Conformal field theory from conformal loop ensembles

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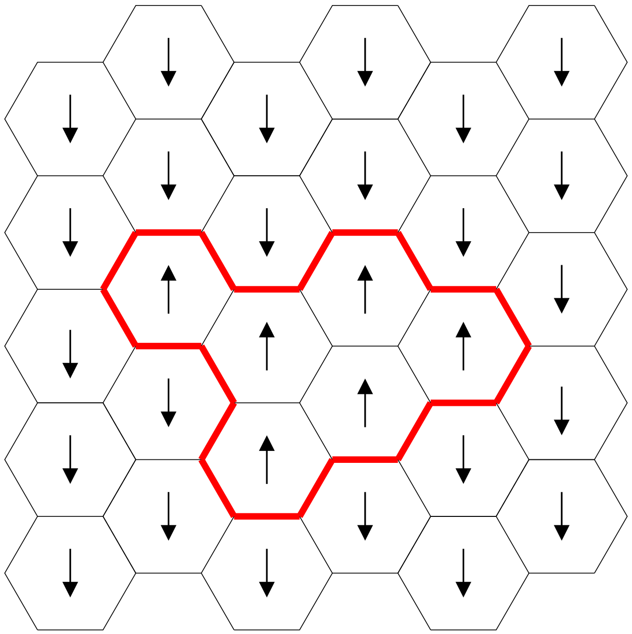
Scaling limits and emergent behaviours

Example: the Ising model



Microscopic model: measure on functions σ from faces of a lattice (ex: hexagonal) to some set (ex: spin $\{\uparrow,\downarrow\} = \{+1,-1\}$), with properties of locality, homogeneity

$$\mu(\sigma) = \exp\left[\beta \sum_{\text{neighbouring faces } j,k} \sigma(j)\sigma(k)\right]$$

Critical point $\beta = \beta_c$: for generic β, locality means that fluctuations are usually uncorrelated at large distances. Small temperatures: all spins tend to align (non-zero average local moment). High temperatures: thermal fluctuations break alignment (zero average local moment). Critical point: combination of both effects gives large-distance correlations of fluctuations.

$\Rightarrow$ Universal emergent correlations

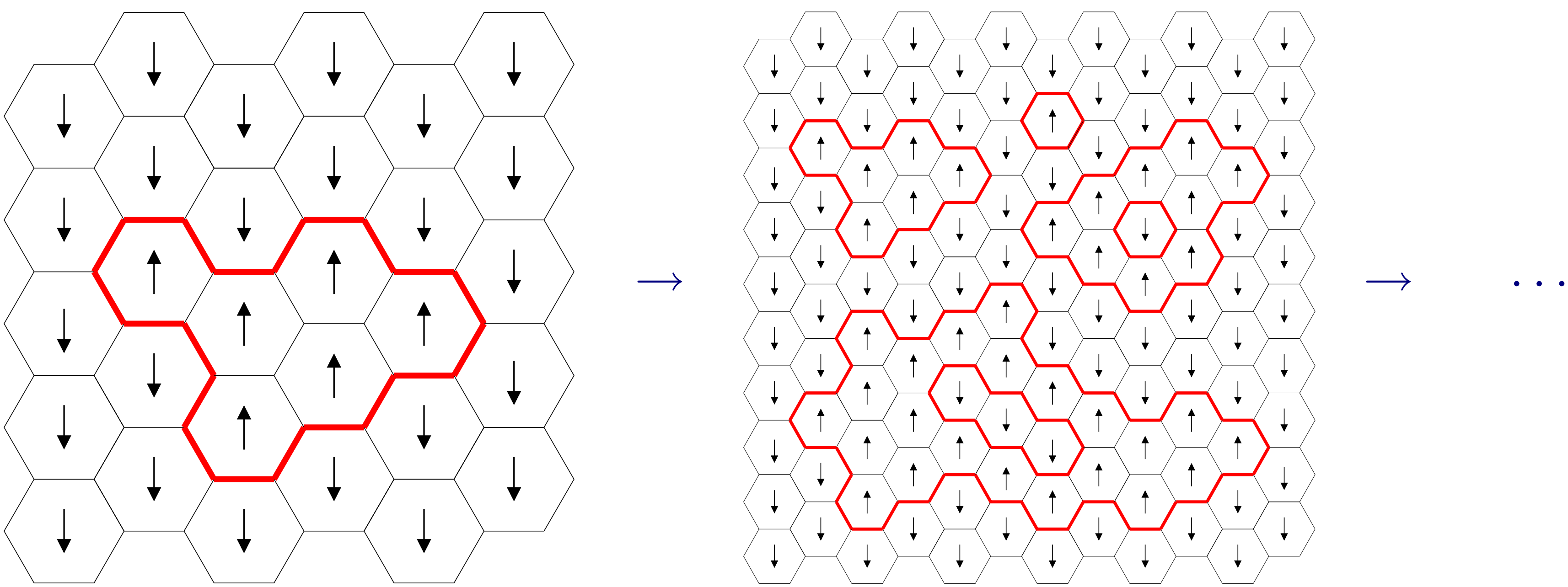
Quantum field theory, a theory for emergent correlations:

