A Perspective on Liouville Quantum Gravity & KPZ

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Conformal maps from probability to physics

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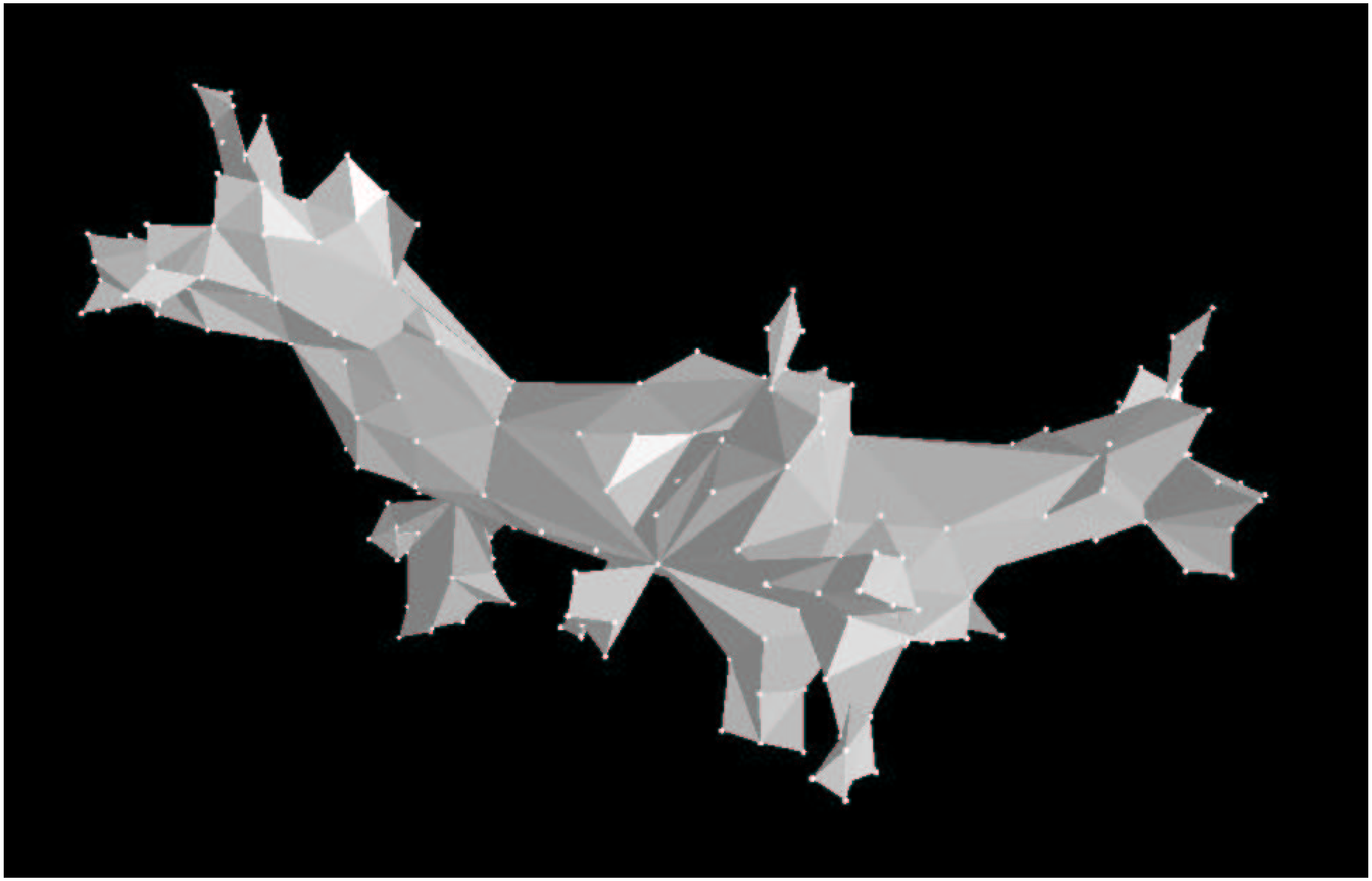
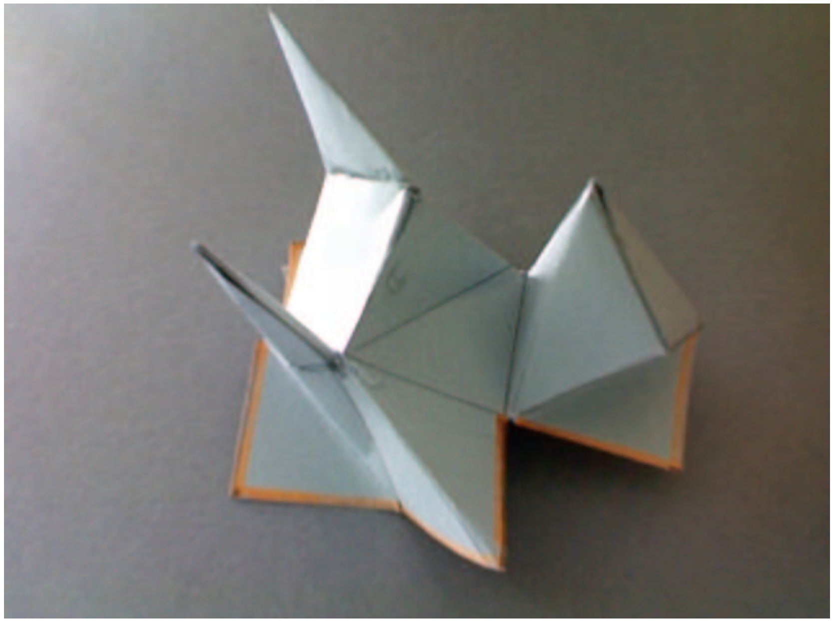
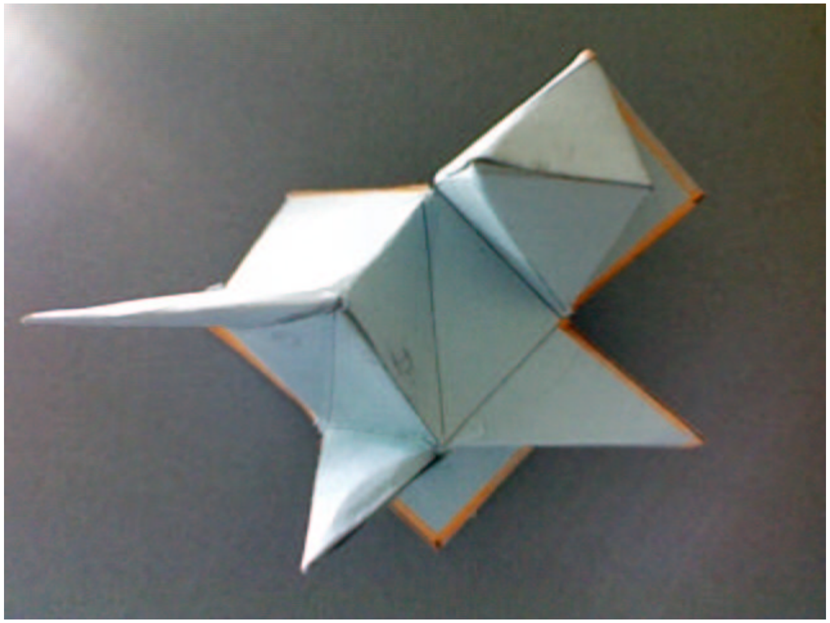
A Random Surface



