

THE CRITICAL Z -INVARIANT ISING MODEL VIA DIMERS

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THE 2-DIMENSIONAL ISING MODEL

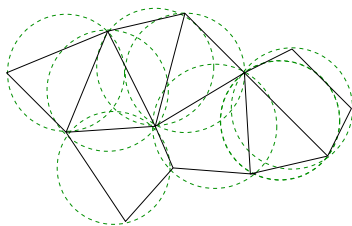
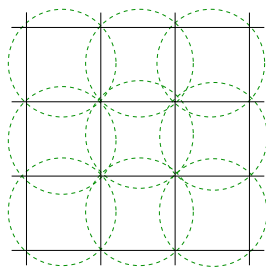
- Planar graph, $G = (V(G), E(G))$.
- Spin Configurations, $\sigma : V(G) \rightarrow \{-1, 1\}$.
- Every edge $e \in E(G)$ has a coupling constant, $J_e > 0$.
- Ising Boltzmann measure (finite graph):

$$P_{\text{Ising}}(\sigma) = \frac{1}{Z_{\text{Ising}}} \exp \left(\sum_{e=uv \in E(G)} J_e \sigma_u \sigma_v \right),$$

where Z_{Ising} is the partition function.

ISORADIAL GRAPHS

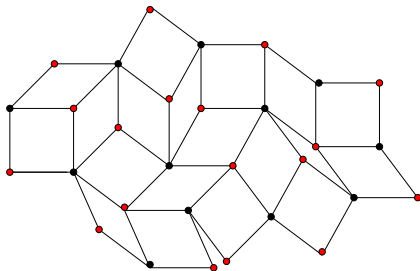
- Graph G **isoradial** : can be embedded in the plane so that every face is inscribable in a circle of radius 1.



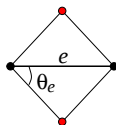
- G isoradial $\Rightarrow G^*$ isoradial : vertices of $G^* =$ circumcenters.

ISORADIAL GRAPHS

- $G^\diamond$: corresponding **diamond graph** $\begin{cases} \text{vertices : } V(G) \cup V(G^*) \\ \text{edges : radii of circles} \end{cases}$



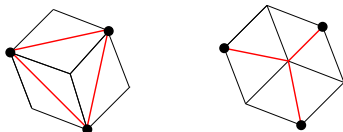
- Edge $e \rightarrow \begin{cases} \text{rhombus} \\ \text{angle } \theta_e \end{cases}$



$\Rightarrow$ Coupling constants: $J_e(\theta_e)$.

THE Z -INVARIANT ISING MODEL [BAXTER]

- Star-triangle transformation on G : preserves isoradiality.



- Z -invariant Ising model : satisfies $\Delta - Y$

$$\Rightarrow \sinh(2J_e(\theta_e)) = \frac{\operatorname{sn}\left(\frac{2K(k)}{\pi}\theta_e|k\right)}{\operatorname{cn}\left(\frac{2K(k)}{\pi}\theta_e|k\right)}, \quad k^2 \in \mathbb{R}.$$

$\operatorname{sn}, \operatorname{cn}$: Jacobi elliptic trigonometric functions.

K : complete elliptic integral of the first kind.

THE CRITICAL Z -INVARIANT ISING MODEL [BAXTER]

- High and low temperature expansion of the partition function
 $\Rightarrow$ measure on contour configurations of G and G^* .
- Generalized form of self-duality $\Rightarrow k = 0$, and

$$J_e(\theta_e) = \frac{1}{2} \log \left(\frac{1 + \sin \theta_e}{\cos \theta_e} \right). \quad (1)$$

- Examples.
 - $G = \mathbb{Z}^2$: $\theta_e = \pi/4$, $J_e(\theta_e) = \log \sqrt{1 + \sqrt{2}}$.
 - G triangular lattice : $\theta_e = \pi/6$, $J_e(\theta_e) = \log(3^{\frac{1}{4}})$.
 - G hexagonal lattice : $\theta_e = \pi/3$, $J_e(\theta_e) = \log \sqrt{2 + \sqrt{3}}$.

Critical temperatures (Kramers, Kramers-Wannier)

(1) are called **critical coupling constants**.

STATISTICAL MECHANICS ON ISORADIAL GRAPHS

- Critical Ising model: [Baxter, Costa-Santos, Mercat, Smirnov, Chelkak & Smirnov].
- Explicit expressions for:
 - the Green's function with weights “ $\tan(\theta_e)$ ” [Kenyon],
 - the inverse of the Dirac operator $\bar{\partial}$ for bipartite graphs, with weights “ $2 \sin \theta_e$ ” [Kenyon],which only depend on the *local geometry* of the graph.
- What is special about this setting ?
 - Z-invariance (integrability).
 - Natural setting for discrete complex analysis [Duffin, Mercat, Chelkak & Smirnov].

