Appendix

CORRECTIONS AND UPDATES – 1st JANUARY, 2003

The book by K. Ohshika has been translated in english [Ohshi–02]. The main chapters are on Gromov's hyperbolic groups, on automatic groups, and on Kleinian groups.

I.B, and random walks on groups.

There is a nice introduction to random walks and diffusion on groups in [Salof–01], starting with a discussion on shuffling cards. A short exposition of Pólya's recurrence theorem can be found in [DymMc–72].

II.21 and VII.38, subgroup growth, and normal subgroup growth.

For further work concerning numbers of subgroups and normal subgroups of finite index in various groups, see among others [LiSMe–00] and [LarLu].

II.24, and a strong Schottky Lemma.

The classical Table-Tennis Lemma, or Schottky Lemma, is often used to show that a pair of isometries g, h of some hyperbolic space have *powers* g^n, h^n which generate a free group. There is a criterion for g, h to generate a free group in [AlFaN].

On free subgroups of isometry groups, see also [Woess–93] and [Karls].

II.25, II.33, and Möbius groups generated by two parabolics which are not free.

Let $\tilde{\Gamma}_z$ denote the subgroup of $SL(2,\mathbb{C})$ generated by $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, so that $\tilde{\Gamma}_z$ is free if $|z| \geq 2$ or if z is transcendental.

Grytczuk and Wójtowicz have shown that $\tilde{\Gamma}_{p/q}$ is *not* free for a set of rational values $z = p/q$ of the parameters which contains infinitely many accumulation points [GryWó–99].

II.28, and arithmeticity of lattices.

In $PSL(2,\mathbb{C})$, all arithmetic lattices which are generated by two elliptic elements and which are not co-compact have been determined [MacMa–01].

II.29$\frac{1}{3}$, more flowers for the herbarium of free groups.

Margulis has discovered a remarkable example of a free subgroup of the affine group of $\mathbb{R}^3$ acting *properly* on $\mathbb{R}^3$ [Margu-83]; an exposition appears in [Drumm–92].

II.29$\frac{2}{3}$, complement on groups with free subgroups.

We reproduce (most of) Problem 12.24 from the Kourovka Notebook.

Given a ring R with identity, the automorphisms of $R[[x]]$ sending x to $x(1 + \sum_{i=1}^{\infty} a_i x^i)$ form a group $N(R)$. We know that $N(\mathbb{Z})$ contains a copy of the free group F_2 of rank 2 (...). Does $N(\mathbb{Z}/p\mathbb{Z})$ contain a copy of F_2 ?

The answer is "yes": see [Camin–97]; it could be a challenge to find a table-tennis proof of this fact. For generalities on these "Nottingham groups" $N(R)$, see [Camin–00].

II.41, a misprint.

There is a misprint in the reference to [Bourb–75], which should be to Chapter VIII, § 2, Exercise 10.

II.41, and dense free subgroups of Lie groups.

The following result [BreGe] answers a question raised by A. Lubotzky and R. Zimmer: *if* Γ *is a dense subgroup of a connected semisimple real Lie group* G*, then* Γ *contains two elements which generate a dense free subgroup of* G*.* Also: in a connected non-solvable real Lie group of dimension d, any finitely generated dense subgroup contains a dense free subgroup of rank $2d$.

II.42, on Tits' alternative.

Let Γ be a subgroup of the group of homeomorphisms of the circle such that the action of Γ on the circle is minimal. Then, either the action is a conjugate of an isometric action, and therefore Γ contains a commutative subgroup of index at most 2, or Γ contains a so-called quasi-Schottky subgroup, which is in particular a non-abelian free subgroup [Margu–00]. A variation (possibly a simplification ?) of Margulis' original ideas appear in Section 5.2 of [Ghys–01].

For a group Γ of orientation preserving $\mathcal{C}^2$-diffeomorphisms of the circle, it is also known that the existence of an exceptional minimal set implies that Γ has non-abelian free subgroups [Navas].

On $Out(F_n)$, see also [BesFe–00].

Tits' alternative holds for automorphism groups of free soluble groups [Licht–95] and for linear groups over rings of fractions of polycyclic group rings [Licht–93], [Licht–99]. It also holds in a strong sense for subgroups of Coxeter groups [NosVi–02].

If Γ is a Bieberbach group, either both its automorphism group and its outer automorphism group are polycyclic, or both contain non-abelian free subgroups. See [MalSz] for precise criteria to decide which situation holds for a given Bieberbach group, in terms of the associated holonomy representation.

III.4, and examples of non-uniform tree lattices.

For the existence of such non-uniform lattices on uniform trees, see the work of L. Carbone [Carbo–01]. For tree lattices in general, see [BasLu–01].

III.$6\frac{1}{2}$, and further examples of finitely-genrated groups.

Let A be a commutative ring which is a finitely-generated $\mathbb{Z}$-module. Then the group A^* of invertible elements in A is a finitely-generated abelian group.

There is a proof in Section 4.7 of [Samue–67]; its main ingredient is Dirichlet's theorem, according to which the group of units in the ring of integers of a number field $\mathbb{K}$ is a direct product $F \times \mathbb{Z}^{r_1+r_2-1}$, where F is a finite group and r_1 [respectively $2r_2$] is the number of real [respectively complex] embeddings of $\mathbb{K}$ in $\mathbb{C}$.

More generally, if B is a commutative ring which is reduced (this means that 0 is the *only* nilpotent element) and finitely generated over $\mathbb{Z}$, then B^* is finitely generated [Samue–66].

III.18.iv, III.20, and residual finiteness.

On residual finiteness and topological dynamics: see also [Egoro–00].

A proof that finitely-generated linear groups are residually finite appears as Proposition III.7.11 in [LynSc–77].

III.21, on Baumslag-Solitar groups which are Hopfian.

For the equivalence between "$\Gamma_{p,q}$ Hopfian" and "p, q meshed" to hold, the definition should be

two integers $p, q \geq 1$ are *meshed* if they have precisely the same prime divisors

and *not* the definition as it reads in [BauSo–62], or on page 57. I am grateful to E. Souche who pointed out this correction to me.

III.21, on actions of Baumslag-Solitar groups on the line.

For any p, q with $p > q \geq 1$, there exists a faithful action of the group $BS(p, q)$ on the line by orientation preserving real-analytic diffeomorphisms. In particular, $Diff_+^\omega(\mathbb{R})$ contains Baumslag-Solitar groups which are not residually finite [FarFr].