The scaling limit of expectations is:

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/4}\, \mathbb{E}^{(\beta=\beta_c-\alpha\varepsilon)}[\sigma(x/\varepsilon)\sigma(y/\varepsilon)] = C^{(\alpha)}(x,y)$$

(here, x, y are in $\mathbb{R}^2$). The coefficient $C^{(\alpha)}(x,y)$ is a correlation function in a QFT

$$C^{(\alpha)}(x,y) = \langle \mathcal{O}(x)\mathcal{O}(y) \rangle^{(\alpha)}$$



The basic ingredients of QFT are

- Local fields $\mathcal{O}(x) \Leftrightarrow$ local variables $1,\ \sigma(k),\ \sigma^2(k),\ \sigma(k)\sigma(\text{neighbour of } k),\ \ldots$
- correlation functions $\langle\cdot\rangle \Leftrightarrow$ expectations of products of local variables $\mathbb{E}[\cdot]$

Questions: 1) Are there emergent random objects? 2) What is the measure theory for them? 3) Can we reproduce the QFT local correlations from this theory? 4) Can we prove that it emerges from the microscopic theory?

Stress-energy tensor

Conformal field theory: with g conformal on a domain D of $\hat{\mathbb{C}}$, there exists a map $\mathcal{O} \mapsto g \cdot \mathcal{O}$ such that

$$\left\langle \prod_i \mathcal{O}_i(z_i) \right\rangle_D = \left\langle \prod_i (g \cdot \mathcal{O}_i)(g(z_i)) \right\rangle_{g(D)}$$

For primary fields, $(g \cdot \mathcal{O})(g(z)) = (\partial g)^h (\bar{\partial}\bar{g})^{\tilde{h}} \mathcal{O}(g(z))$, with $h, \tilde{h} \in \mathbb{R}^+$. Locality and basic QFT concepts: existence of stress-energy tensor $T(w)$, with conformal Ward identities:

$$\left\langle T(w) \prod_i \mathcal{O}(z_i) \right\rangle_D \sim \sum_i \left(\frac{h_i}{(w - z_i)^2} + \frac{1}{w - z_i} \frac{\partial}{\partial z_i} \right) \left\langle \prod_i \mathcal{O}(z_i) \right\rangle_D$$

T is not a primary field, there is a central charge $c \in \mathbb{R}$:

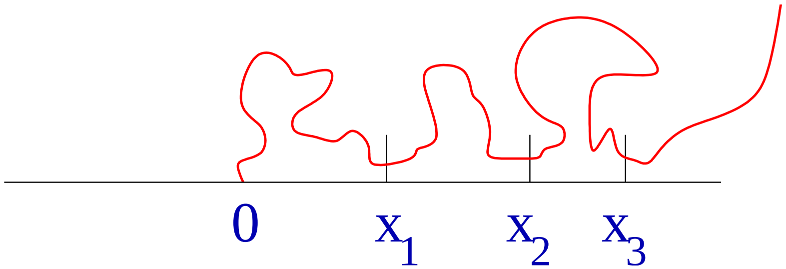
$$(g \cdot T)(g(w)) = g'(w)^2 T(g(w)) + \frac{c}{12} \{g, w\}, \quad \{g, w\} = \left(\frac{\partial^3 g(w)}{\partial g(w)} - \frac{3}{2} \left(\frac{\partial^2 g(w)}{\partial g(w)} \right)^2 \right)$$

$\Rightarrow$ vertex operator algebra formulation [Kac, Lepowsky, ..., Cardy, Zamokodchikov, ...].