THE DIMER MODEL

- Graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ (planar).
- **Dimer configuration**: perfect matching M .
- **Weight function**, $\nu : E(\mathcal{G}) \rightarrow \mathbb{R}^+$
- **Dimer Boltzmann measure** (finite graph) :

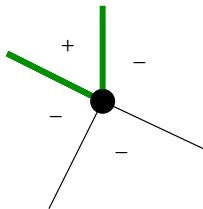
$$P_{\text{dimer}}(M) = \frac{1}{Z_{\text{dimer}}} \prod_{e \in M} \nu_e,$$

where Z_{dimer} is the **dimer partition function**.

ISING-DIMER CORRESPONDENCE [FISHER]

Ising model on toroidal graph G , coupling constants J .

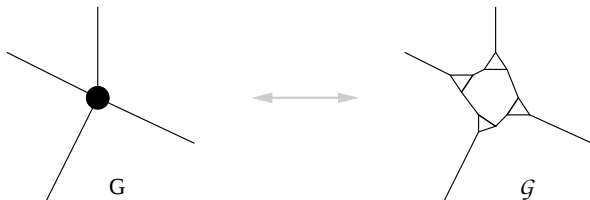
- Low temperature expansion on $G^* \rightarrow$ measure on polygonal contours of G .



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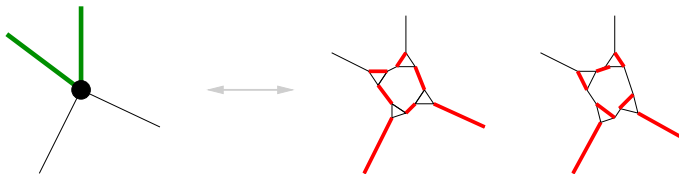


- $\mathcal{G}$: Fisher graph of G .

ISING-DIMER CORRESPONDENCE [FISHER]

Ising model on toroidal graph G , coupling constants J .

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- $\mathcal{G}$: **Fisher graph** of G .
- Correspondence : Contour conf. $\rightarrow 2^{|V(G)|}$ dimer configurations.
- Critical Ising model on $G \leftrightarrow$ critical dimer model on $\mathcal{G}$.

$$\nu_e = \begin{cases} \cot \frac{\theta_e}{2} & \text{original edges} \\ 1 & \text{edges of the decorations.} \end{cases}$$

CRITICAL DIMER MODEL ON INFINITE FISHER GRAPH: PERIODIC CASE

- G : infinite $\mathbb{Z}^2$ -periodic isoradial graph,
 $\mathcal{G}$: corresponding Fisher graph.
- Toroidal exhaustion $\{\mathcal{G}_n\} = \{\mathcal{G}/n\mathbb{Z}^2\}$. $\mathcal{G}_1$: **fundamental domain**.
 Z_n : partition function of $\mathcal{G}_n$, P_n : Boltzmann measure of $\mathcal{G}_n$.

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 Z_n : partition function of $\mathcal{G}_n$, P_n : Boltzmann measure of $\mathcal{G}_n$.
- Our goal is to:
 - compute the **free energy**:

$$f = - \lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n.$$

- obtain an explicit expression for a natural **Gibbs measure**:
probability measure P such that if one fixes a perfect matching
in an annular region, matchings inside and outside of the annulus
are independent, and

$$P(M) \propto \prod_{e \in M} \nu_e,$$

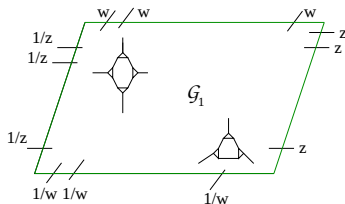
when M is a matching inside of the annulus (DLR conditions).

KASTELEYN MATRIX K OF THE GRAPH $\mathcal{G}$

- **Kasteleyn orientation of the graph:** all elementary cycles are clockwise odd.
- K : corresponding weighted oriented adjacency matrix,

$$K_{u,v} = \begin{cases} \nu_{uv} & \text{if } u \sim v, u \rightarrow v \\ -\nu_{uv} & \text{if } u \sim v, u \leftarrow v \\ 0 & \text{else.} \end{cases}$$

- $K_1(z, w)$: is the Kasteleyn matrix of $\mathcal{G}_1$, with modified weights along edges crossing a dual horizontal and vertical cycle.



