Seoul
National
University

Radial Conformal Field Theory

Joint work with Nikolai G. Makarov

Nam-Gyu Kang

Department of Mathematical Science, Seoul National University

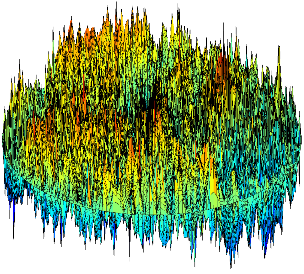
Conformal maps from probability to physics

May 23-28, 2010

Outline

- Gaussian free field and conformal field theory
  - Probabilistic setting for CFT.
  - Calculus of CFT and the source of tensor structures of conformal fields.
  - Fields = certain types of Fock space fields + tensor nature.
  - We use “conformal invariance” to denote consistence with conformal structures.
  - We treat a stress energy tensor in terms of Lie derivatives.
- Radial conformal field theory
  - In radial CFT, several trivial fields are multi-valued.
  - 2 types of radial CFT and relation to SLE.
  - Twisted radial CFT.

Gaussian Free Field Φ
and its approximation Φ_n



- Φ : Gaussian Free Field

$$\Phi = \sum_{n=1}^{\infty} a_n f_n.$$

  - f_n: O.N.B. for $W_0^{1,2}(D)$ with Dirichlet inner product.
  - D: a hyperbolic R.S.
  - a_n: i.i.d. $\sim N(0,1)$.

- $\mathbb{E}[\Phi(z)] = 0.$
- $\mathbb{E}[\Phi(z)\Phi(w)] = 2G(z,w).$
- $\mathrm{var}\,(\Phi(f)) = \iint 2G(z,w)f(z)f(w).$
- $\Phi_n(z) = \sqrt{2}\sum_{j=1}^{n}(G(z,\lambda_j) - G(z,\mu_j)).$

Poles of Φ_n

Ginibre eigenvalues

- $\Phi_n(z) = \sqrt{2}\sum_{j=1}^{n}(G(z,\lambda_j) - G(z,\mu_j))$.
- $\{\lambda_j\}_{j=1}^{n}$: eigenvalues of the Ginibre ensemble, $\{\mu_j\}_{j=1}^{n}$: an independent copy.
- Ginibre ensemble is the $n \times n$ random matrix $(a_{j,k})_{j,k=1}^{n}$.
- $a_{j,k}$: i.i.d. complex Gaussians with mean zero and variance $1/n$.
- $\Phi_n(f) \xrightarrow{law} \Phi(f)$. (Y. Ameur, H. Hedenmalm, and N. Makarov)

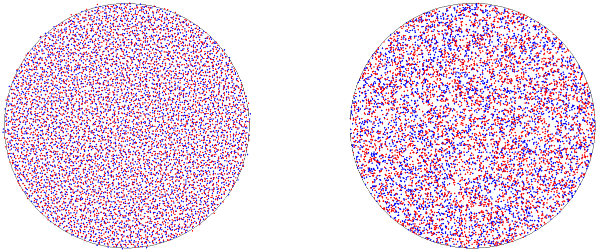
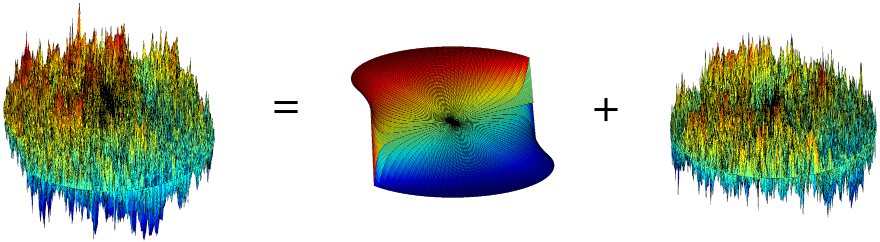


Figure: Ginibre eigenvalues and uniform points ($n = 4096$)