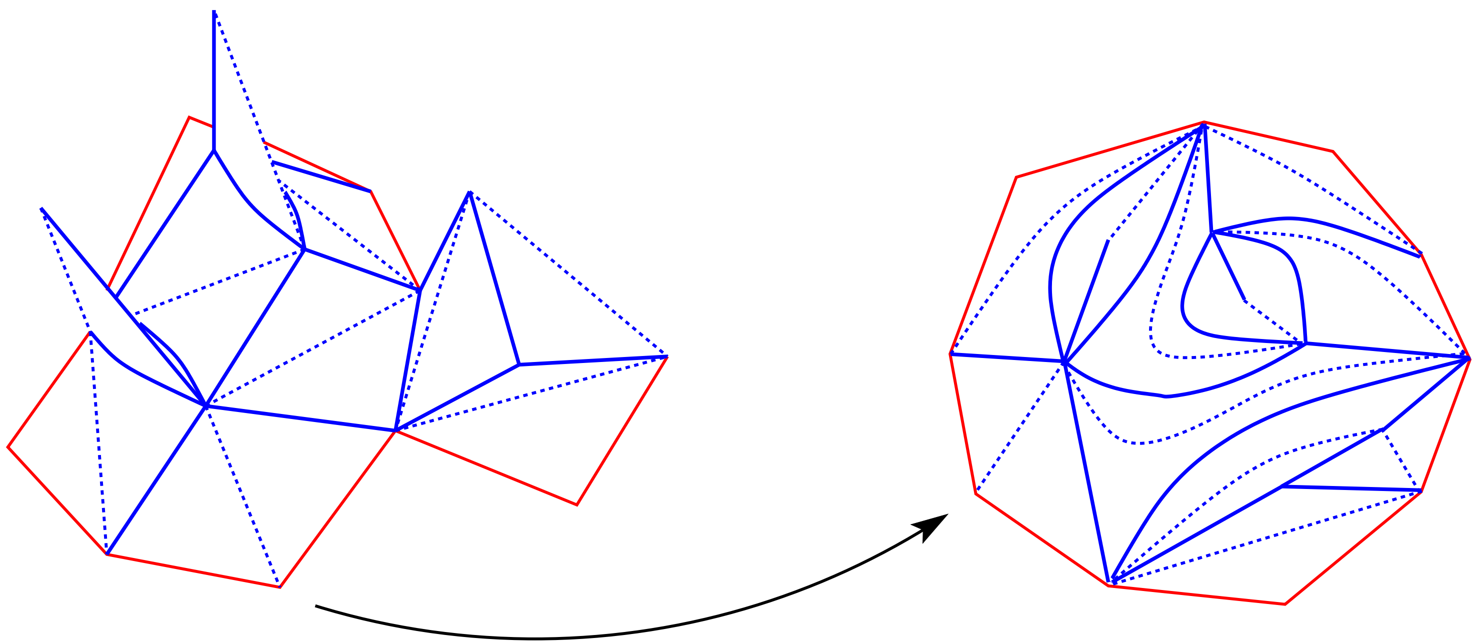
[Courtesy of G. Chapuy (2009)]

A Random Quadrangulation





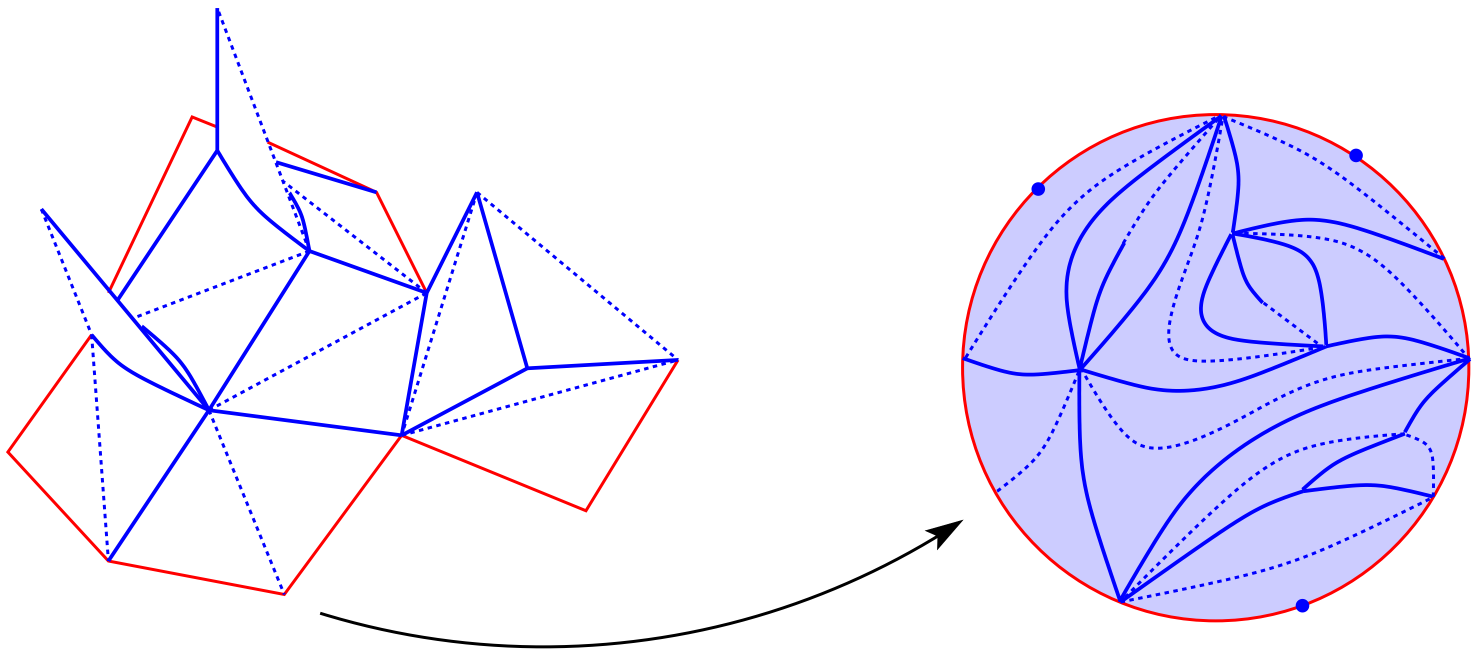
Random Quadrangulation & Random Planar Map



