{-1})\,(P_{\underline{\alpha}}\,\Psi)(\underline{l}) \\ &= \int d\underline{h} \int d\underline{l}\; t(\underline{g}\,\underline{l}^{-1})\, d_{\underline{\alpha}}\, \overline{\chi_{\underline{\alpha}}(\underline{h})}\, \Psi(\underline{l}^{\underline{h}}).\end{aligned} \tag{2.129}$$

Using the variable transformation $\underline{k} := \underline{l}^{\underline{h}}$ this is equal to (by invariance of the group measure)

$$\int d\underline{h} \int d\underline{k}\; t\Big[\underline{g}\,\big(\underline{k}^{\underline{h}^{-1}}\big)^{-1}\Big]\, d_{\underline{\alpha}}\, \overline{\chi_{\underline{\alpha}}(\underline{h})}\, \Psi(\underline{k}). \tag{2.130}$$

Now

$$t\big(\{g_{\mathbf{yx}}\,(h_{\mathbf{y}} k_{\mathbf{yx}} h_{\mathbf{x}}^{-1})^{-1}\}\big) = t\big(\{g_{\mathbf{yx}} h_{\mathbf{x}} k_{\mathbf{yx}}^{-1} h_{\mathbf{y}}^{-1}\}\big) = t\big(\{h_{\mathbf{y}}^{-1} g_{\mathbf{yx}} h_{\mathbf{x}} k_{\mathbf{yx}}^{-1}\}\big) \tag{2.131}$$

by the cyclicity of χ in t, so that

$$\begin{aligned}(T\,\mathcal{P}_{\underline{\alpha}}\,\Psi)(\underline{g}) &= \int d\underline{h} \int d\underline{k}\; t(\underline{g}^{\underline{h}}\,\underline{k}^{-1})\, d_{\underline{\alpha}}\, \overline{\chi_{\underline{\alpha}}(\underline{h})}\, \Psi(\underline{k}) \\ &= \int d\underline{h}\; d_{\underline{\alpha}}\, \overline{\chi_{\underline{\alpha}}(\underline{h})}\, (T\,\Psi)(\underline{g}^{\underline{h}}) \\ &= (\mathcal{P}_{\underline{\alpha}}\, T\,\Psi)(\underline{g}).\end{aligned} \tag{2.132}$$

□

Lemma 2.2 implies that if the state Ψ has a charge $\alpha_{\mathbf{x}}$ at $\mathbf{x}$, so does $\mathcal{T}\Psi$; in other words the charges do not move in time. For this reason they are called static or infinitely heavy. It can be shown [18] that the representation $\underline{\alpha}$ on the Lie algebra is a charge density operator that obeys a Gauss law with the electric field; this provides a physical interpretation of the charges.

2.7 A criterion for confinement at finite temperature

Our treatment of confinement is based on [18, 19].

2.7.1 The static quark potential and Polyakov loops

We define the *partition function at inverse temperature* β *and external charges* $\underline{\alpha}$ as

$$Z_{\underline{\alpha}} := \mathrm{tr}\left[\mathcal{P}_{\underline{\alpha}} e^{-\beta H_\tau}\right]. \tag{2.133}$$

This can be interpreted as the sum with canonical weight over all states with external charges $\underline{\alpha}$. For

$$\beta = \tau N_0 \tag{2.134}$$

we thus have

$$Z_{\underline{\alpha}} = \mathrm{tr}\left[\mathcal{P}_{\underline{\alpha}} \mathcal{T}^{N_0}\right], \tag{2.135}$$

so that the inverse temperature of the lattice gauge theory is equal to the length of the lattice in the time direction. Define further the *free energy* $F_{\underline{\alpha}}$ as

$$Z_{\underline{\alpha}} =: e^{-\beta F_{\underline{\alpha}}}. \tag{2.136}$$

We may now return to the functional integral formulation of our theory. Denoting by $\underline{\alpha} = 0$ the trivial representation we have

Lemma 2.3. *For* $\beta = \tau N_0$ *and periodic boundary conditions in the time direction we have*

$$Z_0 = Z_\Lambda := \int \prod_{b \subset \Lambda} dg_b \, e^{-S(\{g_b\})}. \tag{2.137}$$

In other words, our old partition function (2.76) with periodic boundary conditions in the time direction is recovered when no external charges are present.

Proof. Denoting by τ the kernel of $\mathcal{T}$

$$\mathrm{tr}\left[\mathcal{P}_0 \, \mathcal{T}^{N_0}\right] = \int d\underline{g}_1 \, d\underline{h} \, d\underline{g}_2 \dots d\underline{g}_{N_0} \; \tau(\underline{g}_1^{\underline{h}}, \underline{g}_2) \, \tau(\underline{g}_2, \underline{g}_3) \dots \tau(\underline{g}_{N_0}, \underline{g}_1). \tag{2.138}$$

The integrand is equal to

$$m(\underline{g}_1^{\underline{h}}) \, t(\underline{g}_1^{\underline{h}}, \underline{g}_2) \, m^2(\underline{g}_2) \, t(\underline{g}_2, \underline{g}_3) \dots m^2(\underline{g}_{N_0}) \, t(\underline{g}_{N_0}, \underline{g}_1) \, m(\underline{g}_1), \tag{2.139}$$

which, by gauge invariance of m, is equal to

$$m^2(\underline{g}_1) \, t(\underline{g}_1^{\underline{h}}, \underline{g}_2) \, m^2(\underline{g}_2) \, t(\underline{g}_2, \underline{g}_3) \dots m^2(\underline{g}_{N_0}) \, t(\underline{g}_{N_0}, \underline{g}_1). \tag{2.140}$$

Integrating this gives Z_Λ, as can be seen by fixing a gauge in which all links in the 0-direction are set to unity except between the layers $x^0 = 1$ and $x^0 = 2$: The links in the j'th layer are $\underline{g}_j$ and the links between layers 1 and 2 are $\underline{h}$. □

What happens if we add external charges? To answer this consider a closed loop $L_{\mathbf{x}}$ that winds once around the periodic time dimension at fixed spatial coordinates $\mathbf{x}$ (see Figure 2.9). Denote by $g_{L_{\mathbf{x}}}$ the corresponding ordered product of link variables (with arbitrary starting point). Then we have

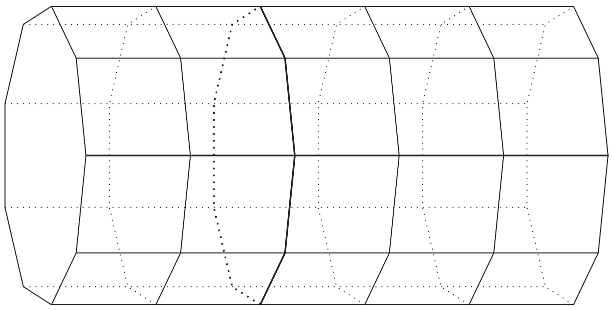

Figure 2.9: *Periodic boundary conditions in the time direction and a loop $L_{\mathbf{x}}$.*

LEMMA 2.4. *Let $\underline{\alpha}$ consist of k external charges $\alpha_{\mathbf{x}_i}$ located at $\mathbf{x}_i$, $i = 1, \dots k$:*

$$\alpha_{\mathbf{x}} = \begin{cases} \alpha_{\mathbf{x}_i}, & \mathbf{x} = \mathbf{x}_i, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.141}$$

Then, assuming periodic boundary conditions in the time direction, we have

$$Z_{\underline{\alpha}} = \int \prod_{b \subset \Lambda} dg_b \, e^{-S(\{g_b\})} \prod_{i=1}^{k} \big(d_{\alpha_{\mathbf{x}_i}} \overline{\chi_{\alpha_{\mathbf{x}_i}}(g_{L_{\mathbf{x}_i}})} \big). \tag{2.142}$$

PROOF. The proof is very similar to that of Lemma 2.3. Using (2.124) we get

$$\operatorname{tr}\left[\mathcal{P}_{\underline{\alpha}} \mathcal{T}^{N_0}\right] = \int d\underline{g}_1 \, d\underline{h} \, d\underline{g}_2 \dots d\underline{g}_{N_0} \times \prod_{i=1}^{k} \big(d_{\alpha_{\mathbf{x}_i}} \overline{\chi_{\alpha_{\mathbf{x}_i}}(h_{\mathbf{x}_i})} \big) \, \tau(\underline{g}_1^{\underline{h}}, \underline{g}_2) \, \tau(\underline{g}_2, \underline{g}_3) \dots \tau(\underline{g}_{N_0}, \underline{g}_1). \tag{2.143}$$

By gauge invariance of m the integrand is equal to

$$\prod_{i=1}^{k} \big(d_{\alpha_{\mathbf{x}_i}} \overline{\chi_{\alpha_{\mathbf{x}_i}}(h_{\mathbf{x}_i})} \big) \, m^2(\underline{g}_1) \, t(\underline{g}_1^{\underline{h}}, \underline{g}_2) \, m^2(\underline{g}_2) \, t(\underline{g}_2, \underline{g}_3) \dots m^2(\underline{g}_{N_0}) \, t(\underline{g}_{N_0}, \underline{g}_1). \tag{2.144}$$

Since $\overline{\chi_{\alpha_{\mathbf{x}_i}}(g_{L_{\mathbf{x}_i}})}$ is gauge invariant we may again fix a gauge in the integration in which all links in the 0-direction are set to unity except between the layers $x^0 = 1$ and $x^0 = 2$. Integrating the above expression yields

$$\int \prod_{b \subset \Lambda} dg_b \, e^{-S(\{g_b\})} \prod_{i=1}^{k} \big(d_{\alpha_{\mathbf{x}_i}} \overline{\chi_{\alpha_{\mathbf{x}_i}}(g_{L_{\mathbf{x}_i}})} \big). \tag{2.145}$$

□

Note that the complex conjugation over the character is merely conventional; it may be removed by taking the loop $L_{\mathbf{x}}$ in the opposite direction.

The observable $\chi_{\alpha_{\mathbf{x}_i}}(g_{L_{\mathbf{x}_i}})$ is called a *Polyakov loop*. Thus to every external static charge corresponds a Polyakov loop.

In order to study confinement at a finite temperature, we consider the physically relevant case $G = \mathrm{U}(n)$ or $G = \mathrm{SU}(n)$. We wish to compute the cost in free energy of adding two opposite static charges ("quarks") at spatial coordinates $\mathbf{x}$ and $\mathbf{y}$. Let thus the external charge configuration be

$$\alpha_{\mathbf{x}'} = \begin{cases} q, & \mathbf{x}' = \mathbf{x}, \\ \overline{q}, & \mathbf{x}' = \mathbf{y}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.146}$$

where q is the fundamental representation of G, $\overline{q}$ its conjugate representation, and 0 the trivial representation.

The *static quark potential* $V_{q\overline{q}}(\mathbf{x}-\mathbf{y})$ is then defined as the difference in free energy between the case when a quark-antiquark pair (with charges q and $\overline{q}$, respectively) is present at coordinates $\mathbf{x}$ and $\mathbf{y}$ and the case when no quarks are present. Note that we have the assumed translation invariance if we take periodic boundary conditions in the spatial directions. Explicitly,

$$V_{q\overline{q}}(\mathbf{x}-\mathbf{y}) := F_{\underline{\alpha}} - F_0, \tag{2.147}$$

with $\underline{\alpha}$ given by (2.146). Therefore from lemmas 2.3 and 2.4 we get

$$V_{q\overline{q}}(\mathbf{x}-\mathbf{y}) = -\frac{1}{\beta}\log\big[d_q^2\, G_\Lambda(\mathbf{x}-\mathbf{y})\big], \tag{2.148}$$

where the "correlation function" $G_\Lambda(\mathbf{x}-\mathbf{y})$ of two Polyakov loops is

$$G_\Lambda(\mathbf{x}-\mathbf{y}) := \frac{1}{Z_\Lambda}\int \prod_{b\subset\Lambda} dg_b\; e^{-S(\{g_b\})}\, \overline{\chi_q(g_{L_{\mathbf{x}}})}\, \chi_q(g_{L_{\mathbf{y}}}). \tag{2.149}$$

2.7.2 The correlation function

We now switch back to our original notation $\underline{g} := \{g_b \;:\; b\subset\Lambda\}$, and abbreviate $P_{\mathbf{x}}(\underline{g}) := \chi_q(g_{L_{\mathbf{x}}})$. We may also consider the more general case of G being any compact group and replace χ_q by any character $\tilde{\chi}$ of G (χ being reserved for the character used in the Wilson action). The two-point function of two Polyakov loops can then be written as

$$G_\Lambda(\mathbf{x}) = \langle \overline{P_{\mathbf{x}}}\, P_{\mathbf{0}}\rangle_\Lambda. \tag{2.150}$$

Explicitly,

$$G_\Lambda(\mathbf{x}) = \frac{1}{Z_\Lambda}\int d\underline{g}\; e^{-S(\underline{g})}\, \overline{\tilde{\chi}\Big(\prod_t^{\uparrow} g^0_{(t,\mathbf{x})}\Big)}\, \tilde{\chi}\Big(\prod_t^{\uparrow} g^0_{(t,\mathbf{0})}\Big), \tag{2.151}$$

where the arrow $\uparrow$ indicates a path ordered product. It is good to keep in mind that the physically relevant case is $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$ and $\tilde{\chi} = \chi_q = \mathrm{tr}$. At the end we take the thermodynamic limit:

$$G(\mathbf{x}) := \lim_{\Lambda_0\to\mathbb{Z}^d} G_\Lambda(\mathbf{x}). \tag{2.152}$$

Note that N_0 remains constant even in the thermodynamic limit since it is fixed by the finite temperature.

We now list some basic properties of $G(\mathbf{x})$. The argument of the previous section makes it clear that $G(\mathbf{x})$ is real. This can also be seen directly by using the variable transformation $\underline{g} \mapsto \underline{h}$:

$$h_{(t_2,\mathbf{x}_2)\,(t_1,\mathbf{x}_1)} := g_{(N_0-t_2,\mathbf{x}_2)\,(N_0-t_1,\mathbf{x}_1)}. \tag{2.153}$$

Then $d\underline{h} = d\underline{g}$ and $S(\underline{h}) = S(\underline{g})$ and we have

$$\begin{aligned}\overline{G_\Lambda(\mathbf{x})} &= \frac{1}{Z_\Lambda} \int d\underline{g}\; e^{-S(\underline{g})}\; \tilde{\chi}\Big(\prod_t^{\uparrow} g^0_{(t,\mathbf{x})}\Big)\; \overline{\tilde{\chi}\Big(\prod_t^{\uparrow} g^0_{(t,\mathbf{0})}\Big)} \\ &= \frac{1}{Z_\Lambda} \int d\underline{h}\; e^{-S(\underline{h})}\; \overline{\tilde{\chi}\Big(\prod_t^{\uparrow} h^0_{(t,\mathbf{x})}\Big)}\; \tilde{\chi}\Big(\prod_t^{\uparrow} h^0_{(t,\mathbf{0})}\Big) \\ &= G_\Lambda(\mathbf{x}).\end{aligned} \tag{2.154}$$

Moreover, (2.148) implies

$$G(\mathbf{x}) \geq 0. \tag{2.155}$$

For $\mathbf{x}$ of the form $\mathbf{x} = (0,\dots,0,x^\mu,0,\dots,0)$ we also give a direct proof. In that case we may find a reflection plane perpendicular to the μ-direction mapping $(0,\mathbf{0})$ to $(0,\mathbf{x})$. Then the nonnegativity of $G_\Lambda(\mathbf{x})$ follows immediately from reflection positivity.

Long range order of two Polyakov loops is measured by the connected two-point function

$$G_c(\mathbf{x}) := \langle \overline{P_\mathbf{x}}\, P_\mathbf{0}\rangle - |\langle P_\mathbf{0}\rangle|^2. \tag{2.156}$$

In the physically relevant case $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$ as well as $\tilde{\chi} = \mathrm{tr}$ the fundamental character, $G_c(\mathbf{x}) = G(\mathbf{x})$. This can be seen by picking a nontrivial $h = e^{i\varphi}\mathbb{1} \in \mathcal{Z}(G)$ in the centre $\mathcal{Z}(G)$ of G. Then $S(\underline{g})$ remains invariant under the rotation $g^0_{(0,\mathbf{x})} \mapsto h g^0_{(0,\mathbf{x})}$ and we have by invariance of the Haar measure

$$\langle P_\mathbf{0}\rangle = e^{i\varphi}\langle P_\mathbf{0}\rangle \tag{2.157}$$

so that $\langle P_\mathbf{0}\rangle = 0$.

2.7.3 Confinement and the correlation function

The *string tension* is defined by

$$\sigma := \lim_{|\mathbf{x}|\to\infty} \frac{V_{q\bar{q}}(\mathbf{x})}{|\mathbf{x}|}. \tag{2.158}$$

By (linear) *confinement* we mean that $\sigma > 0$, so that the free energy required to separate two quarks grows linearly with the distance. From

$$G(\mathbf{x}) = \frac{1}{d_q^2}\, e^{-\beta V_{q\bar{q}}(\mathbf{x})} \tag{2.159}$$

we see that confinement means exponential decay of the correlation function. On the other hand, if we have long range order

$$G(\mathbf{x}) \xrightarrow{|\mathbf{x}|\to\infty} c > 0, \tag{2.160}$$

$V_{q\bar{q}}(\mathbf{x})$ must remain bounded as $|\mathbf{x}|$ increases, so that we have *deconfinement* and $\sigma = 0$.

2.8 A criterion for confinement at zero temperature

We now derive a criterion for confinement at zero temperature due to Wilson [2]. We again restrict ourselves to the physical case $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$ and use the notation of Section 2.5. Take two spatial points $\mathbf{x}, \mathbf{y} \in \Lambda_0$ and a path $\gamma \subset \Lambda_0$ joining them, oriented from $\mathbf{x}$ to $\mathbf{y}$. Define the operator $M^{ab}(\gamma)$ on $\mathscr{H}$ as the multiplication operator by

$$D_q^{ab}(g_\gamma), \tag{2.161}$$

where g_γ is the parallel transporter along γ (an ordered product of link variables), and D_q^{ab} is the matrix element (a,b) of the fundamental representation q. Let $\Psi_0 \in \mathscr{H}_0$ be any chargeless state and consider the state $M^{ab}(\gamma)\Psi_0$. Under the representation (2.121) this state transforms as

$$\begin{aligned}\big(U(\underline{h})M^{ab}(\gamma)\Psi_0\big)(\underline{g}) &= D_q^{ab}\big[\big(\underline{g}^{\underline{h}}\big)_\gamma\big]\Psi_0(\underline{g}^{\underline{h}}) \\ &= D_q^{ab}(h_y^{-1} g_\gamma h_x)\Psi_0(\underline{g}) \\ &= \sum_{c,d} \overline{D_q^{ca}(h_y)} D_q^{db}(h_x) D_q^{cd}(g_\gamma)\Psi_0(\underline{g}).\end{aligned} \tag{2.162}$$

Thus $M^{ab}(\gamma)\Psi_0$ transforms under the representation $\underline{\alpha}$, with

$$\alpha_{\mathbf{x}'} = \begin{cases} q, & \mathbf{x}' = \mathbf{x}, \\ \overline{q}, & \mathbf{x}' = \mathbf{y}, \\ 0, & \mathbf{x}' \neq \mathbf{x}, \mathbf{y}. \end{cases} \tag{2.163}$$

Therefore $M^{ab}(\gamma)$ creates a static quark at $\mathbf{x}$ and a static antiquark at $\mathbf{y}$.

Now take $|\Psi_0\rangle \in \mathscr{H}_0$ as the state $\Psi_0(\underline{g}) = 1$. Consider the two (long) time intervals $S := \tau N_0 > T \in \mathbb{R}$, and define

$$W_{T,S}^{ab}(\gamma) := \frac{\langle \Psi_0 \,|\, e^{-\frac{1}{2}(S-T)H} \big(M^{ab}(\gamma)\big)^* e^{-TH} M^{ab}(\gamma) e^{-\frac{1}{2}(S-T)H} \,|\, \Psi_0\rangle}{\langle \Psi_0 \,|\, e^{-SH} \,|\, \Psi_0\rangle}. \tag{2.164}$$

Then we have

Lemma 2.5.

$$W_{T,S}^{ab}(\gamma) = \frac{1}{d_q^2}\langle W(\gamma, T)\rangle_\Lambda, \tag{2.165}$$

where the Wilson loop $W(\gamma, T) := \chi_q(g_{(\gamma,T)})$ *is the trace of the parallel transporter around the closed path made up of* γ*, the line* $[\mathbf{y}, \mathbf{y} + T\hat{0}]$*,* $\gamma + T\hat{0}$ *reversed, and the line* $[\mathbf{x} + T\hat{0}, \mathbf{x}]$ *(see Figure 2.10). Here* $d_q = n$ *is the dimension of the fundamental representation. The integration in* $\langle\cdot\rangle_\Lambda$ *is performed with free boundary conditions in the time direction.*

Proof. First note that $W_{T,S}^{ab}(\gamma)$ is independent of a, b. This can be seen by choosing a unitary permutation matrix $u \in \mathrm{SU}(n)$ such that, for a', b' given, we have

$$(gu)^{ab} = g^{a'b'}, \qquad \forall g. \tag{2.166}$$

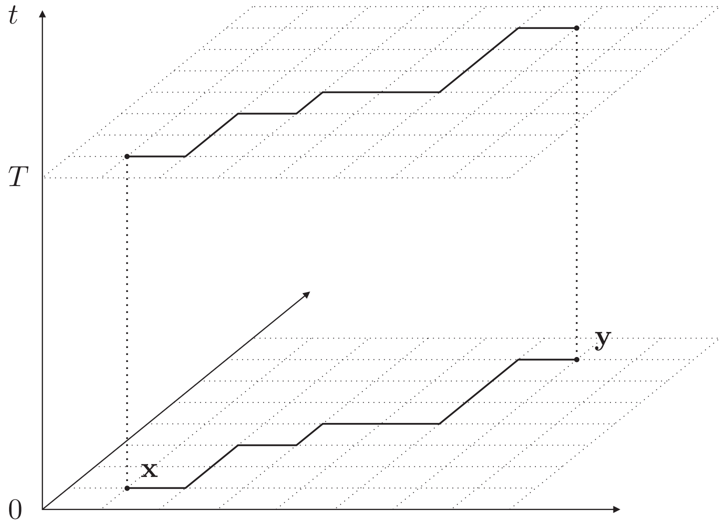

Figure 2.10: *A Wilson loop over the spatial path* γ *joining* $\mathbf{x}$ *and* $\mathbf{y}$.

Use the variable transform $\{g_{yx}\} \mapsto \{h_y^{-1} g_{yx} h_x\}$ in (2.164), where

$$h_{(t,\mathbf{x}')} := \begin{cases} u, & \mathbf{x}' = \mathbf{x}, \\ \mathbb{1}, & \mathbf{x}' \neq \mathbf{x}. \end{cases} \tag{2.167}$$

The desired result then follows by invariance of the Haar measure and gauge invariance of t and m.

Therefore

$$W_{T,S}^{a_0 b_0}(\gamma) = \frac{1}{d_q^2} \sum_{a,b} W_{T,S}^{ab}(\gamma). \tag{2.168}$$

The denominator of (2.164) is simply Z_Λ with free boundary conditions, as can be seen by fixing a gauge in which *all* temporal links are set to unity. Similarly, the numerator summed over a, b is equal to

$$\int \prod_{\langle x,y \rangle \subset \Lambda} dg_{yx}\ \chi_q(g_{(\gamma,T)})\, e^{-S(\{g_{yx}\})}. \tag{2.169}$$

This can again be seen in the temporal gauge, whereby we note that $\big(M^{ab}(\gamma)\big)^*$ is the multiplication operator by $D_q^{ba}(g_{\gamma^{-1}})$. □

To understand the physical meaning of $W_{T,S}^{ab}(\gamma)$ study first the limit $S \to \infty$, then $T \to \infty$. Let E_0 be the smallest eigenvalue of H on $\mathscr{H}_0$, corresponding to an eigenstate[3] $|0\rangle$ ("vacuum"). From (2.164) we see that in the limit $S \to \infty$ the eigenstate $|0\rangle$

[3] Note that by the Perron-Frobenius theorem [17] such a state is unique.

dominates[4], so that

$$W^{ab}_{T,S}(\gamma) \sim \frac{e^{-(S-T)E_0}\,\langle \Psi_0 \,|\, \big(M^{ab}(\gamma)\big)^* e^{-TH} M^{ab}(\gamma) \,|\, \Psi_0\rangle}{e^{-SE_0}\langle \Psi_0 \,|\, \Psi_0\rangle}, \tag{2.170}$$

where $\sim$ denotes asymptotic ($S \to \infty$) equality. This limit is the thermodynamic limit in the time direction. Define E_γ as the smallest eigenvalue of H on the space generated by $M^{ab}(\gamma)\mathscr{H}_0$. This corresponds the the ground state energy of the quark-antiquark pair at the points $\partial\gamma$. Then in the limit $T \to \infty$ we have

$$W^{ab}_{T,S}(\gamma) \sim e^{-T(E_\gamma - E_0)} \frac{\|M^{ab}(\gamma)\Psi_0\|^2}{\|\Psi_0\|^2}. \tag{2.171}$$

Since $M^{ab}(\gamma)$ is bounded (uniformly in γ) we get

$$\langle W(\gamma, T)\rangle \sim K\, e^{-T(E_\gamma - E_0)}. \tag{2.172}$$

If γ is the shortest path joining $\mathbf{x}$ and $\mathbf{y}$ we may thus define the *static quark potential*

$$V_{q\overline{q}}(\mathbf{x}-\mathbf{y}) := -\lim_{T\to\infty} \frac{1}{T} \log\langle W(\gamma, T)\rangle, \tag{2.173}$$

which describes the energy difference between the ground states with and without an external quark-antiquark pair located at the points $\mathbf{x}$ and $\mathbf{y}$.

As above, confinement means that

$$\lim_{|\mathbf{x}-\mathbf{y}|\to\infty} \frac{V_{q\overline{q}}(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} = c > 0, \tag{2.174}$$

and thus, for $L := |\mathbf{x}-\mathbf{y}|$

$$\langle W(\gamma, T)\rangle \sim K\, e^{-cTL}, \tag{2.175}$$

i.e. the Wilson loop has *area decay*. If we have *perimeter decay*

$$\langle W(\gamma, T)\rangle \sim K\, e^{-\tilde{c}T}, \tag{2.176}$$

$V_{q\overline{q}}(\mathbf{x}-\mathbf{y})$ is bounded and we have deconfinement.

We may of course consider more general Wilson loops of the form $\tilde{\chi}(g_\gamma)$, where the gauge group G can be any compact group, $\tilde{\chi}$ is some character of G and γ is a closed loop.

[4] We assume here that $\langle \Psi_0 \,|\, 0\rangle \neq 0$.

3 Confinement at finite temperature

In this part we investigate confinement at a finite fixed temperature. In Section 3.1 we show that in $2+1$ dimensions and $G = \mathrm{U}(n)$ we always have confinement. In Section 3.2 we show that for a strong coupling constant g we have confinement in all dimensions and any gauge group G. Finally, in Section 3.3 we show that in $3+1$ or higher dimensions, for weak coupling g and large temperatures there is deconfinement: the static quarks are "liberated".

3.1 Confinement in $2+1$ dimensions

The goal of this section is to prove a Mermin-Wagner-type theorem: In $2+1$ dimensions a *continuous* symmetry cannot be broken. We take some nontrivial homomorphism

$$h : \mathrm{U}(1) \mapsto G, \tag{3.1}$$

which is a symmetry of S under a global rotation of the link variables

$$g_{yx} \mapsto h\, g_{yx}. \tag{3.2}$$

So, in addition to gauge invariance, S has a further continuous symmetry. Our proof is based on the fact that the Polyakov loop, although gauge invariant, is *not* invariant under the above symmetry. For such a symmetry to exist, G must have a continuous centre. For example, our result is applicable to $G = \mathrm{U}(n)$ but not to $G = \mathrm{SU}(n)$, since $\mathcal{Z}\big(\mathrm{SU}(n)\big) \cong \mathbb{Z}_n$, a discrete group.

For the following parametrise $\mathrm{U}(1)$ by $e^{i\varphi}$, $\varphi \in \mathbb{R}$, and pick some fixed "temperature" J.