III.24, on maximal subgroups.

In "familiar" uncountable groups, maximal subgroups cannot be countable. More precisely, Pettis [Petti–52] has shown that, if G is a second category[1] nondiscrete Hausdorff group containing a countable everywhere dense subset, then any proper subgroup H of G lies in an uncountable proper sugroup H_+ of G; if H is countable, H_+ can be taken to be everywhere dense as well.

In their work on maximal subgroups of infinite index in finitely generated linear groups (exluding extensions of solvable groups by finite kernels), Margulis and Soifer have shown that such a group Γ contains a free (*infinitely* generated) subgroup F_∞ which maps *onto* any finite quotient of Γ; they deduce from this that any maximal subgroup of Γ which contains F_∞ is necessarily of infinite index. Soifer and Venkataramana have shown the following result: if Γ is an arithmetic subgroup of a non-compact linear semi-simple group G such that the associated simply connected algebraic group over $\mathbb{Q}$ has the so-called congruence subgroup property, for example if $\Gamma = SL(n, \mathbb{Z})$ with $n \geq 3$, then Γ contains a *finitely generated* free subgroup which maps onto any finite quotient of Γ [SeiVe–00].

III.24 and VIII.39. The Grigorchuk group has the following property: any maximal subgroup in it is of finite index [Pervo–00]. The same property holds for any group commensurable with Γ [GriWi].

III.B, an additional problem: does $SO(3)$ act non-trivially on $\mathbb{Z}$? (Ulam's problem).

I do not know which uncountable groups can act faithfully on a countable set. Of course, the group $Sym(\mathbb{N})$ of all permutations of $\mathbb{N}$ is itself uncountable, and it has received attention at least since [SchUl–33]. Here is a sketch to show that $\mathbb{R}$, viewed as a discrete group, acts faithfully on $\mathbb{N}$; in other and somehow biased words, this produces "a continuous flow on a discrete space". I am most grateful to Tim Steger for several helpful conversations on this material.

Choose a basis (e_t) of $\mathbb{R}$ as a vector space over $\mathbb{Q}$ which is indexed by the open interval $]0, 1[$ of the line. Let C denote the countable set of pairs (a, b) of rational numbers such that $0 < a < b < 1$. For each $(a, b) \in C$, the map

$$\phi_{a,b} \, : \, \mathbb{R} \ni \sum_{t \in (a,b)} x_t e_t \quad \longmapsto \quad \sum_{t \in]a,b[} x_t \, \in \, \mathbb{Q}$$

is well-defined, $\mathbb{Q}$-linear and onto. Observe that, for any $x \neq 0$ in $\mathbb{R}$, there exists $(a, b) \in C$ such that $\phi_{a,b}(x) \neq 0$. Now $\mathbb{N}$ is in bijection with the disjoint union $\bigsqcup_{(a,b) \in C} \mathbb{Q}_{a,b}$ of copies of $\mathbb{Q}$ indexed by C. Define an action ϕ of $\mathbb{R}$ on this union which leaves each $\mathbb{Q}_{a,b}$ invariant and for which $x \in \mathbb{R}$ transforms $q \in \mathbb{Q}_{a,b}$ to $q + \phi_{a,b}(x)$. This ϕ is a faithful action. [Even if it is not important for our argument, observe that the product over $(a, b) \in C$ of the $\phi_{a,b}$ is a $\mathbb{Q}$-linear bijection from $\mathbb{R}$ onto a subspace of the vector space which is a direct product over C of copies of $\mathbb{Q}$.]

The group $\mathbb{R}/\mathbb{Z}$ is a direct sum of the torsion group $\mathbb{Q}/\mathbb{Z}$, which is countable, and a group isomorphic to $\mathbb{R}$ (a $\mathbb{Q}$-vector space of dimension the power of the continuum). It follows from the previous construction that there exists an injective homomorphism from $\mathbb{R}/\mathbb{Z}$ into $Sym(\mathbb{N})$.

[1] Recall that a topological space X is "second category" (= non-meager) if it is *not* the union of countably many subsets whose closures have empty interiors ("ensembles rares"). Baire's theorem shows that locally compact spaces and complete metric spaces are second category, indeed are Baire spaces (= spaces in which countable unions of closed subspaces with empty interiors have empty interiors).

In 1960, Ulam asked if the compact group $SO(3)$ of rotations of the usual space, viewed as a discrete group, can act on a countable set (see Section V.2 in [Ulam–60]). As far as I know, this is still open. Previous observations are possibly near what Ulam had in mind when writing his comments in Section II.7 of [Ulam–60].

III.38 and III.D, on finite quotients of the modular group.

For more on which finite simple groups are quotients of $PSL(2, \mathbb{Z})$, see the exposition of [Shale–01].

III.45, uncountably many finitely-generated groups with pairwise non-isomorphic von Neumann algebras.

Let Γ be a torsion-free Gromov-hyperbolic group which is not cyclic. Building up on results of Gromov, Ol'shanskii has shown that Γ has an uncountable family $(\Gamma_\iota)_{\iota\in I}$ of pairwise non-isomorphic quotient groups, all of which are simple and icc [Ol's–93]. N. Ozawa [Ozawa] has shown that, for any given separable factor M of type II_1, the set of those $\iota \in I$ for which the unitary group $\mathcal{U}(M)$ has a subgroup isomorphic to Γ_ι is a countable set. In particular, the set of von Neumann algebras of the groups Γ_ι (which are factors of type II_1) contains uncountably many isomorphism classes.

III.46, on groups with two generators.

It has been shown that two randomly chosen elements of a finite simple group G generate G with probability 1 as $|G| \to \infty$ (work of Dixon, Kantor-Lubotzky, Liebeck-Shalev, see [Shale–01]).

IV.1 and VI.1, on infinite generating sets and related word lengths.

Consider an integer $n \geq 2$, the group $\Gamma = SL(n, \mathbb{Z})$, and the infinite subset S of Γ consisting of those matrices of the form $I + kE_{i,j}$, with $k \in \mathbb{Z}$, $i, j \in \{1, \ldots, n\}$, $i \neq j$, and $E_{i,j}$ the matrix with all entries 0 except one 1 at the intersection of the ith row and the jth column.

As stated in Item III.2, the diameter of Γ with respect to the corresponding S-word length is finite as soon as $n \geq 3$.