Constructions in SLE$_{8/3}$:

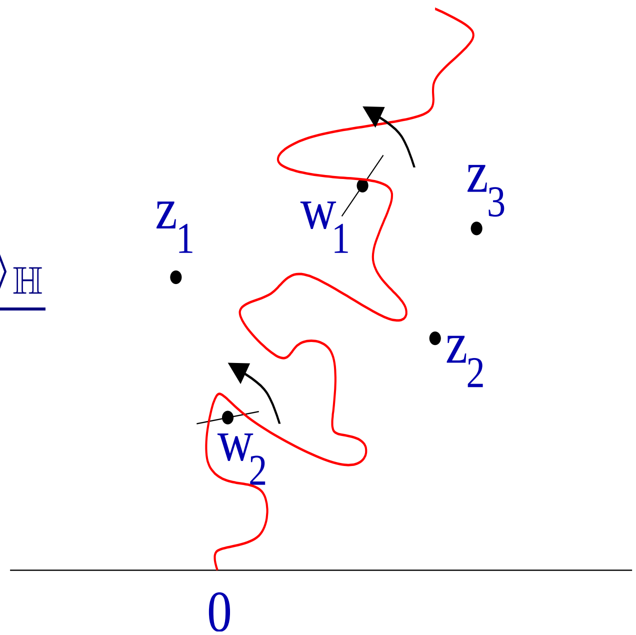
Friedrich and Werner (2002,2003): boundary correlation functions

$$\frac{\langle \phi_{12}(\infty) T(x_1) \cdots T(x_k) \phi_{12}(0) \rangle_{\mathbb{H}}}{\langle \phi_{12}(\infty) \phi_{12}(0) \rangle_{\mathbb{H}}}$$

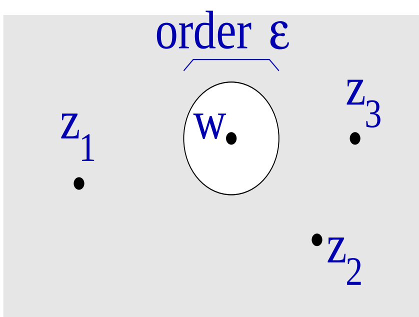


D., Riva and Cardy (2006): bulk correlation functions with “Schramm fields” inserted

$$\frac{\langle \phi_{12}(\infty) T(w_1) \cdots T(w_j) \mathcal{O}(z_1) \cdots \mathcal{O}(z_k) \phi_{12}(0) \rangle_{\mathbb{H}}}{\langle \phi_{12}(\infty) \phi_{12}(0) \rangle_{\mathbb{H}}}$$



Fundamental principle: conformal restriction of Lawler, Schramm, Werner (2002). The calculation also uses the Joukowsky-like tranform $z' = z + \varepsilon^2 e^{2i\theta}/(w-z)$. Schematically,



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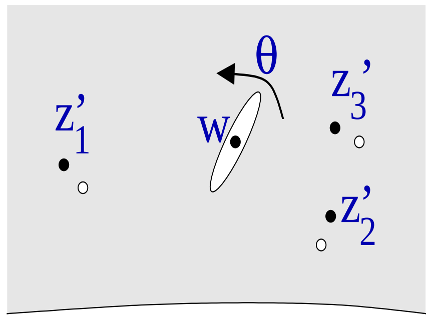
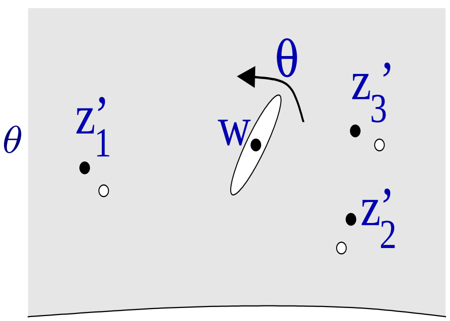
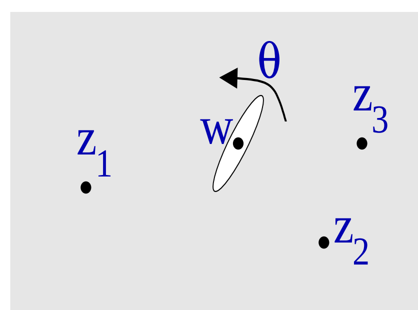