3.1.1 Gibbs states are G-invariant

Consider the infinite lattice $\mathbb{Z}^2 \times \mathbb{Z}_{N_0}$ and a finite rectangular subset Λ. We also denote by Λ the set of links between the sites of Λ. Let $\underline{g}$ be the infinite collection of all link variables in the lattice, $\underline{g}_\Lambda$ all the links in Λ and $\underline{g}_{\Lambda^c}$ the remaining links. Instead of first computing expectation values on the finite Λ and then taking the thermodynamic limit we consider here the full lattice. A *state* μ is a probability measure on the space of all configurations $\underline{g}$. The average of some observable f is then as usual

$$\langle f \rangle_\mu := \int d\mu(\underline{g})\ f(\underline{g}). \tag{3.3}$$

We are interested in states that are the thermodynamic limit of a "Boltzmannian" state

$$d\mu(\underline{g}) = \frac{1}{Z}\, e^{-J\, S(\underline{g})}\ d\underline{g} \tag{3.4}$$

which is well defined for a finite lattice. Assume for a while that the whole lattice is finite. Then (3.4) is the desired state. It can be characterised by its restriction to Λ:

$$\mu_\Lambda(B_\Lambda) := \mu\big(B_\Lambda \times \underbrace{G \times \cdots \times G}_{|\Lambda^c|}\big), \tag{3.5}$$

where B_Λ is some measurable subset of the configuration space $\{\underline{g}_\Lambda\}$. In other words,

$$d\mu_\Lambda(\underline{g}_\Lambda) = \int_{\{\underline{g}_{\Lambda^c}\}} d\mu(\underline{g}_\Lambda, \underline{g}_{\Lambda^c}). \tag{3.6}$$

Note that μ_Λ is a probability measure. Split the action S into a term regrouping all the plaquettes touching Λ and another term with the remaining plaquettes that does not depend on $\underline{g}_\Lambda$:

$$S(\underline{g}_\Lambda, \underline{g}_{\Lambda^c}) = S(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c}) + S_{\Lambda^c}(\underline{g}_{\Lambda^c}), \tag{3.7}$$

where

$$S(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c}) := -\sum_{p,\, \partial p \cap \Lambda \neq \emptyset} \big(\mathrm{Re}\,\chi(g_p) - \chi(\mathbb{1})\big) \tag{3.8}$$

and

$$S_{\Lambda^c}(\underline{g}_{\Lambda^c}) := -\sum_{p,\, \partial p \cap \Lambda = \emptyset} \big(\mathrm{Re}\,\chi(g_p) - \chi(\mathbb{1})\big). \tag{3.9}$$

Then

$$\begin{aligned} d\mu_\Lambda(\underline{g}_\Lambda) &= \frac{1}{Z}\, d\underline{g}_\Lambda \int d\underline{g}_{\Lambda^c}\; e^{-J\, S(\underline{g}_\Lambda, \underline{g}_{\Lambda^c})} \\ &= \frac{1}{Z}\, d\underline{g}_\Lambda \int d\underline{g}_{\Lambda^c}\; e^{-J\, S_{\Lambda^c}(\underline{g}_{\Lambda^c})}\, e^{-J\, S(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})}. \end{aligned} \tag{3.10}$$

Defining the measure

$$d\rho_{\Lambda^c}(\underline{g}_{\Lambda^c}) := \frac{1}{Z}\, e^{-J\, S_{\Lambda^c}(\underline{g}_{\Lambda^c})}\, d\underline{g}_{\Lambda^c} \tag{3.11}$$

on the space $\{\underline{g}_{\Lambda^c}\}$ we thus have

$$d\mu_\Lambda(\underline{g}_\Lambda) = d\underline{g}_\Lambda \int d\rho_{\Lambda^c}(\underline{g}_{\Lambda^c})\; e^{-J\, S(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})}. \tag{3.12}$$

This property may be taken over to the case of the infinite lattice: We say a state μ is a *Gibbs state* for the action S if its restriction upon any finite Λ is of the form (3.12) for some finite measure ρ_{Λ^c} on the space $\{\underline{g}_{\Lambda^c}\}$. Thus we have for a Gibbs state μ and an observable $f = f(\underline{g}_\Lambda)$

$$\langle f\rangle_\mu = \int d\mu_\Lambda(\underline{g}_\Lambda)\, f(\underline{g}_\Lambda) = \int d\rho_{\Lambda^c}(\underline{g}_{\Lambda^c}) \int d\underline{g}_\Lambda\; e^{-J\, S(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})}\, f(\underline{g}_\Lambda) \tag{3.13}$$

Note that the Gibbs states form a convex set K. Of special interest are the extremal Gibbs states which are the extremal points of K.

Let $\varphi \in [0, 2\pi)$ be some rotation. For a state μ we define the *rotated state* μ_φ as

$$d\mu_\varphi(\underline{g}) := d\mu(R_\varphi^{-1}\underline{g}) \tag{3.14}$$

where R_φ rotates the temporal links at $t = 0$ by φ:

$$R_\varphi(g^\mu_{(t,\mathbf{x})}) := \begin{cases} h(\varphi)\, g^\mu_{(t,\mathbf{x})}, & t = 0,\ \mu = 0 \\ g^\mu_{(t,\mathbf{x})}, & \text{otherwise.} \end{cases} \tag{3.15}$$

In the following we shall abbreviate $\theta^\mu_{(0,\mathbf{x})}$ by $\theta^\mu_{\mathbf{x}}$.

The main result is

THEOREM 3.1. *A Gibbs state μ is invariant under rotation:*

$$\mu_\varphi = \mu, \qquad \forall \varphi \in [0, 2\pi). \tag{3.16}$$

PROOF. The proof is based on a physical argument of Herring and Kittel [20] showing the absence of spontaneous magnetisation for the Heisenberg model in two dimensions. A similar proof was found for spin systems by Dobrushin and Shlosman [21] and generalised by Pfister [22]. Denote by Λ_l the sublattice $\{-l, \ldots, l\}^2 \times \mathbb{Z}_{N_0}$. The first step is to show that for any configuration $\underline{g}$, any rotation φ and any spatial lattice size l there is a configuration $\underline{\hat{g}}$ such that (see Figure 3.1)

$$\text{(i)}\ \ \hat{g}^0_{\mathbf{x}} = h(\varphi)\, g^0_{\mathbf{x}}, \qquad |\mathbf{x}| \le l, \tag{3.17}$$

$$\text{(ii)}\ \ \hat{g}^\mu_x = g^\mu_x, \qquad x \in \Lambda^c_{l+L} \quad \text{for some } L = L(l), \tag{3.18}$$

$$\text{(iii)}\ S(\underline{\hat{g}}) - S(\underline{g}) \le K, \qquad K \text{ independent of } \underline{g},\ \varphi,\ l. \tag{3.19}$$

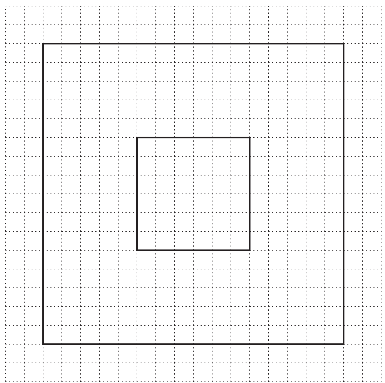

Figure 3.1: *The lattice at $t = 0$ with the boundaries $|\mathbf{x}| = l$, $|\mathbf{x}| = l + L$.*

Set $\psi := \varphi - 2\pi \in [-2\pi, 0)$. Define

$$\varphi_n \in [0, 2\pi),$$
$$\psi_n \in [-2\pi, 0)$$

for $0 < n < L$ as

$$\varphi_n := \varphi \frac{L - n}{L}, \tag{3.20a}$$

$$\psi_n := \psi \frac{L - n}{L}. \tag{3.20b}$$

Thus $\psi_n = \frac{\psi}{\varphi}\, \varphi_n$. Furthermore define the two candidate states for $\underline{\hat{g}}$, $\underline{g}^1$ and $\underline{g}^2$, as

$$(g^1)^0_{\mathbf{x}} := h(\varphi_{\mathbf{x}})\, g^0_{\mathbf{x}} \tag{3.21}$$

and all other components of $\underline{g}^1$ are left equal to those of $\underline{g}$. Here the rotation is

$$\varphi_{\mathbf{x}} := \begin{cases} \varphi, & |\mathbf{x}| \leq l \\ \varphi_n, & |\mathbf{x}| = l+n,\ 0 < n < L \\ 0, & |\mathbf{x}| \geq l+L, \end{cases} \tag{3.22}$$

The definition of $\underline{g}^2$ is identical except φ, φ_n are replaced by ψ, ψ_n.

The map $t \mapsto \rho\big(h(t)\, g\big)$, being a finite-dimensional representation of U(1), is smooth. Therefore

$$s_g : t \mapsto \operatorname{Re}\chi\big(h(t)\, g\big) \tag{3.23}$$

is a smooth function and $|s_g''(t)|$ is uniformly (in t and g) bounded by some constant C, since $s'' : G \times [0, 2\pi] \mapsto \mathbb{R}$ is continuous on a compact domain. Let us concentrate first on $\underline{g}^1$. Since $\underline{g}^1$ and $\underline{g}$ differ only on a finite set $S(\underline{g}^1) - S(\underline{g})$ is finite and

$$\begin{aligned} S(\underline{g}^1) - S(\underline{g}) &= -\sum_{\mathbf{x}\in\mathbb{Z}^2}\sum_{\mu=1,2}\big[\operatorname{Re}\chi\big((g^1)_p\big) - \operatorname{Re}\chi(g_p)\big] \\ &= -\sum_{\mathbf{x}\in\mathbb{Z}^2}\sum_{\mu=1,2}\big[s_{g_p}(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\hat{\mu}}) - s_{g_p}(0)\big], \end{aligned} \tag{3.24}$$

where $p = p^{0\mu}_{(0,\mathbf{x})}$. By expansion we get

$$\begin{aligned} S(\underline{g}^1) - S(\underline{g}) = &-\sum_{\mathbf{x}\in\mathbb{Z}^2}\sum_{\mu=1,2} s'_{g_p}(0)\,(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\hat{\mu}}) \\ &-\sum_{\mathbf{x}\in\mathbb{Z}^2}\sum_{\mu=1,2} \frac{1}{2}\, s''_{g_p}(\xi)\,(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\hat{\mu}})^2 \end{aligned} \tag{3.25}$$

for some ξ between 0 and $\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\hat{\mu}}$. The quadratic term is bounded by

$$\begin{aligned} \sum_{\mathbf{x}\in\mathbb{Z}^2}\sum_{\mu=1,2}\frac{C}{2}\,(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\hat{\mu}})^2 &\leq \sum_{\mathbf{x}\in\Lambda_{l+L}}\sum_{\mu=1,2}\frac{C}{2}\frac{\varphi^2}{L^2} \\ &\leq C\,4\pi^2\,\frac{(2l+2L+1)^2}{L^2} \\ &\leq K \end{aligned} \tag{3.26}$$

for some K independent of l provided that $L = L(l)$ is large enough.

An identical estimate holds for $S(\underline{g}^2) - S(\underline{g})$ with φ replaced by ψ. In that case the linear term of the above estimate is

$$-\sum_{\mathbf{x}\in\mathbb{Z}^2}\sum_{\mu=1,2} s'_{g_p}(0)\,(\psi_{\mathbf{x}} - \psi_{\mathbf{x}+\hat{\mu}}). \tag{3.27}$$

Now since

$$(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\hat{\mu}}) = \underbrace{\frac{\varphi}{\psi}}_{\leq 0}\,(\psi_{\mathbf{x}} - \psi_{\mathbf{x}+\hat{\mu}}) \tag{3.28}$$

the linear terms have opposite signs, and we choose $\underline{\hat{g}} := \underline{g}^i$, with $i = 1, 2$ such that the linear term is negative. Then we have

$$S(\underline{\hat{g}}) - S(\underline{g}) \leq K \tag{3.29}$$

with K independent of $\underline{g}$, φ, l.

In a second step we show the G-invariance of Gibbs states. Since any Gibbs state can be expressed as a convex combination of extremal Gibbs states, it suffices to consider an extremal Gibbs state μ. Let f be a positive observable depending only on the links in some finite subset $\Omega \subset \mathbb{Z}^2 \times \mathbb{Z}_{N_0}$. Choose l such that $\Omega \subset \Lambda_l$. Let $\Lambda := \Lambda_{l+L}$ with L large enough for the estimate (3.19) to hold. Since μ is a Gibbs state, we have

$$\langle f \rangle_\mu = \int d\rho_{\Lambda^c}(\underline{g}_{\Lambda^c}) \int d\underline{g}_\Lambda \, e^{-J\, S(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \, f(\underline{g}_\Lambda), \tag{3.30a}$$

$$\langle f \rangle_{\mu_\varphi} = \int d\rho_{\Lambda^c}(\underline{g}_{\Lambda^c}) \int d\underline{g}_\Lambda \, e^{-J\, S(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \, f(R_\varphi \, \underline{g}_\Lambda). \tag{3.30b}$$

Fix the boundary terms $\underline{g}_{\Lambda^c}$ and define the two transformations T_1, T_2 on $[0, 2\pi)^\Lambda$ as

$$T_i \, \underline{g}_\Lambda := \underline{g}^i_\Lambda, \tag{3.31}$$

with $\underline{g}^i$ as defined above. Note that the uniform measure $d\underline{g}_\Lambda$ is left invariant by T_i. We have proven that for any $\underline{g}_\Lambda$ there is an $i = 1, 2$ such that

$$S(T_i \, \underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c}) - S(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c}) \leq K. \tag{3.32}$$

Partition $G^\Lambda = \Omega_1 \cup \Omega_2$ into two disjoint sets such that

$$S(T_i \, \underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c}) - S(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c}) \leq K, \qquad \forall \underline{g}_\Lambda \in \Omega_i \tag{3.33}$$

and denote by χ_i the characteristic function of Ω_i. Then

$$\begin{aligned}
&\int d\underline{g}_\Lambda \, e^{-JS(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \, f(R_\varphi \, \underline{g}_\Lambda) \\
&= \sum_{i=1,2} \int d\underline{g}_\Lambda \, e^{-JS(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \, \chi_i(\underline{g}_\Lambda) \, f(T_i \, \underline{g}_\Lambda) \\
&= \sum_{i=1,2} \int d\underline{g}_\Lambda \, e^{-JS(T_i \, \underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \, \chi_i(\underline{g}_\Lambda) \, f(T_i \, \underline{g}_\Lambda) \, e^{JS(T_i \, \underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c}) - JS(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \\
&\leq e^K \sum_{i=1,2} \int d\underline{g}_\Lambda \, e^{-JS(T_i \, \underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \, \chi_i(\underline{g}_\Lambda) \, f(T_i \, \underline{g}_\Lambda) \\
&\leq e^K \sum_{i=1,2} \int d\underline{g}_\Lambda \, e^{-JS(T_i \, \underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \, f(T_i \, \underline{g}_\Lambda) \\
&\leq e^K \sum_{i=1,2} \int d\underline{g}_\Lambda \, e^{-JS(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \, f(\underline{g}_\Lambda) \\
&= 2e^K \int d\underline{g}_\Lambda \, e^{-JS(\underline{g}_\Lambda \,|\, \underline{g}_{\Lambda^c})} \, f(\underline{g}_\Lambda).
\end{aligned} \tag{3.34}$$

Integrating this with respect to $d\rho_{\Lambda^c}(\underline{g}_{\Lambda^c})$ gives

$$\langle f\rangle_{\mu_\varphi} \leq \tilde{K}\,\langle f\rangle_\mu, \tag{3.35}$$

where $\tilde{K} = 2e^K$ is independent of f, μ and φ. In particular if we set $\varphi \mapsto 2\pi - \varphi$, $\mu \mapsto \mu_\varphi$ we get

$$\tilde{K}^{-1}\,\langle f\rangle_{\mu_\varphi} \leq \langle f\rangle_\mu \leq \tilde{K}\,\langle f\rangle_{\mu_\varphi}. \tag{3.36}$$

Since this holds in particular for f the characteristic function of any finite set, μ and μ_φ are absolutely continuous with respect to one other:

$$\mu \sim \mu_\varphi. \tag{3.37}$$

Since μ (and therefore μ_φ) is extremal it follows that $\mu = \mu_\varphi$, i.e. μ is G-invariant. □

A consequence is the absence of long-range order, i.e. confinement, which we prove for the U(n)-case:

Corollary 3.2. *For $G = \mathrm{U}(n)$, $\tilde{\chi} = \mathrm{tr}$ the character of the fundamental representation, and a Gibbs state μ,*

$$G(\mathbf{x}) \xrightarrow{|\mathbf{x}|\to\infty} 0. \tag{3.38}$$

Proof. Assume the contrary. Since $\langle \overline{P_{\mathbf{x}}}\,P_{\mathbf{0}}\rangle_\mu \leq n^2$ is bounded in $\mathbf{x}$ there exists a sequence $(\mathbf{x}_j)_{j\in\mathbb{N}}$ such that

$$\begin{aligned} |\mathbf{x}_j| &\longrightarrow \infty \\ \langle \overline{P_{\mathbf{x}_j}}\,P_{\mathbf{0}}\rangle_\mu &\longrightarrow c > 0. \end{aligned} \tag{3.39}$$

The Gibbs state μ can be expressed as a convex combination of extremal Gibbs states $\{\mu_\sigma\}_{\sigma\in\Sigma}$:

$$\mu = \int d\rho(\sigma)\;\mu_\sigma, \tag{3.40}$$

where ρ is a nonnegative measure of volume 1 on Σ. Since μ_σ is extremal it clusters for $|\mathbf{x}_j| \to \infty$ and we have, for any local observables a and b,

$$\langle a\;\tau_{\mathbf{x}_j} b\rangle_{\mu_\sigma} - \langle a\rangle_{\mu_\sigma}\,\langle \tau_{\mathbf{x}_j} b\rangle_{\mu_\sigma} \longrightarrow 0, \tag{3.41}$$

where $\tau_{\mathbf{x}}$ denotes translation by $\mathbf{x}$.

Now there must be a $\sigma \in \Sigma$ such that

$$\langle \overline{P_{\mathbf{x}_j}}\,P_{\mathbf{0}}\rangle_{\mu_\sigma} \not\longrightarrow 0, \tag{3.42}$$

since otherwise we would have by dominated convergence

$$\lim_{j\to\infty} \int d\rho(\sigma)\;\langle \overline{P_{\mathbf{x}_j}}\,P_{\mathbf{0}}\rangle_{\mu_\sigma} = 0 \tag{3.43}$$

contradicting the assumption. Therefore

$$\langle \overline{P_{\mathbf{x}_j}}\rangle_{\mu_\sigma}\,\langle P_{\mathbf{0}}\rangle_{\mu_\sigma} \not\longrightarrow 0 \tag{3.44}$$

and thus $\langle P_{\mathbf{0}} \rangle_{\mu_\sigma} \neq 0$

Define the imbedding h as

$$h(\varphi) := e^{i\varphi}\, \mathbb{1}. \tag{3.45}$$

By the theorem μ_σ is rotation invariant:

$$\langle P_{\mathbf{0}} \rangle_{(\mu_\sigma)_\varphi} = \langle P_{\mathbf{0}} \rangle_{\mu_\sigma}, \qquad \forall \varphi. \tag{3.46}$$

However, by an explicit calculation for $\varphi \neq 0$

$$\langle P_{\mathbf{0}} \rangle_{(\mu_\sigma)_\varphi} = e^{i\varphi} \langle P_{\mathbf{0}} \rangle_{\mu_\sigma} \neq \langle P_{\mathbf{0}} \rangle_{\mu_\sigma}, \tag{3.47}$$

which is the desired contradiction. □

3.1.2 Power law decay for the correlation function

We give another, more elementary, proof of Corollary 3.2 based on an idea of McBryan and Spencer [23]. Its advantage, aside from its simplicity, is the fact that it actually gives a power law upper bound for $G(\mathbf{x})$.

We again restrict ourselves to $G = \mathrm{U}(n)$. A matrix $g \in \mathrm{U}(n)$ can be written as[5]

$$g = e^{i\theta} u, \tag{3.48}$$

where $\theta \in [0, 2\pi)$ and $u \in \mathrm{SU}(n)$. Then

$$dg = d\theta\, du \tag{3.49}$$

since both sides are normalised and left-invariant. Here $d\theta$ is the uniform measure on $[0, 2\pi)$ and du the uniform measure on $\mathrm{SU}(n)$.

THEOREM 3.3. *For $G = \mathrm{U}(n)$ and $\chi = \tilde{\chi} = \mathrm{tr}$ the trace of the fundamental representation, the correlation function $G(\mathbf{x})$ decays as a power law in $|\mathbf{x}|$ for all J. More precisely, there is a universal constant c such that*

$$G(\mathbf{x}) \leq n^2\, |\mathbf{x}|^{-\gamma}, \qquad \gamma = \frac{cN_0}{n(J+1)}. \tag{3.50}$$

PROOF. Consider first the finite lattice $\Lambda := \{-N, \dots, N\}^2 \times \mathbb{Z}_{N_0}$ where N is chosen large enough: $N > |\mathbf{x}|$ so that $(0, \mathbf{x}) \in \Lambda$. Take periodic boundary conditions in all directions. Then the correlation function for the lattice Λ is

$$G_{\Lambda,J}(\mathbf{x}) := \frac{1}{Z_{\Lambda,J}} \int d\underline{\theta} d\underline{u}\; e^{-i\theta_{\mathbf{x}}}\, \overline{\mathrm{tr}\, u_{\mathbf{x}}}\; e^{i\theta_{\mathbf{0}}}\; \mathrm{tr}\, u_{\mathbf{0}} \\ \times \exp\Big(\frac{J}{2} \sum_{y,\, \mu<\nu} e^{i\theta_p}\; \mathrm{tr}\, u_p + e^{-i\theta_p}\, \overline{\mathrm{tr}\, u_p} \Big), \tag{3.51}$$

where p labels the plaquette $p_y^{\mu\nu}$ and θ_p, u_p the corresponding plaquette variables; $\theta_{\mathbf{x}}$ and $u_{\mathbf{x}}$ label the product of the links in the Polyakov loop at $\mathbf{x}$. The partition function is

$$Z_{\Lambda,J} := \int d\underline{\theta} d\underline{u}\; \exp\Big(\frac{J}{2} \sum_{y,\, \mu<\nu} e^{i\theta_p}\; \mathrm{tr}\, u_p + e^{-i\theta_p}\, \overline{\mathrm{tr}\, u_p} \Big). \tag{3.52}$$

[5]Note that this parametrisation is not injective.

The key trick in the proof is the following variable transformation:

$$\theta_y^\mu = \theta_y'^\mu + i\beta_y, \qquad \beta_y := \begin{cases} -\alpha \log(|y|_2+1), & |y|_2 < |\mathbf{x}| \\ -\alpha \log(|\mathbf{x}|+1), & |y|_2 \geq |\mathbf{x}| \end{cases}, \tag{3.53}$$

where $\alpha = \alpha(n, J) \leq 1$ is some positive constant to be chosen later and $|y|_2 := \max\{y^1, y^2\}$ is the "spatial" norm of y.

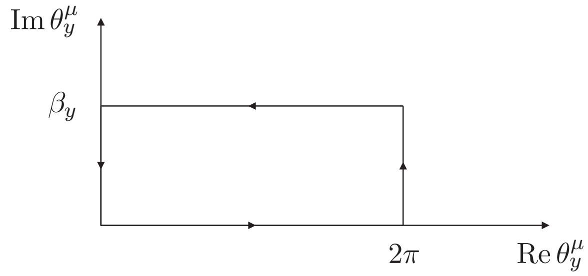

Figure 3.2: *The integration contour in the complex θ_y^μ-plane.*

For each θ_y^μ, the integrand of (3.51) is analytic on whole $\mathbb{C}$ and 2π-periodic in the real direction. Cauchy's theorem applied to the integration contour shown in Figure 3.2 therefore yields

$$\begin{aligned} G_{\Lambda,J}(\mathbf{x}) = \frac{1}{Z_{\Lambda,J}} \int & d\underline{\theta}' d\underline{u}\; e^{-i\theta'_{\mathbf{x}} - N_0\,\alpha\,\log(|\mathbf{x}|+1)}\, \overline{\operatorname{tr} u_{\mathbf{x}}}\, e^{i\theta'_{\mathbf{0}}}\, \operatorname{tr} u_{\mathbf{0}} \\ & \times \exp\Big(\frac{J}{2} \sum_{y,\,\mu<\nu} e^{i\theta'_p - (\beta_{y+\hat\mu} - \beta_{y+\hat\nu})}\, \operatorname{tr} u_p + e^{-i\theta'_p + (\beta_{y+\hat\mu} - \beta_{y+\hat\nu})}\, \overline{\operatorname{tr} u_p}\Big), \end{aligned} \tag{3.54}$$

Writing $\beta_p := \beta_{y+\hat\mu} - \beta_{y+\hat\nu}$ we may therefore bound $G_{\Lambda,J}(\mathbf{x})$ from above by

$$\begin{aligned} & \frac{n^2\, e^{-N_0\,\alpha\,\log(|\mathbf{x}|+1)}}{Z_{\Lambda,J}} \int d\underline{\theta}' d\underline{u}\; \exp\Big(\frac{J}{2} \sum_{y,\,\mu<\nu} e^{i\theta'_p - \beta_p}\, \operatorname{tr} u_p + e^{-i\theta'_p + \beta_p}\, \overline{\operatorname{tr} u_p}\Big) \\ = & \frac{n^2\, e^{-N_0\,\alpha\,\log(|\mathbf{x}|+1)}}{Z_{\Lambda,J}} \int d\underline{\theta}' d\underline{u}\; \exp\Big(\frac{J}{2} \sum_{y,\,\mu<\nu} \Big[e^{i\theta'_p}\, \operatorname{tr} u_p + e^{-i\theta'_p}\, \overline{\operatorname{tr} u_p}\Big]\Big) \\ & \qquad \times \exp\Big(\frac{J}{2} \sum_{y,\,\mu<\nu} \Big[\big(e^{-\beta_p} - 1\big)\, e^{i\theta'_p}\, \operatorname{tr} u_p + \big(e^{\beta_p} - 1\big)\, e^{-i\theta'_p}\, \overline{\operatorname{tr} u_p}\Big]\Big). \end{aligned} \tag{3.55}$$

Therefore

$$G_{\Lambda,J}(\mathbf{x}) \leq n^2 e^{-N_0\,\alpha\,\log(|\mathbf{x}|+1)} \exp\Big(nJ \sum_{y,\mu<\nu} \big[\cosh(\beta_{y+\hat\mu} - \beta_{y+\hat\nu}) - 1\big]\Big). \tag{3.56}$$

Now

$$\begin{aligned} |\beta_{y+\hat\mu} - \beta_{y+\hat\nu}| &\leq |\beta_{y+\hat\mu} - \beta_y| + |\beta_{y+\hat\nu} - \beta_y| \\ &\leq 2\,\alpha\,[\log(|y|_2+1) - \log(|y|_2)] \end{aligned} \tag{3.57}$$

for $0 < |y|_2 \le |\mathbf{x}|$. By the mean value theorem we have therefore

$$|\beta_{y+\hat{\mu}} - \beta_{y+\hat{\nu}}| \le \frac{2\,\alpha}{|y|_2}, \qquad 0 < |y|_2 \le |\mathbf{x}|. \tag{3.58a}$$

Furthermore

$$|\beta_{y+\hat{\mu}} - \beta_{y+\hat{\nu}}| \le \alpha\,\log 2, \qquad |y|_2 = 0 \tag{3.58b}$$

and

$$|\beta_{y+\hat{\mu}} - \beta_{y+\hat{\nu}}| = 0, \qquad |y|_2 > |\mathbf{x}|. \tag{3.58c}$$

Now

$$\cosh\xi - 1 \le \xi^2, \qquad \forall \xi :\ |\xi| \le 2, \tag{3.59}$$

so that

$$\begin{aligned}
&\exp\Big(nJ \sum_{y,\mu<\nu} \big[\cosh(\beta_{y+\hat{\mu}} - \beta_{y+\hat{\nu}}) - 1\big]\Big) \\
=\ &\exp\Big(nJ \sum_{|y|_2=0}\ \sum_{\mu<\nu} \big[\cosh(\beta_{y+\hat{\mu}} - \beta_{y+\hat{\nu}}) - 1\big]\Big) \\
&\times \exp\Big(nJ \sum_{0<|y|_2\le|\mathbf{x}|}\ \sum_{\mu<\nu} \big[\cosh(\beta_{y+\hat{\mu}} - \beta_{y+\hat{\nu}}) - 1\big]\Big) \\
\le\ & e^{3nJN_0\,[\cosh(\alpha\,\log 2)-1]}\ \exp\Big(3nJN_0 \sum_{\substack{\mathbf{y}\in\mathbb{Z}^2\\ 0<|\mathbf{y}|\le|\mathbf{x}|}} \Big[\cosh\frac{2\alpha}{|\mathbf{y}|} - 1\Big]\Big) \\
=\ & e^{3nJN_0\,[\cosh(\alpha\,\log 2)-1]}\ \exp\Big(3nJN_0 \sum_{l=1}^{|\mathbf{x}|} 8l\Big[\cosh\frac{2\alpha}{l} - 1\Big]\Big) \\
\le\ & e^{3nJN_0\alpha^2\log^2 2}\ \exp\Big(3nJN_0 \sum_{l=1}^{|\mathbf{x}|} 8l\frac{4\alpha^2}{l^2}\Big) \\
=\ & \exp\Big[3nJN_0\alpha^2\big(\log^2 2 + 32\sum_{l=1}^{|\mathbf{x}|}\frac{1}{l}\big)\Big].
\end{aligned} \tag{3.60}$$

Now there is a constant k such that

$$\log^2 2 + 32 \sum_{l=1}^{|\mathbf{x}|} \frac{1}{l} \le k\,\log|\mathbf{x}| \tag{3.61}$$

and therefore

$$\begin{aligned}
G_{\Lambda,J}(\mathbf{x}) &\le n^2 e^{-N_0\,\alpha\,\log(|\mathbf{x}|+1)}\ e^{3nJN_0\alpha^2 k\log|\mathbf{x}|} \\
&\le n^2 e^{N_0\,(3kn(J+1)\alpha^2-\alpha)\log|\mathbf{x}|} \\
&= n^2\,|\mathbf{x}|^{-\gamma},
\end{aligned} \tag{3.62}$$

where

$$\gamma = N_0(\alpha - 3kn(J+1)\alpha^2). \tag{3.63}$$

For α small enough γ is positive. Choose α so that γ is maximised:

$$\alpha(J,n) := \frac{1}{6kn(J+1)} \tag{3.64}$$

which yields

$$\gamma = \frac{N_0}{12kn(J+1)}, \tag{3.65}$$

so that

$$c = \frac{1}{12k}. \tag{3.66}$$

Since this estimate is independent of N as long as $N > |\mathbf{x}|$, the result holds in the thermodynamic limit $N \to \infty, \quad G_{\Lambda,J}(\mathbf{x}) \to G_J(\mathbf{x})$. □

3.2 Exponential clustering for strong coupling

In this section we show that whenever the coupling g is sufficiently strong, or J sufficiently small, $G_c(\mathbf{x}) \le e^{-c|\mathbf{x}|}$. The result is valid for all groups G and characters χ, $\tilde{\chi}$. If we choose $G = \mathrm{U}(n)$, $\mathrm{SU}(n)$ and $\tilde{\chi} = \mathrm{tr}$ the character of the fundamental representation $G_c(\mathbf{x}) = G(\mathbf{x})$ and we have confinement.