RS & Random Matrices *BIPZ '78; Ambjørn, Durhuus, Fröhlich, Jonsson '83-85; David '85; Boulatov, Kazakov, Kostov, Migdal '85...*

Bijective Combinatorics *Cori, Vauquelin '81; Schaeffer '97; Angel, Schramm '03; Bouttier, Di Francesco, Guitter '04; Le Gall, Miermont...*

Random Quadrangulation & Conformal Map to $\mathbb{D}$

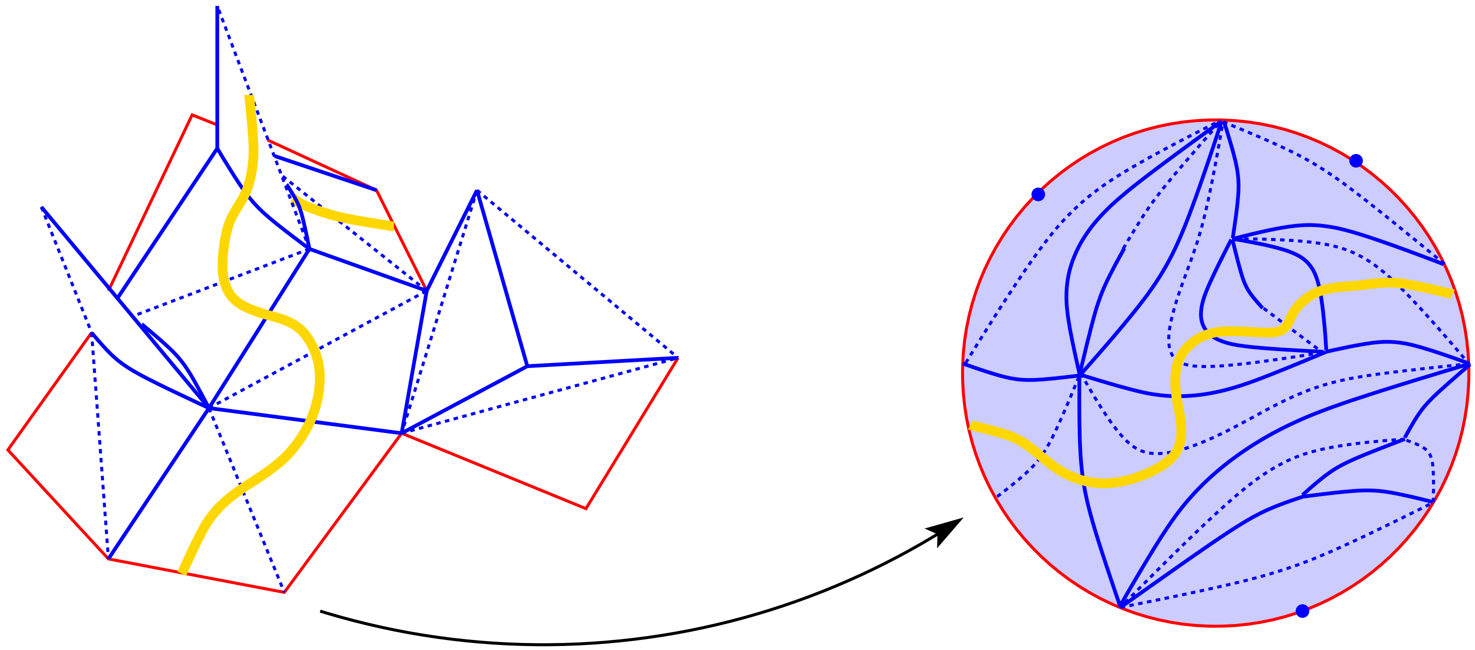


In the continuum scaling limit: **Liouville Quantum Gravity**

A.M. Polyakov '81

Correlation Functions *Seiberg, '90; Goulian, Li '91; Ginsparg, Moore '93; Dorn, Otto '94; Takhtajan '95; Teschner '95; Zamolodchikov^2 '96; Fateev-ZZ '00; Ponsot, Teschner '02; Kostov, Ponsot, Serban '04...*

Random Quadrangulation & Random Model



Ising, SAW, $O(N)$ & Potts models: **Random Matrix Models**

Kazakov '86; D. & Kostov '88; Daul; Eynard, Zinn-Justin[2]; Staudacher...

Bijective Combinatorics *Chassaing & Schaeffer '02;*

Bousquet-Mélou & Schaeffer '02; BDFG '02; Bernardi & B.-M. '09...

Continuum limit: **Liouville Gravity & Conformal Ansatz**

Liouville Field Theory (Polyakov '81)

$$S_{\mathcal{L}} = \frac{1}{2}\int d^2z\sqrt{\hat{g}}\left(\hat{g}^{ab}\partial_a h\partial_b h + Q\hat{R}h + \lambda e^{\gamma h}\right) [+\mathrm{CFT}]$$

Background metric $\hat{g}$ & curvature $\hat{R}$

Random metric $g_{ab} = e^{\gamma h}\hat{g}_{ab}$

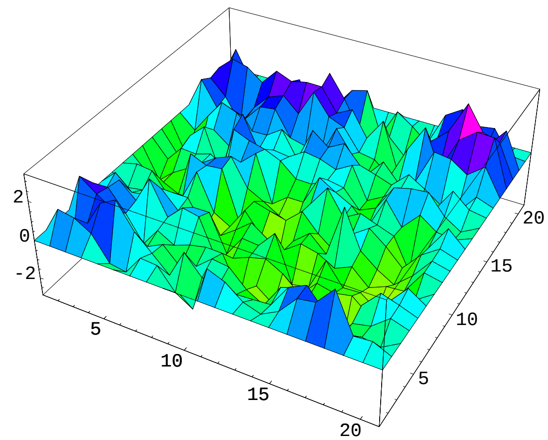
Quantum area $\mathcal{A} = \int d^2z\sqrt{\hat{g}}e^{\gamma h}$