IV.3.viii, on stable length: a correction.

The subadditivity

$$\tau(\gamma\gamma') \leq \tau(\gamma) + \tau(\gamma')$$

holds for *commuting* elements $\gamma, \gamma' \in \Gamma$ (as correctly stated by Gersten and Short).

For example, if γ, γ' are the two standard generators of the infinite dihedral group, then $\tau(\gamma\gamma') > 0$ and $\tau(\gamma) = \tau(\gamma') = 0$.

IV.24.i, and values of the indices for subgroups: a question.

Consider the following property of a group Γ: whenever two subgroups Γ_1, Γ_2 of finite indices are abstractly isomorphic, the indices $[\Gamma : \Gamma_1]$ and $[\Gamma : \Gamma_2]$ are equal.

Finitely generated free groups and fundamental groups of closed surfaces have this property, by an easy argument using Euler characteristics.

More generally, it would be interesting to know which groups have this property and which groups don't.

IV.25.vii, a quasi-isometry criterion for existence of lattices.

B. Chaluleau and C. Pittet [ChaPi–01] have answered one of the questions there and have shown:

Let N be a graded simply connected nilpotent real Lie group. If there exists a finitely-generated group which is quasi-isometric to N, then N has lattices.

IV.25, and examples of quasi-isometries.

(x) Say that a metric space X is *quasi-isometrically incompressible* if any quasi-isometric embedding from X into itself is a quasi-isometry. E. Souche [Souch] has shown that finitely generated nilpotent groups and uniform lattices in simple connected real Lie groups are quasi-isometrically incompressible, but that finitely-generated free groups and Baumslag-Solitar groups are not.

(xi) A finitely-generated group cannot be quasi-isometric to an infinite dimensional Hilbert space. Indeed, such a space has the following quasi-isometric-invariant property: for any positive number r, there exists a positive number R such that a ball of radius R contains infinitely many pairwise disjoint balls of radius r; and a finitely-generated group does not have this property.

IV.27, groups which are commensurable up to finite kernels.

Another terminology for commensurable up to finite kernels is *weakly commensurable* subgroups. See § 5.5 in [GorAn–93]; these authors also point out the following fact.

If M is a manifold on which some Lie group act transitively, then $\pi_1(M)$ contains a subgroup of finite index which is isomorphic to a discrete subgroup of a connected Lie group; if M is also compact, then $\pi_1(M)$ contains a subgroup of finite index which is isomorphic to a uniform lattice in some connected Lie group.

IV.29.v and VII.26, and the classification of lattices up to commensurability in some nilpotent Lie groups.

Y. Semenov has classified $\mathbb{Q}$-forms of some real nilpotent Lie algebras, and thus the commensurability classes of lattices in the corresponding nilpotent Lie groups [Semen]. It seems that the following question is open:

does there exist a finite dimensional real nilpotent Lie algebra of which the number k of $\mathbb{Q}$-forms (up to isomorphism) is such that $1 < k < \infty$?

IV.34 & 35, and commensurability. The following exercise is taken from [Gabor–02] and is clearly missing just before IV.34.

Exercise. (i) Show that two groups Γ_1, Γ_2 are commensurable if and only if they have commuting free actions on a set X such that both quotients $\Gamma_1\backslash X, \Gamma_2\backslash X$ are finite.

[Hint for one direction. Let Γ'_j be a subgroup of finite index in Γ_j, $j = 1, 2$, such that there exists an isomorphism $\varphi : \Gamma'_1 \longrightarrow \Gamma'_2$. Set $\Delta = \{ (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \mid \gamma_1 \in \Gamma'_1 , \gamma_2 = \varphi(\gamma_1)\}$. Consider the natural actions of Γ_1 and Γ_2 on $(\Gamma_1 \times \Gamma_2)/\Delta$.

Hint for the other direction. Choose $x_0 \in X$. Consider the natural action of Γ_1 on $\Gamma_2\backslash X$ and the canonical projection $[x_0]_2$ of x_0 in $\Gamma_2\backslash X$. Let Γ'_1 be the isotropy subgroup of Γ_1 defined by $[x_0]_2$ and set $\gamma_1 x_0 = \varphi(\gamma_1)x_0$. Check that φ is a well-defined group homomorphism $\Gamma'_1 \longrightarrow \Gamma_2$ which is injective and whose image is of finite index in Γ_2.]

(ii) Assume that Γ_1, Γ_2 have commuting free actions on X such that both $\Gamma_1\backslash X, \Gamma_2\backslash X$ are finite, and let Γ'_1, Γ'_2 be as in the previous hints. Check that

$$\frac{[\Gamma_1 : \Gamma'_1]}{[\Gamma_2 : \Gamma'_2]} = \frac{|\Gamma_2\backslash X|}{|\Gamma_1\backslash X|} .$$

IV.36, on commensurability and torsion.

G. Levitt has observed that a group Γ with infinitely many torsion conjugacy classes can have a subgroup of finite index Γ_0 which is torsion-free.

Indeed, let first Γ_0 be the wreath product $\mathbb{Z} \wr \mathbb{Z} = (\oplus_{i\in\mathbb{Z}}\mathbb{Z}a_i) \rtimes \mathbb{Z}$, where the generator t of $\mathbb{Z}$ acts on the direct sum by a shift; this group is torsion-free. Then let Γ be the semi-direct product of Γ_0 with the automorphism ϕ of Γ_0 of order 2 defined by $\phi(a_i) = -a_i$ for all $i \in \mathbb{Z}$ and $\phi(t) = t$; and let $s \in \Gamma$ denote the element of order 2 which implements ϕ on the subgroup Γ_0. For $v, v' \in \oplus_{i\in\mathbb{Z}}\mathbb{Z}a_i$, the elements sv, sv' are on the one hand of order 2; on the other hand, they are conjugate in Γ if and only if there exist $\epsilon \in \{\pm 1\}$, $k \in \mathbb{Z}$, and $w \in \oplus_{i\in\mathbb{Z}}\mathbb{Z}a_i$ such that $v' = \epsilon t^k v t^{-k} + 2w$; it follows that the conjugacy classes in Γ of $s(a_1 + \cdots + a_n)$ are pairwise distinct ($n \geq 0$).

A. Erschler has shown that a torsion-free group can be quasi-isometric to a group having torsion of unbounded order [Ersch–b].