3.2.1 High-temperature expansions

The proof is based on a *high-temperature expansion*. Let us first consider the partition function Z_Λ for a finite rectangular sublattice Λ with periodic boundary conditions. We take a general action of the form

$$S(\underline{g}) = \sum_p S_p(g_p), \tag{3.67}$$

where the sum ranges over all positively oriented plaquettes $p \subset \Lambda$. We assume that S_p is positive, uniformly bounded in g, p and[6] J, and vanishes for $J \to 0$:

$$S_p(g) \le c\,J, \tag{3.68}$$

for some $c > 0$. Note that the Wilson action is a special case, with $c = 2\,\chi(\mathbb{1})$. The partition function is then

$$Z_\Lambda = \int d\underline{g}\, \prod_p e^{-S_p(g_p)}. \tag{3.69}$$

We wish to expand in the small parameter J. Our goal is (i) find an expansion for $\log Z_\Lambda$ and (ii) show convergence for J small enough. Knowing $\log Z_\Lambda$ we may easily compute various observables by adding perturbations to Z_Λ.

Since J is the small parameter, define

$$\rho_p(g) := e^{-S_p(g)} - 1. \tag{3.70}$$

Then (3.68) implies that $\rho_p(g)$ vanishes uniformly in p and g for $J \to 0$:

$$|\rho_p(g)| \le r(J), \tag{3.71}$$

with

$$r(J) := 1 - e^{-cJ}. \tag{3.72}$$

Then we may rewrite the partition function

$$Z_\Lambda = \int d\underline{g}\, \prod_p \big(1 + \rho_p(g_p)\big) = \sum_{\underline{n}} \int d\underline{g}\, \prod_p \rho_p(g_p)^{n_p}, \tag{3.73}$$

where the sum ranges over all multi-indices $\underline{n} = \{n_p \;:\; p \subset \Lambda\}$, where $n_p = 0, 1$. For a given set of plaquettes defined by $P_{\underline{n}} := \{p \;:\; n_p = 1\} = Y_1 \cup \cdots \cup Y_n$ we find a

[6] For simplicity of notation we take here $J_{\mathrm{E}} = J_{\mathrm{M}} = J$, although the following proof is also valid for separate electric and magnetic couplings, provided that both are smaller than the bound we shall find for J.

decomposition into connected components Y_j, whereby two plaquettes are connected if they share a link. The integral above then factors for each connected component and we have

$$Z_\Lambda = \sum_{\underline{n}} z(P_{\underline{n}}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{Y_1,\ldots,Y_n \\ \text{disjoint}}} \prod_{j=1}^{n} z(Y_j), \tag{3.74}$$

where the *activity* $z(P)$ of a set of plaquettes is

$$z(P) := \int d\underline{g} \prod_{p\in P} \rho_p(g_p). \tag{3.75}$$

The sum ranges over all sets of disjoint sets of connected plaquettes Y_j, which has to be normalised by $1/n!$ to compensate the overcounting caused by permutations (which do not lead to different $P_{\underline{n}}$). We call a connected set of plaquettes Y a *polymer*. Disjoint polymers are also called *compatible*. So the above sum ranges over all compatible polymers.

At this point we generalise the discussion: we have some set $\mathscr{Y}$ of polymers Y as well as a symmetric function $g : \mathscr{Y} \times \mathscr{Y} \mapsto \{-1, 0\}$, such that $g(Y, Y) = -1$. We say Y_1 and Y_2 are compatible (disjoint) if $g(Y_1, Y_2) = 0$ and incompatible (they intersect) otherwise. Let the partition function be of the form (3.74) for some activity $z : \mathscr{Y} \mapsto \mathbb{R}$. Then we may re-express the compatibility condition as

$$Z_\Lambda = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1,\ldots,Y_n} \prod_{j=1}^{n} z(Y_j) \prod_{i<j} \left(1 + g^{\underline{Y}}_{(i,j)}\right), \tag{3.76}$$

where $\underline{Y} = \{Y_1, \ldots Y_n\}$ and $g^{\underline{Y}}_{(i,j)} := g(Y_i, Y_j)$. Explicitly,

$$g^{\underline{Y}}_{(i,j)} = \begin{cases} 0, & Y_i \text{ and } Y_j \text{ disjoint} \\ -1, & Y_i \text{ and } Y_j \text{ intersect.} \end{cases} \tag{3.77}$$

The evaluation of the above expression requires somewhat tedious combinatorics. We use a method based on the *Taylor forest formula*. First we need some basic facts about forests.

For a set $\{1, \ldots, n\}$ of (ordered) vertices define the set $\mathcal{P}_n$ of (unordered) links on the vertices $\{1, \ldots, n\}$; then $|\mathcal{P}_n| = \binom{n}{2}$. A *forest* $\mathscr{F} \subset \mathcal{P}_n$ is a collection of links that contains no closed loops. We shall occasionally use *ordered forests* in which the links are labelled; for each forest $\mathscr{F}$ there are $|\mathscr{F}|!$ ordered forests. A *tree* $\mathscr{T} \subset \mathcal{P}_n$ is a connected forest, so that each forest can be uniquely broken down into trees (connected components). We consider a lone vertex to form a tree with zero links. In the following $\mathscr{F} \subset \mathcal{P}_n$ shall always denote a forest (and not any subset of $\mathcal{P}_n$) and similarly $\mathscr{T} \subset \mathcal{P}_n$ a tree.

Lemma 3.4 (Taylor forest formula). *Let $H = H(\underline{x})$ be a function of the real variables $\underline{x} = \{x_l\}_{l\in\mathcal{P}_n}$. Then*

$$H(\underline{x}) = \sum_{\mathscr{F}} \left(\prod_{l\in\mathscr{F}} \int_0^1 dh_l \right) \left(\prod_{l\in\mathscr{F}} x_l \frac{\partial}{\partial x_l} \right) H(\underline{X}^{\mathscr{F}}(h_1, \ldots, h_k)), \tag{3.78}$$

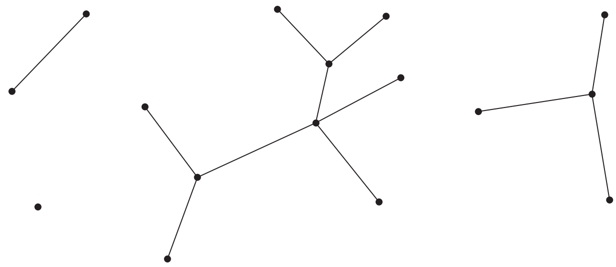

Figure 3.3: *A forest consisting of 4 trees.*

where the sum ranges over all forests $\mathscr{F}$ *on* $\mathcal{P}_n$, $k = |\mathscr{F}|$ *and*

$$X_l^{\mathscr{F}}(h_1, \dots, h_k) := \begin{cases} 0, & \gamma_{\mathscr{F}}(l) = \emptyset, \\ x_l \min\{h_{l'}\}_{l' \in \gamma_{\mathscr{F}}(l)}, & \textit{otherwise}, \end{cases} \tag{3.79}$$

where $\gamma_{\mathscr{F}}(l)$ *denotes the (unique) path in* $\mathscr{F}$ *that connects the endpoints of* l*.*

Proof. The proof is essentially a repeated application of Taylor's theorem to the zeroth order with an integral remainder. We begin by interpolating between 0 and 1:

$$x_l(h_1) := h_1 x_l. \tag{3.80}$$

Then

$$\begin{aligned} H(\underline{x}) &= H(\underline{0}) + \int_0^1 dh_1 \, \frac{d}{dh_1} \, H(\underline{x}(h_1)) \\ &= H(\underline{0}) + \sum_{l_1} \int_0^1 dh_1 \, \left(x_{l_1} \frac{\partial}{\partial x_{l_1}} \right) H(\underline{x}(h_1)). \end{aligned} \tag{3.81}$$

In a second step we interpolate between 0 and h_1:

$$x_l(h_1, h_2) := \begin{cases} x_l(h_1), & l = l_1, \\ h_2 x_l, & l \neq l_1. \end{cases} \tag{3.82}$$

Thus the integrand above is equal to

$$\left(x_{l_1} \frac{\partial}{\partial x_{l_1}} \right) H(\underline{x}(h_1, 0)) + \sum_{l_2 \neq l_1} \int_0^{h_1} dh_2 \, \left(x_{l_1} \frac{\partial}{\partial x_{l_1}} \right) \left(x_{l_2} \frac{\partial}{\partial x_{l_2}} \right) H(\underline{x}(h_1, h_2)). \tag{3.83}$$

so that

$$\begin{aligned} H(\underline{x}) =& H(\underline{0}) + \sum_{l_1} \int_0^1 dh_1 \, \left(x_{l_1} \frac{\partial}{\partial x_{l_1}} \right) H(\underline{x}(h_1, 0)) \\ &+ \sum_{(l_1, l_2) \text{ o-forest}} \int_0^1 dh_1 \int_0^{h_1} dh_2 \, \left(x_{l_1} \frac{\partial}{\partial x_{l_1}} \right) \left(x_{l_2} \frac{\partial}{\partial x_{l_2}} \right) H(\underline{x}(h_1, h_2)), \end{aligned} \tag{3.84}$$

where we rewrote the sum over $(l_1, l_2), l_1 \neq l_2$ as a sum over ordered forests with two links. We continue in this manner. Let $l_1, \ldots, l_k$ already be chosen. Define recursively

$$x_l(h_1, \ldots, h_k, h_{k+1}) := \begin{cases} x_l(h_1, \ldots, h_k), & (l_1, \ldots, l_k, l) \text{ do not form a forest} \\ h_{k+1} x_l, & \text{otherwise.} \end{cases} \tag{3.85}$$

(Note that if a link appears twice in a graph, it is understood that this leads to a loop and the graph is therefore not a forest.) Then by induction (repeat the above argument k-times) we see that

$$\begin{aligned}
&H(\underline{x}) = \\
&H(\underline{0}) + \sum_{l_1 \text{ o-forest}} \int_0^1 dh_1 \left(x_{l_1} \frac{\partial}{\partial x_{l_1}} \right) H(\underline{x}(h_1, 0)) \\
&+ \sum_{(l_1, l_2) \text{ o-forest}} \int_0^1 dh_1 \int_0^{h_1} dh_2 \left(x_{l_1} \frac{\partial}{\partial x_{l_1}} \right) \left(x_{l_2} \frac{\partial}{\partial x_{l_2}} \right) H(\underline{x}(h_1, h_2, 0)) \\
&+ \sum_{(l_1, l_2, l_3) \text{ o-forest}} \int_0^1 dh_1 \int_0^{h_1} dh_2 \int_0^{h_2} dh_3 \left(x_{l_1} \frac{\partial}{\partial x_{l_1}} \right) \left(x_{l_2} \frac{\partial}{\partial x_{l_2}} \right) \left(x_{l_3} \frac{\partial}{\partial x_{l_3}} \right) H(\underline{x}(h_1, h_2, h_3, 0)) \\
&\vdots \\
&+ \sum_{(l_1, \ldots, l_k) \text{ o-forest}} \int_0^1 dh_1 \ldots \int_0^{h_{k-1}} dh_k \left(x_{l_1} \frac{\partial}{\partial x_{l_1}} \right) \ldots \left(x_{l_k} \frac{\partial}{\partial x_{l_k}} \right) H(\underline{x}(h_1, \ldots h_k, 0)) \\
&\ddots
\end{aligned} \tag{3.86}$$

The series ends with maximal trees to which no link can be added to create a new forest; in that case $\underline{x}(h_1, \ldots, h_k, h_{k+1}) = \underline{x}(h_1, \ldots, h_k, 0)$ is a constant in h_{k+1} and this is the last term.

We now show that $x_l(h_1, \ldots h_k, 0) = X_l^{\mathscr{F}}(h_1, \ldots h_k)$. Obviously, if $(l_1, \ldots, l_k, l)$ is a forest $x_l(h_1, \ldots h_k, 0) = X_l^{\mathscr{F}}(h_1, \ldots h_k) = 0$. Let therefore $(l_1, \ldots, l_k, l)$ contain a (unique) loop. Since $(l_1, \ldots, l_k)$ is a forest the loop must contain l, i.e. the loop is of the form $l \cup \gamma_l$, where γ_l is an ordered set of links of the forest that form a path. To compute $x_l(h_1, \ldots h_k, 0)$, we apply (3.85) recursively. Each application removes the latest link (i.e. the link with the highest index) from the forest. The recursion terminates when we have removed the loop. Let l_j be the link whose removal kills the loop; then $x_l(h_1, \ldots h_k, 0) = h_j x_l$. Now l_j is the latest link in γ_l, so that h_j is the smallest h in γ_l.

Finally, to return to unordered forests, we note that

$$\{(h_1, \ldots, h_k) \ : \ h_j \in [0, 1]\} \tag{3.87}$$

is equal to (up to a set of volume 0)

$$\bigcup_{\sigma \in S_k} \{(h_{\sigma(1)}, \ldots, h_{\sigma(k)}) \ : \ h_j \in [0, 1], \ h_1 \geq h_2 \geq \cdots \geq h_k\}. \tag{3.88}$$

Now for a given forest $\mathscr{F}$ of k links all $k!$ permutations of the links appear in the above sum over ordered forests, so that we integrate over each of the $k!$ subsets of (3.88) and thus recover (3.78). □

We now apply the Taylor forest formula to

$$H(\{x_l\}) = \prod_{l \in \mathcal{P}_n} \left(1 + x_l \, g_l^{\underline{Y}}\right), \tag{3.89}$$

the set of vertices $\{1, \dots, n\}$ corresponding to the polymers $\underline{Y}$. Using

$$\left(\prod_{l \in \mathscr{F}} \frac{\partial}{\partial x_l}\right) H(\{x_l\}) = \left(\prod_{l \in \mathscr{F}} g_l^{\underline{Y}}\right) \left(\prod_{l \notin \mathscr{F}} (1 + x_l \, g_l^{\underline{Y}})\right) \tag{3.90}$$

we have

$$Z_\Lambda = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1, \dots, Y_n} \prod_{j=1}^{n} z(Y_j) \sum_{\mathscr{F} \subset \mathcal{P}_n} \left(\prod_{l \in \mathscr{F}} g_l^{\underline{Y}} \int_0^1 dh_l\right) \left(\prod_{l \notin \mathscr{F}} (1 + X_l^{\mathscr{F}} \, g_l^{\underline{Y}})\right) \tag{3.91}$$

We want to rewrite this in terms of the trees of $\mathscr{F}$. To this end note that, since $\left(1 + X_l^{\mathscr{F}} \, g_l^{\underline{Y}}\right) = 1$ for l connecting two different trees, we may factorise

$$\prod_{l \notin \mathscr{F}} (1 + X_l^{\mathscr{F}} \, g_l^{\underline{Y}}) = \prod_{\mathscr{T} \subset \mathscr{F}} \prod_{l \notin \mathscr{T}} (1 + X_l^{\mathscr{T}} \, g_l^{\underline{Y}^{\mathscr{T}}}), \tag{3.92}$$

where $\mathscr{T}$ ranges over all trees of $\mathscr{F}$, $l \notin \mathscr{T}$ over all links connecting the vertices of $\mathscr{T}$ excluding those belonging to $\mathscr{T}$, and $\underline{Y}^{\mathscr{T}}$ is the set of polymers whose corresponding vertices are contained in $\mathscr{T}$. Note that $X_l^{\mathscr{T}}$ only depends on h_l if $l \in \mathscr{T}$. Therefore each term in the sum over forests of (3.91) factors over the trees. Decomposing every forest $\mathscr{F}$ into k trees $\mathscr{T}_1, \dots, \mathscr{T}_k$ with n_i vertices each we have

$$Z_\Lambda = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathscr{F} \subset \mathcal{P}_n} f_{n_1}(\mathscr{T}_1) \dots f_{n_k}(\mathscr{T}_k), \tag{3.93}$$

where

$$f_n(\mathscr{T}) := \sum_{Y_1, \dots, Y_n} \prod_{j=1}^{n} z(Y_j) \left(\prod_{l \in \mathscr{T}} g_l^{\underline{Y}} \int_0^1 dh_l\right) \left(\prod_{l \notin \mathscr{T}} \left(1 + X_l^{\mathscr{T}} \, g_l^{\underline{Y}}\right)\right). \tag{3.94}$$

We rewrite the sum over forests $\mathscr{F}$ by indexing forests as follows: First the number of trees k; second the k numbers $n_1, \dots, n_k$, $n_i \geq 1$, $\sum_i n_i = n$ defining the number of vertices in each tree; third a partition of $\{1, \dots, n\}$ into k subsets with n_i elements each; fourth a sum over all trees within all k subsets. Note that, since the summand $f_{n_i}(\mathscr{T}_i)$ is trivially independent of which n_i vertices were chosen from the n vertices, all terms of the third sum are equal, so that it may be replaced by a combinatorial coefficient: the number of partitions of $\{1, \dots, n\}$ into k subsets with n_i vertices each, i.e. $\frac{n!}{n_1!\ldots n_k!}$. Since any permutation of $(n_1,\ldots,n_k)$ leads to an identical set of partitions, we compensate the overcounting by a factor $1/k!$. Putting all this together we get

$$\begin{aligned}
Z_\Lambda &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=1}^{n}\sum_{\substack{n_1,\ldots,n_k\\ \sum_i n_i=n}}\frac{n!}{n_1!\ldots n_k!}\frac{1}{k!}\sum_{\mathscr{T}_1\subset\mathcal{P}_{n_1}}\cdots\sum_{\mathscr{T}_k\subset\mathcal{P}_{n_k}} f_{n_1}(\mathscr{T}_1)\ldots f_{n_k}(\mathscr{T}_k)\\
&= 1+\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{n_1,\ldots,n_k}\frac{1}{n_1!\ldots n_k!}\sum_{\mathscr{T}_1\subset\mathcal{P}_{n_1}}\cdots\sum_{\mathscr{T}_k\subset\mathcal{P}_{n_k}} f_{n_1}(\mathscr{T}_1)\ldots f_{n_k}(\mathscr{T}_k)\\
&= 1+\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{i=1}^{k}\left(\sum_{n_i}\frac{1}{n_i!}\sum_{\mathscr{T}_i\subset\mathcal{P}_{n_i}} f_{n_i}(\mathscr{T}_i)\right)\\
&= 1+\sum_{k=1}^{\infty}\frac{1}{k!}\left(\sum_{n}\frac{1}{n!}\sum_{\mathscr{T}\subset\mathcal{P}_{n}} f_{n}(\mathscr{T})\right)^k. \qquad (3.95)
\end{aligned}$$

Therefore

$$\log Z_\Lambda = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\mathscr{T}\subset\mathcal{P}_n} f_n(\mathscr{T}). \qquad (3.96)$$

Inserting (3.94) we get

$$\boxed{\log Z_\Lambda = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{Y_1,\ldots,Y_n}\phi^T(Y_1,\ldots,Y_n)\prod_{j=1}^{n} z(Y_j),} \qquad (3.97)$$

with the *Ursell function*

$$\phi^T(Y_1,\ldots,Y_n) := \sum_{\mathscr{T}\subset\mathcal{P}_n}\left(\prod_{l\in\mathscr{T}} g_l^Y\int_0^1 dh_l\right)\left(\prod_{l\notin\mathscr{T}}\left(1+X_l^{\mathscr{T}} g_l^Y\right)\right). \qquad (3.98)$$

Note that $\phi^T(Y_1,\ldots,Y_n)$ vanishes unless there is a tree connecting all the polymers $Y_1,\ldots Y_n$. Surprisingly, the original sum over disjoint polymers has become a sum over connected polymers.

For many applications a bound for $|\phi^T(Y_1,\ldots,Y_n)|$ suffices. Since $\left|1+X_l^{\mathscr{T}} g_l^Y\right|\le 1$ and the integration in (3.98) has total volume 1, we may bound

$$|\phi^T(Y_1,\ldots,Y_n)| \le \sum_{\mathscr{T}\subset\mathcal{P}_n}\prod_{l\in\mathscr{T}}\left|g_l^Y\right|. \qquad (3.99)$$

3.2.2 Convergence

We now proceed to show that, for J small enough, the high-temperature expansion (3.97) for a lattice gauge theory converges, and the "pressure"

$$\frac{1}{|\Lambda|}\log Z_\Lambda \le \text{const.} \qquad (3.100)$$

is uniformly bounded in the lattice size $|\Lambda|$.

For the following we need some basic facts about trees. For a given tree $\mathscr{T}$ on $\mathcal{P}_n$ define the *coordination number* $d_j = 1, 2, 3, \dots$ of a vertex j as the number of links connecting to j; we abbreviate $\underline{d} := \{d_1, \dots d_n\}$. A vertex j is called a *leaf* if $d_j = 1$.

Lemma 3.5.

$$\sum_{j=1}^{n} d_j = 2(n-1). \tag{3.101}$$

Proof. Each link in $\mathscr{T}$ contributes a total of 2 in the sum, so that

$$\sum_{j=1}^{n} d_j = 2\,|\mathscr{T}|. \tag{3.102}$$

To compute $|\mathscr{T}|$, we start at a leaf and count the number of vertices along linked paths. Each link gives raise to exactly one vertex (a vertex may not be counted twice since this would imply the existence of a closed loop). Thus

$$|\mathscr{T}| = n - 1. \tag{3.103}$$

□

Lemma 3.6 (Cayley's theorem). *The number $t(n, \underline{d})$ of trees on $\mathcal{P}_n$ with coordination numbers $\underline{d}$ is*

$$t(n, \underline{d}) = \frac{(n-2)!}{\prod_j (d_j - 1)!}. \tag{3.104}$$

Proof. Since each tree contains a leaf we may assume that, after a relabelling of the vertices, $d_1 \geq d_2 \geq \dots \geq d_n = 1$. We proceed by induction on n. The assertion obviously holds for $n = 2$. Now the leaf n has to be connected to one of the remaining $n-1$ vertices with coordination number greater than 1, so that

$$t(n, \{d_1, \dots d_n\}) = \sum_{j<n,\, d_j>1} t(n-1, \{d_1, \dots, d_j - 1, \dots, d_n\}). \tag{3.105}$$

Assuming the assertion holds for $n-1$, we have

$$\begin{aligned} \sum_{j<n,\, d_j>1} t(n-1, \{d_1, \dots, d_j - 1, \dots, d_n\}) &= \sum_{j<n,\, d_j>1} \frac{(n-3)!}{(d_j-2)! \prod_{i\neq j} (d_i - 1)!} \\ &= \frac{(n-2)!}{\prod_i (d_i - 1)!} \sum_{j<n} \frac{d_j - 1}{n-2}, \end{aligned} \tag{3.106}$$

since j's with $d_j = 1$ give a zero contribution to the sum. Using Lemma 3.5, the sum is equal to

$$\sum_{j\leq n} \frac{d_j - 1}{n-2} = \frac{2(n-1) - n}{n-2} = 1. \tag{3.107}$$

□

As a byproduct we also get the total number $t(n)$ of trees on $\mathcal{P}_n$:

$$t(n) = \sum_{\substack{\{d_j\},\, d_j \geq 1 \\ \sum_j d_j = 2(n-1)}} t(n, \{d_j\}) \overset{c_j = d_j - 1}{=} \sum_{\substack{\{c_j\},\, c_j \geq 0 \\ \sum_j c_j = n-2}} \frac{(n-2)!}{\prod_j c_j!} = n^{n-2}. \tag{3.108}$$

Define the size $|Y|$ of the polymer Y as the number of plaquettes in Y, and choose some plaquette $p_0 \subset \Lambda$. Then convergence of the high-temperature expansion can be inferred from

Lemma 3.7. *If*

$$\sum_{\substack{Y \in \mathscr{Y} \\ Y \text{ connected to } p_0}} |z(Y)|\, e^{|Y|} \leq 1, \tag{3.109}$$

then the expansion (3.97) *converges absolutely and*

$$\frac{1}{|\Lambda|} \log Z_\Lambda \leq 2. \tag{3.110}$$

Proof. The proof is a model for all the other convergence proofs we will need. Using the bound (3.99) we have

$$\log Z_\Lambda \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathscr{T} \subset \mathcal{P}_n} \sum_{Y_1, \ldots, Y_n} \prod_{l \in \mathscr{T}} \left| g_l^Y \right| \prod_{j=1}^{n} |z(Y_j)|. \tag{3.111}$$

We may rewrite the sum over trees by first specifying the coordination numbers $\underline{d}$ and then summing over all trees with the specified $\underline{d}$. Using Cayley's theorem 3.6 we get the bound

$$\begin{aligned} \log Z_\Lambda &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{d_1, \ldots, d_n} \sum_{\mathscr{T} \text{ with } \underline{d}} \sum_{Y_1, \ldots, Y_n} \prod_{l \in \mathscr{T}} \left| g_l^Y \right| \prod_{j=1}^{n} |z(Y_j)| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n(n-1)} \sum_{d_1, \ldots, d_n} \sup_{\mathscr{T} \text{ with } \underline{d}} \sum_{Y_1, \ldots, Y_n} \prod_{l \in \mathscr{T}} \left| g_l^Y \right| \prod_{j=1}^{n} \frac{|z(Y_j)|}{(d_j - 1)!}. \end{aligned} \tag{3.112}$$

(The factor $\frac{1}{n(n-1)}$ is understood to equal 1 for $n = 1$.) Pick some fixed tree $\mathscr{T}$ on $\mathcal{P}_n$ with coordination numbers $\underline{d}$. We wish to bound

$$\sum_{Y_1, \ldots, Y_n} \prod_{l \in \mathscr{T}} \left| g_l^Y \right| \prod_{j=1}^{n} \frac{|z(Y_j)|}{(d_j - 1)!}. \tag{3.113}$$

The ensuing bound will be independent of the choice of $\mathscr{T}$, so that it also holds for the supremum. Choose a *root* $r \in \{1, \ldots, n\}$ on $\mathscr{T}$. Then we have a hierarchy on $\mathscr{T}$: each vertex $j \neq r$ has $d_j - 1$ children and a single parent. Let us start the summing in (3.113) from a leaf $l \in \{1, \ldots, n\}$ with parent j. Then Y_l must connect to Y_j and the sum over Y_l yields

$$\begin{aligned} \sum_{Y_l \text{ conn. } Y_j} \frac{|z(Y_l)|}{(d_l - 1)!} &= \sum_{Y_l \text{ conn. } Y_j} \frac{|z(Y_l)|\, |Y_l|^{d_l - 1}}{(d_l - 1)!} \\ &\leq |Y_j| \sum_{Y_l \ni p_0} \frac{|z(Y_l)|\, |Y_l|^{d_l - 1}}{(d_l - 1)!}, \end{aligned} \tag{3.114}$$

whereby in the last step we used that any polymer Y_l connecting to Y_j must do so on at least one plaquette of Y_j, as well as translation invariance. Now move to the vertex j with parent i. Each of the $d_j - 1$ children produces an above factor, so that all the terms corresponding to j and its children are bounded by

$$|Y_i| \sum_{Y_j \ni p_0} \frac{|z(Y_j)|\,|Y_j|^{d_j-1}}{(d_j-1)!} \prod_{l \text{ child of } j} \left(\sum_{Y_l \ni p_0} \frac{|z(Y_l)|\,|Y_l|^{d_l-1}}{(d_l-1)!} \right). \tag{3.115}$$

We continue recursively in this manner until we reach the root. Then the entire sum (3.113) is bounded by

$$\begin{aligned} &\sum_{Y_r} \frac{|z(Y_r)|\,|Y_r|^{d_r-1}}{(d_r-1)!} \prod_{j \neq r} \left(\sum_{Y_j \ni p_0} \frac{|z(Y_j)|\,|Y_j|^{d_j-1}}{(d_j-1)!} \right) \\ &\leq |\Lambda| \sum_{Y_r \ni p_0} \frac{|z(Y_r)|\,|Y_r|^{d_r-1}}{(d_r-1)!} \prod_{j \neq r} \left(\sum_{Y_j \ni p_0} \frac{|z(Y_j)|\,|Y_j|^{d_j-1}}{(d_j-1)!} \right) \\ &= |\Lambda| \prod_{j=1}^{n} \left(\sum_{Y_j \ni p_0} \frac{|z(Y_j)|\,|Y_j|^{d_j-1}}{(d_j-1)!} \right) \end{aligned} \tag{3.116}$$

so that

$$\begin{aligned} \log Z_\Lambda &\leq |\Lambda| \sum_{n=1}^{\infty} \frac{1}{n(n-1)} \prod_{j=1}^{n} \left(\sum_{Y_j \ni p_0} \sum_{d_j} \frac{|z(Y_j)|\,|Y_j|^{d_j-1}}{(d_j-1)!} \right) \\ &= |\Lambda| \sum_{n=1}^{\infty} \frac{1}{n(n-1)} \left(\sum_{Y \ni p_0} |z(Y)|\, e^{|Y|} \right)^n . \end{aligned} \tag{3.117}$$

By assumption, the sum converges and is uniformly bounded by[7] $1 + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 2$. □

To complete the proof of convergence we therefore need

Lemma 3.8. *The convergence condition* (3.109) *is satisfied if*

$$J \leq -\frac{1}{c} \log\left(1 - \frac{1}{16de}\right), \tag{3.118}$$

where $d+1$ *is the dimension of space-time and* $e = 2.71828\ldots$.