Boundary Conditions
Chordal case



$$H_\lambda(z) = \sqrt{2}\lambda(\arg(1+z) - \arg(1-z))$$

Level Lines of GFF$_n$

Chordal case

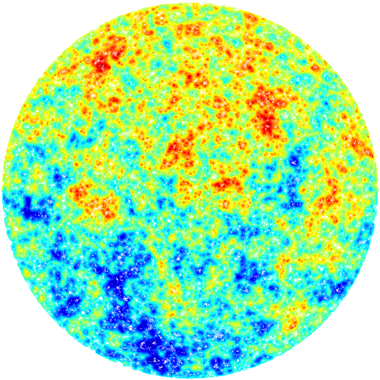


Figure: $\Phi_n(z) + H_{(\lambda=1)}(z) = 0.$

Conjecture: Zero Set = Chordal SLE_4
motivated by O. Schramm and S. Sheffield's

Seoul National University

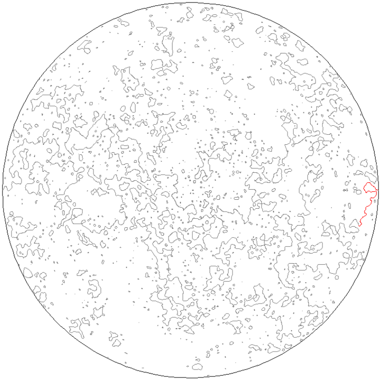
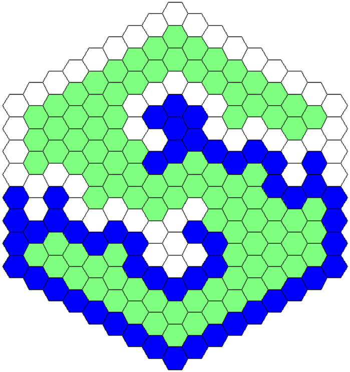


Figure: $\Phi_n(z) + H_{(\lambda=1)}(z) = 0.$

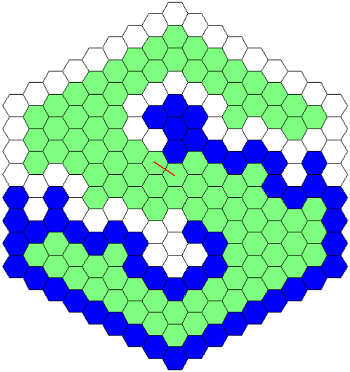
Harmonic Explorer
Radial case

Seoul
National
University



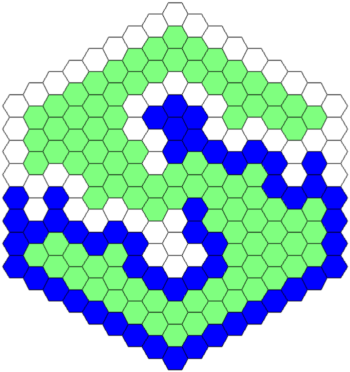
Harmonic Explorer
Radial case

Seoul
National
University



Harmonic Explorer
Radial case

Seoul National University



Harmonic Explorer

Radial case

Seoul National University

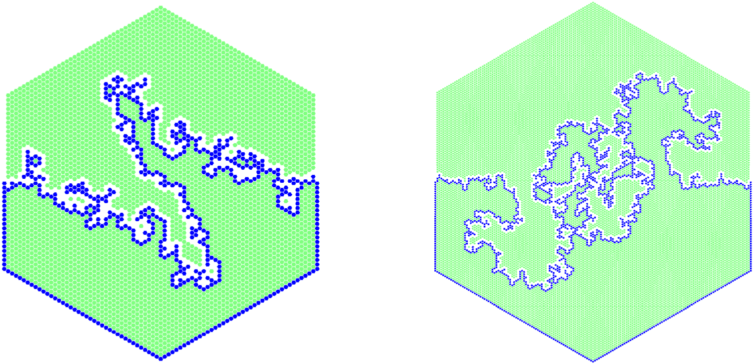


Figure: As the mesh gets finer, does the HE converge?

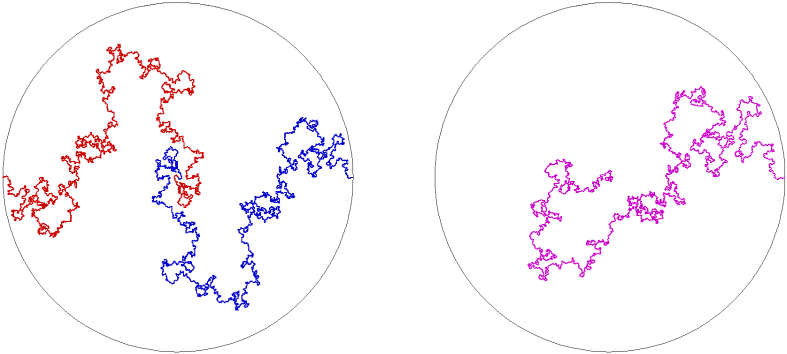
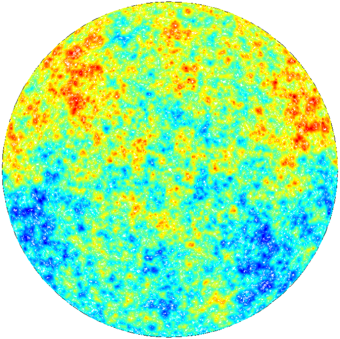
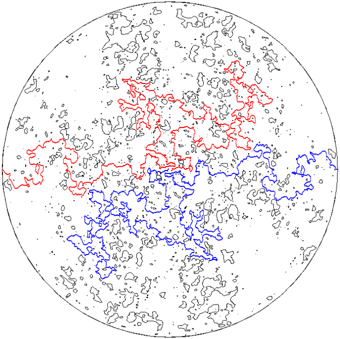


Figure: As the mesh gets finer, the HE converges to radial SLE$_4$.