- $\det K_1(z, w)$ is the **dimer characteristic polynomial**.

FREE ENERGY AND GIBBS MEASURE

THEOREM (BOUTILLIER,DT)

- The free energy of the dimer model on $\mathcal{G}$ is:

$$f = -\frac{1}{2(2\pi i)^2} \iint_{\mathbb{T}^2} \log(\det K_1(z, w)) \frac{dz}{z} \frac{dw}{w}$$

- The weak limit of the Boltzmann measures P_n defines a Gibbs measure P_{dimer} on $\mathcal{G}$. The probability of the subset of edges $E = \{e_1 = u_1 v_1, \dots, e_k = u_k v_k\}$ being in a dimer configuration of $\mathcal{G}$ is given by:

$$\mathcal{P}(e_1, \dots, e_k) = \left(\prod_{i=1}^k K_{u_i, v_i} \right) \text{Pf}((K^{-1})_E), \quad \text{where}$$

$$K_{(v,x,y)(v',x',y')}^{-1} = \frac{1}{(2\pi i)^2} \iint_{\mathbb{T}^2} \frac{\text{Cof}(K_1(z,w))_{v,v'}^t}{\det K_1(z,w)} z^{x'-x} w^{y'-y} \frac{dz}{z} \frac{dw}{w}.$$

IDEA OF THE PROOF (GIBBS MEASURE)

[Cohn, Kenyon, Propp; Kenyon, Okounkov, Sheffield]

THEOREM (KASTELEYN, KENYON, ...)

The Boltzmann measure $P_n(e_1, \dots, e_k)$ is equal to:

$$\frac{\left(\prod_{i=1}^k K_{u_i, v_i}\right)}{2Z_n} \left(-\text{Pf}(K_n^{00})_{EC} + \text{Pf}(K_n^{10})_{EC} + \text{Pf}(K_n^{01})_{EC} + \text{Pf}(K_n^{11})_{EC}\right)$$

- Use Jacobi's formula: $\text{Pf}((K_n^{\theta\tau})_{EC}) = \text{Pf}(K_n^{\theta\tau})\text{Pf}((K_n^{\theta\tau})_E^{-1})$.
- Use Fourier to block diagonalize $K_n^{\theta\tau}$.
- Obtain Riemann sums. Show that they converge on a subsequence to the corresponding integral.
- Use Sheffield's theorem, which proves a priori existence of the limit.

PROOF, CONTINUED

- Sheffield's theorem does not hold for non bipartite graphs.
- K_n^{00} is never invertible $\Rightarrow$ delicate estimate, $\frac{\text{Pf}(K_n^{00})_{EC}}{\mathcal{Z}^n} = O\left(\frac{1}{n}\right)$.
- To show convergence of the Riemann sums, need to know what are the zeros of $\det K_1(z, w)$ on the torus $\mathbb{T}^2$.

CHARACTERISTIC POLYNOMIAL AND THE LAPLACIAN

- G : infinite $\mathbb{Z}^2$ -periodic isoradial graph.
- **Laplacian** on G , with weights $\tan(\theta_e)$ is represented by the matrix Δ :

$$\Delta_{u,v} = \begin{cases} \tan(\theta_{uv}) & \text{if } u \sim v \\ -\sum_{w \sim u} \tan(\theta_{uw}) & \text{if } u = v. \end{cases}$$

- $\Delta_1(z, w)$: Laplacian matrix on G_1 with additional weights z and w .
- **Laplacian characteristic polynomial**: $\det(\Delta_1(z, w))$.

THEOREM (BOUTILLIER,DT)

- *There exists a constant c such that:*

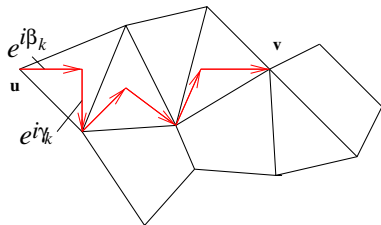
$$\det K_1(z, w) = c \det \Delta_1(z, w).$$

- *The curve $\{(z, w) \in \mathbb{C}^2 : \det K_1(z, w) = 0\}$ is a Harnack curve and $\det K_1(z, w)$ admits a unique double zero $(1, 1)$ on $\mathbb{T}^2$.*

LOCAL EXPRESSION FOR K^{-1} , GENERAL CASE

- Discrete exponential function [Mercat, Kenyon]