Proof. Since

$$|z(Y)| \leq \int d\underline{g} \prod_{p \in Y} |\rho_p(g_p)| \tag{3.119}$$

we get from (3.71)

$$|z(Y)| \leq r(J)^{|Y|}. \tag{3.120}$$

[7] To show this define $f(x) := \sum_{m=0}^{\infty} \frac{x^{m+2}}{(m+1)(m+2)}$ so that $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lim_{x\to 1} f(x)$. Then f is smooth on $[0,1)$ and $f''(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$. Using $f(0) = f'(0) = 0$ we find $f(x) = x + (1-x)\log(1-x)$ and therefore $\lim_{x\to 1} f(x) = 1$.

The number of polymers with length $|Y|$ connecting to p_0 can be bounded as follows. To a given plaquette we may connect at most $8d$ plaquettes, $d+1$ being the dimension of Λ. Indeed we may choose to connect at any of the 4 links; each link is contained in $2d$ plaquettes. If we let the polymer "grow" from p_0, we see that we have at most $(8d)^{|Y|}$ possible polymers. Therefore

$$\sum_{\substack{Y\in\mathscr{Y}\\ Y \text{ connected to } p_0}} |z(Y)|\, e^{|Y|} \le \sum_{|Y|=1}^{\infty} (8d\, r(J)\, e)^{|Y|} = \frac{8d\, r(J)\, e}{1-8d\, r(J)\, e} \le 1, \tag{3.121}$$

if $r(J) \le \frac{1}{16de}$, i.e.

$$J \le -\frac{1}{c} \log\Big(1 - \frac{1}{16de}\Big). \tag{3.122}$$

□

For example for the Wilson action in $3+1$ dimensions we have convergence if $J \le 0.1308\ldots/\chi(\mathbb{1})$.

Note that the proof of Lemma 3.8 actually implies the stronger bound

$$\sum_{\substack{Y\in\mathscr{Y}\\ Y \text{ connected to } p_0}} \big[r(J)e\big]^{|Y|} < 1 \tag{3.123}$$

for J small enough.

3.2.3 Observables: expectations and correlation functions

Our goal is to compute the compute the expectation of an observable A:

$$\langle A\rangle_\Lambda = \frac{d}{d\lambda}\bigg|_{\lambda=0} \log \int d\underline{g}\, \big(1+\lambda A(\underline{g})\big)\, e^{-S(\underline{g})}, \tag{3.124}$$

as well as connected two-point functions:

$$\begin{aligned}\langle A_1\,;\,A_2\rangle_\Lambda &:= \langle A_1A_2\rangle_\Lambda - \langle A_1\rangle_\Lambda\langle A_2\rangle_\Lambda\\ &= \frac{\partial^2}{\partial\lambda_1\,\partial\lambda_2}\bigg|_{\lambda_1=\lambda_2=0} \log\int d\underline{g}\, \big(1+\lambda_1A_1(\underline{g})\big)\big(1+\lambda_2A_2(\underline{g})\big)\, e^{-S(\underline{g})}.\end{aligned} \tag{3.125}$$

Generally, we consider a connected m-point function of the m observables $A_1,\ldots,A_m$:

$$\langle A_1\,;\,\ldots\,;\,A_m\rangle_\Lambda = \frac{\partial^m}{\partial\lambda_1\ldots\partial\lambda_m}\bigg|_{\lambda_1=\cdots=\lambda_m=0} \log Z_\Lambda(\{\lambda_iA_i\}), \tag{3.126}$$

with the modified partition function

$$Z_\Lambda(\{A_i\}) := \int d\underline{g}\, \prod_{i=1}^m \big(1+A_i(\underline{g})\big)\, e^{-S(\underline{g})}. \tag{3.127}$$

Let $\operatorname{supp} A_i$ be the support of A_i, i.e. the set of links in Λ on which A_i depends. We assume all observables to have pairwise disjoint support: $\operatorname{supp} A_i \cap \operatorname{supp} A_j = \emptyset$ for $i \neq j$.

The expansion method of the preceding sections can be almost immediately taken over to this case, after a modification of the set of polymers and the definition of connectedness. The modified partition function is

$$Z_\Lambda(\underline{A}) = \int d\underline{g} \prod_{i=1}^{m} \big(1 + A_i(\underline{g})\big) \prod_{p\subset\Lambda} \big(1 + \rho_p(g_p)\big). \tag{3.128}$$

Denote the set of plaquettes by $P := \{p \;:\; p \subset \Lambda\}$ and define the set of "generalised plaquettes"

$$Q := P \cup \{1, \dots, m\}, \tag{3.129}$$

as well as, for $q \in Q$, the function

$$\rho_q(\underline{g}) := \begin{cases} \rho_p(g_p), & q = p \in P, \\ A_i(\underline{g}), & q = i \in \{1, \dots, m\}. \end{cases} \tag{3.130}$$

Then we may rewrite (3.128) as

$$\begin{aligned} Z_\Lambda(\underline{A}) &= \int d\underline{g} \prod_{q\in Q} \big(1 + \rho_q(\underline{g})\big) \\ &= \sum_{\underline{n}} \int d\underline{g} \prod_{q\in Q} \rho_q(\underline{g})^{n_q}, \end{aligned} \tag{3.131}$$

where the sum ranges over all multi-indices $\underline{n} = \{n_q \;:\; q \in Q\}$, where $n_q = 0, 1$. In view of factoring the integral into connected components, we define connectedness as follows: Two elements $q_1, q_2 \in Q$ are connected if (i) q_1 and q_2 are both plaquettes and they share a link, *or* (ii) one is a plaquette and the other an observable such that the support of the observable and the plaquette share a link. Otherwise q_1 and q_2 are disconnected. Now we proceed exactly as in Section 3.2.1. A polymer is a connected subset $Y \subset Q$. The set of polymers is $\mathscr{Y}$ and the connection function is g, as given by (3.77). Then we may decompose the set $Q_{\underline{n}} := \{q \in Q \;:\; n_q = 1\} = Y_1 \cup \cdots \cup Y_n$ into connected components. Thus the integral in (3.131) factors and we have

$$Z_\Lambda(\underline{A}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{Y_1,\dots,Y_n \\ \text{disjoint}}} \prod_{j=1}^{n} z_{\underline{A}}(Y_j), \tag{3.132}$$

where the activity $z_{\underline{A}}(Y)$ of a polymer is

$$z_{\underline{A}}(Y) := \int d\underline{g} \prod_{q\in Y} \rho_q(\underline{g}). \tag{3.133}$$

and may now depend on the observables $\underline{A}$. Using g we rewrite this as

$$Z_\Lambda(\underline{A}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1,\dots,Y_n} \prod_{j=1}^{n} z_{\underline{A}}(Y_j) \prod_{i<j} \big(1 + g^{Y}_{(i,j)}\big). \tag{3.134}$$

Unleashing the abstract machinery of Section 3.2.1 we find from (3.97)

$$\log Z_\Lambda(\underline{A}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1,\dots,Y_n} \phi^T(Y_1,\dots,Y_n) \prod_{j=1}^{n} z_{\underline{A}}(Y_j), \tag{3.135}$$

with the Ursell function ϕ^T given by (3.98).

The study of convergence is almost identical to Section 3.2.2. The main trick is to scale all observables A_i by a small factor λ_i so that the activity $z_{\underline{A}}(Y)$ can be made as small as necessary for all polymers Y. Note that λ_i may be as small as we want since we only need the derivative at 0. Convergence of the expansion for $\log Z_\Lambda(\{\lambda_i A_i\})$ is shown precisely as Lemma 3.7. The only required modification to its proof arises from the breaking of translation symmetry by the observables. This does not matter, however, if we choose $q_0 \in Q$ (corresponding to p_0 in the proof of Lemma 3.7) to be the element with the largest number of connected neighbour elements (either a plaquette or the observable with the largest support, whichever has more links). Then all bounds in the proof are valid (if somewhat crude). Again denoting by $|Y|$ the number of elements in Y, we thus have

Lemma 3.9. *If*

$$\sum_{\substack{Y\in\mathscr{Y} \\ Y \text{ connected to } q_0}} |z_{\{\lambda_i A_i\}}(Y)|\, e^{|Y|} \leq 1, \tag{3.136}$$

then the expansion for $\log Z_\Lambda(\{\lambda_i A_i\})$ *converges absolutely and*

$$\frac{1}{|\Lambda|} \log Z_\Lambda(\{\lambda_i A_i\}) \leq 2. \tag{3.137}$$

The proof is then completed by

Lemma 3.10. *The convergence condition* (3.136) *is fulfilled if*

$$J \leq -\frac{1}{c} \log\left(1 - \frac{1}{4d|q_0|e}\right), \tag{3.138}$$

$|q_0|$ *being the number of links in* q_0*, and for* $\lambda_1,\dots,\lambda_m$ *small enough:*

$$\lambda_i \leq \frac{r(J)}{\|A_i\|}, \qquad i = 1,\dots,m, \tag{3.139}$$

where $\|\cdot\| := \|\cdot\|_\infty$ *denotes the supremum norm.*

Proof. We only sketch the modifications required in the proof of Lemma 3.8. For a plaquette $p \in P$ we have

$$\big|\rho_p(g_p)\big| \leq r(J). \tag{3.140}$$

Furthermore from (3.139) we get

$$\big|\rho_i(\underline{g})\big| = \big|\lambda_i A_i(\underline{g})\big| \leq \lambda_i \|A_i\| \leq r(J), \tag{3.141}$$

so that $\big|\rho_q(\underline{g})\big| \leq r(J)$ for any $q \in Q$ and therefore

$$|z(Y)| \leq r(J)^{|Y|}. \tag{3.142}$$

For the rest of the proof we note that to a given element $q \in Q$ we may connect at most $2d|q_0|$ elements, so that

$$\sum_{\substack{Y \in \mathscr{Y} \\ Y \text{ connected to } q_0}} |z(Y)|\, e^{|Y|} \leq 1, \tag{3.143}$$

if

$$J \leq -\frac{1}{c} \log\left(1 - \frac{1}{4d|q_0|e}\right). \tag{3.144}$$

□

3.2.4 Tree graph decay for the connected m-point function

Consider now the connected m-point function for $m \geq 2$:

$$\langle A_1 \,;\, \dots \,;\, A_m\rangle_\Lambda = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1,\dots,Y_n} \phi^T(Y_1,\dots,Y_n) \frac{\partial^m}{\partial\lambda_1 \dots \partial\lambda_m}\bigg|_{\lambda_1=\dots=\lambda_m=0} \prod_{j=1}^{n} z_{\{\lambda_i A_i\}}(Y_j). \tag{3.145}$$

The derivative filters out exactly the terms that are multilinear in $\lambda_1,\dots,\lambda_m$, so that

$$\langle A_1 \,;\, \dots \,;\, A_m\rangle_\Lambda = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1,\dots,Y_n}{}' \phi^T(Y_1,\dots,Y_n) \prod_{j=1}^{n} z_{\underline{A}}(Y_j), \tag{3.146}$$

Where $\sum'$ denotes a sum over all terms multilinear in $A_1,\dots,A_m$, i.e. over all sets $\{Y_1,\dots,Y_n\}$ that contain each observable exactly once.

Convergence of the above expansion follows directly from the fact that it is the derivative at 0 of a convergent power series. In view of later applications, we also provide a direct proof. Concentrate first on $\prod_{j=1}^n |z_{\underline{A}}(Y_j)|$. Now the set $\{Y_1,\dots,Y_m\}$ in the sum above must contain all the observables exactly once and consequently m observables and at least $m-1$ plaquettes. Denoting $|\underline{Y}| := |Y_1| + \dots + |Y_n|$ we get

$$\prod_{j=1}^{n} |z_{\underline{A}}(Y_j)| \leq r(J)^{|\underline{Y}|-m} \prod_{i=1}^{m} \|A_i\|. \tag{3.147}$$

Now, since $|\underline{Y}| \geq 2m-1$, we have

$$r(J)^{|\underline{Y}|-m} \leq r(J)^{\alpha|\underline{Y}|}, \tag{3.148}$$

for $\alpha = \frac{m-1}{2m-1}$ (for example), so that

$$\prod_{j=1}^{n} |z_{\underline{A}}(Y_j)| \leq \prod_{i=1}^{m} \|A_i\| \prod_{j=1}^{n} r(J)^{\alpha|Y_j|}. \tag{3.149}$$

Therefore

$$|\langle A_1 \,;\, \dots \,;\, A_m\rangle_\Lambda| \leq \prod_{i=1}^{m} \|A_i\| \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1,\dots,Y_n}{}' |\phi^T(Y_1,\dots,Y_n)| \prod_{j=1}^{n} r(J)^{\alpha|Y_j|}. \tag{3.150}$$

The convergence of this is shown precisely as Lemma 3.7 with the minor modifications described in the previous section. In the proof we bound $|z(Y_j)|$ by $r(J)^{\alpha|Y_j|}$. Finally, the bound is uniform in Λ since at least one polymer must connect to a fixed location (the support of A_1 for instance). The required condition is

$$\sum_{\substack{Y\in\mathscr{Y}\\ Y \text{ connected to } q_0}} \big(r(J)^\alpha e\big)^{|Y|} \leq 1, \tag{3.151}$$

which is fulfilled for J small enough.

We may now prove the main result of this section. Denote by $t(\underline{A})$ the number of plaquettes in the smallest tree connecting the supports of $A_1, \ldots, A_n$.

Theorem 3.11 (Tree graph decay). *There is a constant K such that, for J small enough,*

$$|\langle A_1\,;\,\ldots\,;\,A_m\rangle| \leq K \prod_{i=1}^m \|A_i\|\; e^{-m(J)\,t(\underline{A})}, \tag{3.152}$$

where $m(J) := -\frac{\alpha}{2}\log r(J) \geq 0$.

Proof. Choose J small enough that

$$\sum_{n=1}^\infty \frac{1}{n!} {\sum_{Y_1,\ldots,Y_n}}' |\phi^T(Y_1,\ldots,Y_n)| \prod_{j=1}^n r(J)^{\frac{\alpha}{2}|Y_j|} \tag{3.153}$$

converges as shown above, and call this bound K. Then from (3.150) we get

$$|\langle A_1\,;\,\ldots\,;\,A_m\rangle_\Lambda| \leq \prod_{i=1}^m \|A_i\| \sum_{n=1}^\infty \frac{1}{n!} {\sum_{Y_1,\ldots,Y_n}}' |\phi^T(Y_1,\ldots,Y_n)| \prod_{j=1}^n r(J)^{\frac{\alpha}{2}|Y_j|}\, r(J)^{\frac{\alpha}{2}|Y_j|}. \tag{3.154}$$

Now $\sum_{j=1}^n |Y_j| \geq t(\underline{A})$ for each term in the sum above so that

$$r(J)^{\frac{\alpha}{2}\sum_{j=1}^n |Y_j|} \leq r(J)^{\frac{\alpha}{2}t(\underline{A})} \tag{3.155}$$

and therefore

$$\begin{aligned}|\langle A_1\,;\,\ldots\,;\,A_m\rangle_\Lambda| &\leq \prod_{i=1}^m \|A_i\|\, r(J)^{\frac{\alpha}{2}t(\underline{A})} \sum_{n=1}^\infty \frac{1}{n!} {\sum_{Y_1,\ldots,Y_n}}' |\phi^T(Y_1,\ldots,Y_n)| \prod_{j=1}^n r(J)^{\frac{\alpha}{2}|Y_j|} \\ &\leq K \prod_{i=1}^m \|A_i\|\; e^{-m(J)\,t(\underline{A})}.\end{aligned} \tag{3.156}$$

Since K is uniformly bounded in $|\Lambda|$ taking the thermodynamic limit finishes the proof. □

A direct consequence is confinement at strong couplings:

Corollary 3.12. *For any compact group G, characters χ, $\tilde\chi$ and J small enough there is a constant $m > 0$ such that*

$$G_c(\mathbf{x}) \leq G_c(\mathbf{0})\, e^{-m\,|\mathbf{x}|}. \tag{3.157}$$

3.3 The deconfining transition in $3+1$ or more dimensions

In this section we show that, for $d+1 \geq 3+1$ and J_{E} large, there is long range order: $G_c(\mathbf{x})$ approaches some nonzero constant as $\mathbf{x}$ approaches infinity; this implies deconfinement. The proof is based on the method of *infrared bounds*, introduced by Fröhlich, Simon and Spencer [24].

In the following we take $\tilde{\chi} := \chi$, i.e. identical characters for the Polyakov loop and the Wilson action.

3.3.1 The method of infrared bounds

Consider the lattice $\mathbb{Z}^d$ and its Brillouin zone $B := [-\pi, \pi]^d$. Define the Fourier transform

$$\hat{G}(\mathbf{k}) := \sum_{\mathbf{x}} G(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.158}$$

The inverse Fourier transform $\check{}$ is then given by

$$G(\mathbf{x}) = \int_B \frac{d\mathbf{k}}{(2\pi)^d}\, \hat{G}(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{3.159}$$

The strategy is as follows. Assume the correlation function in $\mathbf{k}$-space is of the form

$$\hat{G}(\mathbf{k}) = c\,(2\pi)^d\, \delta(\mathbf{k}) + m(\mathbf{k}), \tag{3.160}$$

with m absolutely continuous. Note that $\hat{G}(\mathbf{k})$ is positive definite and therefore $c \geq 0$ and $m(\mathbf{k}) \geq 0$; we postpone this proof to the end of the section. An *infrared bound* is of the form

$$0 \leq m(\mathbf{k}) \leq \frac{D}{J\,|\mathbf{k}|^2}. \tag{3.161}$$

Integrating over $\mathbf{k}$ we get

$$G(\mathbf{0}) \leq c + \frac{D}{J}\, I(d), \qquad I(d) := \frac{1}{(2\pi)^d} \int_B \frac{d\mathbf{k}}{|\mathbf{k}|^2}. \tag{3.162}$$

If $d \geq 3$ then $I(d) < \infty$ and $m \in L^1(B)$. Thus by the Riemann-Lebesgue lemma $\check{m}(\mathbf{x}) \to 0$ for $|\mathbf{x}| \to \infty$ and we have

$$G(\mathbf{x}) \xrightarrow{|\mathbf{x}|\to\infty} c. \tag{3.163}$$

Now

$$c \geq G(\mathbf{0}) - \frac{D}{J}\, I(d), \tag{3.164}$$

is nonzero if $G(\mathbf{0}) > 0$ and for J large enough.

As explained in Section 2.7.2, for $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$ and χ the fundamental character, $G_c(\mathbf{x}) = G(\mathbf{x})$ and $c > 0$ implies long-range order and hence deconfinement.

Proof of $\hat{G} \geq 0$. For any smooth test function $\psi(\mathbf{k}) = \hat{\varphi}(\mathbf{k})$ we have ($*$ denotes convolution)

$$\begin{aligned}
\int \frac{d\mathbf{k}}{(2\pi)^d}\, \hat{G}(\mathbf{k})\, |\psi(\mathbf{k})|^2 &= \int \frac{d\mathbf{k}}{(2\pi)^d}\, \overline{\hat{\varphi}(\mathbf{k})}\, \widehat{(G * \varphi)}(\mathbf{k}) \\
&= \sum_{\mathbf{x}} \overline{\varphi(\mathbf{x})}\, (G * \varphi)(\mathbf{x}) \\
&= \sum_{\mathbf{x},\mathbf{y}} \overline{\varphi(\mathbf{x})}\, G(\mathbf{x} - \mathbf{y})\, \varphi(\mathbf{y}) \\
&= \Big\langle \Big| \sum_{\mathbf{x}} \varphi(\mathbf{x})\, P_{\mathbf{x}} \Big|^2 \Big\rangle \geq 0.
\end{aligned} \tag{3.165}$$

□

The first required bound, positivity of $G(\mathbf{0})$, follows from

3.3.2 A lower bound for $G(\mathbf{0})$

Lemma 3.13. $G(\mathbf{0}) \geq 1$.

Proof. Let $\rho : G \mapsto \mathrm{U}(n)$ be the representation corresponding to χ. Define the representation σ of G on $\mathbb{C}^n \otimes \mathbb{C}^n$ by

$$\sigma(g) := \overline{\rho(g)} \otimes \rho(g), \tag{3.166}$$

so that

$$\overline{\chi(g)}\, \chi(g) = \operatorname{tr} \sigma(g). \tag{3.167}$$

Decompose σ into irreducible representations:

$$\operatorname{tr} \sigma(g) = \sum_{\tau \in \widehat{G}} n_\tau\, \chi_\tau(g), \tag{3.168}$$

where $\widehat{G}$ denotes the set of irreducible representations of G, n_τ are nonnegative integers and χ_τ is the character of the representation τ.

Now the number n_0 of times the trivial representation occurs in σ is

$$n_0 = \int dg\, \overline{\chi_0(g)}\, \operatorname{tr} \sigma(g) = \int dg\, \overline{\chi(g)}\, \chi(g), \tag{3.169}$$

which is, by orthonormality of the characters of irreducible representations, the sum of the squares of the multiplicities of the irreducible representations in ρ and consequently at least 1. That the trivial representation occurs in σ can also be seen from the fact that the span of $\sum_i e_i \otimes e_i$, where $\{e_i\}_{i=1}^n$ is an orthonormal basis of $\mathbb{C}^n$, bears the trivial representation.

Therefore

$$\overline{\chi(g)}\, \chi(g) = 1 + \sum_{\tau \in \widehat{G}} n'_\tau\, \chi_\tau(g) \tag{3.170}$$

with $n'_\tau \geq 0$. We thus only need to show that

$$\langle \operatorname{tr} \upsilon(g_{\mathbf{0}}) \rangle \geq 0 \tag{3.171}$$

for any representation υ, where $g_{\mathbf{0}}$ denotes the product of the link variables in the Polyakov loop. Then we are done, since

$$G(\mathbf{0}) = \langle \overline{\chi(g_{\mathbf{0}})}\, \chi(g_{\mathbf{0}}) \rangle. \tag{3.172}$$

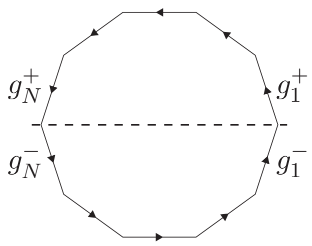

Figure 3.4: *The Polyakov loop for N_0 even.*

The bound (3.171) can be shown using reflection positivity. If $N_0 = 2N$ is even take a site-reflection about π as shown in Figure 3.4 and write

$$g_{\mathbf{0}} = g_1^- \cdots g_N^-\, g_N^+ \cdots g_1^+. \tag{3.173}$$

Then

$$\operatorname{tr} \upsilon(g_{\mathbf{0}}) = \sum_{i,j} s_{ij}(\underline{g}_+)\; \Theta s_{ij}(\underline{g}_-), \tag{3.174}$$

where

$$s_{ij}(\underline{g}_+) := \big(\upsilon(g_N^+ \cdots g_1^+)\big)_{ij}. \tag{3.175}$$

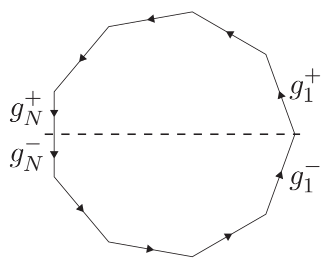

Figure 3.5: *The Polyakov loop for N_0 odd.*

If $N_0 = 2N - 1$ is odd the argument is similar (see Figure 3.5):

$$g_{\mathbf{0}} = g_1^- \cdots g_N^-\, g_N^+ \cdots g_1^+, \tag{3.176}$$

where we split the temporal link crossing π as $g_N^-\, g_N^+$. □

3.3.3 The transfer matrix

For the proof of an infrared bound we follow the appendix of [24] as well as [18].

As usual, we take a space-time lattice of the form $\Lambda = \mathbb{Z}_{N_0} \times \cdots \times \mathbb{Z}_{N_d}$ with periodic boundary conditions in all directions. The action is

$$S = -\sum_p J_p \left(\operatorname{Re}\chi(g_p) - \chi(\mathbb{1})\right), \tag{3.177}$$

where J_p is defined to be either J_{E} or J_{M}, depending on whether p is magnetic or electric.