The main ingredient of the proof is the construction, for any finitely-generated group A, of another finitely generated group $W^\infty(A)$, using an iterated wreath product construction and an HNN-extension. On the one hand, if A, B are Lipschitz equivalent groups, then $W^\infty(A), W^\infty(B)$ are Lipschitz equivalent; on the other hand, if A is torsion-free and if B has torsion, then $W^\infty(A)$ is torsion-free and $W^\infty(B)$ has torsion of unbounded order. One example is provided by $A = \mathbb{Z}$ and $B = \mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})$.

IV.40, and groups quasi-isometric to abelian groups.

Some of Shalom's ideas are now available in [Shalo].

IV.41, on groups of classes of quasi-isometries.

J. Taback has studied the quasi-isometry groups of $PSL_2(\mathbb{Z}[1/p])$, for p prime. These quasi-isometry groups are all isomorphic to $PSL_2(\mathbb{Q})$, even though the groups are not quasi-isometric for different values of the prime p. For this and other results, see [Tabac–00].

IV.43, and quasi-isometries of Baumslag-Solitar groups.

For the results of K. Whyte quoted from [Whyte–a], see now [Whyte–01].

IV.46, and Lipschitz equivalence.

Here is a question of B. Bowditch. (Private communication, March 2000. See also Item 1.A' in [Gromo–93].) Consider a Penrose tiling of the plane with two prototiles D and K (dart and kite), more precisely a tiling $\mathbb{R}^2 = \bigsqcup_{j\in J} T_j$ with each T_j given together with an isometry onto either D or K. This defines a cell decomposition X of the plane, of which the 0-skeleton $X^{(0)}$ is a discrete subset of the plane.

Is $X^{(0)}$ Lipschitz equivalent to a lattice in $\mathbb{R}^2$?

IV.47.vi, on costs and ℓ^2–Betti numbers.

For a group Γ with cost $\mathcal{C}(\Gamma)$ and ℓ^2–Betti numbers $\beta_j^{(2)}(\Gamma)$, we have always

$$\mathcal{C}(\Gamma) - 1 \;\geq\; \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma).$$

Moreover, for a large class of groups (including all groups for which both terms are known), the two terms are indeed equal. See [Gabor], in particular Corollary 3.22.

IV.50, geometric properties and weakly geometric properties.

Following [Ersch–b], it can be useful to be more precise in the terminology concerning a property $(\mathcal{P})$ of finitely generated groups. She suggests the following definitions.

Say $(\mathcal{P})$ is *geometric* if, for a pair (Γ_1, Γ_2) of finitely-generated groups which are quasi-isometric, Γ_1 has Property $(\mathcal{P})$ if and only if Γ_2 is commensurable to a group which has Property $(\mathcal{P})$.

Say $(\mathcal{P})$ is *weakly geometric* if, for a pair (Γ_1, Γ_2) of finitely-generated groups which are quasi-isometric, Γ_1 has Property $(\mathcal{P})$ if and only if Γ_2 is commensurable up to finite kernels to a group which has Property $(\mathcal{P})$.

An example of a property which is weakly geometric and which is not geometric is "being a lattice in $Spin(2,5)$"; see III.18.vi, III.18.x, and IV.42.

V.18, on the group of a remarkable simple closed curve.

It has been shown by Anna Erschler Dyubina that the group of V.18 is not finitely generated. Finding a proof is proposed as Problem 10 835 in the American Mathematical Monthly [DyuHa–00].

Problem. *Let Γ be the group defined by the presentation which has an infinite sequence $b_0, b_1, b_2, \dots$ of generators and an infinite sequence $b_1 b_0 b_1^{-1} = b_2 b_1 b_2^{-1} = b_3 b_2 b_3^{-1} = \dots$ of relations. Show that Γ is not finitely generated.*

We would like to add a comment and our solution. The nice solution of S.M. Gagola has appeared in the *Monthly*, November 2002.

Comment. In a short paper on wild knots, R.H. Fox discovered *A remarkable simple closed curve* (Annals of Math. **50**, 1949, pages 264–265) which is almost unknotted, a fact that Fox thinks "should be obvious to anyone who has ever dropped a stitch". The fundamental group Γ of the complement of this curve in 3-space has the presentation described above.

For other fundamental groups of complements of wild knots, see [Myers–00].

Our solution. Observe first that there is a homomorphism $\Gamma \longrightarrow \mathbb{Z}$ mapping b_k onto 1 for each $k \geq 0$; hence b_k is of infinite order in Γ for each $k \geq 0$. Observe also that there is a homomorphism σ from Γ onto the symmetric group $\langle x, y \mid x^2 = y^2 = (xy)^3 = 1 \rangle$ such that $\sigma(b_{2j}) = x$ and $\sigma(b_{2j+1}) = y$ for all $j \geq 0$; hence $b_k b_{k+1} \neq b_{k+1} b_k$ for all $k \geq 0$.

For each $n \geq 0$, there is a homomorphism $\phi_n : \Gamma \longrightarrow \Gamma$ such that $\phi_n(b_k) = b_{k+n}$ for all $k \geq 0$. Since the first relation of the presentation defining Γ can be written as $b_0 = b_1^{-1} b_2 b_1 b_2^{-1} b_1$ and since the other relations do not involve b_0, the group Γ has another presentation with generators b_k and relations $b_{k+1} b_k b_{k+1}^{-1} = b_{k+2} b_{k+1} b_{k+2}^{-1}$ for $k \geq 1$. Similarly, for each $n \geq 0$, the group Γ has a presentation with generators b_k and relations $b_{k+1} b_k b_{k+1}^{-1} = b_{k+2} b_{k+1} b_{k+2}^{-1}$ for $k \geq n$, so that ϕ_n is an automorphism of Γ.

Assume now by contradiction that Γ is finitely generated, and therefore generated by $b_0, b_1, \dots, b_{n+1}$ for some $n \geq 0$. Using again the relations $b_{k+1} b_k b_{k+1}^{-1} = b_{k+2} b_{k+1} b_{k+2}^{-1}$, this time for $0 \leq k \leq n-1$, we see that Γ is generated by $\{b_n, b_{n+1}\}$. Thus Γ is also generated by $\{b_0, b_1\} = \phi_n^{-1} (\{b_n, b_{n+1}\})$, as well as by $\{b_1, b_2\} = \phi_1 (\{b_0, b_1\})$.

