Discrete parafermions and Coulomb gas in the square-lattice $\mathrm{O}(n)$ model

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Ascona

This talk is based on :

- YI, J. Cardy, J. Phys. A **42**, 102001 (2009)
  M. Rajabpour, J. Cardy, J. Phys. A **42**, 14703 (2007)
  V. Riva, J. Cardy, J. Stat. Mech P12001 (2006)
- S. Smirnov, ICM vol. II, 1421 (2006)
- S.0. Warnaar, M.T. Batchelor and B. Nienhuis, J. Phys. A **25**, 3077 (1992)
  H.W.J. Blöte, B. Nienhuis, J. Phys. A **22**, 1415 (1989)

Plan

The square-lattice $\mathrm{O}(n)$ model

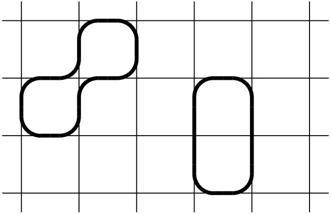
Integrable models on a regular rhombic lattice

Discrete parafermions

The square-lattice $\mathrm{O}(n)$ model

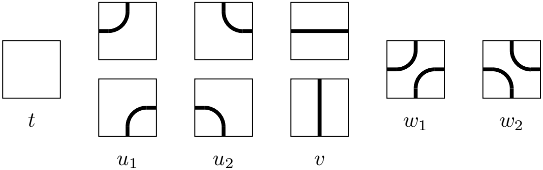
1. The model

- Partition function



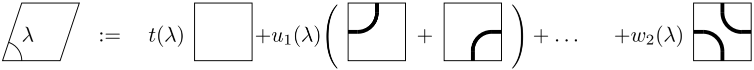
$$\Pi(G) = n^{\#\mathrm{loops}(G)} \prod_{\mathrm{site}\ j} \omega(G,j)\,, \qquad Z = \sum_{\mathrm{subgraph}\ G} \Pi(G)$$

- Local Boltzmann weights

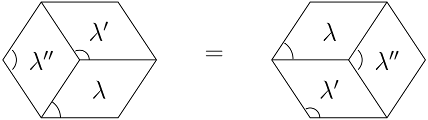


2. Yang-Baxter Equations (YBE)

- Plaquette diagram ("$\check{R}$-matrix")

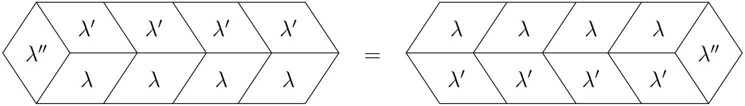


- Yang-Baxter Equations



with $\lambda'' = \lambda - \lambda' - \frac{3(\pi-\theta)}{4}$

- Commutation of transfer matrices



3. Solution of the Yang-Baxter Equations

[Nienhuis]

$$
\begin{aligned}
n &= -2\cos 2\theta \\
t(\lambda) &= -\cos 2\lambda + \sin\frac{5\theta}{2} - \sin\frac{3\theta}{2} - \sin\frac{\theta}{2} \\
u_1(\lambda) &= 2\sin\theta\cos\left(\frac{3\theta-\pi}{4} - \lambda\right) \\
u_2(\lambda) &= 2\sin\theta\cos\left(\frac{3\theta-\pi}{4} + \lambda\right) \\
v(\lambda) &= -\left(\cos 2\lambda + \sin\frac{3\theta}{2}\right) \\
w_1(\lambda) &= -\left[\cos(\theta - 2\lambda) + \sin\frac{\theta}{2}\right] \\
w_2(\lambda) &= -\left[\cos(\theta + 2\lambda) + \sin\frac{\theta}{2}\right] .
\end{aligned}
$$

4. Three physical regimes

[Nienhuis *et al.*]

- Regime I : $0 < \theta < \pi$
  - Central charge : $c_{\mathrm{eff}} = 1 - \frac{6(1-g)^2}{g}\,, \quad g = \frac{2\theta}{\pi}$
  - Conformal dimensions : $h, \bar{h} = \frac{1}{4}\left(\frac{e}{\sqrt{g}} \pm m\sqrt{g}\right)^2\,, \quad e, m \in \mathbb{Z}$

  Simple Coulomb gas (= compactified GFF)

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  Simple Coulomb gas (= compactified GFF)
- Regime II : $-\pi < \theta < -\frac{\pi}{3}$
  - Central charge : $c_{\mathrm{eff}} = \frac{3}{2} - \frac{6(1/2-2g)^2}{g}\,, \quad g = \frac{\pi+\theta}{2\pi}$
  - Conformal dimensions :

$$
\begin{aligned}
h &\in \left\{ \frac{\left(e/\sqrt{2g}+m\sqrt{2g}\right)^2}{8}, \frac{\left(e/\sqrt{2g}+m\sqrt{2g}\right)^2}{8} + \frac{1}{2} \right\}\,, && e \equiv m\ [2] \\
h &= \frac{\left(e/\sqrt{2g}+m\sqrt{2g}\right)^2}{8} + \frac{1}{16}\,, && e \equiv m+1\ [2]
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  Coulomb gas + Ising

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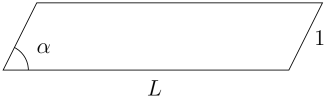
  Coulomb gas + Ising
- Regime III : $-\frac{\pi}{3} < \theta < 0$
  - Coupling of CG and Ising ?
  - Full low-energy spectrum is not known