The proof relies on the existence of a transfer matrix in a *spatial* direction. Pick a direction, say $\mu = 1$, and denote by Λ_1 the "sliced" lattice $\mathbb{Z}_{N_0} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_d}$, whose points we index using α, β, etc. We also denote by Λ_1 the set $\{b = \langle \alpha, \beta \rangle\}$ of links between sites of Λ_1, with corresponding link variables $\underline{g} = \{g_{\beta\alpha}\}$. Everything in sections 2.5 and 2.6 may then be directly taken over to this case:

Define the Hilbert space $\mathscr{H} := L^2(G^{\Lambda_1}, d\underline{g})$ of square-integrable functions of the link variables $\underline{g}$. A gauge transformation is a function $\underline{h} = \{h_\alpha\}$ of the lattice sites into G. As before we write the action of $\underline{h}$ on a field configuration $\underline{g}$ as

$$\underline{g}^{\underline{h}} := \{h_\beta^{-1}\, g_{\beta\alpha}\, h_\alpha\}. \tag{3.178}$$

This defines a unitary representation U of $\mathscr{G} := \prod_{\alpha \in \Lambda_1} G$ on $\mathscr{H}$:

$$\big(U(\underline{h})\psi\big)(\underline{g}) := \psi(\underline{g}^{\underline{h}}). \tag{3.179}$$

We are interested in the subspace $\mathscr{H}_0$ of $\mathscr{H}$ that bears the trivial representation: the space of observables. The orthogonal projection $\mathcal{P}_0$ onto $\mathscr{H}_0$ is given by [see (2.124)]

$$\mathcal{P}_0 = \int d\underline{h}\, U(\underline{h}), \tag{3.180}$$

effectively averaging over all gauges. The transfer matrix $\mathcal{T} \in \mathscr{J}_1(\mathscr{H})$ is again of the form

$$\mathcal{T} := MTM. \tag{3.181}$$

Now M is the multiplication operator by

$$m(\underline{g}) := \exp\left[\frac{1}{2} \sum_p J_p \left(\operatorname{Re}\chi(g_p) - \chi(\mathbb{1})\right)\right], \tag{3.182}$$

where the sum ranges over all positively oriented plaquettes in Λ_1, and T is the convolution operator with kernel

$$t(\underline{g}) := \exp\left[\sum_b J_b \left(\operatorname{Re}\chi(g_b) - \chi(\mathbb{1})\right)\right], \tag{3.183}$$

where the sum ranges over all positively oriented links b in Λ_1. Here J_b is equal to J_{E} if b is in the time direction and to J_{M} otherwise.

As shown in Section 2.5, $\mathcal{T}$ is self-adjoint, trace class (and thus compact) and strictly positive. Furthermore Lemma 2.3 translates to

Lemma 3.14. *The partition function for the lattice* Λ *and Wilson's action is equal to*

$$Z_\Lambda = \operatorname{tr}\left[\mathcal{P}_0\,\mathcal{T}^{N_1}\right]. \tag{3.184}$$

The proof of Lemma 3.14 makes it clear that we may also compute expected values of observables with a similar computation, provided that the observable does not depend on the links in the 1-direction. To remove the partition function from such averages, we normalise $\mathcal{T}$ such that

$$\operatorname{tr}\left[\mathcal{P}_0\,\mathcal{T}^{N_1}\right] = 1. \tag{3.185}$$

3.3.4 The infrared bound, part one

We split the spatial coordinates as $\mathbf{x} = (x^1, x')$, so that $\alpha = (t, x')$. Recall that a Polyakov loop is an observable on Λ defined as (for convenience we slightly modify our notation)

$$p(\mathbf{x}) = \chi\Big(\prod_t^{\uparrow} g^0_{(t,\mathbf{x})}\Big). \tag{3.186}$$

Choose some smearing function $h : \Lambda_1 \mapsto \mathbb{C}$, and define the Polyakov loop smeared in the transverse directions:

$$p_h(x^1) := \sum_{x'} h(x')\,p(x^1, x'). \tag{3.187}$$

This induces a smeared correlation function (for the lattice Λ)

$$G_h(x^1) := \langle\overline{p_h(x^1)}\,p_h(0)\rangle = \sum_{x',y'} \overline{h(x')}\,h(y')G_\Lambda(x^1, x' - y'). \tag{3.188}$$

Note that in close analogy to lemma 3.14 we may compute $G_h(x^1)$ using the transfer matrix:

$$G_h(x^1) = \operatorname{tr}\left[\mathcal{P}_0\,\overline{P(h)}\,\mathcal{T}^{x^1}\,P(h)\,\mathcal{T}^{N_1 - x^1}\right], \tag{3.189}$$

where $P(h)$ is the (bounded) multiplication operator by $p(h)$; $p(h)$ is the smeared Polyakov loop on Λ_1. Explicitly, we define the observable on Λ_1

$$p(x') := \chi\Big(\prod_t^{\uparrow} g^0_{(t,x')}\Big). \tag{3.190}$$

as well as its smeared value

$$p(h) := \sum_{x'} h(x')\,p(x'). \tag{3.191}$$

Useful for later computations is the fact that $\mathcal{P}_0$ "commutes with everything":

Lemma 3.15.

$$[\mathcal{P}_0, T] = [\mathcal{P}_0, M] = [\mathcal{P}_0, P(h)] = 0. \tag{3.192}$$

Proof. $\mathcal{P}_0$ commutes with $P(h)$ since $p(h)$ is gauge invariant. Everything else follows from the proof of Lemma 2.2. □

After these preparations we may do the first step of the proof:

LEMMA 3.16.

$$(1-\cos k_1)\,\hat{G}_h(k_1) \leq \operatorname{tr}\left\{\mathcal{P}_0\left[\overline{P(h)},\,[P(h),\,\mathcal{T}]\right]\mathcal{T}^{N_1-1}\right\}, \tag{3.193}$$

where $\hat{G}_h$ is the (one-dimensional discrete) Fourier transform of G_h:

$$\hat{G}_h(k_1) := \sum_{x^1=0}^{N_1-1} G_h(x^1)\,e^{ik_1x^1}, \qquad k_1 \in \frac{2\pi}{N_1}\,\mathbb{Z}. \tag{3.194}$$

For the proof we need the elementary

LEMMA 3.17. *If $0 \leq r \leq 1$ and k_1 is a multiple of $2\pi/N_1$ then*

$$(1-\cos k_1)\sum_{x^1=0}^{N_1-1}(r^{x^1}+r^{N_1-x^1})\,e^{ik_1x^1} \leq 2\big[(1+r^{N_1})-(r+r^{N_1-1})\big]. \tag{3.195}$$

PROOF. If $r=0$ or $r=1$ the proof is obvious. Let therefore $0<r<1$. Use

$$\begin{aligned}\sum_{x^1=0}^{N_1-1}(r^{x^1}+r^{N_1-x^1})\,e^{ik_1x^1} &= \frac{1-r^{N_1}e^{ik_1N_1}}{1-re^{ik_1}} + r^{N_1}\frac{1-r^{-N_1}e^{ik_1N_1}}{1-e^{ik_1}r^{-1}} \\ &= (1-r^{N_1})\left[\frac{1}{1-re^{ik_1}} - \frac{1}{1-e^{ik_1}r^{-1}}\right] \\ &= (1-r^{N_1})\frac{1-r^2}{|1-e^{ik_1}r|^2}\end{aligned} \tag{3.196}$$

and

$$(1+r)(1-r^{N_1-1}) = (1-r^{N_1})+(r-r^{N_1-1}) \geq 1-r^{N_1} \tag{3.197}$$

as well as

$$\begin{aligned}2\frac{|1-e^{ik_1}r|^2}{(1+r)^2} &= 2\frac{1+r^2}{(1+r)^2} - \frac{4r}{(1+r)^2}\cos k_1 \\ &\geq 1-\cos k_1\end{aligned} \tag{3.198}$$

to get

$$\begin{aligned}(1-\cos k_1)\sum_{x^1=0}^{N_1-1}(r^{x^1}+r^{N_1-x^1})\,e^{ik_1x^1} &\leq 2(1+r)(1-r^{N_1-1})(1+r)^{-2}(1-r^2) \\ &= 2(1-r^{N_1-1})(1-r) \\ &= 2\big[(1+r^{N_1})-(r+r^{N_1-1})\big].\end{aligned} \tag{3.199}$$

□

Proof of lemma 3.16. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of eigenvectors of $\mathcal{T}$ and $\mathcal{P}_0$ corresponding to eigenvalues λ_n of $\mathcal{T}$, ordered so that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is nonincreasing. Note that complex conjugation commutes with $\mathcal{T}$ (since m and t are real) and with $\mathcal{P}_0$ (trivial). Thus we may choose the eigenfunctions $e_n(\underline{g})$ to be real. In this basis $\mathcal{P}_0$ is given by the eigenvalues $\zeta_n = 0, 1$ and $P(h)$ by the symmetric matrix

$$c_{mn} := \langle e_m\,, P(h) e_n\rangle. \tag{3.200}$$

Then we get, using $\mathcal{P}_0^2 = \mathcal{P}_0$ and lemma 3.15,

$$\begin{aligned}
\operatorname{tr}\left[\mathcal{P}_0\,\overline{P(h)}\,\mathcal{T}^{x^1}P(h)\,\mathcal{T}^{N_1-x^1}\right] &= \operatorname{tr}\left[\mathcal{P}_0\,\overline{P(h)}\,\mathcal{P}_0\,\mathcal{T}^{x^1}P(h)\,\mathcal{T}^{N_1-x^1}\right]\\
&= \sum_{m,n}\langle e_m\,, \mathcal{P}_0\overline{P(h)}e_n\rangle\,\langle e_n\,, \mathcal{P}_0\mathcal{T}^{x^1}P(h)\mathcal{T}^{N_1-x^1}e_m\rangle\\
&= \sum_{m,n}\zeta_m\,\overline{c_{mn}}\,\zeta_n\,\lambda_n^{x^1}\,c_{nm}\,\lambda_m^{N_1-x^1}\\
&= \sum_{m,n}\zeta_m\,\zeta_n\,|c_{mn}|^2\,\lambda_n^{x^1}\,\lambda_m^{N_1-x^1}.
\end{aligned} \tag{3.201}$$

Therefore

$$\begin{aligned}
\hat{G}_h(k_1) &= \sum_{x^1=0}^{N_1-1}\operatorname{tr}\left[\mathcal{P}_0\,\overline{P(h)}\,\mathcal{T}^{x^1}P(h)\,\mathcal{T}^{N_1-x^1}\right]e^{ik_1x^1}\\
&= \sum_{x^1=0}^{N_1-1}\sum_{m\neq n}\zeta_m\,\zeta_n\,|c_{mn}|^2\,\lambda_n^{x^1}\,\lambda_m^{N_1-x^1}\,e^{ik_1x^1} + \sum_{x^1=0}^{N_1-1}\sum_{n}p_n^2\,|c_{nn}|^2\,\lambda_n^{N_1}\,e^{ik_1x^1}\\
&= \sum_{x^1=0}^{N_1-1}\sum_{m<n}\zeta_m\,\zeta_n\,|c_{mn}|^2\,\lambda_m^{N_1}\left[\left(\frac{\lambda_n}{\lambda_m}\right)^{x^1} + \left(\frac{\lambda_n}{\lambda_m}\right)^{N_1-x^1}\right]e^{ik_1x^1}.
\end{aligned} \tag{3.202}$$

Lemma 3.17 with $r = \lambda_n/\lambda_m$ therefore yields

$$\begin{aligned}
(1-\cos k_1)\,\hat{G}_h(k_1) &\leq 2\sum_{m<n}\zeta_m\,\zeta_n\,|c_{mn}|^2\,\lambda_m^{N_1}\,(1 + r^{N_1} - r - r^{N_1-1})\\
&= 2\sum_{m,n}\zeta_m\,\zeta_n\,|c_{mn}|^2\,(\lambda_m^{N_1} - \lambda_n\,\lambda_m^{N_1-1})\\
&= \sum_{m,n}\Big[2\,\zeta_m\,\overline{c_{mn}}\,\zeta_n\,c_{nm}\,\lambda_m^{N_1} - \zeta_m\,\overline{c_{mn}}\lambda_n\,\zeta_n\,c_{nm}\lambda_m^{N_1-1}\\
&\qquad\qquad - \zeta_m\,c_{mn}\lambda_n\,\zeta_n\,\overline{c_{nm}}\lambda_m^{N_1-1}\Big]\\
&= \operatorname{tr}\Big\{2\,\mathcal{P}_0\,\overline{P(h)}\,P(h)\,\mathcal{T}^{N_1} - \mathcal{P}_0\,\overline{P(h)}\,\mathcal{T}\,P(h)\,\mathcal{T}^{N_1-1}\\
&\qquad\qquad - \mathcal{P}_0\,P(h)\,\mathcal{T}\,\overline{P(h)}\,\mathcal{T}^{N_1-1}\Big\}\\
&= \operatorname{tr}\Big\{\mathcal{P}_0\,\big[\overline{P(h)},\,[P(h),\,\mathcal{T}]\big]\,\mathcal{T}^{N_1-1}\Big\}.
\end{aligned} \tag{3.203}$$

□

The rest of the proof consists in estimating $\big[\overline{P(h)},\,[P(h),\,\mathcal{T}]\big]$. This is done by considering the unitary representation matrices of g.

3.3.5 Formulation in $\mathbb{C}^{n^2}$

Since the representation ρ corresponding to χ is injective, we may as well work with the unitary matrices $u_{yx} := \rho(g_{yx}) \in \mathrm{U}(n)$ instead of the group elements g_{yx}. With this identification in mind, define the measure $d\mu(u)$ on $\mathbb{C}^{n^2}$ as the image of dg, the Haar measure on G, under ρ. Therefore

$$\int_G dg\ f(\rho(g)) = \int_{\mathbb{C}^{n^2}} d\mu(u)\ f(u). \tag{3.204}$$

Note that $d\mu(u) = du$, the Haar measure on $\mathrm{U}(n)$, if and only if ρ is surjective. So our Hilbert space is isomorphic to

$$\mathscr{H} = \bigotimes_{b\in\Lambda_1} L^2(\mathbb{C}^{n^2}, d\mu). \tag{3.205}$$

By the above identification we may immediately reformulate the transfer matrix in terms of the u-variables:

$$m(\underline{u}) = \exp\left[\frac{1}{2}\sum_p J_p\,(\mathrm{Re}\,\mathrm{tr}\,u_p - \mathrm{tr}\,\mathbb{1})\right], \tag{3.206}$$

as well as

$$t(\underline{u},\underline{u}') = \exp\left[-\sum_b \frac{J_b}{2}\,\|u_b - u'_b\|^2\right], \tag{3.207}$$

where $\|\cdot\|$ denotes the norm with respect to the Hilbert-Schmidt scalar product $\langle a\,, b\rangle := \mathrm{tr}\,a^*b$. T is now the integral operator with kernel t with respect to the measure $\prod_{b\in\Lambda_1} d\mu(u_b)$.

We may further replace $d\mu$ by the [$(2n^2)$-dimensional] Lebesgue measure $d\lambda$. To this end, we introduce a measure density function $f^2(u)$ for some $f(u) \geq 0$ and approximate $d\mu(u)$ by $f^2(u)\,d\lambda(u)$. Note that since $d\mu$ is left- and right-invariant, i.e. $d\mu(\rho(g)\,u\,\rho(g')) = d\mu(u)$, we may also choose f to be right- and left-invariant. The results for $f^2(u)\,d\lambda(u)$ then carry over to $d\mu(u)$ by a limiting procedure on f. Denote by F the multiplication operator by $\prod_{b\in\Lambda_1} f(u_b)$ and define the transfer matrix

$$\tilde{T}^f := F\tilde{M}\tilde{T}\tilde{M}F \tag{3.208}$$

on

$$\tilde{\mathscr{H}} := \bigotimes_{b\in\Lambda_1} L^2(\mathbb{C}^{n^2}, d\lambda), \tag{3.209}$$

with $\tilde{\ }$ denoting the natural extension from $\mathscr{H}$ to $\tilde{\mathscr{H}}$: the integration in $\tilde{T}$ is now performed with respect to $d\lambda$. Unsurprisingly, $\tilde{T}^f$ is self-adjoint and positive.

Proof. Self-adjointness is obvious. We only need to show that $\tilde{T}$ is positive. This follows from (write $e(u) := e^{-J_p/2\,\|u\|^2}$)

$$\int d\lambda(u)\,d\lambda(u')\ \overline{\phi(u)}e(u-u')\phi(u') = \int \frac{d\lambda(k)}{(2\pi)^{2n^2}}\ |\hat{\phi}(k)|^2\,\hat{e}(k) \geq 0. \tag{3.210}$$

□

3.3.6 The infrared bound, part two

Note that $\tilde{T}$ can be decomposed as a product:

$$\tilde{T} = \bigotimes_{b\in\Lambda_1} \tilde{T}_b, \tag{3.211}$$

where $\tilde{T}_b$ is the convolution operator with

$$t_b(u) = \exp\Big[-\frac{J_p}{2}\,\|u\|^2\Big], \tag{3.212}$$

with J_b equal to J_{E} or J_{E} depending on whether b is temporal or spatial. We separate the product (3.211) into two parts:

$$\tilde{T} = \tilde{T}_{\mathrm{M}} \otimes \tilde{T}_{\mathrm{E}}, \tag{3.213}$$

where $\tilde{T}_{\mathrm{M}}$ contains all the links in the spatial directions, and $\tilde{T}_{\mathrm{E}}$ all the temporal links. Since $p(h)$ only involves temporal links $\tilde{P}(h)$ commutes with $\tilde{T}_{\mathrm{M}}$ and

$$\big[\overline{\tilde{P}(h)},\, [\tilde{P}(h),\, \tilde{\mathcal{T}}^f]\big] = F\tilde{M}\tilde{T}_{\mathrm{M}}\big[\overline{\tilde{P}(h)},\, [\tilde{P}(h),\, \tilde{T}_{\mathrm{E}}]\big]\tilde{M}F. \tag{3.214}$$

Here, as in the following, we identify $\tilde{T}_{\mathrm{E}}$ with $\mathbb{1}\otimes\tilde{T}_{\mathrm{E}}$ and so on to keep things looking simple. Decompose further

$$\tilde{T}_{\mathrm{E}} = \bigotimes_{x'} \tilde{T}_{x'}, \tag{3.215}$$

with $\tilde{T}_{x'}$ depending on the temporal links at spatial coordinates x'. For $x' \neq y'$, $\tilde{T}_{x'}$ clearly commutes with $\tilde{P}(y')$, the multiplication operator by $p(y')$. Furthermore $\tilde{P}(x')$ commutes with $\tilde{T}_{\mathrm{E}}\tilde{T}_{x'}^{-1}$, so that we have, for $x' \neq y'$,

$$\big[\overline{\tilde{P}_{x'}}, [\tilde{P}_{y'}, \tilde{T}_{\mathrm{E}}]\big] = -[\overline{\tilde{P}_{x'}}, \tilde{T}_{x'}]\,\tilde{T}_{x'}^{-1}\,\tilde{T}_{\mathrm{E}}\,\tilde{T}_{y'}^{-1}\,[\tilde{T}_{y'}, \tilde{P}_{y'}]. \tag{3.216}$$

The occurrence of unbounded operators $\tilde{T}_{x'}^{-1}$ means that such identities are valid on a dense subspace of $\tilde{\mathscr{H}}$; since both sides are bounded, the identity may be uniquely extended to $\tilde{\mathscr{H}}$. Using (3.216) we get

$$\begin{aligned}
\big[\overline{\tilde{P}(h)},\, [\tilde{P}(h),\, \tilde{T}_{\mathrm{E}}]\big] &= \sum_{x',y'} \overline{h(x')}\,h(y')\,\big[\overline{\tilde{P}_{x'}}, [\tilde{P}_{y'}, \tilde{T}_{\mathrm{E}}]\big] \\
&= -\sum_{x',y'} \overline{h(x')}\,h(y')\,[\overline{\tilde{P}_{x'}}, \tilde{T}_{x'}]\,\tilde{T}_{x'}^{-1}\,\tilde{T}_{\mathrm{E}}\,\tilde{T}_{y'}^{-1}\,[\tilde{T}_{y'}, \tilde{P}_{y'}] \\
&\quad + \sum_{x'} |h(x')|^2 \Big([\overline{\tilde{P}_{x'}}, \tilde{T}_{x'}]\,\tilde{T}_{x'}^{-1}\,\tilde{T}_{\mathrm{E}}\,\tilde{T}_{x'}^{-1}\,[\tilde{T}_{x'}, \tilde{P}_{x'}] + \big[\overline{\tilde{P}_{x'}}, [\tilde{P}_{x'}, \tilde{T}_{\mathrm{E}}]\big]\Big) \\
&= -\Big(\sum_{x'} h(x')\,\tilde{T}_{x'}^{-1}\,[\tilde{T}_{x'}, \tilde{P}_{x'}]\Big)^{*}\,\tilde{T}_{\mathrm{E}}\,\Big(\sum_{x'} h(x')\,\tilde{T}_{x'}^{-1}\,[\tilde{T}_{x'}, \tilde{P}_{x'}]\Big) \\
&\quad + \sum_{x'} |h(x')|^2 \big(\tilde{T}_{x'}\,\overline{\tilde{P}_{x'}}\,\tilde{T}_{x'}^{-1}\,\tilde{P}_{x'}\,\tilde{T}_{\mathrm{E}} - \tilde{P}_{x}\,\tilde{T}_{\mathrm{E}}\,\overline{\tilde{P}_{x'}}\big).
\end{aligned} \tag{3.217}$$

Since $\tilde{T}_{\mathrm{E}}$ is positive, this means that

$$\big[\overline{\tilde{P}(h)},\,[\tilde{P}(h),\,\tilde{T}_{\mathrm{E}}]\big] \leq \sum_{x'} |h(x')|^2 \,\big(\tilde{T}_{\mathrm{E}}\,\overline{\tilde{P}_{x'}}\,\tilde{T}_{\mathrm{E}}^{-1}\,\tilde{P}_{x'}\,\tilde{T}_{\mathrm{E}} - \tilde{P}_x\,\tilde{T}_{\mathrm{E}}\,\overline{\tilde{P}_{x'}}\big). \tag{3.218}$$

Denote by $u_{(t,x')} := u^0_{(t,x')}$ the temporal link matrix at (t, x') and define the multiplication operator $\tilde{U}^{ab}_{(t,x')}$ as multiplication by $u^{ab}_{(t,x')}$. Then we may rewrite the above expression by expanding the matrix product in $p(x')$:

$$\begin{aligned}
&\tilde{T}_{\mathrm{E}}\,\overline{\tilde{P}_{x'}}\,\tilde{T}_{\mathrm{E}}^{-1}\,\tilde{P}_{x'}\,\tilde{T}_{\mathrm{E}} - \tilde{P}_x\,\tilde{T}_{\mathrm{E}}\,\overline{\tilde{P}_{x'}} \\
&\qquad = \sum_{\{a_t\},\{b_t\}} \prod_{t=0}^{N_0-1} \prod_{y'\neq x'} T_{(t,y')}\,\tilde{T}_{(t,x')}\,\overline{\tilde{U}^{a_{t+1}a_t}_{(t,x')}}\,\tilde{T}^{-1}_{(t,x')}\,\tilde{U}^{b_{t+1}b_t}_{(t,x')}\tilde{T}_{(t,x')} \\
&\qquad - \sum_{\{a_t\},\{b_t\}} \prod_{t=0}^{N_0-1} \prod_{y'\neq x'} T_{(t,y')}\,\tilde{U}^{a_{t+1}a_t}_{(t,x')}\,\tilde{T}_{(t,x')}\,\overline{\tilde{U}^{b_{t+1}b_t}_{(t,x')}},
\end{aligned} \tag{3.219}$$

where $\tilde{T}_{(t,x')}$ is equal to $\tilde{T}_b$ for $b = \langle (t, x'), (t + 1, x')\rangle$.

We therefore need to evaluate

$$\tilde{t}\,\overline{\tilde{U}^{aa'}}\,\tilde{t}^{-1}\,\tilde{U}^{bb'}\,\tilde{t}, \quad \text{and} \quad \tilde{U}^{bb'}\,\tilde{t}\,\overline{\tilde{U}^{aa'}} \tag{3.220}$$

on $L^2(\mathbb{C}^{2n^2}, d\lambda)$, where $\tilde{t}$ is given by convolution with

$$t_{\mathrm{E}}(u) := e^{-\frac{J_{\mathrm{E}}}{2}\,\|u\|^2}. \tag{3.221}$$

This can be done by Fourier transformation $u \mapsto k$ (this is reason for introducing $\mathscr{H}$). Then $\tilde{t}$ is multiplication by

$$e^{-\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2} \tag{3.222}$$

(any multiplicative constants will drop out of the calculation) and $\tilde{U}^{ab}$, being multiplication by $\operatorname{Re} u^{ab} + i \operatorname{Im} u^{ab}$, is given by the differential operator (defined on a suitable dense subspace)

$$\frac{1}{i}\Big(\frac{\partial}{\partial \operatorname{Re} k_{ab}} + i\frac{\partial}{\partial \operatorname{Im} k_{ab}}\Big) = \frac{2}{i}\frac{d}{d\overline{k_{ab}}}. \tag{3.223}$$

Similarly, $\overline{\tilde{U}^{ab}}$ is given by

$$\frac{2}{i}\frac{d}{dk_{ab}}. \tag{3.224}$$

Then

$$\begin{aligned}
&e^{-\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2}\,\frac{2}{i}\frac{d}{dk_{aa'}}\,e^{\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2}\,\frac{2}{i}\frac{d}{d\overline{k_{bb'}}}\,e^{-\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2} \\
&= e^{-\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2}\,\frac{2}{i}\frac{d}{dk_{aa'}}\Big(\frac{2}{i}\frac{d}{d\overline{k_{bb'}}} - \frac{2}{i}\frac{1}{2J_{\mathrm{E}}}k_{bb'}\Big) \\
&= e^{-\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2}\,\frac{2}{i}\frac{d}{dk_{aa'}}\,\frac{2}{i}\frac{d}{d\overline{k_{bb'}}} - e^{-\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2}\,\frac{2}{i}\frac{1}{2J_{\mathrm{E}}}k_{bb'}\,\frac{2}{i}\frac{d}{dk_{aa'}} + e^{-\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2}\,\frac{2}{J_{\mathrm{E}}}\,\delta_{aa'}\,\delta_{bb'} \\
&= \frac{2}{i}\frac{d}{d\overline{k_{bb'}}}\,e^{-\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2}\,\frac{2}{i}\frac{d}{dk_{aa'}} + e^{-\frac{1}{2J_{\mathrm{E}}}\,\|k\|^2}\,\frac{2}{J_{\mathrm{E}}}\,\delta_{aa'}\,\delta_{bb'}
\end{aligned} \tag{3.225}$$

implies

$$\tilde{t}\,\overline{\tilde{U}^{aa'}}\,\tilde{t}^{-1}\,\tilde{U}^{bb'}\,\tilde{t} = \tilde{U}^{bb'}\,\tilde{t}\,\overline{\tilde{U}^{aa'}} + \frac{2}{J_{\mathrm{E}}}\,\delta_{aa'}\,\delta_{bb'}\tilde{t}. \tag{3.226}$$

This can be used to expand (3.219) in powers of $\frac{2}{J_{\mathrm{E}}}$:

$$\tilde{T}_{\mathrm{E}}\,\overline{\tilde{P}_{x'}}\,\tilde{T}_{\mathrm{E}}^{-1}\,\tilde{P}_{x'}\,\tilde{T}_{\mathrm{E}} - \tilde{P}_x\,\tilde{T}_{\mathrm{E}}\,\overline{\tilde{P}_{x'}} =: \sum_{k=1}^{N_0}\left(\frac{2}{J_{\mathrm{E}}}\right)^k \tilde{A}_k. \tag{3.227}$$

Note that the constant terms cancel out and the highest power of $\frac{2}{J_{\mathrm{E}}}$ is N_0. The expansion (3.227) is essentially a binomial expansion, and can be described as follows. Define

$$\begin{aligned}\tilde{A}_0 &:= \sum_{\{a_t\},\{b_t\}}\prod_{t=0}^{N_0-1}\prod_{y'\neq x'} T_{(t,y')}\,\tilde{U}^{a_{t+1}a_t}_{(t,x')}\,\tilde{T}_{(t,x')}\,\overline{\tilde{U}^{b_{t+1}b_t}_{(t,x')}} \\ &= \sum_{\{a_t\},\{b_t\}}\prod_{t=0}^{N_0-1}\tilde{U}^{a_{t+1}a_t}_{(t,x')}\,\tilde{T}_t\,\overline{\tilde{U}^{b_{t+1}b_t}_{(t,x')}},\end{aligned} \tag{3.228}$$

with $\tilde{T}_t := \bigotimes_{x'} T_{(t,x')}$. Then $\tilde{A}_k$ is the sum of the $\binom{N_0}{k}$ terms obtained from (3.228) by replacing k of the N_0 factors $\tilde{U}^{a_{t+1}a_t}_{(t,x')}\,\tilde{T}_t\,\overline{\tilde{U}^{b_{t+1}b_t}_{(t,x')}}$ with

$$\frac{2}{J_{\mathrm{E}}}\,\delta_{a_t b_t}\,\delta_{a_{t+1}b_{t+1}}\,\tilde{T}_t. \tag{3.229}$$

Defining

$$\tilde{B}_k := F\,\tilde{M}\,\tilde{T}_{\mathrm{M}}\,\tilde{A}_k\,\tilde{M}\,F \tag{3.230}$$

we finally get ($\tilde{T}_{\mathrm{M}}$ is positive)

$$\left[\overline{\tilde{P}(h)},\,[\tilde{P}(h),\,\tilde{\mathcal{T}}^f]\right] \leq \sum_{x'}|h(x')|^2\sum_{k=1}^{N_0}\left(\frac{2}{J_{\mathrm{E}}}\right)^k\tilde{B}_k, \tag{3.231}$$

so that

$$\mathrm{tr}\left\{\tilde{\mathcal{P}}_0\left[\overline{\tilde{P}(h)},\,[\tilde{P}(h),\,\tilde{\mathcal{T}}^f]\right](\tilde{\mathcal{T}}^f)^{N_1-1}\right\} \leq \sum_{x'}|h(x')|^2\sum_{k=1}^{N_0}\left(\frac{2}{J_{\mathrm{E}}}\right)^k \mathrm{tr}\left\{\tilde{\mathcal{P}}_0\,\tilde{B}_k\,(\tilde{\mathcal{T}}^f)^{N_1-1}\right\}. \tag{3.232}$$

Since $[F,\tilde{\mathcal{P}}_0]=0$, we get in the limit $f^2(u)\,d\lambda(u)\to d\mu(u)$:

$$\mathrm{tr}\left\{\mathcal{P}_0\left[\overline{P(h)},\,[P(h),\,\mathcal{T}]\right]\mathcal{T}^{N_1-1}\right\} \leq \sum_{x'}|h(x')|^2\sum_{k=1}^{N_0}\left(\frac{2}{J_{\mathrm{E}}}\right)^k \mathrm{tr}\left\{\mathcal{P}_0\,B_k\,\mathcal{T}^{N_1-1}\right\}, \tag{3.233}$$

where

$$B_k := M\,T_{\mathrm{M}}\,A_k\,M \tag{3.234}$$

and A_k defined precisely as $\tilde{A}_k$ with all ˜'s omitted.