Conformal invariance & CFT central charge

$$\gamma = \frac{1}{\sqrt{6}}\left(\sqrt{25-c} - \sqrt{1-c}\right) \left(= \sqrt{\kappa \wedge 16/\kappa}, \quad \mathrm{SLE}_\kappa\right)$$

$$\gamma \leqslant 2, \quad c \leqslant 1; \qquad Q = \frac{2}{\gamma} + \frac{\gamma}{2}$$

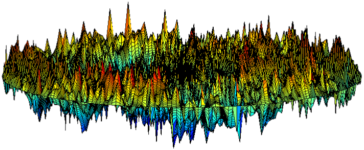
Gaussian Free Field (GFF)



Distribution h with *Gaussian weight* $\exp\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$(f_1, f_2)_\nabla := (2\pi)^{-1}\int_D \nabla f_1(z)\cdot\nabla f_2(z)\,d^2z$$
$$= \mathrm{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big)$$

$\diamondsuit$ STARRING THE GFF! *(Courtesy of N.-G. Kang)* $\diamondsuit$



Liouville QG

Random Measure

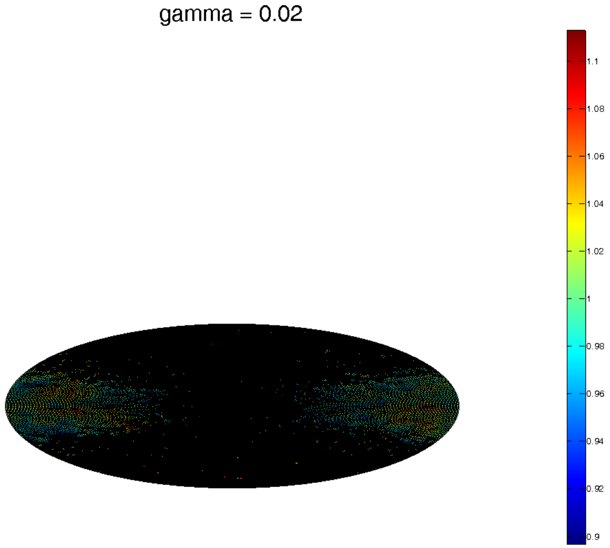
$$d\mu = \text{“}e^{\gamma h} d^2 z\text{”}$$

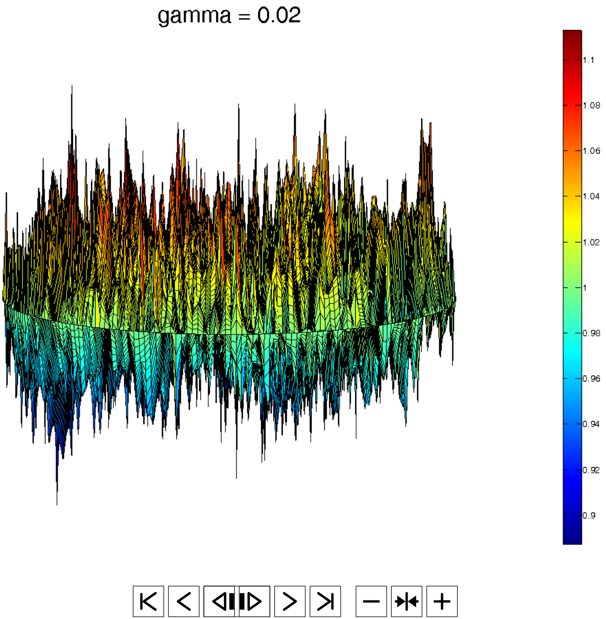
$\Diamond$

The Emergence of Quantum Gravity

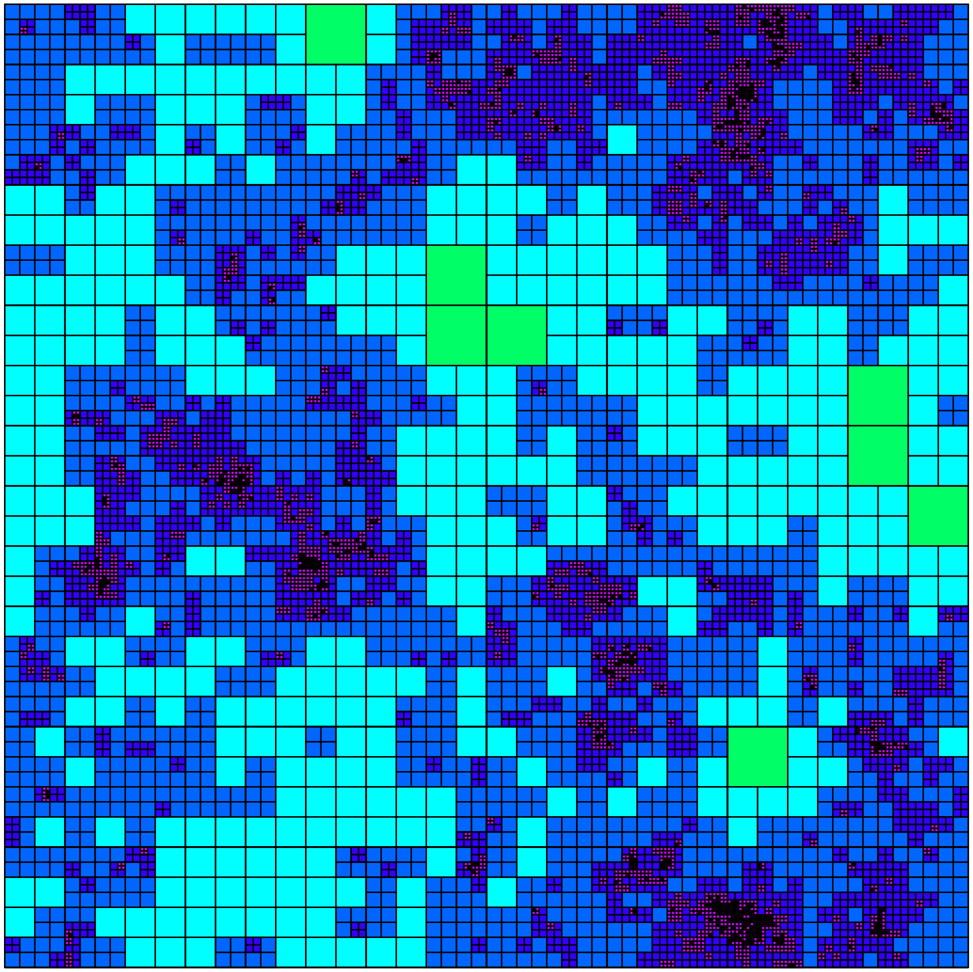
(Courtesy of N.-G. Kang)