Integrable models on a regular rhombic lattice

5. Transfer matrix for rhombic lattice

- One-row transfer matrix (periodic transverse BCs)



- Scaling limit

$$T_L(\alpha) \sim \mathrm{const}^L \;\; \exp(-\sin\alpha \;\, H) \;\; \exp(i\cos\alpha \;\, P)$$

- Conformal invariance

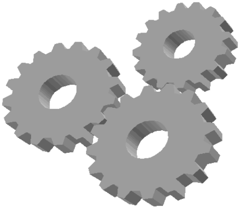
$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12})\,, \qquad P = \frac{2\pi}{L}(L_0 - \bar{L}_0)\,,$$

where $L_n, \bar{L}_n$ are Virasoro generators

- CFT prediction for eigenvalues of $T_L(\alpha)$

$$-\log \Lambda_L(\alpha) \simeq L f_\infty - \frac{2\pi}{L}\left[i e^{i\alpha}\left(h - \frac{c}{24}\right) - i e^{-i\alpha}\left(\bar{h} - \frac{c}{24}\right)\right]$$

6. Relation between α and λ

- YBE $\longrightarrow$  Bethe Ansatz $\longrightarrow$ asymptotics of Λ_L
- Result

$$-\log \Lambda \simeq L f_\infty + \frac{2\pi}{L}\left[e^{i\rho(\theta)\lambda}\left(h - \frac{c}{24}\right) + e^{-i\rho(\theta)\lambda}\left(\bar{h} - \frac{c}{24}\right)\right]$$

For example, in regime I, $\rho(\theta) = 2\pi/(3\pi - 3\theta)$.

- Simple relation

$$\alpha = \frac{\pi}{2} - \rho(\theta)\lambda\,, \qquad |\lambda| < \frac{\pi}{2\rho}$$

Discrete parafermions

7. Discretely holomorphic functions

- Morera's theorem (in the continuum) :
  If F is continuous and $\forall C$ closed circuit, $\oint_C F(z)\, dz = 0$,
  then F is holomorphic.
- Discrete version :
  Let F be defined on the edges of the lattice $\mathcal{L}$. We say that F is discretely holomorphic on $\mathcal{L}$ iff, for every plaquette $\mathcal{P}$ with corners $\{z_i\}$:

$$\sum_{\langle i\ j\rangle \in \partial\mathcal{P}} (z_i - z_j)\ F\left(\frac{z_i + z_j}{2}\right) = 0\,.$$

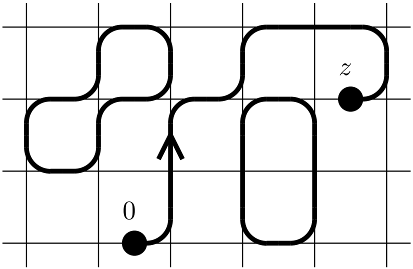
8. Definition of the discrete parafermion

[Smirnov, Cardy-Rajabpour-Riva-YI]

- Introduce a pair of defects at 0 and z, and let

$$\psi_s(z) = \frac{1}{Z} \sum_{G|\ [0 \text{ and } z \text{ carry defects}]} \Pi(G)\ e^{-isW(z)}$$

- Example configuration ($W(z) = -\pi$)



9. Discrete holomorphicity equations

- Around a plaquette : $\sum \langle \psi(z) \rangle \delta z = 0$.
- Linear equations on the Boltzmann weights

$$\underline{\underline{A}}(\theta, s, \alpha) \begin{pmatrix} t \\ \vdots \\ w_2 \end{pmatrix} = 0$$

- Singularity condition : $\det \underline{\underline{A}}(\theta, s, \alpha) = 0 \;\Leftrightarrow\; s = \frac{3\theta - \pi}{2\pi}$
- Solution for Boltzmann weights = solution of YBE !
  One recovers the relation $\lambda \leftrightarrow \alpha$

10. Relation to SLE

[Smirnov]

- Prove (or assume) convergence of $\langle \psi_s \rangle$ to an analytic function
- In the upper half plane $\mathbb{H}$, $\langle \psi_s \rangle_{\mathbb{H}}$ solves a boundary value problem :

$$\mathrm{Arg}\ \langle \psi_s \rangle_{\mathbb{H}} = \frac{\pi s}{2} \mathrm{sgn}(z) \qquad \text{for real } z$$

$$\Rightarrow \langle \psi_s \rangle_{\mathbb{H}} = \frac{\mathrm{const}}{z^s}$$

- If g_t is the conformal map $\qquad g_t : \ \mathbb{H} \backslash \gamma_t \to \mathbb{H}$
  then $(g_t'/g_t)^s$ is a martingale
- Consequence : the driving function W_t is Brownian
  $W_t = \sqrt{\kappa} B_t$, with $s = \frac{6-\kappa}{2\kappa}$

Discussion and Conclusion

- The square $\mathrm{O}(n)$ model is integrable, with three physical regimes
- In regimes I and II, effective degrees of freedom were determined by Bethe Ansatz $+$ asymptotic calculation
- Regime III is analytically and numerically harder
- We found a discrete parafermion for all regimes
- Smirnov's argument (*modulo* convergence) connects the model to SLE :
  In regimes II and III, how to describe fermions in the SLE formalism ?

Thank you for your attention !