Decompose $\mathcal{T} = \mathcal{T}_0 \otimes \cdots \otimes \mathcal{T}_{N_0-1}$, where $\mathcal{T}_t$ only involves link variables between the time layers t and $t+1$. Then

$$\mathrm{tr}\left\{\mathcal{P}_0\, B_k\, \mathcal{T}^{N_1-1}\right\} \tag{3.235}$$

arises from

$$\begin{aligned} &\sum_{\{a_t\}\{b_t\}} \mathrm{tr}\left\{\mathcal{P}_0 \prod_{t=0}^{N_0-1} \left[U^{a_{t+1}a_t}_{(t,x')}\, \mathcal{T}_t\, \overline{U^{b_{t+1}b_t}_{(t,x')}}\right] \mathcal{T}^{N_1-1}\right\} \\ = &\sum_{\{a_t\}\{b_t\}} \mathrm{tr}\left\{\mathcal{P}_0 \prod_{t=0}^{N_0-1} \left[U^{a_{t+1}a_t}_{(t+1,x')(t,x')}\, \mathcal{T}_t\, U^{b_t b_{t+1}}_{(t,x')(t+1,x')}\right] \mathcal{T}^{N_1-1}\right\} \end{aligned} \tag{3.236}$$

by replacing exactly k of the factors $\left[U^{a_{t+1}a_t}_{(t+1,x')(t,x')}\, \mathcal{T}_t\, U^{b_t b_{t+1}}_{(t,x')(t+1,x')}\right]$ by $\left[\mathcal{T}_t\, \delta_{a_t b_t}\, \delta_{a_{t+1}b_{t+1}}\right]$ and summing over all $\binom{N_0}{k}$ possibilities.

Such nasty expressions are best studied graphically. For example, if $k = 0$, the above expression is the average of the product of two Polyakov loops in opposite directions separated in the 1-direction by one lattice spacing (see Figure 3.6).

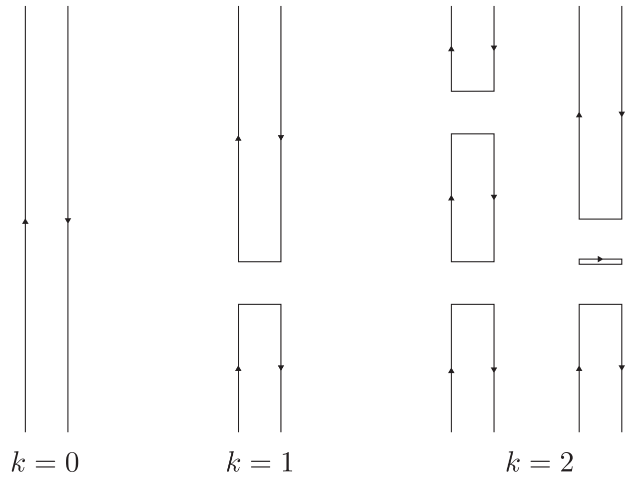

Figure 3.6: *A few examples of the breaking of two Polyakov loops. Shown are* $k = 0, 1, 2$.

Replacing a single factor at time t will connect the points $(t, 0, x')$, $(t, 1, x')$ as well as $(t+1, 0, x')$, $(t+1, 1, x')$ in the above matrix multiplication by unit matrices. Then the trace yields the expected value of the corresponding Wilson loop. This can be seen by fixing a gauge in which all links in the 1-direction are set to unity except between the layers $x^1 = 0$ and $x^1 = 1$, where integration is performed by $\mathcal{P}_0$ (by normalisation and invariance of the Haar measure).

For higher k we split in k different locations and get therefore a product of k separate Wilson loops (see Figure 3.6). All that matters to us is the ensuing bound. Since $|\chi(g)| \leq \chi(\mathbb{1})$ we get

$$\mathrm{tr}\left\{\mathcal{P}_0\, B_k\, \mathcal{T}^{N_1-1}\right\} \leq \binom{N_0}{k} \chi(\mathbb{1})^k. \tag{3.237}$$

With lemma 3.16 and (3.233) we get

$$\begin{aligned}(1-\cos k_1)\,\hat{G}_h(k_1) &\leq \sum_{x'} |h(x')|^2 \sum_{k=1}^{N_0} \left(\frac{2}{J_\mathrm{E}}\right)^k \binom{N_0}{k} \chi(\mathbb{1})^k \\ &= \sum_{x'} |h(x')|^2 \left[\left(1+\frac{2\chi(\mathbb{1})}{J_\mathrm{E}}\right)^{N_0} - 1\right] \end{aligned} \tag{3.238}$$

Now choose some k' such that $k_\mu \in \frac{2\pi}{N_\mu}\mathbb{Z}$ and define $h(x') := e^{-ik'\cdot x'}$. Then, writing $\mathbf{k} = (k_1, k')$ and $\mathbf{x} = (x^1, x')$, we have

$$G_h(x^1) = \sum_{y',z'} e^{ik'\cdot(y'-z')}\, G_\Lambda(x^1, y'-z') = \sum_{y'}\sum_{x'} e^{ik'\cdot x'}\, G_\Lambda(x^1, x') \tag{3.239}$$

and consequently

$$\hat{G}_h(k_1) = \sum_{y'}\sum_{x^1}\sum_{x'} e^{i\mathbf{k}\cdot\mathbf{x}}\, G_\Lambda(\mathbf{x}) = \sum_{y'} \hat{G}_\Lambda(\mathbf{k}), \tag{3.240}$$

where $\hat{G}_\Lambda$ is the discrete Fourier transform of G_Λ on the finite lattice Λ. So we have proven

$$(1-\cos k_1)\,\hat{G}_\Lambda(\mathbf{k}) \leq \left[\left(1+\frac{2\chi(\mathbb{1})}{J_\mathrm{E}}\right)^{N_0} - 1\right]. \tag{3.241}$$

The space direction $\mu = 1$ was arbitrary and the entire procedure can be repeated in all other directions, yielding

$$\sum_{\mu=1}^{d} (1-\cos k_\mu)\,\hat{G}_\Lambda(\mathbf{k}) \leq d\left[\left(1+\frac{2\chi(\mathbb{1})}{J_\mathrm{E}}\right)^{N_0} - 1\right]. \tag{3.242}$$

This estimate is independent of the spatial lattice dimensions and carries over to the thermodynamic limit, $\hat{G}_\Lambda(\mathbf{k})$ becoming $\hat{G}(\mathbf{k})$, the (continuous) Fourier transform of G and k varying continuously in B. Note that the factor in front of $\hat{G}(\mathbf{k})$ is of the form $|\mathbf{k}|^2$ and this is therefore the desired infrared bound. The result is summarised in

Lemma 3.18 (Infrared bound). *$\hat{G}(\mathbf{k})$ is absolutely continuous for $\mathbf{k} \neq 0$ and is bounded by*

$$0 \leq \hat{G}(\mathbf{k}) \leq d\left[\left(1+\frac{2\chi(\mathbb{1})}{J_E}\right)^{N_0} - 1\right]\left(\sum_{\mu=1}^{d}(1-\cos k_\mu)\right)^{-1}, \qquad \mathbf{k} \neq 0. \tag{3.243}$$

3.3.7 Deconfinement

We now have all that is needed to show long range order using the argument given at the beginning of the section. We know $\hat{G}(\mathbf{k})$ is of the form (3.160) with $m(\mathbf{k})$ obeying an infrared bound (3.243) for all $\mathbf{k}$.

Define

$$I(d) := \frac{1}{(2\pi)^d}\int_B d\mathbf{k}\left(\sum_{\mu=1}^{d}(1-\cos k_\mu)\right)^{-1} \tag{3.244}$$

For $d \geq 3$ $I(d)$ is finite. For example, $I(3) = 0.5054620197\ldots$ [24]. We have proven:

Theorem 3.19. *For $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$ with χ the fundamental character, the long-range order parameter*

$$c = \lim_{|\mathbf{x}|\to\infty} G(\mathbf{x}) \tag{3.245}$$

obeys

$$c \geq 1 - \left[\left(1 + \frac{2n}{J_E}\right)^{N_0} - 1\right] d\,I(d). \tag{3.246}$$

In particular, there is long-range order if

$$J_E > 2n\left[\left(1 + \frac{1}{d\,I(d)}\right)^{1/N_0} - 1\right]^{-1}. \tag{3.247}$$

We therefore have deconfinement for large J_{E} and small N_0, i.e. weak coupling and high temperature. From corollary 3.12 we know that for strong coupling there is confinement. We thus expect a phase transition from a confining phase into a deconfining phase as the coupling is decreased or the temperature increased.

4 Confinement at zero temperature

In this section we investigate confinement at zero temperature. Instead of using the Wilson loop criterion of Section 2.8 to probe confinement, we use another approach due to 't Hooft [3]. First, however, we need some formalism.

4.1 Cells, forms and duality

We expand upon the short discussion in Section 2.3.2. The idea is to introduce forms on the lattice in analogy to differential forms on the continuum. Let Λ be some sublattice of[8] $\mathbb{Z}^d$ with trivial homology, for example a finite rectangular subset or $\mathbb{Z}^d$ itself.

A *k-cell* $c_k \equiv c \subset \Lambda$ is a k-dimensional oriented hypercube with side length 1. A 0-cell is a lattice site, a 1-cell an oriented link, a 2-cell an oriented plaquette, etc. To every k-cell c corresponds a k-cell c^{-1} obtained from c by reversing its orientation. We choose a positive orientation of k-cells: a cell from each pair $\{c, c^{-1}\}$ to count each k-cell only once. Every k-cell c has a *boundary* ∂c, a set of $(k-1)$-cells.

To the lattice $\mathbb{Z}^d$ we associate the *dual lattice* $\left(\mathbb{Z}^d\right)^* := \left(\mathbb{Z} + \frac{1}{2}\right)^d$. There is then a natural bijective map $\cdot^*$ of k-cells in $\mathbb{Z}^d$ into $(d-k)$-cells in $\left(\mathbb{Z}^d\right)^*$: $c \mapsto c^*$ such that the centres of c and c^* coincide.

A *k-form* α is a function that maps k-cells into $\mathbb{Z}$ or $\mathbb{R}$, such that $\alpha(c^{-1}) = -\alpha(c)$. The set of k-forms on Λ is denoted $\Omega^k(\Lambda) \equiv \Omega^k$. On Ω^k we have a natural scalar product:

$$\langle \alpha \, , \beta \rangle := \sum_{c_k \subset \Lambda} \alpha(c_k)\, \beta(c_k), \tag{4.1}$$

with the sum ranging over all positively oriented k-cells c_k. If Λ is infinite we define $\Omega^k(\Lambda)$ to be the set of square-summable forms, so that the norm induced by $\langle \cdot \, , \cdot \rangle$ is well defined for all $\alpha \in \Omega^k(\Lambda)$.

On Ω^k we may define the *exterior derivative* $d : \Omega^k \mapsto \Omega^{k+1}$:

$$(d\alpha)(c_{k+1}) := \sum_{c_k \in \partial c_{k+1}} \alpha(c_k), \tag{4.2}$$

as well the *exterior coderivative* $\delta : \Omega^k \mapsto \Omega^{k-1}$:

$$(\delta\alpha)(c_{k-1}) := \sum_{c_k :\, c_{k-1} \in \partial c_k} \alpha(c_k). \tag{4.3}$$

A simple rearranging of terms shows that d and δ are the adjoints of each other. Furthermore $d^2 = 0$ and $\delta^2 = 0$ since each term occurs twice with opposite signs. The (negative) Laplacean is defined as

$$\Delta := \delta d + d\delta, \tag{4.4}$$

a self-adjoint positive operator.

[8] Note that from now on d stands for the dimension of space-time and not of space alone.

A k-form α is called *harmonic* if $\Delta\alpha = 0$. Note that

$$\begin{aligned}\Delta\alpha = 0 \;&\Leftrightarrow\; \langle \alpha\,, (\delta d + d\delta)\alpha \rangle = 0 \\ &\Leftrightarrow\; \langle d\alpha\,, d\alpha \rangle + \langle \delta\alpha\,, \delta\alpha \rangle = 0 \\ &\Leftrightarrow\; d\alpha = \delta a = 0. \end{aligned} \tag{4.5}$$

As in the continuous case we have

LEMMA 4.1 (Hodge decomposition). *For any k-form ω there is a $(k-1)$-form α, a $(k+1)$-form β and a harmonic k-form h such that*

$$\omega = d\alpha + \delta\beta + h. \tag{4.6}$$

PROOF. See for instance [25]. □

We shall always assume Dirichlet boundary conditions on Λ, i.e. k-forms vanish on k-cells connected to $\partial\Lambda$. Now by assumption Λ has trivial homology, which implies that there are no harmonic forms, i.e. Δ is injective. Then Lemma 4.1 immediately gives

LEMMA 4.2 (Poincaré). *For $\alpha \in \Omega^k$ we have*

$$d\alpha = 0 \quad \Longrightarrow \quad \exists \beta \in \Omega^{k-1} : \; \alpha = d\beta; \tag{4.7a}$$

$$\delta\alpha = 0 \quad \Longrightarrow \quad \exists \beta \in \Omega^{k+1} : \; \alpha = \delta\beta. \tag{4.7b}$$

If α has values in $\mathbb{Z}$, β can be chosen to also have values in $\mathbb{Z}$.

Finally, using the duality $\cdot^*$ of cells we may introduce a Hodge duality of forms:

$$* : \Omega^k \mapsto \left(\Omega^*\right)^{d-k}, \tag{4.8}$$

(Ω^* denotes forms on the dual lattice) defined by

$$*\alpha(c^*) := \alpha(c). \tag{4.9}$$

Then we have[9]

$$\delta = *d* \,. \tag{4.10}$$

4.2 Disorder observables

We assume Λ as above and finite. We take a general action depending only on the plaquette variables g_p:

$$S(\underline{g}) = \sum_{p \subset \Lambda} S_p(g_p). \tag{4.11}$$

The Wilson action is of this form, as is the Villain action introduced in the next section.

[9] Note that this relation is only correct up to a sign. Since this plays no role in the following we shall ignore it.

Take some *external flux* $\zeta \in \mathcal{Z}(G)$. We first discuss the case $d = 3$. Consider two *vortices* $x_1, x_2 \in \Lambda^*$ of the dual lattice, and some path $\gamma \subset \Lambda^*$ connecting them: $\partial\gamma = \{x_1, x_2\}$. For a link $b' \subset \Lambda^*$ define the flux variable

$$\zeta_{b'} := \begin{cases} \zeta, & b' \in \gamma \\ \mathbb{1}, & b' \notin \gamma \end{cases} . \tag{4.12}$$

Define the perturbed partition function

$$Z_\Lambda(\zeta, \{x_1, x_2\}) := \int d\underline{g} \prod_p e^{-S_p(g_p \zeta_{p^*})}, \tag{4.13}$$

which is the unperturbed partition function

$$Z_\Lambda := \int d\underline{g} \prod_p e^{-S_p(g_p)} \tag{4.14}$$

with an added external flux along the *flux tube* γ connecting x_1 and x_2. We are interested in the *monopole variable*

$$D_\Lambda(\zeta, \{x_1, x_2\}) := \frac{Z_\Lambda(\zeta, \{x_1, x_2\})}{Z_\Lambda}. \tag{4.15}$$

That $D_\Lambda(\zeta, \{x_1, x_2\})$ only depends on $\partial\gamma$ and not γ itself follows from the fact that we may find a transformation of the link variables that leaves (4.13) invariant but shifts the path γ. Take a function $\{\alpha_b\}$ of the links with $\alpha_b \in \{\mathbb{1}, \zeta\}$. Then (4.13) is invariant under the substitution $g_b \mapsto \alpha_b\, g_b$. This, however, has the effect of shifting γ whenever $\alpha_b \neq \mathbb{1}$ (see Figure 4.1). The following calculations will give us an explicitly γ-independent expression for $D_\Lambda(\zeta, \{x_1, x_2\})$.

The 4-dimensional case is similar: Consider a loop $\mathscr{L} \subset \Lambda^*$ in the dual lattice as well as some surface $\Sigma \subset \Lambda^*$ such that $\partial\Sigma = \mathscr{L}$. For a plaquette $p' \subset \Lambda^*$ define the flux variable

$$\zeta_{p'} := \begin{cases} \zeta, & p' \in \Sigma \\ \mathbb{1}, & p' \notin \Sigma \end{cases} . \tag{4.16}$$

As above we define a perturbed partition function $Z_\Lambda(\zeta, \mathscr{L})$ and call

$$D_\Lambda(\zeta, \mathscr{L}) := \frac{Z_\Lambda(\zeta, \mathscr{L})}{Z_\Lambda}, \tag{4.17}$$

a *'t Hooft loop*. Again, by redistributing the perturbation ζ_{p^*} across the link variables, we see that $D_\Lambda(\zeta, \mathscr{L})$ depends only on $\partial\Sigma = \mathscr{L}$ and not Σ itself.

As always, we are interested in the thermodynamic limit

$$D := \lim_{\Lambda \to \mathbb{Z}^d} D_\Lambda. \tag{4.18}$$

't Hooft [3] interpreted the disorder observables as describing the potential between monopoles and antimonopoles (in analogy to the Wilson loop that relates to the potential between static charges). The details are discussed in [18, 16]. In the following we show that the disorder observables obey a perimeter law for $G = \mathrm{U}(1)$ (Section 4.3) and all couplings, as well as for any compact group G if the coupling is strong enough (Section 4.4). This expresses deconfinement of monopoles.

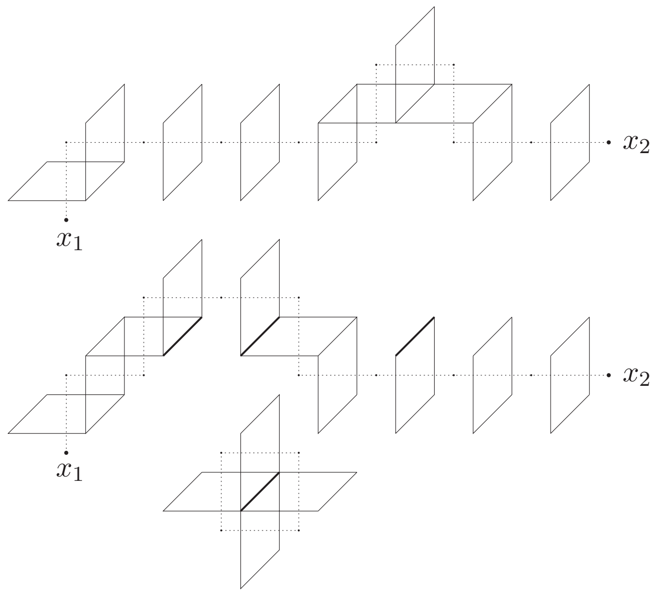

Figure 4.1: *A flux tube connecting x_1 and x_2. Shown above is some path γ and below another path γ' obtained from γ by setting $\alpha_b = \zeta$ for the bold links b.*

4.3 Perimeter law in the Abelian case

We now move onto proving the main results of this section: $D(\zeta, \partial\Sigma) \geq e^{-c\,J\,|\partial\Sigma|}$, in 3 ($\Sigma$ is a path) and 4 dimensions (Σ is a surface), i.e. the disorder observables obey a perimeter law. We first prove this for all J for the Abelian case. In a second stage we prove it for small J and any group G using a high-temperature expansion.

First consider the case $G = \mathrm{U}(1)$. As usual we parametrise $g \in G$ as $g = e^{i\theta}$, $\theta \in \mathbb{R}$. The proofs in 3 and 4 dimensions are almost identical: We perform a *duality transform*, a Fourier transform in the plaquette variables, to reformulate the problem on the dual lattice. Using a *correlation inequality* we may find a lower bound that can be computed using Gaussian integration and gives the desired perimeter decay.

4.3.1 The Villain action

The following calculations are easier if instead of the Wilson action we use the *Villain action*, also called the *heat kernel action*. Define the heat kernel

$$k(\theta) := \sum_{n\in\mathbb{Z}} e^{-\frac{J}{2}(\theta+2\pi n)^2}. \tag{4.19}$$

Then the Villain action is defined as

$$S(\underline{\theta}) := -\sum_{p\subset\Lambda} \log k(\theta_p). \tag{4.20}$$

To study the continuum limit we take a lattice spacing ε in all directions. Then we have

Lemma 4.3. *In the continuum limit $\varepsilon \to 0$ the Villain action becomes the Yang-Mills action* (2.58)*, whenever the lattice gauge field $\{\theta_x^\mu\}$ arises from a smooth continuum gauge field A.*

Proof. The Villain action is equal (up to an irrelevant constant) to

$$-\sum_p \log \frac{k(\theta_p)}{k(0)} = -\sum_p \log\Big(1 + \frac{k(\theta_p) - k(0)}{k(0)}\Big). \tag{4.21}$$

Now $k'(0) = 0$ so that the above expression is equal to

$$-\sum_p \log\Big(1 + \frac{k''(0)}{2k(0)}\theta_p^2 + \mathcal{O}(\theta_p^3)\Big) = -\sum_p \frac{k''(0)}{2k(0)}\theta_p^2 + \mathcal{O}(\theta_p^3) \tag{4.22}$$

From (2.65) we get, for $p = p_x^{\mu\nu}$,

$$\theta_p = i\varepsilon^2 F_{\mu\nu}(x) + \mathcal{O}(\varepsilon^3). \tag{4.23}$$

$S(\underline{\theta})$ then becomes

$$\sum_{x,\,\mu<\nu} \frac{k''(0)}{2k(0)}\,\varepsilon^4 F_{\mu\nu}^2(x) + \mathcal{O}(\varepsilon^6) = \varepsilon^d \sum_x \sum_{\mu<\nu} \varepsilon^{4-d}\,\frac{k''(0)}{2k(0)}\,F_{\mu\nu}^2(x) + \mathcal{O}(\varepsilon^6) \tag{4.24}$$

which is, in leading order of ε, the Riemann sum approximation of

$$\int dx \sum_{\mu<\nu} \varepsilon^{4-d}\,\frac{k''(0)}{2k(0)} F_{\mu\nu}^2(x). \tag{4.25}$$

In other words, we recover the Yang-Mills action if we require

$$-\frac{k''(0)}{k(0)} = \varepsilon^{d-4}\,\frac{1}{g^2}. \tag{4.26}$$

This needs to be solved in J. Using

$$k''(0) = J^2 \sum_{n\in\mathbb{Z}} (2\pi n)^2\, e^{-\frac{J}{2}(2\pi n)^2} - J \sum_{n\in\mathbb{Z}} e^{-\frac{J}{2}(2\pi n)^2} \tag{4.27}$$

we have

$$h(J) := -\frac{k''(0)}{k(0)} = J\left[1 - J\,\frac{\sum_n (2\pi n)^2\, e^{-\frac{J}{2}(2\pi n)^2}}{\sum_n e^{-\frac{J}{2}(2\pi n)^2}}\right]. \tag{4.28}$$

It suffices to study the asymptotic behaviour of (4.28) (see Figure 4.2). For $J \to 0$ the quotient of the sums approaches

$$\frac{\int ds\; s^2\, e^{-\frac{J}{2}s^2}}{\int ds\; e^{-\frac{J}{2}s^2}} = \frac{1}{J} \tag{4.29}$$

so that $h(J) \xrightarrow{J\to 0} 0$ at least quadratically. On the other hand for $J \to \infty$ the same quotient clearly approaches 0 as the term $n = 0$ dominates; therefore $h(J) \sim J$ for $J \to \infty$. We conclude that (4.26) has a solution for all g. □

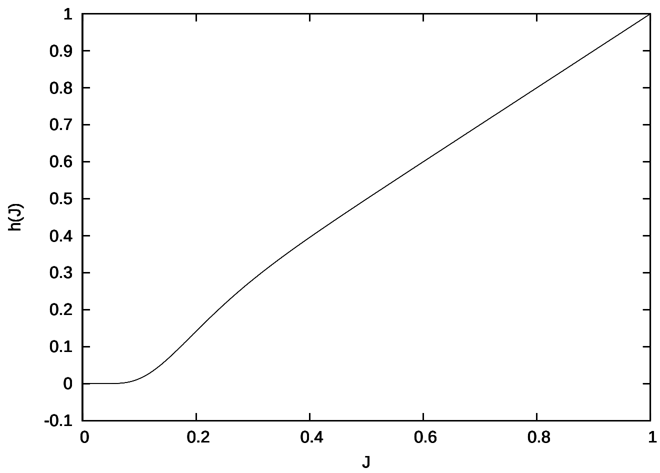

Figure 4.2: *The function $h(J)$.*

4.3.2 Three dimensions: a monopole variable

Let the external flux be $\varphi \in [0, 2\pi)$. Choose $\{x_1, x_2\} = \partial\gamma \subset \Lambda$. Note that the plaquette variables θ_p are the exterior derivative of the link variables: $\theta_p = (d\theta)_p$. Then the perturbed partition function (4.13) is

$$Z_\Lambda(\varphi, \{x_1, x_2\}) = \int d\underline{\theta} \prod_p e^{-S_p\left((d\theta)_p + \varphi_{p^*}\right)}. \tag{4.30}$$

We use the Villain action (4.20), so that

$$Z_\Lambda(\varphi, \{x_1, x_2\}) = \int d\underline{\theta} \prod_p k\big((d\theta)_p + \varphi_{p^*}\big) = \int d\underline{\theta} \prod_p \sum_{n\in\mathbb{Z}} e^{-\frac{J}{2}\left((d\theta)_p + \varphi_{p^*} + 2\pi n\right)^2}. \tag{4.31}$$

Using the Fourier representation of the heat kernel (Poisson resummation formula)

$$\sum_{n\in\mathbb{Z}} e^{-\frac{J}{2}(\theta+2\pi n)^2} = \sqrt{\frac{2\pi}{J}} \sum_{f\in\mathbb{Z}} e^{-\frac{1}{2J}f^2} e^{if\theta} \tag{4.32}$$

$Z_\Lambda(\varphi, \{x_1, x_2\})$ is equal to (up to an irrelevant constant)

$$\begin{aligned} &\int d\underline{\theta} \prod_p \sum_{f\in\mathbb{Z}} e^{-\frac{1}{2J}f^2} e^{if(d\theta)_p} e^{if\varphi_{p^*}} \\ &= \sum_{\{f_p\in\mathbb{Z}\}} \left[\prod_p e^{-\frac{1}{2J}f_p^2} e^{if_p\varphi_{p^*}}\right] \int d\underline{\theta} \prod_p e^{if_p(d\theta)_p}. \end{aligned} \tag{4.33}$$

The integral is equal to

$$\int d\underline{\theta}\, e^{i\langle f, d\theta\rangle} = \int d\underline{\theta}\, e^{i\langle \delta f, \theta\rangle} = \prod_b \int d\theta_b\, e^{i(\delta f)_b\theta_b} = \prod_b \delta\big((\delta f)_b\big), \tag{4.34}$$

where $\delta : \mathbb{Z} \mapsto \{0, 1\}$ is the discrete (Kronecker) delta function, so that $Z_\Lambda(\varphi, \{x_1, x_2\})$ is equal to

$$\sum_{f,\,\delta f=0} \Big[\prod_p e^{-\frac{1}{2J} f_p^2} e^{i f_p \varphi_{p^*}}\Big]. \tag{4.35}$$

Now by the Poincaré lemma $\delta f = 0$ implies that there is a 3-form c such that $f = \delta c$, i.e.

$$*f = *\delta * a = da, \tag{4.36}$$

with $*a := c$, a being a 0-form on Λ^*. Therefore

$$f_p = (*f)_{p^*} = (da)_{p^*}. \tag{4.37}$$

Now d is a one-to-one map between $\mathbb{Z}$-valued 0-forms a and $\mathbb{Z}$-valued 1-forms g such that $dg = 0$. Indeed the Poincaré lemma implies surjectivity. Furthermore, $da = 0$ implies $a = \text{const.} = 0$ because of the boundary conditions, so that d is also injective in this case. We may therefore rewrite the sum as

$$\sum_a e^{-\frac{1}{2J}\langle da\,,da\rangle} e^{i\varphi\langle \eta\,,da\rangle} \tag{4.38}$$

where summation ranges over all $\mathbb{Z}$-valued 0-forms a, and η is the 1-form on Λ^* given by

$$\eta_b := \begin{cases} 1, & b \in \gamma \\ 0, & b \notin \gamma \end{cases}. \tag{4.39}$$

(γ is oriented from x_1 to x_2.) Therefore

$$D_\Lambda(\varphi, \{x_1, x_2\}) = \frac{1}{Z_\Lambda} \sum_a e^{-\frac{1}{2J}\langle da\,,da\rangle} e^{i\varphi\langle \xi\,,a\rangle}, \tag{4.40}$$

with

$$Z_\Lambda = \sum_a e^{-\frac{1}{2J}\langle da\,,da\rangle} \tag{4.41}$$

and $\xi := \delta\eta$. Explicitly,

$$\xi_x = \begin{cases} 1, & x = x_1 \\ -1, & x = x_2 \\ 0, & \text{otherwise} \end{cases}, \tag{4.42}$$

so that $D_\Lambda(\varphi, \{x_1, x_2\})$ manifestly only depends on $\{x_1, x_2\}$.

