Pi may be a normal number

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The Conjecture

Conjecture

π is a normal number.

So what is a normal number?

Definition

A real number (necessarily irrational) is NORMAL in base b if all whole numbers in base b are distributed uniformly in its infinite sequence of digits.

An illustration

Consider a number x that is normal in base 2. The odds of a random digit of x being 0 are 50%, as are the odds of said digit being 1. Moreover, the odds of a random pair of digits being 00 are 1 in 2^2, or 25%, as are the odds of said pair being 01, 10 or 11.

Generally, the odds of a random block of n digits being a given whole number of n digits are 1 in b^n, where b is the base.

Some examples

In base 10 (and possibly all bases?):

- 0.123456789101112...
- 0.23571113171923...
- 0.14916253649648 1100...

The history of normal numbers

[Émile Borel, 1909]: almost all numbers are normal (the non-normal numbers constitute a Lebesgue measure zero set)

[Sierpinski, 1917]: one can specify a normal number

[Becher and Figueira, 2002]: there exist computable numbers that are normal in every base

Something everyone knows but no one can prove

Conjecture

Every irrational algebraic number (including π) is normal.

It should be noted that no irrational algebraic number has been proven to be normal.
Likewise, no irrational algebraic number has been proven to be non-normal.

Why do we think π is normal?

[Bailey et al., 2012] Statistical calculations on the first four trillion base-16 digits of π show it is almost certainly normal in base 16.

[Artacho et al., 2012] Graphical representations, such as this one of 100 billion base-4 digits of π, show similarities with pseudorandom walks.

A fun exercise

Prove that any positive real number is the product of two normal numbers.

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