$=$ restricted probability: $\dfrac{P(X \cap Y)}{P(Y)}$

$$\Rightarrow \quad 0 \quad = \quad \int_0^{2\pi} d\theta \, e^{-2i\theta}$$



$$= \quad \int_0^{2\pi} d\theta e^{-2i\theta} \Bigg($$



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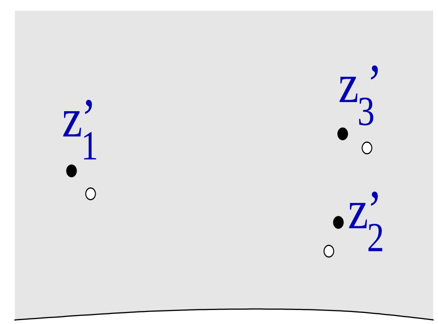


$$\Bigg)$$

One can also derive the transformation property from this, giving $c = 0$, because for small ε, only translation, rotation and scaling affect the rotating ellipse:

$$
\begin{aligned}
&-\lim_{\varepsilon\to 0}\frac{8}{\pi\varepsilon^2}\int_0^{2\pi} d\theta e^{-2i\theta} P(X(z,\ldots)\cap Y(w,\varepsilon,\theta))_D \\
&\quad = -(\partial g(w))^2 \lim_{\varepsilon\to 0}\frac{8}{\pi\varepsilon^2}\int_0^{2\pi} d\theta e^{-2i\theta} P(X(g(z),\ldots)\cap Y(g(w),\varepsilon,\theta)))_{g(D)}
\end{aligned}
$$

Conformal restriction needs to be modified to get $c \neq 0$. We need to consider all cluster boundaries: CLE

The stress-energy tensor and its descendants, in my re-formulation (2010)

Consider a function f on a space Ω where there is action of conformal maps on some domain.

- Conformal derivative: for h holomorphic on some domain,

$$\nabla_h f(\Sigma) = \lim_{\varepsilon \to 0} \frac{f((\mathrm{id} + \varepsilon h)(\Sigma)) - f(\Sigma)}{\varepsilon}$$

- A particular differential operator:

$$\Delta_w^{(n)} = \int_0^{2\pi} d\theta \, e^{-i\theta} \nabla_{z \mapsto e^{i\theta}(w-z)^{n+1}}$$

If f is invariant under conformal maps g on a simply connected domain D, then,

$$(\partial g(w))^2 \Delta_{g(w)}^{(-2)} f(g(\Sigma)) = \Delta_w^{(-2)} f(\Sigma).$$

Consider also $Z = Z(D)$ a Möbius invariant function of simply connected domains D with

$$\Delta_w^{(-2)} \log Z(D) = \frac{c}{12} \{g, w\}$$

for $w \in D$ and $g : D \to \mathbb{D}$.

The vector space spanned by (for $0 \in D$)

$$\bar{\Delta}_0^{(n_1,\ldots,n_k)} f(\Sigma) := Z^{-1} \Delta_0^{(n_1)} \cdots \Delta_0^{(n_k)} Z f(\Sigma) \equiv L_{n_1} \cdots L_{n_k} \mathbf{1}$$