For each $k \geq 0$, let $\tilde{\Gamma}_{k+1}$ the group abstractly defined by $k+2$ generators $b_0, \dots, b_{k+1}$ and k relations $b_1 b_0 b_1^{-1} = \dots = b_{k+1} b_k b_{k+1}^{-1}$. The same argument as above shows that $\tilde{\Gamma}_{k+1}$ has another presentation with 2 generators b_k, b_{k+1} and no relation, hence that $\tilde{\Gamma}_{k+1}$ is free of rank two. As b_0, b_1 do not commute in $\tilde{\Gamma}_{k+1}$, they generate a subgroup of $\tilde{\Gamma}_{k+1}$ which is free of rank two. As this holds for any $k \geq 0$, it follows that the group Γ, generated by b_0 and b_1, is itself free of rank two.

As Γ is free on $\{b_1, b_2\}$, there is a homomorphism $\psi : \Gamma \longrightarrow \mathbb{Z}$ such that $\psi(b_1) = 0$ and $\psi(b_2) = 1$, which is *onto.* On the other hand, as Γ is generated by b_0 and b_1, and as $\psi(b_0) = \psi(b_1^{-1} b_2 b_1 b_2^{-1} b_1) = 0 = \psi(b_1)$, we have $\psi(\Gamma) = \{0\}$. This is a contradiction and ends the proof. $\square$

The group Γ has other straightforward non-finitenes properties. (i) It is not Hopfian, since it is isomorphic to its quotient by the relation $b_0 = 1$. (ii) It maps onto the Baumslag-Solitar group $\langle t, z \mid tzt^{-1} = z^2 \rangle$ by $b_{2n} \longmapsto zt^{-1}$ and $b_{2n+1} \longmapsto t^{-1}$.

V.20, and lattices in Lie groups.

Information on lattices in *complex* Lie groups can be found in [Winke–98].

V.21, and finiteness homological properties of $SL(n, \mathbb{F}_q[T])$.

The finiteness result according to which $SL(n, \mathbb{F}_q[T])$ is of type (F_{n-2}) and not of type (F_{n-1}) for $q \geq 2^{n-2}$ is due independently to H. Abels (as recorded in V.21) and P. Abramenko [Abram–96].

V.22, on commensurability and groups of automorphisms.

G. Levitt has drawn my attention to the fact that, given a group Γ and a subgroup Γ_0 of finite index, there can exist an infinity of automorphisms of Γ which coincide with the identity on Γ_0.

Indeed, let Γ be the infinite dihedral group and let Γ_0 be its infinite cyclic subgroup of index 2. Then the conjugations of Γ by elements of Γ_0 are pairwise distinct.

V.22, on large groups of automorphisms.

The automorphism group of a finitely-generated group is clearly countable. The automorphism group of a countable group need not be; an easy example is provided by an infinite direct sum of copies of any countable group not reduced to one element.

Here is another example, inspired from Ulam and using the notation of the addendum to III.B above. For each $(a, b) \in C$, let $\phi_{a,b} : \mathbb{R} \longrightarrow \mathbb{Q}$ be the homomorphism defined there and let $\mathbb{Q}^2_{a,b}$ be a copy of $\mathbb{Q}$. The mapping

$$\psi_{a,b} \ : \begin{cases} \mathbb{R} \longmapsto \mathrm{Aut}\left(\mathbb{Q}^2_{a,b}\right) \approx GL_2(\mathbb{Q}) \\ x \longmapsto \begin{pmatrix} 1 & \phi_{a,b}(x) \\ 0 & 1 \end{pmatrix} \end{cases}$$

is a homomorphism of groups. Define Γ to be the direct sum, over $(a, b) \in C$, of the groups $\mathbb{Q}^2_{a,b}$; then the direct sum of the homomorphisms $\psi_{a,b}$ is an injection of $\mathbb{R}$ into $\mathrm{Aut}(\Gamma)$.

V.26, and some groups of Richard Thompson.

There are three groups, acting respectively on an interval, the circle, and the Cantor set, denoted by F, T, and V in [CanFP–96], and which appear in many different contexts. For T in the context of Teichmüller theory, see several articles by R.C. Penner, including [Penne–97]; for the isomorphism of Penner's group with T, see [Imber–97]. One interesting byproduct of this circle of ideas is that T can be generated by two elements α, β satisfying $\alpha^4 = \beta^3 = 1$, and other relations, such that the subgroup of T generated by α^2, β is the free product $\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) \approx PSL_2(\mathbb{Z})$; see [LocSc–97].

V.31, and efficiency.

A. Çevik gives in [Çevik–00] a sufficient condition for the efficiency of wreath products of efficient finite groups.

VI.9, an example of spherical growth series which is not monotonic.

On page 161, the last display, the coefficient of z^2 should be 8 not 6. This was pointed out to me by N.J.A. Sloane. Several growth series which appear in the the book appear also in his database of integer sequences: see

http://www.research.att.com/ njas/sequences/

on the web.

VI.19, on groups with the size of spheres not tending to infinity.

Groups in which the size of spheres does not tend to infinity are virtually cyclic (communicated by Anna Erschler Dyubina). More precisely:

Proposition. *If $\sigma(\Gamma, S; k) \leq C$ for infinitely many values of k, then Γ is virtually cyclic.*

Proof. Consider an arbitrary infinite finitely generated group, and let Φ be its inverse growth function, as in VII.32. First, it follows from the definition and from the obvious inequality $\beta(4k) > 2\beta(k)$ that $\Phi(2\beta(k)) \leq 4k$. Then, it follows from the first result quoted in VII.32 that, for an appropriate constant K, we have

$$\frac{\sigma(n)}{\beta(n-1)} \geq \frac{1}{8|S|\Phi(2\beta(n-1))} \geq \frac{1}{8|S|4(n-1)} \geq \frac{1}{Kn},$$

whence

$$\beta(n-1) \leq K\sigma(n)n$$

for all $n \geq 1$.

Assume now that $\sigma(n_j) \leq C$ for some constant C and a strictly increasing infinite sequence $(n_j)_{j\geq 1}$. Thus $\beta(n_j - 1) \leq KCn_j$ for any $j \geq 1$. By the strong form of Gromov's theorem (VII.29) on groups of polynomial growth, which is elementary for linear growth and which is due to Van den Dries and Wilkie [VdDW–84b], this implies that Γ is a group of linear growth and therefore a virtually cyclic group. □

VI.20, and the growth of braid groups for Artin generators.

