Hadamard's formula and couplings of SLE with GFF

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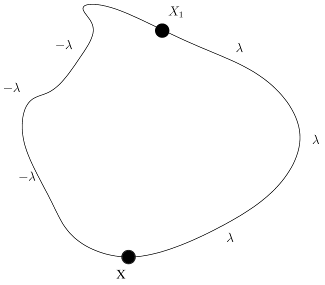
The covariance of the field $C(z_1, z_2) = G(z_1, z_2)$ is a Green's function in Ω (with corresponding homogeneous boundary conditions)

Relations to SLE: level lines

Schramm & Sheffield '2006



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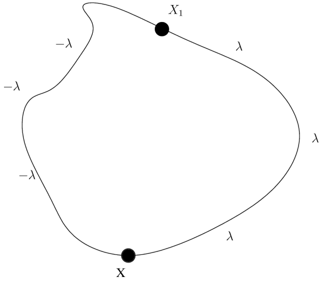

Domains with two marked points x, x_1, with Dirichlet boundary conditions $\pm\lambda = \pm\sqrt{\frac{\pi}{8}}$.

Dirichlet boundary valued Green's function as covariance

Discretize the field, take the mesh to zero

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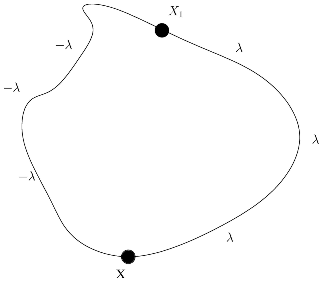

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Dirichlet boundary valued Green's function as covariance

Discretize the field, take the mesh to zero $\Rightarrow$ level lines converge to SLE_4

Soft approach: coupling

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In the continuum: there exists a coupling of SLE_4 and GFF, such that the curve behaves like a level line.

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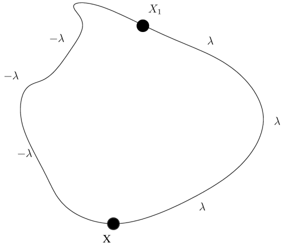
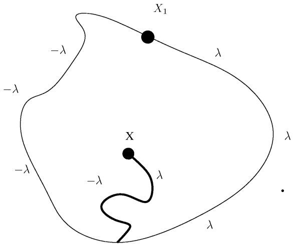
In the continuum: there exists a coupling of SLE_4 and GFF, such that the curve behaves like a level line.

Namely: Conditionally on the curve γ_t, the law of the field is that of the GFF in $\Omega\backslash\gamma_t$, the jump has moved to the tip


X_1
$-\lambda$
λ
$-\lambda$
λ
$-\lambda$
λ
x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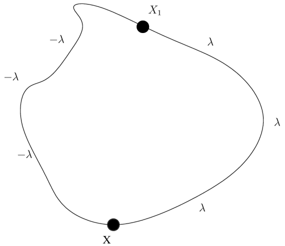
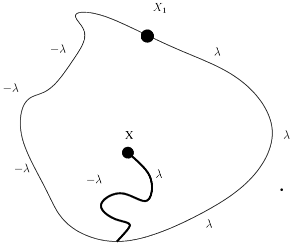
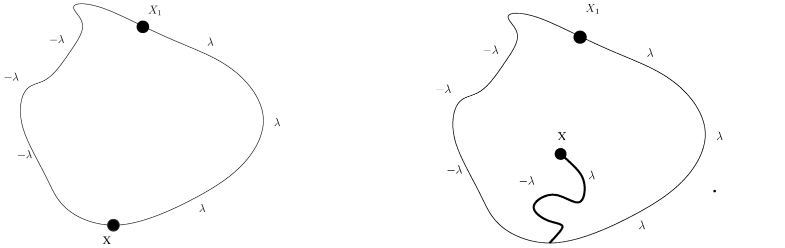




Constructive formulation: sample SLE_4 curve up to time t; sample GFF in $\Omega \setminus \gamma_t$; forget the curve $\Rightarrow$ obtain a new field $\tilde{\Phi}$ in Ω



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which appears to have the same law as Φ.

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Other boundary conditions far away from the curve?

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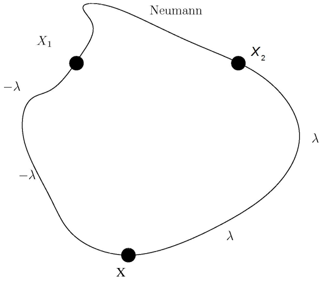
Doubly connected domains?