for $n_j \leq -2$ forms a vertex operator algebra, with

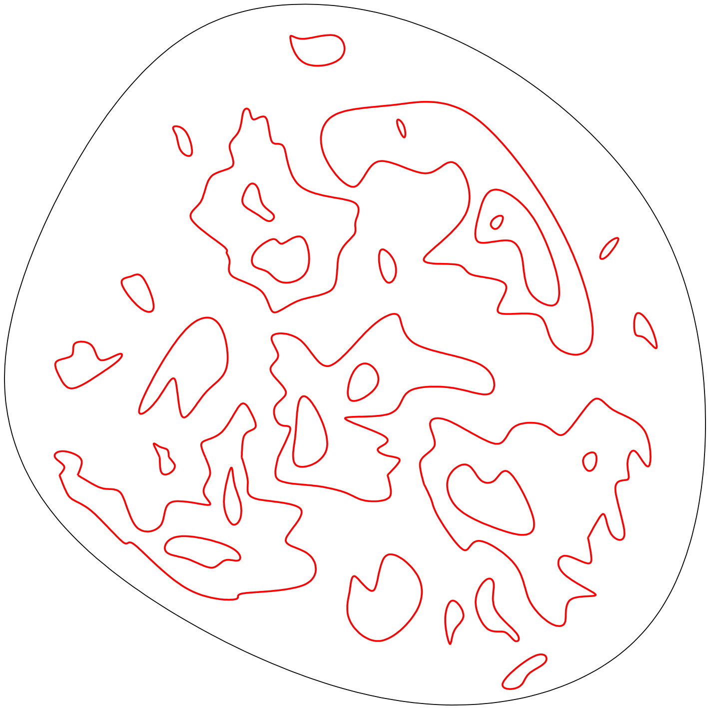
$$\bar{\Delta}_{w_I}^{N_I} \left(\bar{\Delta}_{w_{II}}^{N_{II}} (\cdots f) \right) (\Sigma) = Y(L_{N_I} \mathbf{1}, w_I) Y(L_{N_{II}} \mathbf{1}, w_{II}) \cdots \mathbf{1}.$$

In CFT: $f =$ correlation functions, $Z =$ relative partition function

$$Z = \frac{Z_D Z_{\hat{\mathbb{C}} \setminus \overline{A}}}{Z_{D \setminus \overline{A}}}, \qquad \overline{A} \subset D.$$

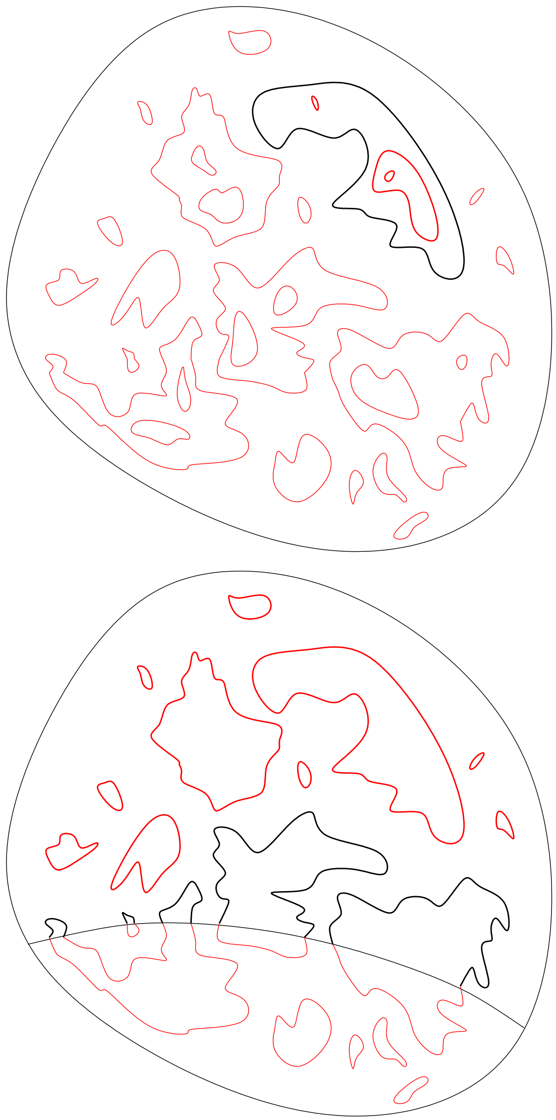
Stress-energy tensor in conformal loop ensembles

Conformal loop ensembles: Consider the set $\mathcal{S}_D$ whose elements are collections of at most a countable infinity of self-avoiding, disjoint loops lying on a simply connected domain D.



A conformal loop ensemble can be seen as a family of measures μ_D on the sets $\mathcal{S}_D$ for all simply connected domains D, with three defining properties.

1. Conformal invariance. For any conformal transformation $f : D \to D'$, we have $\mu_D = \mu_{D'} \cdot f$.
2. Nesting. The measure μ_D restricted on a loop $\gamma \subset D$ and on all loops outside γ is equal to the CLE measure μ_{D_γ} on the domain $D_\gamma \subset D$ delimited by γ.
3. Conformal restriction. Given a domain $B \subset D$ such that $D \setminus B$ is simply connected, consider $\tilde{B}$, the closure of the set of points of B and points that lie inside loops that intersect B. Then the measure on each component C_i of $D \setminus \tilde{B}$, obtained by restriction on loops that intersect B, is μ_{C_i}.