Level Lines and Zero Set

Radial case (2 covers)



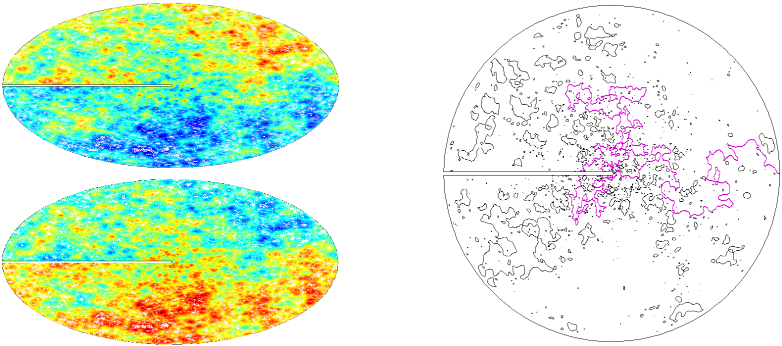


$$\Phi_n^{odd}(z) := \frac{1}{2}(\Phi_n(z) - \Phi_n(-z)).$$

Level Lines and Zero Set

Radial case (Twisted boundary conditions)

Seoul
National
University



$$\Phi_n^{tw}(z) = \sqrt{2}\Phi_n^{odd}(\pm\sqrt{z}) = \pm\sqrt{2}\Phi_n^{odd}(\sqrt{z}).$$

Fock space fields

Fock space fields (*F-fields*) are obtained from GFF by applying the following basic operations:

i. derivatives;
ii. Wick's products;
iii. multiplying by scalar functions and taking linear combinations.

Examples

$$J = \partial\Phi, \quad \Phi \odot \Phi, \quad J \odot \Phi, \quad J \odot J, \quad J \odot \bar{J}.$$

Correlations (at distinct points) are defined for any finite collections of Fock fields: (i) by differentiation; (ii) by Wick's yoga; (iii) by linearity.

Examples

- $\mathbb{E}[J(\zeta)\Phi(z)] = 2\partial_\zeta G(\zeta, z).$
- $\mathbb{E}[\Phi^{\odot 2}(\zeta)\Phi(z_1)\Phi(z_2)] = 2\mathbb{E}[\Phi(\zeta)\Phi(z_1)]\mathbb{E}[\Phi(\zeta)\Phi(z_2)].$
- $\mathbb{E}[e^{\odot\Phi(\zeta)}\big(\sum_{n=0}^{\infty} \dfrac{\alpha^n \Phi^{\odot n}(z)}{n!}\big)] = e^{|\alpha|^2\mathbb{E}[\Phi(\zeta)\Phi(z)]} = e^{2|\alpha|^2 G(\zeta,z)}$

Conformal geometry and conformal invariance

Seoul National University

Definition We consider a non-random field f on a Riemann surface. We say that

a. f is a differential of degree $(d, d^{\#})$ if for two overlapping charts ϕ and $\tilde{\phi}$, we have

$$f = (h')^{d}(\bar{h}')^{d^{\#}}\widetilde{f} \circ h,$$

where f is the notation for $(f\|\phi)$, $\tilde{f}$ for $(f\|\tilde{\phi})$, and h is the transition map.

b. f is a PS-form (pre-Schwarzian form) or 1-from of order μ if

$$f = (h')^{1}\widetilde{f} \circ h + \mu N_h \quad (N_h = h''/h' = (\log h')');$$

c. f is an S-form (Schwarzian form) or 2-from of order μ if

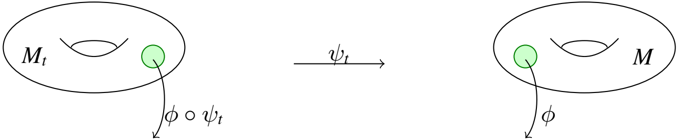
$$f = (h')^{2}\widetilde{f} \circ h + \mu S_h \quad (S_h = N_h' - N_h^2/2).$$

A field X is invariant wrt to some conformal automorphism τ of M if

$$\mathbb{E}[(X\|\phi)\Phi(p_1) \odot \cdots \odot \Phi(p_n)] = \mathbb{E}[(X\|\phi \circ \tau^{-1})\Phi(\tau p_1) \odot \cdots \odot \Phi(\tau p_n)].$$

Conformal invariance allows to define fields in conformally equivalent situations.