For any integer $n \geq 2$, Artin's *braid group* on n strings has presentation

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{r} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (1 \leq i \leq n-2) \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad (1 \leq i,j \leq n-1,\ |i-j| \geq 2) \end{array} \right\rangle$$

[Magnu–73] and is obviously a quotient of the *locally free group of depth* 1 with $n-1$ generators

$$LF_n = \langle f_1, \ldots, f_{n-1} \mid f_if_j = f_jf_i \quad (1 \leq i,j \leq n-1,\ |i-j| \geq 2) \rangle$$

[Versh–90], [Versh–00], [VeNeB–00]. The value of the exponential growth rate of B_n for the generators σ_i is still unknown; however, Vershik and his co-authors have obtained partial results by comparing B_n with LF_n, more precisely by using the fact that LF_n appears both as a group of which B_n is a quotient and as a subgroup of B_n, the image of the injective homomorphism which maps f_i onto σ_i^2 for $i \in \{1, \ldots, n-1\}$.

For example, if ω_n^B, ω_n^{LF} denote respectively the exponential growth rates of B_n, LF_n for the generators discussed here, then

$$\lim_{n\to\infty} \omega_n^{LF} = 7 \qquad \text{and} \qquad \sqrt{7} \leq \omega_n^B \leq 7 \quad \text{for } n \text{ large enough.}$$

VI.B, early papers on growth of groups, and Dye's theorem on orbit equivalence for groups of polynomial growth.

Growth occurs in a paper by Margulis [Margu–67] published one year before those of Milnor ([Miln–68a], [Miln–68b]), where Margulis shows that if a compact three-dimensional manifold admits an Anosov flow, then its fundamental group has exponential growth. For a generalization to higher dimensions, see [PlaTh–72].

Also, between the mid fifties and 1968, some mathematicians in France were aware of the notion of growth of groups. Besides Dixmier (quoted on page 187), Avez had learned this from Arnold in 1965 [Avez–76].

We should also mention the following results of H. Dye. On the one hand, consider the compact abelian group $\prod_{j=0}^{\infty} C_j$, where each C_j is a copy of the group $\{0, 1\}$ of order 2, with its normalised Haar measure μ. Let $T : G \longrightarrow G$ be the adding machine, defined by

$$T(x_0, x_1, x_2, \ldots) = (0, 0, \ldots, 1, x_{j+1}, x_{j+2}, \ldots)$$

where j is the smallest index such that $x_j = 0$, and

$$T(1, 1, 1, 1, \ldots) = (0, 0, 0, 0, \ldots).$$

Then T defines an ergodic action of $\mathbb{Z}$ by measure preserving transformations of the probability space (G, μ). On the other hand, consider any finitely generated group Γ acting by measure preserving transformations on a standard Borel space furnished with a non-atomic probability measure, the action being ergodic.

One of Dye's theorems is that, if Γ is of polynomial growth, then the action of Γ is orbit-equivalent to the odometer action of $\mathbb{Z}$ [Dye–63]; if $\Gamma \approx \mathbb{Z}$, this is already in [Dye–59]. See [Weiss–81] for an exposition, and [OrnWe–80], [CoFeW–81] for related results; in particular, Dye's theorem carries over to *amenable* countable groups.

VI.40, and the functions which are growth functions of semigroups.

Let M be a monoid generated by a finite set S and let $\beta(k) = \beta(M, S; k)$ denote the corresponding growth function (see VI.12). It is obvious that if $\beta(k)$ is unbounded, then $k \prec \beta(k)$; moreover,

$$k \prec \beta(k) \quad \text{and} \quad k \not\approx \beta(k) \qquad \text{imply} \qquad k^2 \prec \beta(k)$$

as has been shown[2] by V.V. Beljaev (reported in [Trofi–80]).

Let $f, g : \mathbb{N} \longrightarrow \mathbb{N}$ be two functions such that $k^2 \prec f(k)$ and $g(k) \prec 2^k$. Then there exists a monoid M generated by a finite set S such that the sets

$$\{k \in \mathbb{N} \mid \beta(M, S; k) \leq f(k)\} \qquad \text{and} \qquad \{k \in \mathbb{N} \mid \beta(M, S; k) \geq g(k)\}$$

are both infinite.

VI.40, and the growth functions of Riemannian manifolds.

For further work after the paper of Grimaldi and Pansu quoted in VI.40, see [GriPa–01] and its bibliography.

VI.42, and growth of groups with respect to weights.

Growth with respect to generating sets and given weights are older than suggested by the references of Chapters VI and VII. In particular, in [PlaTh–76], Plante and Thurston define the growth of a countable group (*not* necessarily of finite type) with respect to a generating set (*not* necessarily finite) and a proper weight on it.

VI.42–43 and VII.35, on relative length functions and relative growth.

In the last line of page 176, read "relative length function" instead of "relative growth function". For relative growth of subgroups of solvable and linear groups, see [Osin–00].

VI.45, on word and Riemannian metrics.

See also [LubMR–00].

[2] This has been shown independently by several other mathematicians.

VI.56, on asymptotics of subadditive functions.

The correct conclusion of (i) should be that the sequence $\left(\frac{\alpha(k)}{k}\right)_{k\geq 1}$ either converges to $\inf_{k\geq 1}\frac{\alpha(k)}{k}$ *or diverges properly to* $-\infty$. (Since sequences appearing in the book are bounded below, the second case does not occur.)

VI.64, and groups of intermediate growth which are not residually finite.

Anna Erschler has shown that there exist uncountably many groups of intermediate growth which are commensurable up to finite kernel with the first Grigorchuk group, but which are not residually finite. She has also shown that there exist groups of intermediate growth which are not commensurable up to finite kernels with any residually finite group. See [Ersch–b].

VII.2, and a version of the Table-Tennis Lemma due to Margulis.