$\Diamond$


gamma = 0.02
1.1
1.08
1.06
1.04
1.02
1
0.98
0.96
0.94
0.92
0.9


gamma = 0.02
1.1
1.08
1.06
1.04
1.02
1
0.98
0.96
0.94
0.92
0.9

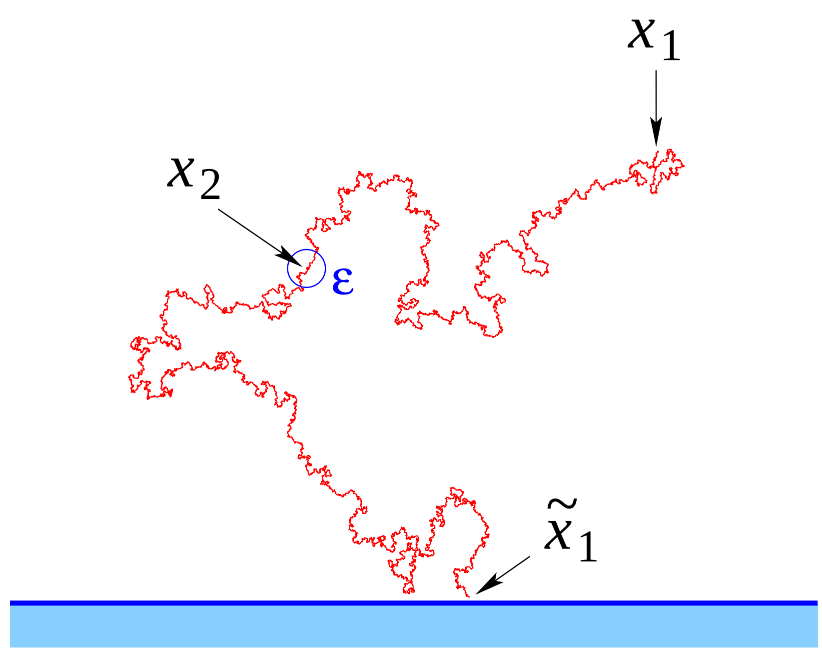
Discrete Quantum Gravity Measure ($\gamma = 3/2$)



Euclidean squares of similar quantum area δ

Scaling Exponents of (Random) Fractals in $\mathbb{H}$

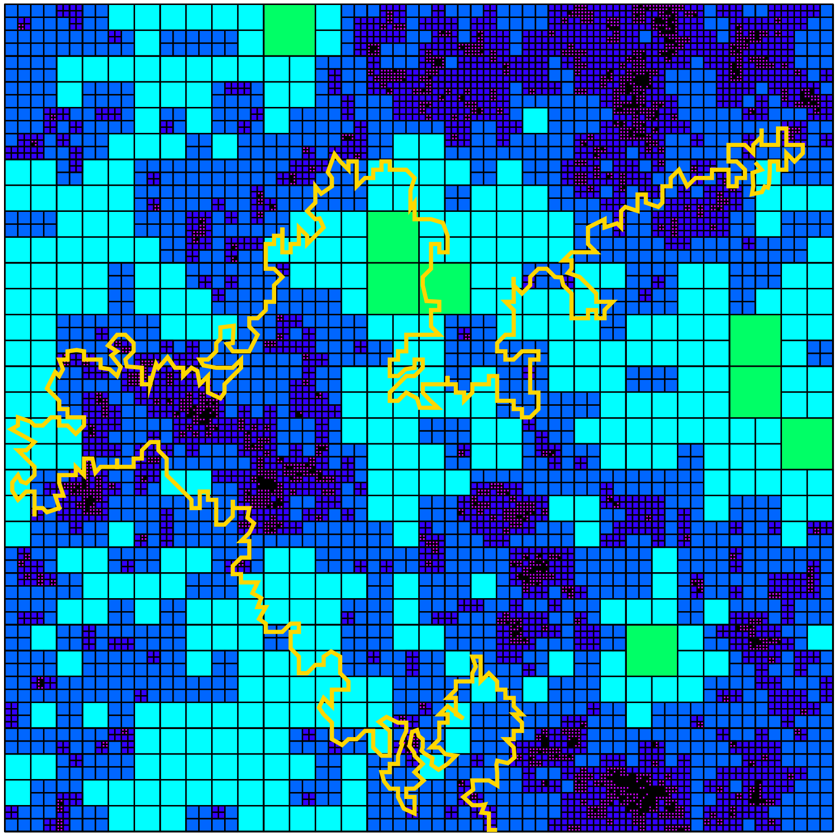
SAW in half plane - 1,000,000 steps



Probabilities & Hausdorff Dimensions (e.g., SLE_κ)

$$\mathbb{P} \asymp \varepsilon^{2x}, \quad \tilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \qquad D = 2 - 2x_2 \qquad (= 1 + \kappa/8)$$

Quantum Gravity Scaling Exponents Δ & $\tilde{\Delta}$



$$\mathbb{P} \asymp \delta^{\Delta}, \qquad \tilde{\mathbb{P}} \asymp \tilde{\delta}^{\tilde{\Delta}}$$

KPZ *(Knizhnik, Polyakov, Zamolodchikov '88)*

x and Δ ($\tilde{x}$ and $\tilde{\Delta}$) are related by the **KPZ formula**

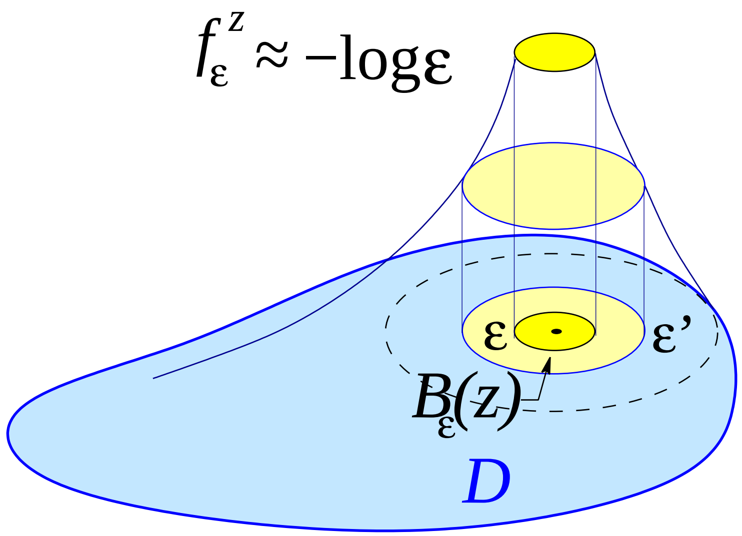
$$x = \left(1 - \frac{\gamma^2}{4}\right)\Delta + \frac{\gamma^2}{4}\Delta^2$$