The zoo of examples: simply-connected case



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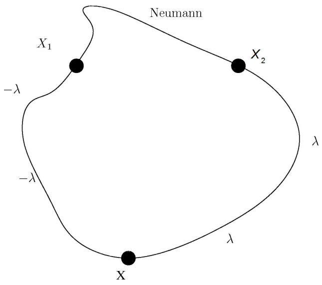

Three arcs, boundary values $-\lambda$, λ, Neumann: dipolar SLE_4.

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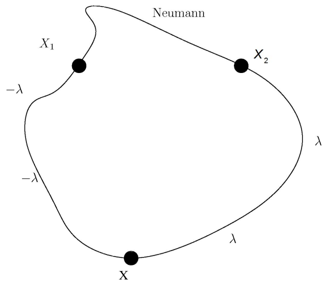

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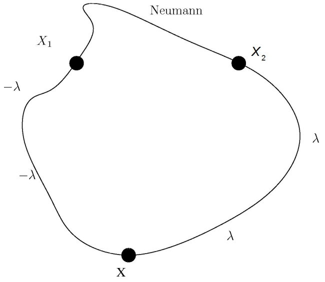
Three arcs, boundary values $-\lambda$, λ, 0: dipolar SLE_4.



Three arcs, boundary values $-\lambda$, λ, Neumann: dipolar SLE_4.

Three arcs, boundary values $-\lambda$, λ, 0: dipolar SLE_4.

Three arcs, boundary values $-\lambda$, λ, Riemann-Hilbert: $\partial_\sigma M(z) = 0$, $\sigma = e^{i\alpha}\tau$: $\mathrm{SLE}_4(\rho)$ with ρ depending on α.

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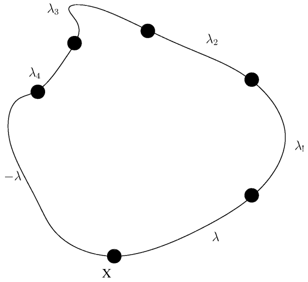

More marked points, jump-Dirichlet boundary conditions: $\mathrm{SLE}_4(\rho_1, \rho_2, \ldots)$ with $\rho's$ proportional to jumps (Schramm & Sheffield, Cardy, Dubédat).

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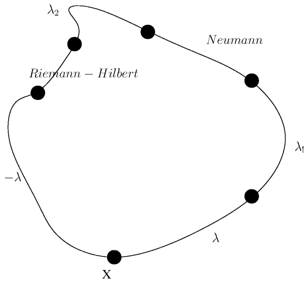

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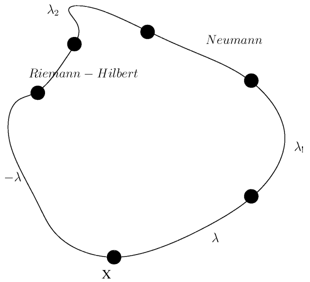
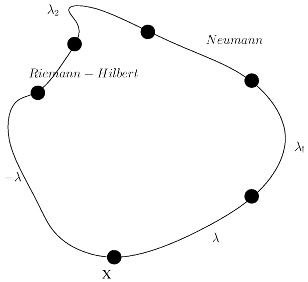


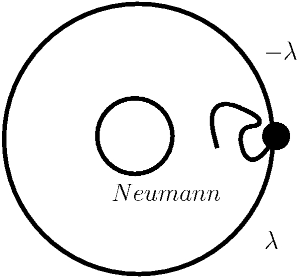
More marked points, mixed boundary conditions: **not** $\mathrm{SLE}_4(\overline{\rho})$!



More marked points, mixed boundary conditions: **not** $\mathrm{SLE}_4(\overline{\rho})$!

But the drift still can be computed. Expression involves derivatives of M and its harmonic conjugate w.r.t marked points.

The zoo of examples: doubly-connected case



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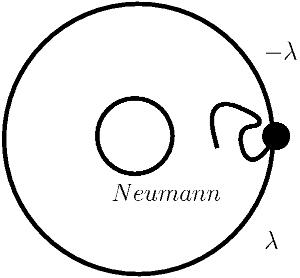
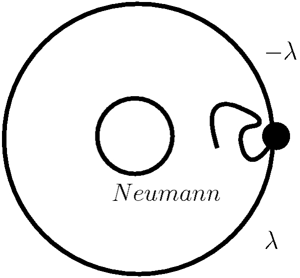


One marked point on the outer boundary with jump $-2\lambda \Rightarrow$ multi-valued mean.