Proposition. *Let* Γ *be a group acting on a set* X *and let* $a, b \in \Gamma$. *Assume that there exists a non-empty subset* U *of* X *such that* $b(U) \cap U = \emptyset$ *and* $ab(U) \cup a^2b(U) \subset U$. *Then the semi-group generated in* Γ *by* ab *and* a^2b *is free; in particular, it is of exponential growth if* Γ *is finitely generated.*

Proof. Inside $U_\emptyset \doteqdot U$, the sets $U_1 = ab(U)$ and $U_2 = a^2b(U)$ are disjoint, since

$$ab(U) \cap a^2b(U) \ = \ a\Big(b(U) \cap U_1\Big) \ \subset \ a\Big(b(U) \cap U\Big) \ = \ \emptyset \, .$$

More generally, for each $n \geq 0$, let J_n denote the set of sequences of length n with elements in $\{1, 2\}$; for each $\underline{j} = (j_1, \ldots, j_n) \in J_n$, define a subset $U_{\underline{j}} = a^{j_1}ba^{j_2}b \ldots a^{j_n}b(U)$ of U. For any $n \geq 1$ and $\underline{j}' \in J_{n-1}$, observe that the sets $U_{(1,\underline{j}')}$ and $U_{(2,\underline{j}')}$ are disjoint, since

$$U_{(1,\underline{j}')} \cap U_{(2,\underline{j}')} \ = \ a\Big(b(U_{\underline{j}'}) \cap U_{(1,\underline{j}')}\Big) \ \subset \ a\Big(b(U) \cap U\Big) \ = \ \emptyset \, ,$$

and that both are inside $U_{\underline{j}'}$. Thus, for two sequences $\underline{j}, \underline{j}' \in \bigcup_{n=0}^{\infty} J_n$, either the corresponding subsets $U_{\underline{j}}, U_{\underline{j}'}$ are disjoint, or one is strictly contained in the other; in other words, their inclusion order is that of the infinite rooted 2-ary tree (see Item VIII.1). The proposition follows. □

This version of the Table-Tennis Lemma was communicated by G.A. Margulis to the authors of [EsMoO–02], see VII.19 below.

VII.13, on tight growth of free groups and hyperbolic groups.

It is easy to show that, for any normal subgroup $N \neq 1$ of F_k and the canonical image $\underline{S}_k$ of S_k in Γ/N, the corresponding exponential growth rates satisfy the strict inequality $\omega(F_k/N, \underline{S}_k) < 2k - 1$. G. Arzhantseva and I.G. Lysenok have shown the following generalization, which answers a question of [GrHa–97]. Let Γ be a non-elementary hyperbolic group, S a finite generating set and N an infinite normal subgroup of Γ; denote by $\underline{S}$ the canonical image of S in the quotient group Γ/N; then $\omega(\Gamma/N, \underline{S}) < \omega(\Gamma, S)$ [ArjLy].

VII.19, on uniformly exponential growth of solvable groups.

D. Osin has shown that any solvable group of exponential growth has uniformly exponential growth [Osin–a], thus solving Problem VII.19.B (see page 297); this has also been shown independently and shortly afterwards by J. Wilson (unpublished). More generally, Osin has shown that any elementary amenable group of exponential growth has uniformly exponential growth [Osin–b].

Also, D. Osin has shown that the uniform Kazhdan constant of an infinite Gromov hyperbolic groups is zero [Osin–c]

John Wilson has discovered *examples of groups which answer the main problem of Item VII.19* [Wilso]. More precisely, there exist groups which are isomorphic to their permutational wreath product with the alternating group on 31 letters. Let $\Gamma \approx \Gamma \wr A_{31}$ be any group of this kind; on the one hand, there exists a sequence $(S_n = \{x_n, y_n\})_{n\geq 1}$ of generating sets of Γ, with $x_n^2 = y_n^3 = 1$ for all $n \geq 1$, such that the limit of the corresponding exponential growth rates is 1, in formula $\lim_{n\to\infty} \omega(\Gamma, S_n) = 1$; on the other hand, for an appropriate choice of Γ, there exist non-abelian free subgroups in Γ, so that in particular Γ is of exponential growth.

VII.19, on uniformly exponential growth of linear groups.

It is a theorem of A. Eskin, S. Mozes and Hee Oh that, given an integer $N \geq 1$ and a field $\mathbb{K}$ of characteristic 0, a finitely generated subgroup of $GL(N, \mathbb{K})$ is of uniformly exponential growth if and only if it is not virtually nilpotent, namely if and only if it is of exponential growth (result of [EsMoO], announced in [EsMoO–02]).

In particular, this *solves Research Problem VII.19.C* (see page 297).

For other progress on uniformly exponential growth, see [BucHa-00], [GrHa–01a], and [GrHa–01b]. For an exposition on uniformly exponential growth, see [Harpe].

If constants measuring exponential growth often have uniform bounds in terms of the generating sets, other constants exhibit the opposite behaviour. For example, T. Gelander and A. Zuk have shown that, in many cases, Kazhdan constants depend in a crucial way on the chosen generating set [GelZu–02].

VII.29, group growth, and Gromov's theorem.

There is a brief survey on group growth and Gromov's theorem by D.L. Johnson [Johns–00].

Concerning polynomial growth for locally compact groups, V. Losert has published a second part to [Loser–87]: see [Loser–01].

VII.29, and growth of double coset classes.

Consider a *Hecke pair* (G, H), namely a group G and a subgroup H such that all orbits of the natural action of H on G/H are finite, or equivalently such that, for each $g \in G$, the indices of $H \cap gHg^{-1}$ in both H and gHg^{-1} are finite. It is a natural counting problem to estimate for each $g \in G$ the cardinality of the H-orbit of gH in G/H, or equivalently the number of one-sided classes g_jH in the double class HgH.

The specific case of the pair $(SL(2, \mathbb{Z}[1/p]), SL(2, \mathbb{Z}))$, p a prime, appears in [BeCuH–02].

VII.34, and the growth of Følner sequences.

A question related to our Problem VII.34.A appears as Problem 14.27 in the Kourovka Notebook [Kouro–95], and has been answered in [Barda–01].

VII.38, and the growth of normal subgroups of finite index.

See [LarLu].

VII.39, growth of conjugacy classes, and growth of pseudogroups.

For growth of conjugacy classes in hyperbolic groups, see [CooKn–02] and [CooKn–b].

Growth of pseudogroups appears in connection with foliations in [Plant–75].

VII.40, and growth of infinitely generated groups.

See [PlaTh–76], and the above comment on Item VI.42.

VII.61, on the set of exponential growth rates.

Part of the problem was solved by Anna Erschler Dyubina, who has shown that *the set* Ω_2 *of exponential growth rates of* 2*-generated groups has the power of the continuum* (see [Ersch–02], [Ersch–a]).

