Self-avoiding walk on the hexagonal lattice

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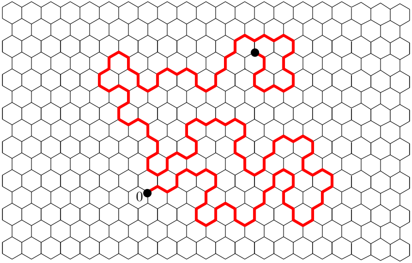
joint work with S. Smirnov

Consider the **hexagonal lattice** $\mathbb{H}$, and define c_n to be the number of self-avoiding walks of length n starting at the origin.



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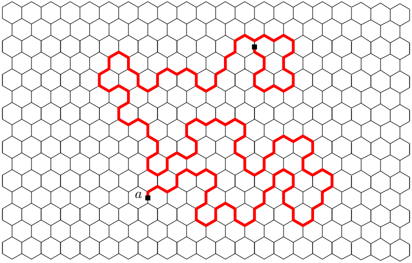

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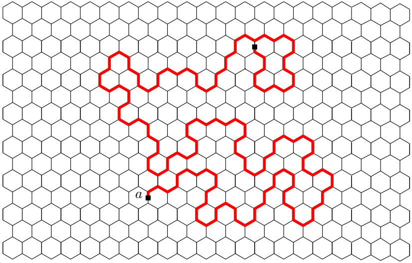

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- **Combinatorial question:** How many such trajectories are there? In other words, what is the **value of** c_n?

Theorem (H. D-C, S. Smirnov, 2010)

The connective constant μ satisfies $\mu := \lim_{n\to\infty} c_n^{\frac{1}{n}} = \sqrt{2+\sqrt{2}}$.

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Strategy : It is sufficient to prove that the **partition function** $Z(x)$ is finite if and only if $x > \sqrt{2+\sqrt{2}} =: x_c$, where

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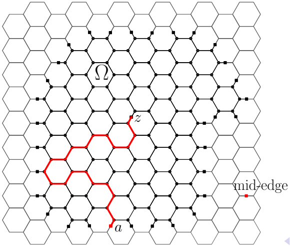
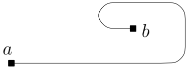
We restrict ourself to *finite domains* Ω and we weight walks by their *winding*.



Definition

The winding $\mathrm{W}_\Gamma(a, b)$ of a curve Γ between a and b is the rotation (in radians) of the curve between a and b.

$\mathrm{W}_\gamma(a,b) = 0$



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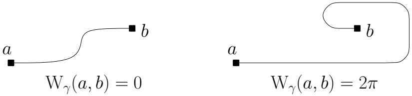

With this definition, we can define the *parafermionic operator* for $a \in \partial\Omega$ and $z \in \Omega$:

$$F(z) = F(a,z,x,\sigma) := \sum_{\gamma \subset \Omega:\ a \to z} \mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)} x^{-\ell(\gamma)}.$$

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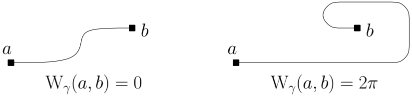

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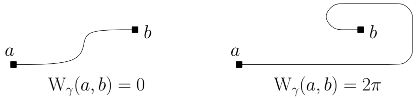

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Lemma (Discrete integrals on elementary contours vanish)

If $x = x_c$ and $\sigma = \frac{5}{8}$, then F satisfies the following relation for every vertex $v \in V(\Omega)$,

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0$$

where p, q, r are the mid-edges of the three edges adjacent to v.

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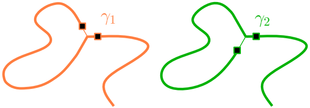

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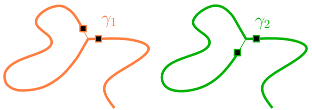

In the first case:

$$c(\gamma_1) + c(\gamma_2) = (q - v)e^{-i\sigma W_{\gamma_1}(a,q)} x_c^{-\ell(\gamma_1)} + (r - v)e^{-i\sigma W_{\gamma_2}(a,r)} x_c^{-\ell(\gamma_2)}$$

$$= (p - v)e^{-i\sigma W_{\gamma_1}(a,p)} x_c^{-\ell(\gamma_1)} \left(e^{i\frac{2\pi}{3}} e^{-i\frac{5}{8}\cdot\frac{-4\pi}{3}} + e^{-i\frac{2\pi}{3}} e^{-i\frac{5}{8}\cdot\frac{4\pi}{3}} \right) = 0$$

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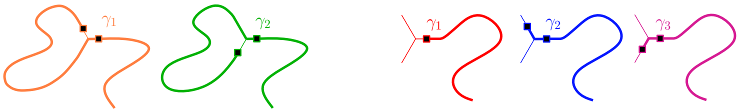

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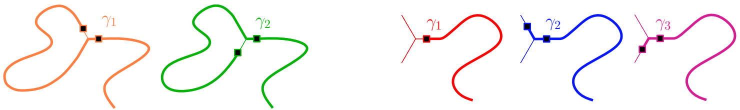

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In the second case:

$$c(\gamma_1)+c(\gamma_2) + c(\gamma_3)$$
$$= (p - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,p)} x_c^{-\ell(\gamma_1)} \left(1 + x_c \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} \mathrm{e}^{-\mathrm{i}\frac{5}{8}\cdot\frac{-\pi}{3}} + x_c \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} \mathrm{e}^{-\mathrm{i}\frac{5}{8}\cdot\frac{\pi}{3}} \right) = 0.$$

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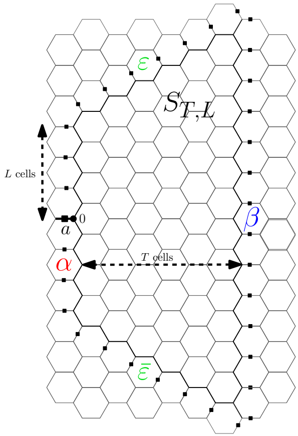