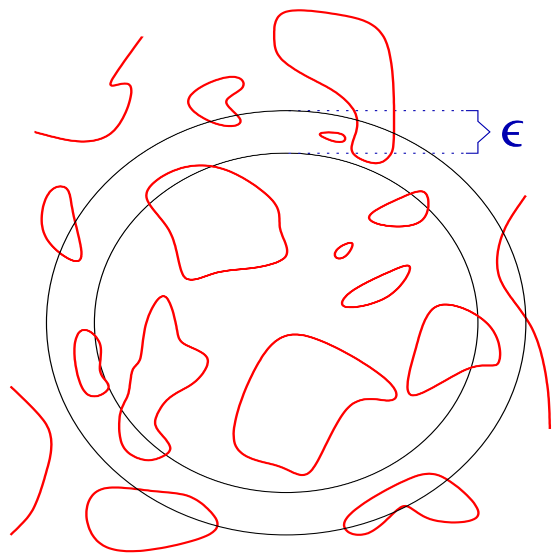


[Sheffield, Werner]

Some renormalised probabilities

Consider a family of events $\mathcal{E}(A, \epsilon)$ characterised by any simply connected domains A and any small enough real numbers $\epsilon > 0$, defined as follows:

- For $A = \mathbb{D}$, it is the event that no loop intersect both $(1-\epsilon)\partial\mathbb{D}$ and $\partial\mathbb{D}$.



- For $A \neq \mathbb{D}$, it is the event $g_A(\mathcal{E}(\mathbb{D}, \epsilon))$, where the conformal transformations g_A is chosen such that $A = g_A(\mathbb{D})$, and such that if $A = G(B)$ for some global conformal transformation G, then there is a global conformal transformation K with $K(B) = B$ such that $g_A = G \circ K \circ g_B$.

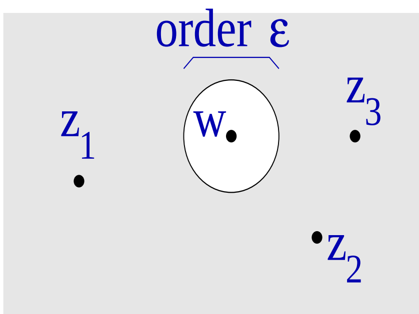
The renormalised probability can be defined by

(for A simply connected and disjoint from the support of X):

$$P^{\rm ren}(X;A)_D = \mathcal{N} \lim_{\epsilon \longrightarrow 0} \frac{P(X \cap \mathcal{E}(A,\epsilon))_D}{P(\mathcal{E}(\mathbb{D},\epsilon))_{2\mathbb{D}}}$$

Theorems:

- The limit exists.
- The ratio $P(X)_{D\setminus\overline{A}} := \dfrac{P^{\rm ren}(X;A)_D}{P^{\rm ren}(A)_D}$ is invariant under maps conformal on $D \setminus \overline{A}$.
- We have $P^{\rm ren}(g\cdot X; g(A))_{g(D)} = f(g,A)P^{\rm ren}(X;A)_D$ with $f(g,A) = 1$ if g is a Möbius map.



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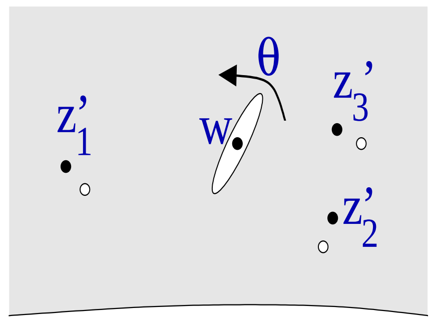
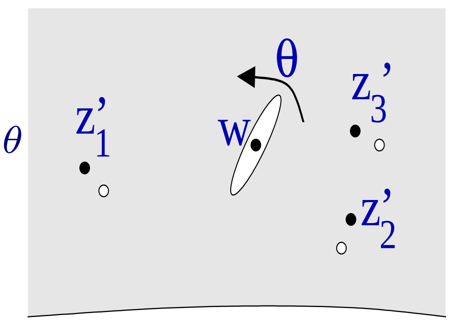
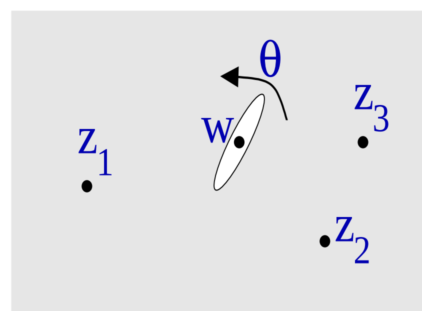