KPZ is a Theorem *[D. & Sheffield, '08]*

arXiv:0808.1560 [math.PR] & PRL **102**, *150603 (2009)*

Kazakov '86; D. & Kostov '88 [Random matrices]
David; Distler & Kawai '88 [Liouville field theory]
Benjamini & Schramm '08; Rhodes & Vargas '08 [Math]
David & Bauer '08

GFF Circular Average & Logarithmic Potential



$$h_\varepsilon(z) \;:=\; (h, f_\varepsilon^z)_\nabla$$

$$f_\varepsilon^z(\cdot) \;:=\; -\log(|\cdot - z| \vee \varepsilon) + G_z(\cdot)$$

$G_z(\cdot)$ *harmonic extension of* $\log|\cdot - z|$ *in* D

Variance

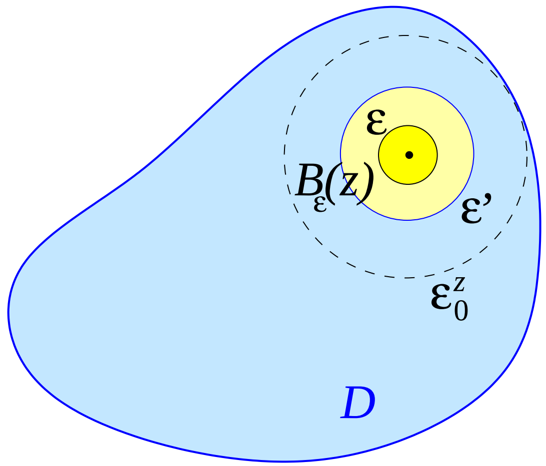
$$\operatorname{Var} h_\varepsilon(z) = f_\varepsilon^z(z) = \log[C(z,D)/\varepsilon]$$

$C(z,D)$ *conformal radius of* D *viewed from* z

$h_\varepsilon(z)$ Gaussian random variable

$$\mathbb{E}\, e^{\gamma h_\varepsilon(z)} = e^{\gamma^2 \operatorname{Var} h_\varepsilon(z)/2} = \left(\frac{C(z,D)}{\varepsilon}\right)^{\gamma^2/2} \qquad \square$$

GFF & Brownian Motion



- $h_\varepsilon(z)$ mean value of h on circle $\partial B_\varepsilon(z)$
- Define $t := -\log \varepsilon$, $\mathcal{B}_t := h_{\varepsilon = e^{-t}}(z)$; for z fixed,

the law of $\mathcal{B}_t$ is **standard Brownian motion** in t

$$\mathrm{Var}\left[(h_\varepsilon - h_{\varepsilon'})(z)\right] = |\log(\varepsilon/\varepsilon')| = |t - t'| = \mathrm{Var}\left[\mathcal{B}_t - \mathcal{B}_{t'}\right] \;\square$$

STOCHASTIC AREA

$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right]\varepsilon^{\gamma^2/2} d^2 z$$

converges to a random measure as $\varepsilon \to 0$ *for*

$$\gamma < 2$$

(Høegh-Krohn '71)

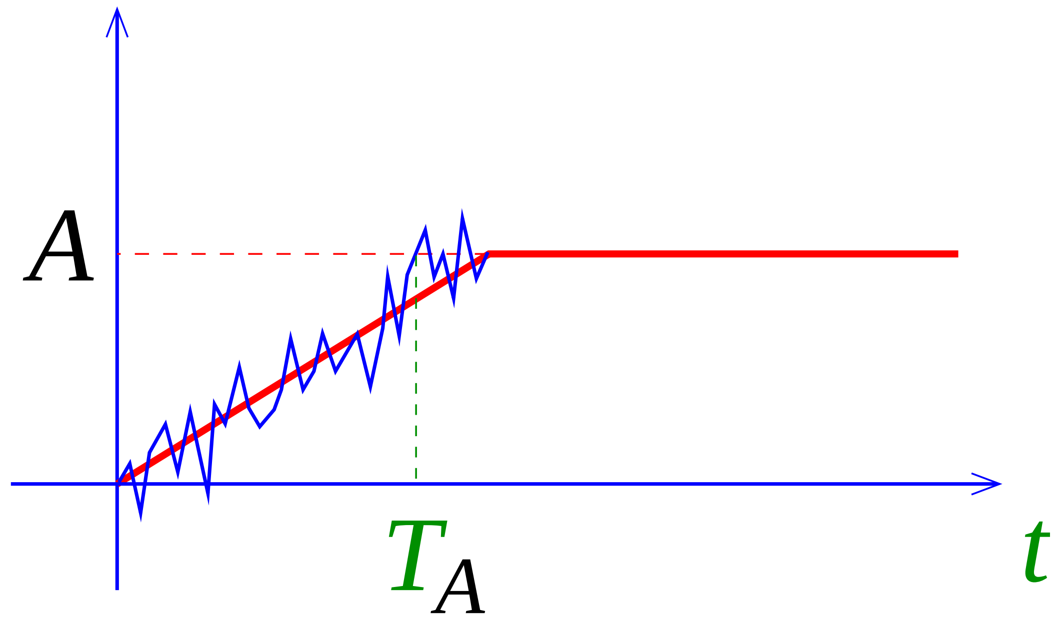
Quantum Ball & Brownian Motion

Quantum area

- $$\delta := \exp\left[\gamma h_\varepsilon(z)\right] \pi \varepsilon^{2+\gamma^2/2}$$

Given z, $h_\varepsilon(z)$ is standard Brownian motion $\mathcal{B}_t$, $t = -\log \varepsilon$, plus the deterministic term: $\gamma f_\varepsilon^z(z) = -\gamma \log \varepsilon = \gamma t$

$$\begin{aligned} \delta &= \exp\left(\gamma \mathcal{B}_t - a t\right), \qquad a := 2 - \gamma^2/2 \ (> 0) \\ -\log \delta &= a t - \gamma \mathcal{B}_t \qquad \square \qquad \text{(B. M. \& drift)} \end{aligned}$$