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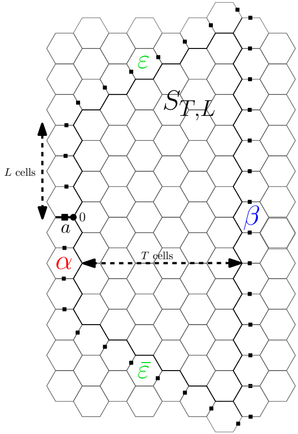

Sum the relation over all vertices in $V(S_{T,L})$ (contour=boundary):

$$0 = -\sum_{z\in\alpha} F(z) + \sum_{z\in\beta} F(z) + e^{i\frac{2\pi}{3}} \sum_{z\in\varepsilon} F(z) + e^{-i\frac{2\pi}{3}} \sum_{z\in\bar{\varepsilon}} F(z)$$

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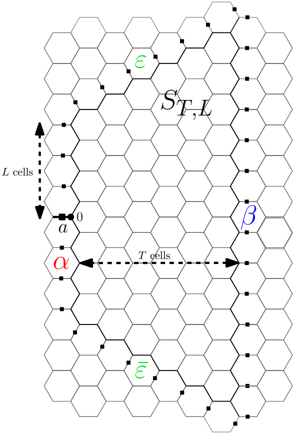
Definition:

$$A^x_{T,L} = \sum_{\gamma \subset S_{T,L}:\ a \to \alpha \setminus \{a\}} x^{-\ell(\gamma)},$$

$$B^x_{T,L} = \sum_{\gamma \subset S_{T,L}:\ a \to \beta} x^{-\ell(\gamma)},$$

$$E^x_{T,L} = \sum_{\gamma \subset S_{T,L}:\ a \to \varepsilon \cup \bar{\varepsilon}} x^{-\ell(\gamma)}.$$



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$$\sum_{z\in\alpha} F(z) = F(a) + \sum_{z\in\alpha\setminus\{a\}} F(z) = 1 - \cos\left(\frac{3\pi}{8}\right) A^x_{T,L}$$

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When $x = x_c$, we have

$$1 = \cos\left(\frac{3\pi}{8}\right) A^{x_c}_{T,L} + B^{x_c}_{T,L} + \cos\left(\frac{\pi}{4}\right) E^{x_c}_{T,L}$$

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(4) It is an easy step to deduce that $B_T^{x_c} \geq c/T$ and therefore

$$Z(x_c) \geq \sum_{T>0} B_T^{x_c} = +\infty.$$

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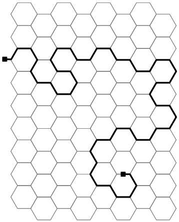

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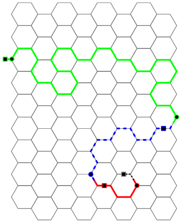

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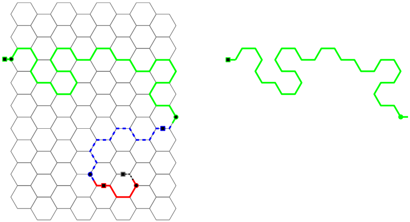

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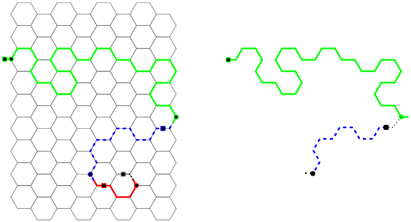

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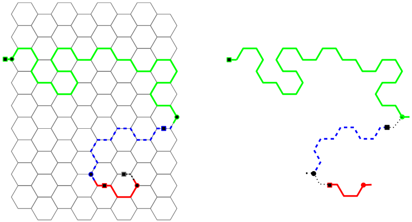

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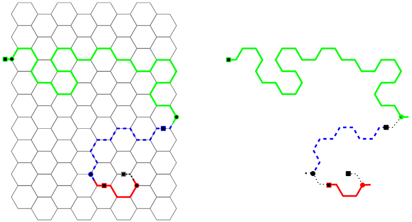

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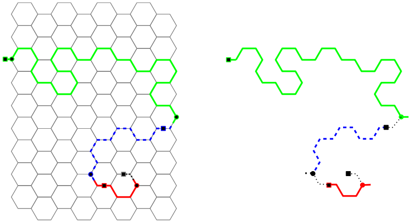

(3) We obtain $$Z(x) \leq 2 \sum_{\substack{T_{-i}<\cdots<T_{-1} \\ T_j<\cdots<T_0}} \left(\prod_{k=-i}^{j} B_{T_k}^x \right) = \prod_{T>0} (1 + B_T^x)^2 < +\infty$$

Conjecture (Nienhuis)

There exists a constant $A \in (0, +\infty)$ such that:

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Conjecture (Lawler, Schramm, Werner)

The scaling limit of the self-avoiding walks is SLE(8/3).

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- similar observable for $O(n)$ models: for $n \in [-2, 2]$, infinite correlation length at $\sqrt{2+\sqrt{2-n}}$.

Extensions of the result

- If we can find the other half of holomorphicity of the parafermionic observable, we can obtain conformal invariance.
- Probably possible to obtain further information: polynomial bounds on the correction, etc...
- similar observable for $O(n)$ models: for $n \in [-2, 2]$, infinite correlation length at $\sqrt{2+\sqrt{2-n}}$.
- similar observable for random-cluster models: second (resp. first) order phase transition for $q \in [1, 4)$ (resp. $q \in (4, +\infty)$), complete conformal invariance when $q = 2$.

Thank you