Neumann boundary conditions on the inner boundary



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Neumann boundary conditions on the inner boundary

Coupled with annular SLE_4.

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Two marked points on the outer boundary (Hagendorf, Bauer, Bernard'09 via partition function): some annulus analogs of $\text{SLE}_4(\rho)$.

On the inner boundary: either Neumann or Dirichlet

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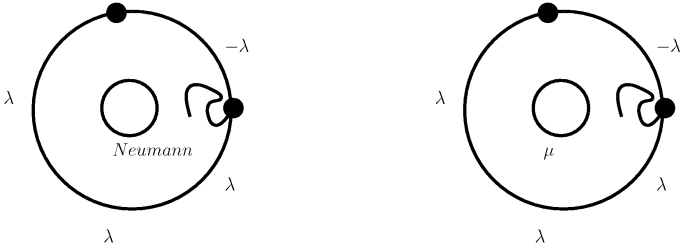

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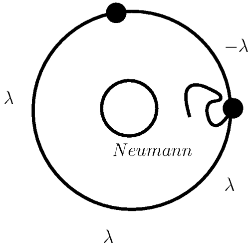
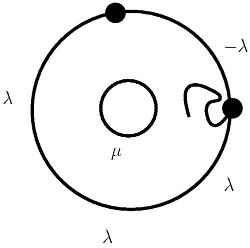
Drifts are computed explicitly (in terms of Schwarz kernels in the annulus), and the existence of couplings is proven.





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Drifts are computed explicitly (in terms of Schwarz kernels in the annulus), and the existence of couplings is proven.

Easily generalizes to many marked points $x_1, x_2, \ldots$ on the outer boundary (of total jump 2λ in Dirichlet case)

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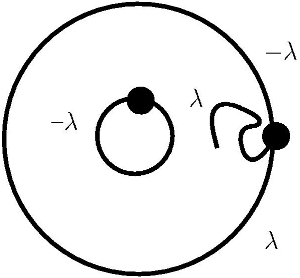

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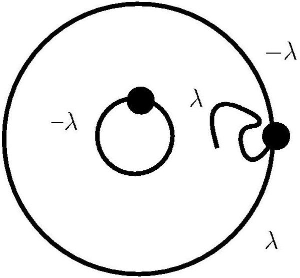

One marked point on the inner boundary; Dirichlet boundary conditions.

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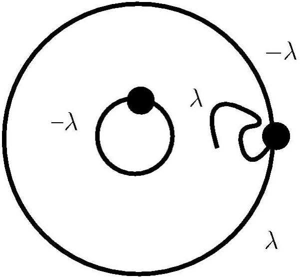

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Still one integer parameter to fix: can add an integer multiple of λ on the inner boundary.

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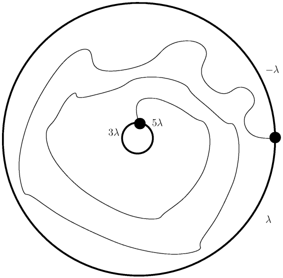

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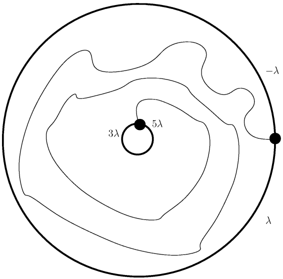

This leads to a curve with a prescribed winding.

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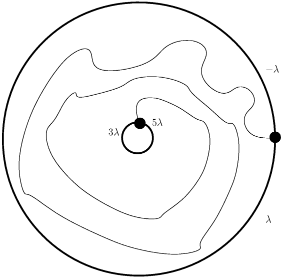

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Indeed; eventually the winding is as it's supposed to be.

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Question: what is the natural coupling of annulus GFF with SLE_κ for $\kappa \neq 4$?.

Multiply connected domains

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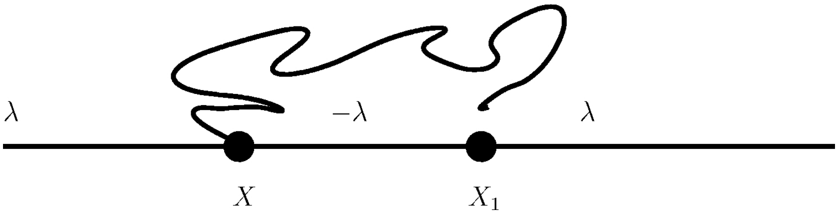
in terms of derivatives of M and $\tilde{M}$ etc w. r. t. marked points and conformal moduli parameters.