$=$ restricted probability: $\dfrac{P^{\mathrm{ren}}(X;E)}{P^{\mathrm{ren}}(E)}$

$$\Rightarrow \quad 0 \quad = \int_0^{2\pi} d\theta\, e^{-2i\theta}$$



$$= \int_0^{2\pi} d\theta e^{-2i\theta} \Bigg($$



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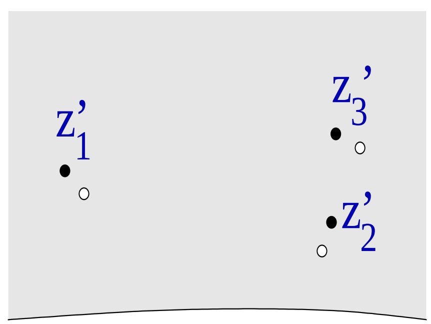

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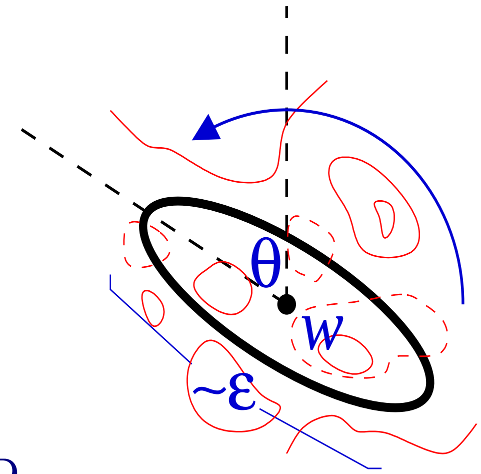
The stress-energy tensor in CLE: [BD 2009-2010]



- $T(w)$: "Random variable" forbidding loops from crossing a small, spin-2 rotating ellipse

$$P^\sharp(\cdot; w)_D = \lim_{\varepsilon \to 0} \frac{8}{\pi \varepsilon^2} \int d\theta\, e^{-2i\theta} P^{\rm ren}(\cdot; E(w,\varepsilon,\theta)_D$$

- Z: Ratio of probabilities where loops are forbidden to cross ∂D

$$\frac{P^{\rm ren}(\hat{\mathbb{C}} \setminus D)_{\hat{\mathbb{C}}}}{P^{\rm ren}(\hat{\mathbb{C}} \setminus D)_{\hat{\mathbb{C}} \setminus \overline{A}}}$$

- I prove conformal Ward identities (and boundary condition, etc.),

$$P^\sharp(X; w)_D = Z^{-1} \Delta_w^{(-2)} Z P(X)_D, \quad \text{in particular } P^\sharp(w)_D = \Delta_w^{-2} \log Z$$

- I prove conformal covariance, equivalently $\Delta_w^{-2} \log Z = c/12\{g, w\}$.

How to get the Schwarzian derivative:

Given a map g conformal in a neighbourhood of $w \neq \infty$ (with $g(w) \neq \infty$), there is a unique Möbius map G such that

$$(G \circ g)(z) = z + a(z-w)^3 + O((z-w)^4)$$

for some a, and this coefficient a is uniquely fixed to

$$a = \{g, w\}/6$$

Universality:

Any “random variable” supported on a point which transforms like the stress-energy tensor and is zero on the unit disk, satisfies the conformal Ward identities (and boundary conditions, etc.), hence is a stress-energy tensor.

For instance: $n(z_1, z_2) =$ number of loops surrounding both z_1 and z_2,

$$T(w) \overset{?}{\propto} \lim_{\substack{|z_1 - z_2| \to 0 \\ (z_1 + z_2)/2 = w}} \partial_{z_1} \partial_{z_2} \left(n(z_1, z_2) - \frac{c}{2} \log |z_1 - z_2| \right)$$