The sum in (4.40) looks like a Gaussian integral. We now show that if we replace the sum with an integral, we obtain a lower bound. To this end, we approximate the sum in (4.40) by a finite measure σ_μ defined on $\mathbb{R}^{|\Lambda^*|}$ by

$$d\sigma_\mu(\underline{a}) := e^{-\frac{1}{2J}\langle da\,,da\rangle} e^{\mu \sum_x \cos(2\pi a_x)} \prod_x da_x. \tag{4.43}$$

We denote by $\langle\cdot\rangle_\mu$ the average with respect to σ_μ. Then we have

$$D_\Lambda(\varphi, \{x_1, x_2\}) = \lim_{\mu\to\infty} \langle e^{i\varphi\langle \xi\,,a\rangle}\rangle_\mu. \tag{4.44}$$

We have therefore transformed the original compact lattice gauge theory into a non-compact spin system on the dual lattice.

Since the measure σ_μ is invariant under the transformation $a \mapsto -a$ we have

$$\langle e^{i\varphi\langle \xi , a\rangle}\rangle_\mu = \big\langle \cos\langle \varphi\, \xi , a\rangle \big\rangle_\mu \tag{4.45}$$

Now

$$\begin{aligned} \frac{d}{d\mu} \big\langle \cos\langle \varphi\, \xi , a\rangle \big\rangle_\mu &= \frac{d}{d\mu} \frac{\int \prod_x da_x\, e^{-\frac{1}{2J}\langle da , da\rangle}\, e^{\mu \sum_x \cos(2\pi a_x)} \cos\langle \varphi\, \xi , a\rangle}{\int \prod_x da_x\, e^{-\frac{1}{2J}\langle da , da\rangle}\, e^{\mu \sum_x \cos(2\pi a_x)}} \\ &= \sum_x \Big[\big\langle \cos(2\pi a_x)\, \cos\langle \varphi\, \xi , a\rangle \big\rangle_\mu - \langle \cos(2\pi a_x)\rangle_\mu \big\langle \cos\langle \varphi\, \xi , a\rangle \big\rangle_\mu \Big] \\ &= \sum_x \big\langle \cos(2\pi a_x)\, ;\, \cos\langle \varphi\, \xi , a\rangle \big\rangle_\mu, \end{aligned} \tag{4.46}$$

with the covariance

$$\langle a\, ;\, b\rangle := \langle ab\rangle - \langle a\rangle\langle b\rangle. \tag{4.47}$$

To show that the above expression is nonnegative we use a correlation inequality, following [26]. With the notation $f\cdot a := \langle f , a\rangle = \sum_x f_x\, a_x$ we have

Lemma 4.4 (Correlation inequality). *For any $f, g : \Lambda^* \mapsto \mathbb{R}$*

$$\langle \cos f\cdot a\, ;\, \cos g\cdot a\rangle_\mu \geq 0. \tag{4.48}$$

Proof. We introduce the identically distributed duplicate random variables $a = \{a_x\}$, $a' = \{a'_x\}$, both distributed according to σ_μ (normalised to a probability measure). We also denote the product expectation $\langle\cdot\rangle \otimes \langle\cdot\rangle'$ by $\langle\cdot\rangle$ (from now on we drop the index μ). The average $\langle\cdot\rangle$ is given by the measure

$$e^{\mu \sum_x \cos(2\pi a_x)}\, dG_C(a), \tag{4.49}$$

where

$$dG_C(a) = e^{-\frac{1}{2J}\langle da , da\rangle} \prod_x da_x \tag{4.50}$$

is a Gaussian measure with some covariance matrix C.

Define new variables $(a, a') \mapsto (\alpha, \beta)$ as

$$a_x = \alpha_x - \beta_x \tag{4.51a}$$

$$a'_x = \alpha_x + \beta_x. \tag{4.51b}$$

The transformation (4.51) is a combination of a rotation and a dilatation by a factor $2^{-1/2}$, so that

$$dG_C(a)\, dG_C(a') = dG_{C/2}(\alpha)\, dG_{C/2}(\beta). \tag{4.52}$$

Then

$$\begin{aligned}\langle \cos f\cdot a\,;\,\cos g\cdot a\rangle &= \big\langle \cos f\cdot a\,\big(\cos g\cdot a - \cos g\cdot a'\big)\big\rangle \\ &= \int dG_C(a)\,dG_C(a')\,\prod_x \Big[e^{\mu\cos(2\pi a_x)+\mu\cos(2\pi a'_x)}\Big] \\ &\quad \times \cos f\cdot a\,\big(\cos g\cdot a - \cos g\cdot a'\big) \\ &= \int dG_{C/2}(\alpha)\,dG_{C/2}(\beta)\,\prod_x \Big[e^{\mu\cos 2\pi(\alpha_x-\beta_x)+\mu\cos 2\pi(\alpha_x+\beta_x)}\Big] \\ &\quad \times \cos f\cdot(\alpha-\beta)\,\big(\cos g\cdot(\alpha-\beta) - \cos g\cdot(\alpha+\beta)\big).\end{aligned} \tag{4.53}$$

Using

$$\cos(u\pm v) = \cos u\cos v \mp \sin u\sin v \tag{4.54}$$

we expand the last line of (4.53):

$$\begin{aligned}&\cos f\cdot(\alpha-\beta)\,\big(\cos g\cdot(\alpha-\beta) - \cos g\cdot(\alpha+\beta)\big) \\ &= (\cos f\cdot\alpha\,\cos f\cdot\beta + \sin f\cdot\alpha\,\sin f\cdot\beta)\,(\sin g\cdot\alpha\sin g\cdot\beta + \sin g\cdot\alpha\sin g\cdot\beta) \\ &= 2\cos f\cdot\alpha\,\sin g\cdot\alpha\,\cos f\cdot\beta\,\sin g\cdot\beta + 2\sin f\cdot\alpha\,\sin g\cdot\alpha\,\sin f\cdot\beta\,\sin g\cdot\beta\end{aligned} \tag{4.55}$$

which is of a linear combination with positive coefficients of expressions of the form $F(\alpha)\,F(\beta)$. The exponential factor in (4.53) is

$$e^{\mu\cos 2\pi(\alpha_x-\beta_x)+\mu\cos 2\pi(\alpha_x+\beta_x)} = e^{2\mu cos(2\pi\alpha_x)\,\cos(2\pi\beta_x)}, \tag{4.56}$$

which yields, after expansion, a linear combination with positive coefficients of terms

$$\cos^k(2\pi\alpha_x)\,\cos^k(2\pi\beta_x), \tag{4.57}$$

which is again of the form $F(\alpha)\,F(\beta)$. Therefore (4.53) is a linear combination with positive coefficients of terms of the form

$$\int dG_{C/2}(\alpha)\,dG_{C/2}(\beta)\;F(\alpha)\,F(\beta) \geq 0. \tag{4.58}$$

□

The lemma implies

$$\frac{d}{d\mu}\,\big\langle \cos\langle\varphi\,\xi\,,a\rangle\big\rangle_\mu \geq 0, \tag{4.59}$$

so that

$$\lim_{\mu\to\infty}\big\langle \cos\langle\varphi\,\xi\,,a\rangle\big\rangle_\mu \geq \big\langle \cos\langle\varphi\,\xi\,,a\rangle\big\rangle_{0<\mu<\infty} \geq \big\langle \cos\langle\varphi\,\xi\,,a\rangle\big\rangle_{\mu=0}, \tag{4.60}$$

i.e.

$$\begin{aligned}D_\Lambda(\varphi,\{x_1,x_2\}) &\geq \frac{1}{Z_\Lambda}\int\prod_x da_x\; e^{-\frac{1}{2J}\,\langle da\,,da\rangle}\;e^{\langle\varphi\,\xi\,,a\rangle} \\ &= \frac{1}{Z_\Lambda}\int\prod_x da_x\; e^{-\frac{1}{2J}\,\langle a\,,K_\Lambda a\rangle}\;e^{\langle\varphi\,\xi\,,a\rangle},\end{aligned} \tag{4.61}$$

where Z_Λ is chosen so that the integrating measure has volume 1 and $K_\Lambda := \delta d$. This is simply a Gaussian integral on $\Omega^0(\Lambda^*)$. Since $\delta = 0$ on Ω^0 we have $D_\Lambda = \Delta$. Note that Δ is the (negative) discrete Laplacean:

$$(Ka)_x = \sum_{b:\, x\in b} (da)_b = \sum_{b:\, x\in b} \sum_{y\in b} a_y = 6a_x - \sum_{\mu=1}^{3} (a_{x+\hat\mu} + a_{x-\hat\mu}). \tag{4.62}$$

The Gaussian integral may be easily computed using the standard result[10]

$$\frac{\int dx\; e^{-\frac{1}{2}\langle x, Ax\rangle}\; e^{i\langle v, x\rangle}}{\int dx\; e^{-\frac{1}{2}\langle x, Ax\rangle}} = e^{-\frac{1}{2}\langle v, A^{-1}v\rangle}, \qquad x, v \in \mathbb{R}^n,\ A \in \mathrm{Gl}_n(\mathbb{R}). \tag{4.63}$$

Therefore

$$D_\Lambda(\varphi, \{x_1, x_2\}) \geq e^{-J\varphi^2/2\langle \xi, K_\Lambda^{-1}\xi\rangle}. \tag{4.64}$$

We now take the thermodynamic limit $\Lambda \to \mathbb{Z}^d$:

$$D(\varphi, \{x_1, x_2\}) \geq e^{-J\varphi^2/2\langle \xi, K^{-1}\xi\rangle} \tag{4.65}$$

where K is the Laplacean on $\Omega^0\left[\left(\mathbb{Z}^d\right)^*\right]$.

Our task is to compute K^{-1}. In Fourier space K^{-1} is given by multiplication with a function $\hat{G}(k)$, so that K^{-1} in the original space is a convolution with the Green function $G(x)$. All that we need is the fact that $G(x)$ is bounded, and in the limit $|x| \to \infty$ $G(x)$ becomes the Green function $1/|x|$ of the continuous Laplacean. Therefore

$$G(x) \leq \frac{c}{1+|x|} \tag{4.66}$$

for some constant $c > 0$. Then

$$\begin{aligned}\langle \xi, K^{-1}\xi\rangle &= |\langle \xi, K^{-1}\xi\rangle| \\ &= \left|\xi(x_1)^2 G(0) + \xi(x_2)^2 G(0) + 2\xi(x_1)\xi(x_2)G(x_1 - x_2)\right| \\ &\leq 2c\left(1 + \frac{1}{1+|x_1 - x_2|}\right).\end{aligned} \tag{4.67}$$

We have proven:

Theorem 4.5. *In 3 dimensions the monopole variable obeys a perimeter law: There is a constant c such that*

$$\begin{aligned}D(\varphi, \{x_1, x_2\}) &\geq e^{-cJ\varphi^2\left(1+\frac{1}{1+|x_1-x_2|}\right)} \\ &\geq e^{-2cJ\varphi^2}.\end{aligned} \tag{4.68}$$

[10] To derive this complete the squares in the exponential and use use Cauchy's theorem to shift the integration back to the real axis.

4.3.3 Four dimensions: a 't Hooft loop

The treatment of the 't Hooft loop in 4 dimensions is similar to the monopole variable in 3 dimensions. Take a loop $\mathscr{L} = \partial\Sigma \subset \Lambda^*$ that is the boundary of some surface Σ. The beginning of the calculation can be directly taken over from the previous section:

$$Z_\Lambda(\varphi, \mathscr{L}) = \sum_{f,\,\delta f=0} \Big[\prod_p e^{-\frac{1}{2J} f_p^2} e^{i f_p \varphi_{p^*}}\Big], \tag{4.69}$$

with the flux now going through a surface:

$$\varphi_{p'} := \begin{cases} \varphi, & p' \in \Sigma \\ 0, & p' \notin \Sigma \end{cases} . \tag{4.70}$$

We again use Poincaré's lemma: $\delta f = 0$ implies that there is a 3-form c such that $f = \delta c$, i.e.

$$*f = *\delta * a = da, \tag{4.71}$$

with $*a := c$, a being a 1-form on Λ^*. Therefore

$$f_p = (*f)_{p^*} = (da)_{p^*}. \tag{4.72}$$

In order to find a one-to-one correspondence between the $\mathbb{Z}$-valued forms $\{a \in \Omega^1(\Lambda^*)\}$ and $\{g = *f \in \Omega^2(\Lambda^*) \; : \; \delta f = 0\}$ we choose a maximal tree $\mathscr{T}$ on $\left(\mathbb{Z}^d\right)^*$ such that

$$\mathscr{L} \cap \mathscr{T} = \emptyset. \tag{4.73}$$

Then the map

$$d : \{a \in \Omega^1(\Lambda^*) \; : \; a_{\mathscr{T}} = 0\} \longmapsto \{g = *f \in \Omega^2(\Lambda^*) \; : \; \delta f = 0\} \tag{4.74}$$

is bijective ($a_{\mathscr{T}}$ denotes the link variables $\{a_b \; : \; b \in \mathscr{T} \cap \Lambda\}$). We may thus rewrite

$$Z_\Lambda(\varphi, \mathscr{L}) = \sum_{a,\, a_{\mathscr{T}}=0} e^{-\frac{1}{2J}\langle da\,, da\rangle} e^{i\varphi\langle \eta\,, da\rangle}, \tag{4.75}$$

where $\eta \in \Omega^2(\Lambda^*)$ given by

$$\eta_p := \begin{cases} 1, & p \in \Sigma \\ 0, & p \notin \Sigma \end{cases} . \tag{4.76}$$

Therefore

$$D_\Lambda(\varphi, \mathscr{L}) = \frac{1}{Z_\Lambda} \sum_{a,\, a_{\mathscr{T}}=0} e^{-\frac{1}{2J}\langle da\,, da\rangle} e^{i\varphi\langle \xi\,, a\rangle}, \tag{4.77}$$

where $\xi := \delta\eta$. Explicitly,

$$\xi_b = \begin{cases} 1, & b \in \mathscr{L} \\ 0, & b \notin \mathscr{L} \end{cases} , \tag{4.78}$$

so that $D_\Lambda(\varphi, \mathscr{L})$ manifestly only depends on $\mathscr{L}$. Again we transform the sum into an integral by a limiting procedure and use the fact that the integrating measure is even in a_b:

$$D_\Lambda(\varphi, \mathscr{L}) = \lim_{\mu\to\infty} \frac{1}{Z_{\Lambda,\mu}} \int \Big[\prod_b da_b\, e^{\mu\cos(2\pi a_b)}\Big] \prod_{b\in\mathscr{T}} \delta(a_b)\, e^{-\frac{1}{2J}\langle da\,, da\rangle} \cos\langle \varphi\xi\,, a\rangle. \tag{4.79}$$

Although the approach of the previous section can be directly applied to this measure to find a lower bound in terms of a Gaussian integral, we take here a longer way round in order to reveal the similarity between the above model and a noncompact Higgs model. In a *Higgs model* we have a bosonic Higgs field $\Phi : x \mapsto \Phi_x \in \mathbb{C}$ and an Abelian gauge field $a : b \mapsto a_b \in \mathbb{R}$. It is convenient to write $\Phi_x =: r_x e^{\theta_x}$, $r_x \geq 0$, $\theta_x \in [-\pi, \pi)$, in polar coordinates.

The action

$$S(a, \Phi) = S_{\text{gauge}}(a) + S_{\text{Higgs}}(\Phi) + S_{\text{int}}(a, \Phi) \tag{4.80}$$

is composed of the free actions of both fields as well as an interaction component S_{int}. The action of the gauge field is

$$S_{\text{gauge}}(a) := \frac{1}{2J} \langle da \, , da \rangle. \tag{4.81}$$

Interpreting $e^{i2\pi a_{yx}}$ as the parallel transporter between the Φ-fibres over the points x and y we define a minimal coupling interaction action

$$\begin{aligned} \frac{1}{2} \sum_{\langle xy \rangle} \left| e^{-i2\pi a_{yx}} \Phi_y - \Phi_x \right|^2 &= \frac{1}{2} \sum_{\langle xy \rangle} \left(r_x^2 + r_y^2 - 2 r_x r_y \cos(\theta_y - 2\pi a_{yx} - \theta_x) \right) \\ &= 2 \sum_x r_x^2 + S_{\text{int}}(a, \Phi), \end{aligned} \tag{4.82}$$

where

$$\begin{aligned} S_{\text{int}}(a, \Phi) &:= - \sum_{\langle xy \rangle} r_x r_y \cos(\theta_y - 2\pi a_{yx} - \theta_x) \\ &= - \sum_{\langle xy \rangle} r_x r_y \cos(-\theta_x + 2\pi a_{xy} + \theta_y) \end{aligned} \tag{4.83}$$

is the interaction action. The remaining term is absorbed in the as yet undetermined self-interaction $S_{\text{Higgs}}(\Phi)$. The measure corresponding to the action S is then

$$d\sigma(a, \Phi) = e^{-\frac{1}{2J} \langle da \, , da \rangle} \, e^{\sum_{\langle xy \rangle} r_x r_y \cos(-\theta_x + 2\pi a_{xy} + \theta_y)} \prod_{b \in \mathscr{T}} \delta(a_b) \prod_b da_b \prod_x d\theta_x \, d\tilde{\rho}_x(r_x), \tag{4.84}$$

with some measure $d\tilde{\rho}_x$ arising from the self-interaction of the Higgs field. If we choose $d\tilde{\rho}_x(r) = \delta(r - \sqrt{\mu})$ and denote the corresponding expectation under $d\sigma$ by $\langle \cdot \rangle_\mu$ we have, by gauge invariance (i.e. the variable transformation $a \mapsto a - d\theta$),

$$\begin{aligned} \big\langle \cos\langle \varphi\xi \, , a \rangle \big\rangle_\mu = \; & \frac{1}{Z_{\Lambda,\mu}} \int \prod_b da_b \prod_x d\theta_x \prod_{b \in \mathscr{T}} \delta(a_b) \, e^{-\frac{1}{2J} \langle da \, , da \rangle} \\ & \qquad \times \, e^{\mu \sum_{\langle xy \rangle} \cos(-\theta_x + 2\pi a_{xy} + \theta_y)} \cos\langle \varphi\xi \, , a \rangle \\ = \; & \frac{1}{Z_{\Lambda,\mu}} \int \prod_b da_b \prod_{b \in \mathscr{T}} \delta(a_b) \, e^{-\frac{1}{2J} \langle da \, , da \rangle} \, e^{\mu \sum_b \cos(2\pi a_b)} \cos\langle \varphi\xi \, , a \rangle, \end{aligned} \tag{4.85}$$

since $\delta\xi = 0$, so that

$$D_\Lambda(\varphi, \mathscr{L}) = \lim_{\mu \to \infty} \langle \cos\langle \varphi\xi \, , a \rangle \rangle_\mu. \tag{4.86}$$

Via the duality transform we have therefore transformed the compact lattice gauge theory into a noncompact Higgs model on the dual lattice; the 't Hooft loop became a Wilson loop.

As in the previous section we have

$$\frac{d}{d\mu}\big\langle \cos\langle\varphi\,\xi\,,a\rangle\big\rangle_\mu = \sum_{\langle xy\rangle}\big\langle \cos(-\theta_x + 2\pi a_{xy} + \theta_y)\,;\, \cos\langle\varphi\,\xi\,,a\rangle\big\rangle_\mu. \tag{4.87}$$

We prove the required correlation inequality in the more general context of the Higgs model, following [26]. For functions $f_b \in \mathbb{R}$ and $n_x \in \mathbb{Z}$ write $f\cdot a := \sum_b f_b a_b$ and $n\cdot\theta := \sum_x n_x\theta_x$. Then

Lemma 4.6 (Correlation inequality). *For the expectation $\langle\cdot\rangle$ of a measure of the form*

$$e^{\sum_{\langle xy\rangle} r_x r_y \cos(-\theta_x+2\pi a_{xy}+\theta_y)}\, dG_C(a) \prod_x d\theta_x\, d\tilde\rho_x(r_x), \tag{4.88}$$

with $dG_C(a)$ a Gaussian measure on $\{a_b, b\subset\Lambda\}$ with mean 0 and covariance C, we have

$$\langle \cos(n\cdot\theta + f\cdot a)\,;\, \cos(m\cdot\theta + g\cdot a)\rangle \geq 0. \tag{4.89}$$

Note that (4.84) is a special case of (4.88).

Proof. The proof is similar to that of Lemma 4.4. We introduce the duplicate identically distributed random variables $a, a', r, r', \theta, \theta'$ with expectations $\langle\cdot\rangle = \langle\cdot\rangle'$ given by the measure (4.88). We also denote the product expectation $\langle\cdot\rangle\otimes\langle\cdot\rangle'$ by $\langle\cdot\rangle$. We introduce new random variables

$$(a, a', r, r', \theta, \theta') \longmapsto (\alpha, \beta, \rho, \lambda, \varepsilon, \delta) \tag{4.90}$$

defined as follows:

$$a_{yx} =: \alpha_{yx} - \beta_{yx}, \tag{4.91a}$$
$$a'_{yx} =: \alpha_{yx} + \beta_{yx}. \tag{4.91b}$$

This is a composition of a rotation and a dilatation by a factor $2^{-1/2}$, so that

$$dG_C(a)\, dG_C(a') = dG_{C/2}(\alpha)\, dG_{C/2}(\beta). \tag{4.92}$$

Furthermore,

$$r_x =: \rho_x + \lambda_x, \tag{4.93a}$$
$$r'_x =: \rho_x - \lambda_x, \tag{4.93b}$$

so that

$$\rho_x = \frac{1}{2}(r_x + r'_x) \geq 0. \tag{4.94}$$

In order to do a similar variable transformation in (θ_x, θ'_x), we let them vary twice over the unit circle: $\theta_x, \theta'_x \in [-2\pi, 2\pi)$. Since all observables are 2π-periodic in θ_x, resp. θ'_x, this does not change the expectation $\langle\cdot\rangle$. Then define

$$\theta_x =: \varepsilon_x - \delta_x, \tag{4.95a}$$
$$\theta'_x =: \varepsilon_x + \delta_x. \tag{4.95b}$$

Thus for an observable $f(\theta_x, \theta'_x)$ we have (see Figure 4.3)

$$\int_{-2\pi}^{2\pi} d\theta_x \int_{-2\pi}^{2\pi} d\theta'_x \; f(\theta_x, \theta'_x) = 4 \int_{-\pi}^{\pi} d\varepsilon_x \int_{-\pi}^{\pi} d\delta_x \; f(\varepsilon_x - \delta_x, \varepsilon_x + \delta_x). \tag{4.96}$$

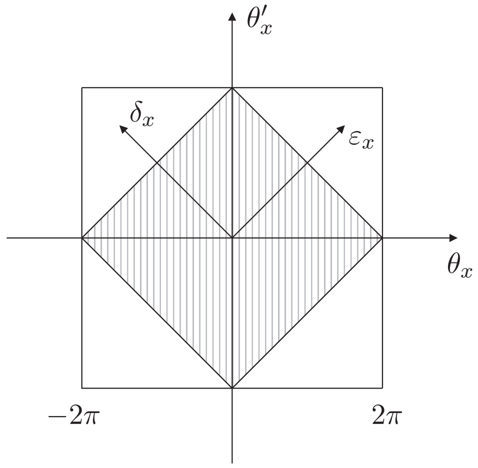

Figure 4.3: *The variable transformation* $(\theta_x, \theta'_x) \mapsto (\varepsilon_x, \delta_x)$. *The shaded area is covered by the* ε_x, δ_x*-integration. Due to the* 2π*-periodicity in both directions of the integrand each of the four white triangles gives the same contribution as a corresponding shaded triangle.*

The above transformations applied to the expression $-\theta_x + 2\pi a_{xy} + \theta_y$ are abbreviated as

$$\chi_{yx} := -\varepsilon_x + 2\pi\alpha_{xy} + \varepsilon_y, \tag{4.97a}$$
$$\Psi_{yx} := -\delta_x + 2\pi\beta_{xy} + \delta_y. \tag{4.97b}$$

Furthermore we have

$$r_x r_y = A_{xy} + B_{xy} \tag{4.98a}$$
$$r'_x r'_y = A_{xy} - B_{xy}, \tag{4.98b}$$

where

$$A_{xy} := \rho_x\rho_y + \lambda_x\lambda_y \tag{4.99a}$$
$$B_{xy} := \rho_x\lambda_y + \rho_y\lambda_x. \tag{4.99b}$$

Now the correlation function is

$$\begin{aligned}
&\langle \cos(n\cdot\theta + f\cdot a)\,;\, \cos(m\cdot\theta + g\cdot a)\rangle \\
&= \langle \cos(n\cdot\theta + f\cdot a)\big[\cos(m\cdot\theta + g\cdot a) - \cos(m\cdot\theta' + g\cdot a')\big]\rangle \\
&= \frac{1}{Z^2} \int dG_C(a)\, dG_C(a') \prod_x \Big[\int_{-2\pi}^{2\pi} d\theta_x \int_{-2\pi}^{2\pi} d\theta'_x\Big] \int \prod_x \Big[d\tilde{\rho}_x(r_x)\, d\tilde{\rho}_x(r'_x)\Big] \\
&\quad \times e^{\sum_{\langle xy\rangle} [r_x r_y \cos(-\theta_x + 2\pi a_{xy} + \theta_y) + r'_x r'_y \cos(-\theta'_x + 2\pi a'_{xy} + \theta'_y)]} \\
&\quad \times \cos(n\cdot\theta + f\cdot a)\big[\cos(m\cdot\theta + g\cdot a) - \cos(m\cdot\theta' + g\cdot a')\big].
\end{aligned} \tag{4.100}$$

Using

$$\begin{aligned} &r_x r_y \cos(-\theta_x + 2\pi a_{xy} + \theta_y) + r'_x r'_y \cos(-\theta'_x + 2\pi a'_{xy} + \theta'_y) \\ &= (A_{xy} + B_{xy}) \cos(\chi_{xy} - \Psi_{xy}) + (A_{xy} - B_{xy}) \cos(\chi_{xy} + \Psi_{xy}) \\ &= 2A_{xy} \cos\chi_{xy} \cos\Psi_{xy} + 2B_{xy} \sin\chi_{xy} \sin\Psi_{xy} \end{aligned} \tag{4.101}$$

the middle line of (4.100) is equal to

$$\prod_{\langle xy \rangle} e^{2A_{xy}\cos\chi_{xy}\cos\Psi_{xy} + 2B_{xy}\sin\chi_{xy}\sin\Psi_{xy}}. \tag{4.102}$$

Define the multiplicative cone $\mathscr{P}$ as the set of linear combinations with positive coefficients of functions of the form $P(\rho,\lambda)\, f(\alpha,\varepsilon)\, f(\beta,\delta)$, where P is some polynomial with positive coefficients and f any function. Then, by (4.97), (4.99) and (4.102), we see that the middle line of (4.100) is in $\mathscr{P}$.