The stochastic area of ball $B_\varepsilon(z)$ equals δ at **stopping time** T_A

$$T_A \;:=\; -\log \varepsilon_A := \inf\{t : at - \gamma \mathcal{B}_t = A\}$$

$$A \;:=\; -\log \delta > 0, \qquad a = 2 - \gamma^2/2 > 0 \quad (\gamma < 2)$$

KPZ Theorem

Stochastic probability & stopping time

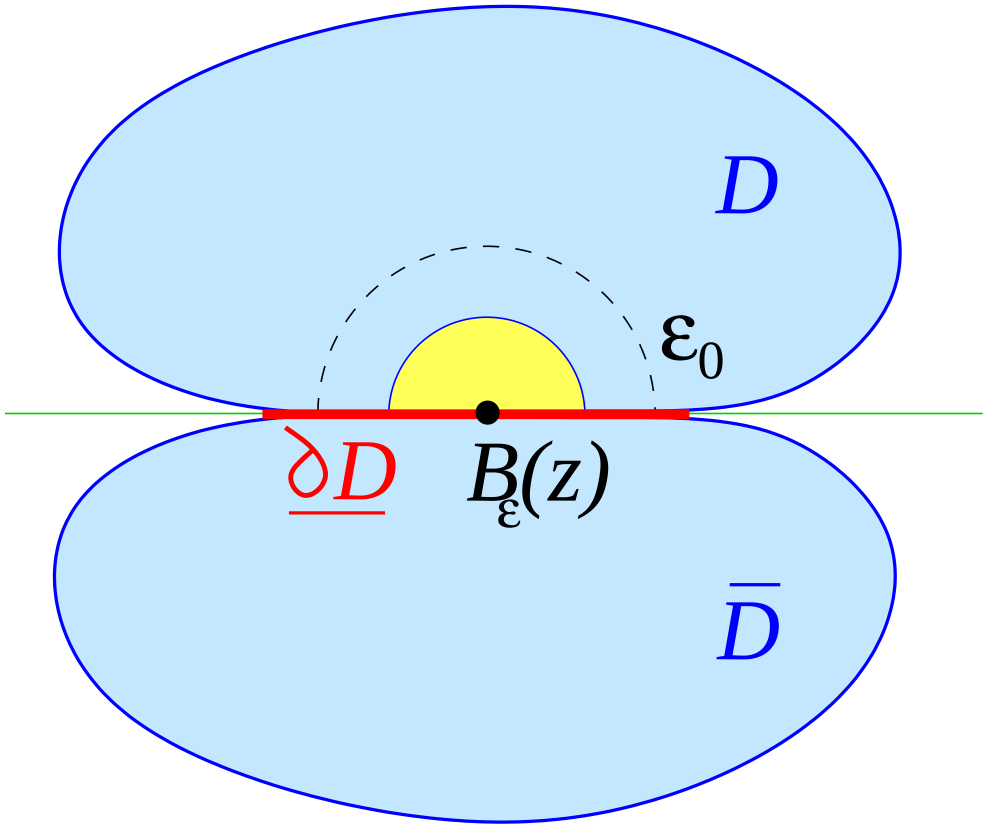
$$-\log \varepsilon_A = T_A = \inf\{t : a t - \gamma \mathcal{B}_t = -\log \delta =: A\}$$

BROWNIAN MARTINGALE & LARGE DEVIATIONS

$$\mathbb{E}\left[{\varepsilon_A}^{2x}\right] = \mathbb{E}\left[e^{-2xT_A}\right] = \exp(-\Delta A) = \delta^{\Delta}$$

$$2x = a\Delta + \frac{\gamma^2}{2}\Delta^2, \qquad a = 2 - \frac{\gamma^2}{2}\,(>0) \qquad \text{(KPZ)} \quad \square$$

Boundary Liouville Quantum Gravity



Free b.c. GFF on $\underline{\partial D}$ *; half-circle averages* $h_\varepsilon(z)$

Quantum Interval & Brownian Motion

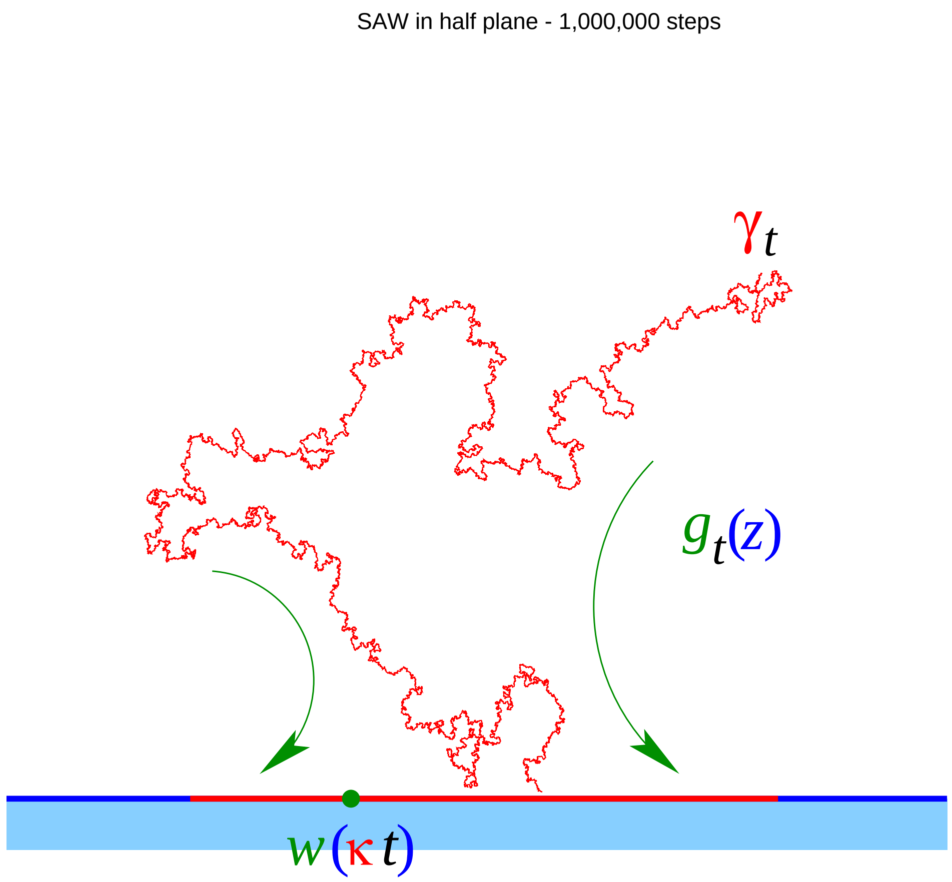
Quantum length

- $$\delta := \exp\left[\gamma h_\varepsilon(z)/2\right]\, 2\,\varepsilon^{1+\gamma^2/4}$$