VIII.7, and the adding machine.

The adding machine on the infinite 2-ary tree $\mathcal{T}^{(2)}$ can be economically (and recursively, compare VIII.9) defined as the element $\tau \in Aut(\mathcal{T}^{(2)})$ such that

$$\tau = a(1\,,\,\tau).$$

Observe that $\tau \neq 1$ since τ exchanges 0 and 1, and that τ is of infinite order since

$$\tau^2 = a(1,\tau)a(1,\tau) = (\tau,1)(1,\tau) = (\tau,\tau).$$

The simple and clever Proposition 20 of [Sidki–00] shows that an element $g \in Aut(\mathcal{T}^{(2)})$ is conjugate to τ if and only if it acts transitively on the set of 2^k vertices of the level $L^{(k)}$ for each $k \geq 0$.

Later, Sidki has shown that a solvable subgroup K of $Aut(\mathcal{T}^{(2)})$ which contains an element such as τ above is an extension of a torsion-free metabelian group by a finite 2-group. If furthermore K is nilpotent then it is torsion-free abelian [Sidki].

VIII.10.ii on automata and finitely generated groups.

This connexion is a very active subject of research; see among others [GriNS–00], [GriZu–a], [GriZu–b], and [Sidki-00].

VIII.31, a result of John Wilson.

At the end of this item, the "recent result" which is quoted was in fact essentially in John Wilson's Ph.D. thesis of 1971, as well as in [Wilso–72]. ("Essentially" in the sense that he did not use the words "branch groups".)

For these, [Grigo] contains comments and a sketchy proof, whereas details can be found in [Wilso].

VIII.32 and VIII.71, and elements of small lengths and large orders in the Grigorchuk group.

Proposition. *For any* $n \in \mathbb{N}$*, there exists* $\gamma \in \Gamma$ *such that*

$$\gamma^{2^n} \neq 1 \qquad \textit{and} \qquad \ell(\gamma) \leq 2^n.$$

Proof (following a sketch of L. Bartholdi). Let $K = \langle abab \rangle^{\Gamma}$ be the normal subgroup of Γ of index 16 defined in VIII.30; recall that K is generated by

$$t = (ab)^2\ , \quad v = (bada)^2\ , \quad w = (abad)^2$$

and that $\psi^{-1}(K \times K)$ is a subgroup of K (the index is 4 by Exercise VIII.81, but we do not use this here). Let σ be the endomorphism of Γ defined in VIII.57. Since $\sigma(a) = aca$, $\sigma(b) = d$, $\sigma(d) = c$, and $\sigma(c) = b$, we have $\psi\sigma(a) = (d,a)$, $\psi\sigma(b) = (1,b)$, $\psi\sigma(c) = (a,c)$, $\psi\sigma(d) = (a,d)$. It follows that $\psi\sigma(x) = (1,x)$ for $x \in \{t,v,w\}$, and therefore for all $x \in K$.

Define inductively a sequence $(x_i)_{i\geq 0}$ by $x_0 = abab$ and $x_{i+1} = a\sigma(x_i)$. Since

$$\psi\left(a\sigma(x_i)a\sigma(x_i)\right) = (x_i, x_i),$$

the order of x_{i+1} is twice that of x_i. As x_0 is of order 8 by Proposition VIII.16, it follows that the order of x_i is 2^{i+3} for all $i \geq 0$.

On the other hand, denote by w_0 the word $abab$ representing x_0; for each $i \geq 0$, let w_{i+1} the word obtained from w_i by

- substitution of aca, d, b, c in place of a, b, c, d respectively,
- deletion of a if it appears as the first letter and addition of a as a prefix letter otherwise,

so that w_{i+1} represents x_{i+1}. Thus $w_1 = cadacad$ and, for each $j \geq 0$,

- w_{2j+1} is a word of length $2\ell(w_{2j}) - 1$ which begins with c and ends with a letter from $\{b, c, d\}$,
- w_{2j+2} is a word of length $2\ell(w_{2j+1})$ which begins with a and ends with a letter from $\{b, c, d\}$;

in particular, $\ell(x_i) \leq \ell(w_i) < 2^{i+2}$ for all $i > 0$. The proposition follows (with $x = x_{n-2}$ for $n \geq 2$). □

VIII.67, and power series with finitely many different coefficients.

Here is a result of Szegö: a power series with finitely many different coefficients that converges inside the unit disk is either a rational function, or has the unit circle as natural boundary [Szegö–22].

VIII.88, complement on commensurability of finitely-generated subgroups.

It is a remarkable result of Grigorchuk and Wilson that any infinite finitely-generated subgroup of the Grigorchuk group Γ is commensurable to Γ [GriWi]. In other words, Γ has exactly two commensurability classes of finitely-generated subgroups: itself and $\{1\}$.

Here are a few examples of other groups for which all commensurability classes of finitely-generated subgroups are known; in case of torsion-free groups, we do not list the class of $\{1\}$.

(i) Free abelian groups $\mathbb{Z}^n$, with $\mathbb{Z}^j$ for $j \in \{1, \ldots, n\}$.

(iii) The Heisenberg group $\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$, with $\mathbb{Z}$, $\mathbb{Z}^2$ and the group itself.

(iii) Non-abelian free groups F_n, with $\mathbb{Z}$ and F_2.

(iv) Virtually free groups, for example $PSL(2, \mathbb{Z})$, with finite subgroups, $\mathbb{Z}$ and F_2.

(v) The fundamental group Γ_g of a closed surface of genus $g \geq 2$, with $\mathbb{Z}$, F_2 and the group itself.

(vi) Olshanskii's "monsters" (see the reference in III.5, as well as [AdyLy–92]), in which any proper subgroup is cyclic.

VIII.87, on complex linear representations of the Grigorchuk group.

For each $k \geq 0$, let Γ_k denote as in VIII.35 the finite quotient of the Grigorchuk group which acts naturally on the level $L(k)$ of the binary tree. Choose some point in $L(k)$ and denote by P_k the corresponding isotropy subgroup of Γ_k. Then (Γ_k, P_k) is a *Gelfand pair,* and the natural linear representation of Γ_k on the space $\mathbb{C}^{L(k)}$ splits as a direct sum of $k+1$ pairwise inequivalent irreducible representations, of dimensions $1, 1, 2, 4, \ldots, 2^{k-1}$ [BeHaG].

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