Furthermore from

$$\cos(n{\cdot}\theta + f{\cdot}a) = \cos(n{\cdot}\varepsilon + f{\cdot}\alpha)\cos(n{\cdot}\delta + f{\cdot}\beta) + \sin(n{\cdot}\varepsilon + f{\cdot}\alpha)\sin(n{\cdot}\delta + f{\cdot}\beta) \tag{4.103}$$

and

$$\cos(m{\cdot}\theta + g{\cdot}a) - \cos(m{\cdot}\theta' + g{\cdot}a') = 2\sin(m{\cdot}\varepsilon + g{\cdot}\alpha)\sin(m{\cdot}\delta + g{\cdot}\beta) \tag{4.104}$$

we get that the last line of (4.100) is also in $\mathscr{P}$. Therefore, using (4.92) and (4.96), the correlation function is a sum of terms of the form

$$\begin{aligned} &\frac{1}{Z^2} \int dG_{C/2}(\alpha)\, dG_{C/2}(\beta) \prod_x \Big[\int_{-\pi}^{\pi} d\varepsilon_x \int_{-\pi}^{\pi} d\delta_x \Big] \\ &\times \int \prod_x \Big[d\tilde{\rho}_x(r_x)\, d\tilde{\rho}_x(r'_x) \Big]\, P(\rho,\lambda)\, f(\alpha,\varepsilon)\, f(\beta,\delta), \end{aligned} \tag{4.105}$$

where P is a polynomial with positive coefficients. Performing the (α, ε)- and (β, δ)-integrations, we get

$$c \int d\mu(\rho,\lambda)\ P(\rho,\lambda), \tag{4.106}$$

with $c \geq 0$ and

$$d\mu(\rho,\lambda) := \prod_x \Big[d\tilde{\rho}_x(r_x)\, d\tilde{\rho}_x(r'_x) \Big] \tag{4.107}$$

Note that the measure $d\mu(\rho,\lambda)$ is invariant under the transformation

$$r_x \mapsto r'_x, \qquad r'_x \mapsto r_x, \tag{4.108}$$

i.e.

$$\rho_x \mapsto \rho_x, \qquad \lambda_x \mapsto -\lambda_x, \tag{4.109}$$

for all x. Therefore

$$\int d\mu(\rho,\lambda)\ P(\rho,\lambda) = \int d\mu(\rho,\lambda)\ P_{\mathrm{e}}(\rho,\lambda), \tag{4.110}$$

where P_{e} is the part of P that is even in λ. Since $\rho_x \geq 0$ for all x and P_{e} is a polynomial with positive coefficients, even in λ_x, we have $P_{\mathrm{e}}(\rho,\lambda) \geq 0$ for all ρ, λ, completing the proof. □

Note that the above proof can be easily generalised to show

$$\langle P(r)\,\cos(n\cdot\theta + f\cdot a)\,;\,Q(r)\,\cos(m\cdot\theta + g\cdot a)\rangle \geq 0, \tag{4.111}$$

where P and Q polynomials with positive coefficients.

The correlation inequality tells us that

$$\frac{d}{d\mu}\big\langle\cos\langle\varphi\,\xi\,,a\rangle\big\rangle_{\mu} \geq 0 \tag{4.112}$$

and therefore

$$D_\Lambda(\varphi,\mathscr{L}) \geq \langle\cos\langle\varphi\xi\,,a\rangle\rangle_{\mu=0} = \frac{1}{Z_{\Lambda,\mu}}\int\prod_b da_b \prod_{b\in\mathscr{T}}\delta(a_b)\,e^{-\frac{1}{2J}\langle da\,,da\rangle}\,e^{i\varphi\langle\xi\,,a\rangle}, \tag{4.113}$$

which is the desired lower bound. Defining the space

$$V_\Lambda := \{a\in\Omega^1(\Lambda^*)\ :\ a_b\in\mathbb{R}, a_{\mathscr{T}} = 0\} \tag{4.114}$$

we see from (4.73) that $\xi\in V_\Lambda$. Furthermore, $K_\Lambda := \delta d : V_\Lambda \mapsto V_\Lambda$ is bijective. Using (4.63) we therefore have

$$D_\Lambda(\varphi,\mathscr{L}) \geq \frac{1}{Z_{\Lambda,\mu}}\int\prod_{b\notin\mathscr{T}} da_b\,e^{-\frac{1}{2J}\langle da\,,da\rangle}\,e^{i\varphi\langle\xi\,,a\rangle} = e^{-J\varphi^2/2\langle\xi\,,K_\Lambda^{-1}\xi\rangle}. \tag{4.115}$$

The thermodynamic limit $\Lambda\to\mathbb{Z}^d$ yields

$$D(\varphi,\mathscr{L}) \geq e^{-J\varphi^2/2\langle\xi\,,K^{-1}\xi\rangle}, \tag{4.116}$$

where $K := \delta d$ is defined on $\Omega^1\big[(\mathbb{Z}^d)^*\big]$. The choice of the inverse $K^{-1}\xi$ is immaterial since

$$\ker K = \big\{d\alpha\ :\ \alpha\in\Omega^0\big[(\mathbb{Z}^d)^*\big]\big\}, \tag{4.117}$$

and $\langle\xi\,,d\alpha\rangle = 0$. Unfortunately in 4 dimensions K is no longer the Laplacean, but mixes links in different directions. This can be corrected using the following trick: Define the subspace

$$D := \{a\in\Omega^1\big[(\mathbb{Z}^d)^*\big]\ :\ \delta a = 0\}. \tag{4.118}$$

Then $\xi\in D$ and

$$K|_D = \delta d + d\delta. \tag{4.119}$$

Now $\Delta = \delta d + d\delta$ is the (negative) discrete Laplacean. Indeed, for $b = b_x^\mu$ and some $a\in\Omega^1$, we have

$$(Ka)_b = \sum_{p:\,b\subset\partial p}(da)_p + \sum_{x\subset\partial b}(\delta a)_x = \sum_{p:\,b\subset\partial p}\sum_{b'\subset\partial p} a_{b'} + \sum_{x\in\partial b}\sum_{b':\,x\in\partial b'} a_{b'}. \tag{4.120}$$

This is best evaluated graphically. Figure 4.4 shows a two-dimensional slice of the links contributing to $(Ka)_b$. Let the sliced direction be ν. Then

$$\begin{aligned}(Ka)_b &= \sum_{\nu\neq\mu}(2a_x^\nu - a_{x+\hat\mu}^\nu - a_{x-\hat\mu}^\nu) + 2a_x^\nu - a_{x+\hat\nu}^\nu - a_{x-\hat\nu}^\nu\\ &= \sum_\nu(2a_x^\nu - a_{x+\hat\mu}^\nu - a_{x-\hat\mu}^\nu),\end{aligned} \tag{4.121}$$

the discrete Laplacean.

Now $\Delta : D\mapsto D$ is bijective.

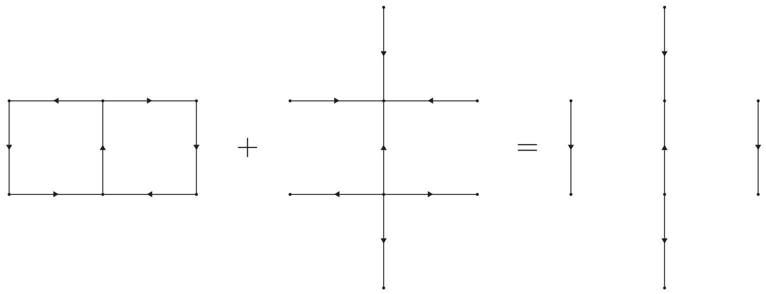

Figure 4.4: *A two-dimensional slice of the links contributing to* $(Ka)_b$.

PROOF. First note that, for $a \in D$, $\delta\Delta a = \delta^2 da + \delta d\delta a = 0$, so that $\Delta a \in D$. Furthermore, $D \perp \ker K$ so that Δ is injective. (In fact, as explained in Section 4.1, Δ is injective on the whole space $\Omega^1\left[\left(\mathbb{Z}^d\right)^*\right]$.) To show surjectivity, choose some $\omega \in D$. Then by the Poincaré lemma $\omega = \delta\upsilon$ for some υ. Now by Hodge decomposition $\upsilon = d\alpha + \delta\beta$, so that $\omega = \delta d\alpha$. Decompose further $\alpha = d\alpha_1 + \delta\alpha_2$ and define $a := \delta a_2 \in D$. Then $\omega = \delta da = \Delta a$. □

Therefore K^{-1} is given by convolution with the Green function $G(x)$, which is bounded and asymptotically equal to $1/|x|^2$, so that

$$G(x) \leq \frac{c}{1+|x|^2}. \tag{4.122}$$

For simplicity we assume that we are dealing with the physically relevant case: $\mathscr{L}$ is a rectangle with dimensions $L \times T$ in the $1,2$-plane with vertices $(0,0)$, $(L,0)$, $(0,T)$ and (L,T) (see Figure 4.5).

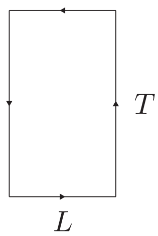

Figure 4.5: *A 't Hooft loop* $\mathscr{L}$ *with side lengths* L *and* T.

Then

$$\langle \xi\,, K^{-1}\xi\rangle = \sum_{xy,\,\mu} \xi_x^\mu\, G(x-y)\, \xi_y^\mu, \tag{4.123}$$

so that the sum over the two pairs of parallel sides decouples. The contribution of the sides of length L is bounded by

$$\sum_x \sum_y \frac{c}{1+|x-y|^2}, \tag{4.124}$$

where the sum ranges over all positions $(0,0),\dots,(L-1,0),(0,T),\dots,(L-1,T)$. This is bounded by

$$\sum_x \bigg(\sum_{n\in\mathbb{Z}} \frac{c}{1+n^2} + \sum_{n\in\mathbb{Z}} \frac{c}{1+T^2+n^2} \bigg). \tag{4.125}$$

The first term in the brackets is bounded by a constant and the second term is bounded in T and becomes, in the limit $T\to\infty$,

$$\int dx\, \frac{c}{T^2+x^2} = \frac{\text{const.}}{T}. \tag{4.126}$$

Therefore the total contribution of the sides of length L is bounded by

$$L\,(\text{const.} + \frac{\text{const.}}{T+1}). \tag{4.127}$$

An identical analysis may be done for the other pair of sides, so that we have

Theorem 4.7. *In 4 dimensions the 't Hooft loop obeys a perimeter law: There is a constant c such that*

$$\begin{aligned} D(\varphi,\mathscr{L}) &\geq e^{-cJ\varphi^2\left[L\left(1+\frac{1}{T+1}\right)+T\left(1+\frac{1}{L+1}\right)\right]} \\ &\geq e^{-2cJ\varphi^2(L+T)}. \end{aligned} \tag{4.128}$$

4.4 Perimeter law for strong coupling

We now show that for small J the disorder observable $D(\zeta,\partial\Sigma)$ obeys a perimeter law. In four dimensions Σ is a surface and in three dimensions a path. Let G be any compact group, and the action of the form (3.67). In particular, we may take the Wilson action with any character χ. In order to compute

$$D_\Lambda(\zeta,\partial\Sigma) = \frac{Z_\Lambda(\zeta,\partial\Sigma)}{Z_\Lambda}, \tag{4.129}$$

we take the logarithm:

$$\log D_\Lambda(\zeta,\partial\Sigma) = \log Z_\Lambda(\zeta,\partial\Sigma) - \log Z_\Lambda, \tag{4.130}$$

and perform a high-temperature expansion on both terms. From (3.97) we get thus

$$\log D_\Lambda(\zeta,\partial\Sigma) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1,\dots,Y_n} \phi^T(Y_1,\dots,Y_n) \left[\prod_{j=1}^{n} z_{\zeta,\partial\Sigma}(Y_j) - \prod_{j=1}^{n} z(Y_j) \right], \tag{4.131}$$

for J satisfying

$$J < -\frac{1}{c}\log\left(1 - \frac{1}{16(d-1)e}\right) \tag{4.132}$$

(see Lemma 3.8). Here ϕ^T is the Ursell function (3.98), $z(Y)$ the activity (3.75), and the "perturbed" activity is

$$z_{\zeta,\partial\Sigma}(Y) := \int d\underline{g} \prod_{p\in Y} \rho_p(g_p\,\zeta_{p^*}), \tag{4.133}$$

with ρ_p given by (3.70). Of fundamental importance for the following estimates is

LEMMA 4.8.

$$z_{\zeta,\partial\Sigma}(Y) = z(Y) \tag{4.134}$$

unless Y is a closed surface linked to $\partial\Sigma$.

PROOF. If a plaquette p has nonzero flux ζ going through it, a multiplication of one of link variables g_b, $b \in \partial p$, by ζ will remove the flux through p but create a flux in the other plaquette containing b; we can thus move the flux around at will on the surface Y. If Y is not closed it has a nonempty boundary ∂Y. Using the above procedure, we may move the flux to a plaquette p' on the boundary. Choose $b \in p' \cap \partial Y$. Then multiplying $g_{p'}$ by ζ is equivalent to multiplying g_b by ζ, since b is contained in no other plaquette. We may thus reduce the effect of multiplying a plaquette variable g_p by ζ to multiplying certain link variables by ζ, whereby the links for a "path" connecting the plaquette p to the boundary (see Figure 4.6). However, by invariance of the group measure, multiplication of a set of link variables by a constant has no effect and $z_{\zeta,\partial\Sigma}(Y) = z(Y)$.

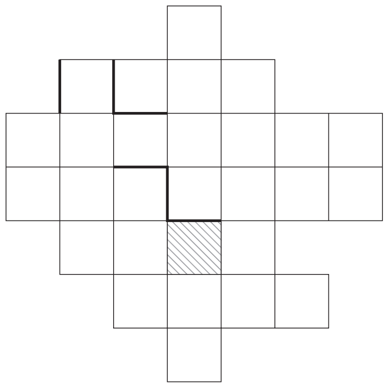

Figure 4.6: *A polymer Y cut open and flattened. The shaded plaquette contains a nontrivial flux ζ. By multiplying the link variables corresponding to the bold links by ζ we absorb the effect of the flux.*

If Y does not link with $\partial\Sigma$ the net flux through the closed surface Y vanishes. By the procedure described above we may shift the flux of all plaquettes onto a single plaquette p, so that all plaquettes have vanishing flux (see Figure 4.7).

□

Now

$$|\log D_\Lambda(\zeta,\partial\Sigma)| \le \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1,\ldots,Y_n} |\phi^T(Y_1,\ldots,Y_n)| \left| \prod_{j=1}^{n} z_{\zeta,\partial\Sigma}(Y_j) - \prod_{j=1}^{n} z(Y_j) \right|. \tag{4.135}$$

Using the bound (3.99) for the Ursell function we have

$$|\log D_\Lambda(\zeta,\partial\Sigma)| \le \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathscr{T}\subset\mathcal{P}_n} \sum_{Y_1,\ldots,Y_n} \prod_{l\in\mathscr{T}} \left| g_l^Y \right| \left| \prod_{j=1}^{n} z_{\zeta,\partial\Sigma}(Y_j) - \prod_{j=1}^{n} z(Y_j) \right|. \tag{4.136}$$

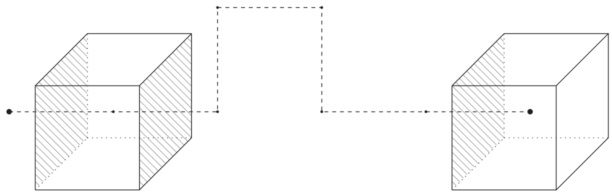

Figure 4.7: *A flux tube and two polymers. The polymer on the right links with $\partial\Sigma$ and has a nontrivial total flux. The leftmost polymer does not link with $\partial\Sigma$ and therefore has a vanishing total flux.*

Using Lemma 4.8 and

$$\left|\prod_{j=1}^{n} z_{\zeta,\partial\Sigma}(Y_j) - \prod_{j=1}^{n} z(Y_j)\right| \le 2\prod_{j=1}^{n} r(J)^{|Y_j|} \tag{4.137}$$

this may be bounded by

$$|\log D_\Lambda(\zeta,\partial\Sigma)| \le \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\mathscr{T}\subset\mathcal{P}_n}\sum_{Y_1,\dots,Y_n}{}' \prod_{l\in\mathscr{T}}\left|g_l^Y\right| 2\prod_{j=1}^{n} r(J)^{|Y_j|}, \tag{4.138}$$

where $\sum'$ means a sum over all sets of polymers $Y_1,\dots Y_n$ such that at least one Y_j is closed and links with $\partial\Sigma$. This sum is bounded by a sum over all indices $j=1,\dots,n$ and then requiring that Y_j links with $\partial\Sigma$. After a choice of some root $r\in\{1,\dots,n\}$ and a relabelling of the indices we thus have

$$|\log D_\Lambda(\zeta,\partial\Sigma)| \le \sum_{n=1}^{\infty}\frac{2n}{n!}\sum_{\mathscr{T}\subset\mathcal{P}_n}\sum_{\substack{Y_1,\dots,Y_n\,:\,Y_r\text{ closed}\\ \text{and links with }\partial\Sigma}}\prod_{l\in\mathscr{T}}\left|g_l^Y\right|\prod_{j=1}^{n} r(J)^{|Y_j|}. \tag{4.139}$$

Following the proof of Lemma 3.7 (almost) to the letter, we may bound this to get

$$\begin{aligned}|\log D_\Lambda(\zeta,\partial\Sigma)| &\le \sum_{n=1}^{\infty}\frac{2}{n-1}\left(\sum_{\substack{Y_r\text{ closed and}\\ \text{links with }\partial\Sigma}}\big[r(J)e\big]^{|Y_r|}\right)\left(\sum_{Y\ni p_0}\big[r(J)e\big]^{|Y|}\right)^{n-1}\\ &\le \tilde{C}\left(\sum_{\substack{Y\text{ closed and}\\ \text{links with }\partial\Sigma}}\big[r(J)e\big]^{|Y|}\right),\end{aligned} \tag{4.140}$$

provided we choose

$$\sum_{n=1}^{\infty}\frac{2}{n-1}\left(\sum_{Y\ni p_0}\big[r(J)e\big]^{|Y|}\right)^{n-1} \le \tilde{C}. \tag{4.141}$$

Note there is such a $\tilde{C}$, independent of Λ, since the above sum is convergent by (3.123) and uniformly bounded by a constant in Λ.

We therefore need a bound for

$$\sum_{\substack{Y \text{ closed and} \\ \text{links with } \partial\Sigma}} \big[r(J)e\big]^{|Y|} = \sum_{n=1}^{\infty} \sum_{\substack{Y:\,|Y|=n,\text{ closed} \\ \text{and links with } \partial\Sigma}} \big[r(J)e\big]^{n}. \tag{4.142}$$

The factor

$$|\{Y \,:\, |Y| = n,\, Y \text{ closed and links with } \partial\Sigma\}| \tag{4.143}$$

may be evaluated as follows. We find such Y's by first specifying a plaquette p_0 and then counting all Y's that contain p_0. If we choose enough plaquettes p_0, we recover all polymers in (4.143). Note that any closed polymer consisting of n plaquettes has a diameter bounded by Kn, where K is some constant. For each cell (a point in 3 dimensions, a link in 4 dimensions) $c \in \partial\Sigma$ choose some plaquette $p_{c,1} \in \partial(c^*)$. Construct a sequence of plaquettes $p_{c,2}, p_{c,3}, \ldots,$ by continuing along the line away from c in the direction defined by $p_{c,1}$ (see Figure 4.8). We stop when we have Kn plaquettes. Take as the set of plaquettes

$$P_0 := \bigcup_{c\in\partial\Sigma} \{p_{c,1}, \ldots, p_{c,Kn}\}. \tag{4.144}$$

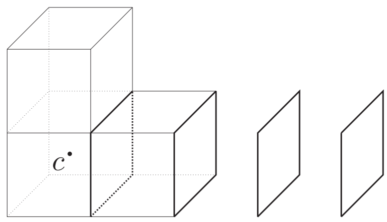

Figure 4.8: *A line of plaquettes away from $c \in \partial\Sigma$ in 3 dimensions. Shown is a polymer containing $p_{c,2}$ that links with c.*

Now if Y links with $\partial\Sigma$, there is a cell $c \in \partial\Sigma$ such that the distance between any plaquette in Y and c is smaller than Kn. Since Y is closed it follows that $p_{c,j} \in Y$ for some j. Since there are at most $\big[8(d-1)\big]^n$ polymers that connect to a given plaquette p_0 we have

$$\begin{aligned} |\{Y \,:\, |Y| = n,\, Y \text{ closed and links with } \partial\Sigma\}| &\le |P_0|\big[8(d-1)\big]^n \\ &= |\partial\Sigma|\, Kn \big[8(d-1)\big]^n. \end{aligned} \tag{4.145}$$

Therefore

$$\sum_{\substack{Y \text{ closed and} \\ \text{links with } \partial\Sigma}} \big[r(J)e\big]^{|Y|} \le K\,|\partial\Sigma| \sum_{n=1}^{\infty} n \big[8(d-1)r(J)e\big]^n. \tag{4.146}$$

For J small enough the sum converges and[11]

$$\sum_{n=1}^{\infty} n \big[8(d-1)r(J)e\big]^n = \frac{8(d-1)r(J)e}{\big(1-8(d-1)r(J)e\big)^2} \le \text{const.}\, r(J). \tag{4.147}$$

[11] Use $\sum_{n=1}^{\infty} na^n = a\frac{d}{da}\sum_{n=1}^{\infty} a^n = a\frac{d}{da}\frac{1}{1-a} = \frac{a}{(1-a)^2}$.

Therefore there is some universal constant C such that, for J small enough, we have

$$|\log D_\Lambda(\zeta, \partial\Sigma)| \leq C\,|\partial\Sigma|\,J. \tag{4.148}$$

Taking the thermodynamic limit we therefore have

THEOREM 4.9. *For J small enough the disorder observable obeys a perimeter law:*

$$D(\zeta, \partial\Sigma) \geq e^{-C\,|\partial\Sigma|\,J}. \tag{4.149}$$

A Some useful group theory

In the following we consider a compact group G with normalised invariant measure dg. Let $\widehat{G}$ be the set of irreducible representations of G. For each $\tau \in \widehat{G}$ we choose a concrete realisation: a unitary matrix $D^{(\tau)}_{ij}$ on a finite-dimensional complex vector space V_τ. We also denote by $\chi_\tau := \operatorname{tr} D^{(\tau)}$ the character of τ and by $d_\tau := \chi_\tau(e)$ the dimension of V_τ.

LEMMA A.1. *Let ρ be a unitary representation of G on a complex vector space W, $\tau \in \widehat{G}$ an irreducible representation. Then*

$$P_\tau := d_\tau \int dg\, \overline{\chi_\tau(g)}\, \rho(g) \tag{A.1}$$

is the orthogonal projector onto the subspace W_τ of W that carries the the irreducible representation τ.

PROOF. (i) P_τ is a projector.

$$\begin{aligned} P_\tau^2 &= d_\tau^2 \int dg \int dh\, \overline{\chi_\tau(g)}\, \overline{\chi_\tau(h)}\, \rho(gh) \\ &\overset{gh=k}{=} d_\tau^2 \int dk \left(\int dh\, \overline{\chi_\tau(kh^{-1})}\, \overline{\chi_\tau(h)} \right) \rho(k). \end{aligned} \tag{A.2}$$

Moreover,

$$\begin{aligned} \int dh\, \chi_\tau(kh^{-1})\, \chi_\tau(h) &= \int dh \left(\sum_{m,n} D^{(\tau)}_{mn}(k)\, \overline{D^{(\tau)}_{mn}(h)} \right) \left(\sum_l D^{(\tau)}_{ll}(h) \right) \\ &= \sum_{m,n,l} D^{(\tau)}_{mn}(k) \underbrace{\int dh\, \overline{D^{(\tau)}_{mn}(h)}\, D^{(\tau)}_{ll}(h)}_{=\frac{1}{d_\tau}\delta_{ml}\delta_{nl}} \\ &= \frac{1}{d_\tau} \sum_l D^{(\tau)}_{ll}(k) \\ &= \frac{1}{d_\tau} \chi_\tau(k). \end{aligned} \tag{A.3}$$

Thus $P_\tau^2 = P_\tau$.

(ii) P_τ is orthogonal.

$$\begin{aligned} P_\tau^* &= d_\tau \int dg\, \chi_\tau(g)\, \rho^*(g) \\ &= d_\tau \int dg\, \overline{\chi_\tau(g^{-1})}\, \rho(g^{-1}) \\ &= P_\tau. \end{aligned} \tag{A.4}$$

(iii) P_τ is the identity on W_τ and vanishes on its orthogonal complement $W_\tau^\perp$.

From (i) and (ii) we get that P_τ is diagonalisable with eigenvalues $0, 1$. Let W' be any finite-dimensional subspace of W_τ such that

$$W' \cong \underbrace{V_\tau \oplus \cdots \oplus V_\tau}_{n \text{ times}}. \tag{A.5}$$

Then

$$\begin{aligned} \operatorname{tr} P_\tau|_{W'} &= d_\tau \int dg\ \overline{\chi_\tau(g)}\ \operatorname{tr} \rho|_{W'}(g) \\ &= d_\tau \int dg\ \overline{\chi_\tau(g)}\ n\chi_\tau(g) \\ &= n\, d_\tau \\ &= \dim W', \end{aligned} \tag{A.6}$$

i.e. $P_\tau|_{W'} = \mathbb{1}$.

By an identical argument we find that $P_\tau|_{W'} = 0$ for any finite-dimensional subspace W' of $W_\tau^\perp$:

$$W' \cong V_{\sigma_1} \oplus \cdots \oplus V_{\sigma_n}, \quad \sigma_j \neq \tau \ \ \forall j. \tag{A.7}$$

Orthonormality of the characters yields $\operatorname{tr} P_\tau|_{W'} = 0$, which finishes the proof. □

The Peter-Weyl theorem states that the set

$$\{\sqrt{d_\tau}\, D_{ij}^{(\tau)}\}_{\tau \in \widehat{G},\ 1 \leq i,j \leq d_\tau} \tag{A.8}$$

is an orthonormal basis of $L^2(G, dg)$. A useful consequence applies to the space of square-integrable class functions. In this case we have

Lemma A.2. *The set*

$$\{\chi_\tau\}_{\tau \in \widehat{G}} \tag{A.9}$$

is an orthonormal basis of

$$\{f \in L^2(G, dg)\ :\ f(hgh^{-1}) = f(g)\ \forall h \in G\}. \tag{A.10}$$

Proof. Orthonormality follows from the orthonormality of characters of irreducible representations. Take a square-integrable class function f. Then, by the Peter-Weyl theorem,

$$f(g) = \sum_{\tau;\, i,j} f_{\tau,ij}\ D_{ij}^{(\tau)}(g). \tag{A.11}$$

Thus

$$f(h) = \int dg\ f(ghg^{-1}) = \sum_{\tau;\, i,j} f_{\tau,ij} \int dg\ D_{ij}^{(\tau)}(ghg^{-1}). \tag{A.12}$$

The integral in the last expression is equal to

$$\begin{aligned}\int dg\, D^{(\tau)}_{ij}(ghg^{-1}) &= \sum_{k,l} D^{(\tau)}_{kl}(h) \int dg\, D^{(\tau)}_{ik}(g)\, \overline{D^{(\tau)}_{jl}(g)} \\ &= \sum_{k,l} D^{(\tau)}_{kl}(h)\, \frac{1}{d_\tau}\delta_{ij}\delta_{kl} \\ &= \frac{1}{d_\tau}\chi_\tau(h)\,\delta_{ij}. \end{aligned} \tag{A.13}$$

We have proven that

$$f(h) = \sum_\tau \left(\frac{1}{d_\tau} \sum_{i=1}^{d_\tau} f_{\tau,ii} \right) \chi_\tau(h). \tag{A.14}$$

□

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