$\text{Exp} : V(G) \times V(G) \times \mathbb{C} \rightarrow \mathbb{C}$,



$$\text{Exp}_{u,u}(\lambda) = 1$$

$$\text{Exp}_{u,u_{k+1}}(\lambda) = \text{Exp}_{u,u_k}(\lambda) \frac{(\lambda + e^{i\beta_k})(\lambda + e^{i\gamma_k})}{(\lambda - e^{i\beta_k})(\lambda - e^{i\gamma_k})}.$$

LOCAL EXPRESSION FOR K^{-1}

THEOREM (BOUTILLIER,DT)

The inverse of the Kasteleyn matrix K on $\mathcal{G}$ has the following local expression:

$$K_{u,v}^{-1} = \frac{1}{(2\pi i)^2} \oint_{C_{uv}} f_u(\lambda) f_v(\lambda) \text{Exp}_{\mathbf{u},\mathbf{v}} \log(\lambda) d\lambda + c_{u,v},$$

where

- C_{uv} is a closed contour containing all poles of the integrand, and avoiding a half-line d_{uv} .
- $c_{u,v} = \begin{cases} 0 & \text{if } \mathbf{u} \neq \mathbf{v} \\ \pm \frac{1}{4} & \text{else.} \end{cases}$

[Comm. Math. Phys. 2010]

PROOF (SKETCH)

- Idea [Kenyon]
- $f_v(\lambda)\text{Exp}_{\mathbf{u},\mathbf{v}}(\lambda)$ is in the kernel of K .
- Use singularities of the log: define contours of integrations in such a way that:

$$(KK^{-1})(u, v) = \begin{cases} 0 & \text{if } u \neq v \\ 1 & \text{if } u = v. \end{cases}$$

GIBBS MEASURE, LOCAL EXPRESSION

THEOREM (BOUTILLIER,DT)

$$\mathcal{P}(e_1, \dots, e_k) = \prod_{i=1}^k K(u_i, v_i) \text{Pf}((K^{-1})_E),$$

defines a Gibbs measure in dimer configurations of $\mathcal{G}$.

PROOF.

- [dT]: every finite, simply connected subgraph of a rhombus tiling can be completed by rhombi in order to become a periodic rhombus tiling of the plane.
- Convergence of the Boltzmann measures in the periodic case.
- Locality of the inverse Kasteleyn matrix.
- Uniqueness of the inverse Kasteleyn matrix in the periodic case.
- Kolmogorov's extension theorem.

CONSEQUENCES

- THEOREM (BAXTER'S FORMULA)

Let G be a periodic isoradial graph. Then, the free energy of the critical Ising model on G is:

$$f_{\text{Ising}} = -\frac{\log 2}{2} - \frac{1}{|V(G_1)|} \sum_{e \in E(G_1)} \frac{\theta_e}{\pi} \log(\theta_e) + \frac{1}{\pi} \left(L(\theta_e) + L\left(\frac{\pi}{2} - \theta_e\right) \right),$$

where $L(\theta) = -\int_0^\theta \log(2 \sin(t)) dt$ is Lobachevski's function.

- $$P_{\text{Ising}} \left(\begin{array}{c} + \quad \quad - \\ \diagup \quad \diagdown \\ \theta_e \\ \diagdown \quad \diagup \\ J_e \\ \diagup \quad \diagdown \end{array} \right) = \frac{1}{4} - \frac{\theta_e}{2\pi \sin \theta_e}.$$

- Spin/spin correlations are local.
- Asymptotics computations for the dimer Gibbs measure.

CRITICAL ISING MODEL AND CRSFs

- The dimer characteristic polynomial $\det K_1(z, w)$ is related to Ising configurations.
- The Laplacian characteristic polynomial $\det \Delta_1(z, w)$ counts CRSFs.

THEOREM (dT)

There exists an explicit correspondence between weighted “double-dimer” configurations of $\mathcal{G}_1$ counted by $\det K_1(z, w)$, and CRSFs counted by $\det \Delta_1(z, w)$.

PROOF.

- Matrix-tree theorem for the Kasteleyn matrix K :

$$\det K_1(z, w) = \sum_{F \in \mathcal{F}(\mathcal{G}_1)} \left(\prod_{e=(x,y) \in F} f_x \bar{f}_y K_{x,y} \right) \prod_{T \in F} (1 - z^{h(T)} w^{v(T)}).$$

- To every CRSF of G_1 corresponds a family of CRSFs of $\mathcal{G}_1$.
Everything is explicit.