Proof: simple case



Domain: half-plane $\mathbb{H}$; two marked points x, x_1

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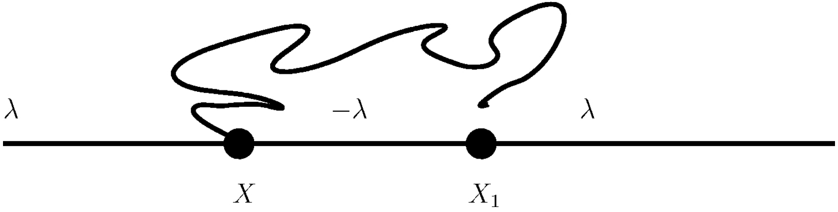

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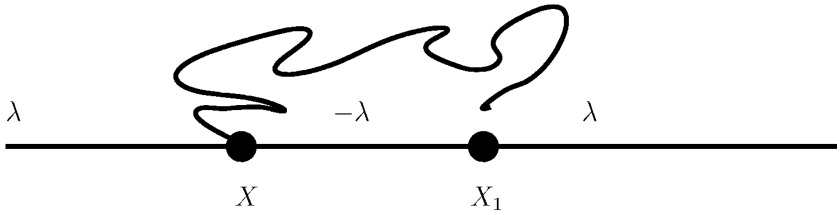

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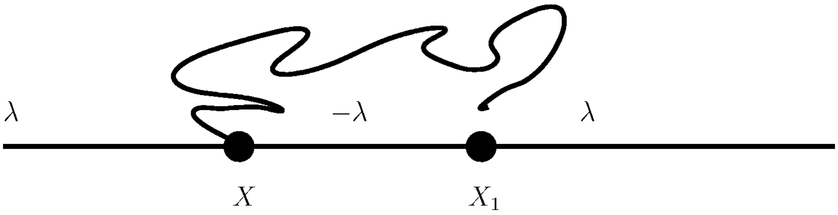

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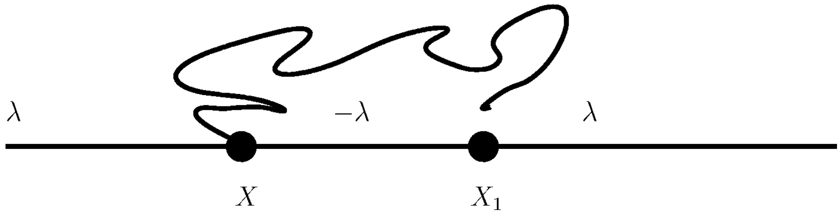

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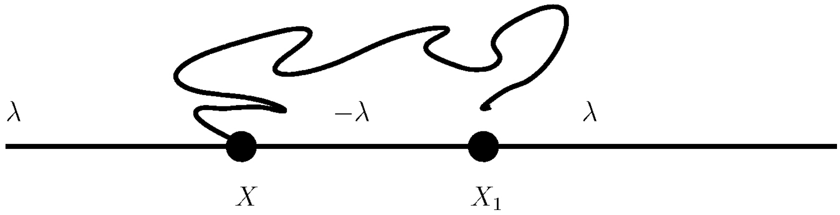

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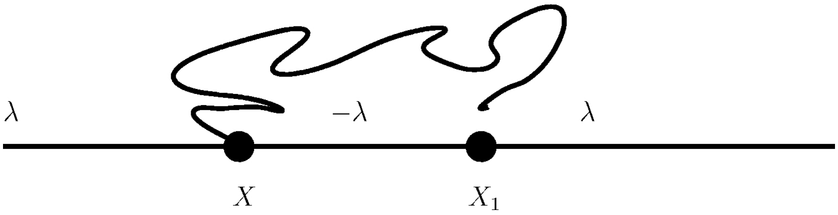

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$$dM(X_t, g_t(x_1), g_t(z)) =$$
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$$\Im \kappa \partial_x F dB_t |_{x, x_1, z \to X_t, g_t(x_1), g_t(z)}$$

Proof: Two-point function is a martingale

Second equation:

$$dG(g_t(z_1), g_t(z_2)) = -d[M(\ldots, \ldots, g_t(z_1))M(\ldots, \ldots, g_t(z_2))]$$

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The equation above is Hadamard's formula; easily generalizes to other cases.

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There exists a unique D_t that cancels it out!

Thank you!