Given $z \in \mathbb{R}$, $h_\varepsilon(z)$ is standard Brownian motion $\mathcal{B}_{2t}$, with $t := -\log\varepsilon$, *plus the deterministic term:* $-\gamma\log\varepsilon = \gamma t$

$$\begin{aligned}
\delta &= \exp\left(\frac{1}{2}\gamma\mathcal{B}_{2t} - \frac{1}{2}at\right), \qquad a := 2-\gamma^2/2 \\
-\log\delta &= \frac{1}{2}at - \frac{1}{2}\gamma\mathcal{B}_{2t} \qquad \square \qquad \text{(B. M. \& drift)}
\end{aligned}$$

Stochastic Löwner Evolution (SLE$_\kappa$) (Schramm)



$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}$$

Take z on the *real axis*, $z = x \in \mathbb{R} \setminus \{0\}$, and consider the *stopping time* T_y, first time t such that $g_t(x) - \sqrt{\kappa} B_t = y \in \mathbb{R}$.
Derivatives' moments [LSW]:

$$\mathbb{E}\left([g'_{T_y}(x)]^b | T_y < T_0 \right) = (x/y)^{\Delta}$$

Fact: Δ is the *KPZ quantum exponent* corresponding to the *Euclidean exponent* b:

$$\Delta(b) = \frac{1}{2\kappa}\left[\sqrt{16\kappa b + (\kappa - 4)^2} + \kappa - 4\right]$$

obeying KPZ for $\gamma := \sqrt{\kappa}$

$$b = \left(1 - \gamma^2/4\right)\Delta + (\gamma^2/4)\Delta^2$$

[Δ standard quantum exponent for $\kappa < 4, \gamma < 2$, dual one for $\kappa > 4, \gamma > 2$]

Boundary SLE & Liouville Quantum Gravity

Write on the real axis

$$X_t := g_t(x) - \sqrt{\kappa} B_t, \; X_t' := g_t'(x)$$

SDE:

$$\begin{aligned} dX_t &= \frac{2}{X_t} dt - \sqrt{\kappa}\, dB_t \\ d\log X_t' &= -\frac{2}{X_t^2} dt \end{aligned}$$

X_t is a *real Bessel process*. For $x > 0$, $X_t > 0$ for $t < T_0$; from Itô's formula:

$$d\log X_t = \left(2 - \frac{\kappa}{2}\right) \frac{dt}{X_t^2} - \sqrt{\kappa} \frac{dB_t}{X_t}$$

Define the *random time* $d\tau := \frac{2}{X_t^2} dt$, so that $\langle \frac{dB_t}{X_t}, \frac{dB_t}{X_t} \rangle = \frac{1}{2} d\tau$ and $\frac{dB_t}{X_t} = \frac{1}{2} d\hat{B}_{2\tau}$, where $\hat{B}_{2\tau}$ is *standard Brownian motion* with respect to 2τ. *Time changed* chordal SLE $\hat{X}_\tau := X_t$:

$$
\begin{aligned}
d \log \hat{X}_\tau &= \left(2 - \frac{\kappa}{2}\right) \frac{1}{2} d\tau - \frac{1}{2} \sqrt{\kappa}\, d\hat{B}_{2\tau} \\
-d \log \hat{X}'_\tau &= d\tau
\end{aligned}
$$

For $\gamma = \sqrt{\kappa}$, *this coincides with the equations for the boundary quantum length* δ *in terms of the Euclidean length* ε*:*

$$
\begin{aligned}
-\log \delta &= \left(2 - \frac{\gamma^2}{2}\right) \frac{1}{2} t - \frac{1}{2} \gamma \mathcal{B}_{2t} \\
-\log \varepsilon &= t
\end{aligned}
$$

$$\begin{aligned} \log \hat{X}_\tau &\equiv -\log \delta \\ \tau := -\log \hat{X}'_\tau &\equiv -\log \varepsilon =: t \end{aligned}$$

Brownian expectations at stopping times formally coincide:

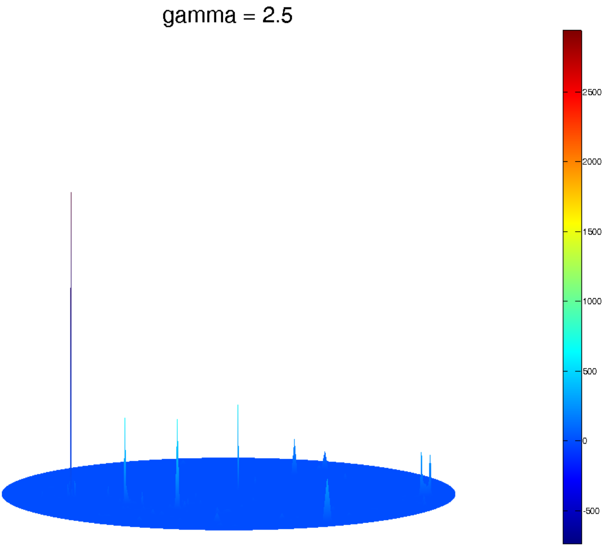
$$\begin{aligned} \mathbb{E}\left([g'_{T_y}(x)]^b\right) &= \mathbb{E}\left([\hat{X}'_{\tau_y}]^b\right) = (x/y)^\Delta \\ &\equiv \mathbb{E}\left[\varepsilon^b\right] = \delta^\Delta, \end{aligned}$$

hence the quantum gravity exponents $\square$

The *boundary quantum length* δ is the analog of (the inverse of) the *forward SLE distance* $\hat{X}_\tau$ to the origin, while the role of the *boundary Euclidean length* ε is played by the *SLE boundary derivative* $\hat{X}'_\tau$ $\square$

Liouville Quantum Duality

$$\gamma > 2, \gamma' = 4/\gamma < 2$$


gamma = 2.5
2500
2000
1500
1000
500
0
-500

LIOUVILLE QUANTUM DUALITY

Baby-Universes *Das, Dhar, Sengupta, Wadia '90; Jain & Mathur '92; Korchemsky '92; Alvarez-Gaumé, Barbón, Crnković '93; Durhuus '94; Ambjørn, Durhuus, Jonsson '94*

The Other Branch of Gravity *(Klebanov '95)*

Dual Dimensions

$$\begin{aligned} \gamma > 2, \gamma' &= 4/\gamma < 2 \\ \Delta_\gamma - 1 &= \frac{4}{\gamma^2}(\Delta_{\gamma'} - 1) \end{aligned}$$

D. & Sheffield, PRL **102**, *150603 (2009)*

Perspectives

- *Scaling limits of discrete models on random planar graphs*
- SLE$_\kappa$ *& Liouville quantum gravity*, $\gamma = \sqrt{\kappa}$
- $\gamma\gamma' = 4$ *duality*
- *Conformal welding*
- *Geodesics & random metrics*

$\diamondsuit \quad \diamondsuit \quad \diamondsuit$