Lie derivative



Suppose M is a Riemann surface. Consider a (local) flow of a vector field v on M

$$\psi_t : M \to M, \qquad \dot{\psi}_0(z) = v(z).$$

Suppose X is a field in M and v is holomorphic in $U = U_v \subset M$. By definition, the field X_t in U is

$$(X_t(z) \parallel \phi) = (X(\psi_t z) \parallel \phi \circ \psi_{-t}).$$

Definition

$$L_v X = \frac{d}{dt}\Big|_{t=0} X_t.$$

Seoul National University

- A pair of quadratic differentials $W = (A_+, A_-)$ is called a stress tensor for X if "residue form of Ward's identity" holds:

$$L_v X(z) = \frac{1}{2\pi i} \oint_{(z)} v A_+ X(z) - \frac{1}{2\pi i} \oint_{(z)} \bar{v} A_- X(z).$$

  Notation: $\mathcal{F}(W)$ is the family of fields with stress tensor $W = (A_+, A_-)$.

- For $A = -\frac{1}{2} J \odot J$, $W = (A, \bar{A})$ is a stress tensor for GFF Φ.

- If $X, Y \in \mathcal{F}(W)$, then $\partial X, X * Y \in \mathcal{F}(W)$.

Seoul National University

Definition Let $W = (A, \bar{A})$ be a stress tensor. A Fock space field T is the *Virasoro field* for the family $\mathcal{F}(W)$ if

- $T - A$ is a non-random holomorphic Schwarzian form,
- $T \in \mathcal{F}(W)$.

Example Twisted Radial CFT

- $G(\zeta, z) = \log \left| \dfrac{1 - \sqrt{\zeta \bar{z}}}{1 + \sqrt{\zeta \bar{z}}} \dfrac{1 + \sqrt{\zeta / z}}{1 - \sqrt{\zeta / z}} \right|$
- $T = -\dfrac{1}{2} J * J = A + S$, where $A = -\dfrac{1}{2} J \odot J$.
- $\mathbb{E}[J(\zeta)J(z)] = -\dfrac{1}{2} \dfrac{w'(\zeta)w'(z)}{(w(\zeta) - w(z))^2} \left(\sqrt{w(\zeta)/w(z)} + \sqrt{w(z)/w(\zeta)} \right).$
- $\mathbb{E}[J(\zeta)J(z)] = -\dfrac{1}{(\zeta - z)^2} - \dfrac{1}{6} S(\zeta, z), \quad S = T - A = \dfrac{3}{4} \dfrac{w'^{\,2}}{w^2} + S_w.$

Seoul National University

Proposition (Ward equation)

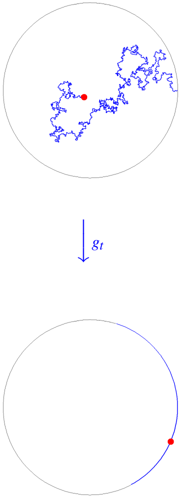
In $\mathbb{D}$, for a string $\mathcal{X}$ of differentials in $\mathcal{F}(W)$, we obtain

$$\mathbb{E}[T(\zeta)\mathcal{X}] = \mathbb{E}[T(\zeta)]\mathbb{E}[\mathcal{X}] + \frac{1}{2\zeta^2}\sum_j \left(z_j \frac{\zeta + z_j}{\zeta - z_j}\partial_j + d_j \frac{\zeta^2 + 2\zeta z_j - z_j^2}{(\zeta - z_j)^2} \right)\mathbb{E}[\mathcal{X}]$$
$$+ \frac{1}{2\zeta^2}\sum_j \left(\bar{z}_j \frac{\bar{\zeta}^* + \bar{z}_j}{\bar{\zeta}^* - \bar{z}_j}\bar{\partial}_j + d_j^{\#} \frac{\bar{\zeta}^{*2} + 2\bar{\zeta}^*\bar{z}_j - \bar{z}_j^2}{(\bar{\zeta}^* - \bar{z}_j)^2} \right)\mathbb{E}[\mathcal{X}].$$

- Consider a vector field $v_\zeta(z) = z\dfrac{\zeta + z}{\zeta - z}$.
- The reflection of a vector field in $\partial\mathbb{D}$ is defined by $v^{\#}(z) = -\overline{v(1/\bar{z})}z^2$.
- $v_\zeta^{\#} = v_{\zeta^*}$ and $\zeta^* := 1/\bar{\zeta}$ is the symmetric point of ζ with respect to $\partial\mathbb{D}$.
- $\bar{\partial}v_\zeta = -2\pi\zeta^2\delta_\zeta$.

SLE numerology

Seoul National University



- SLE$_\kappa$ map $g_t(z)$: $D_t \to \mathbb{D}$

$$\partial g_t(z) = g_t(z) \frac{1 + k(t) g_t(z)}{1 - k(t) g_t(z)}, \quad g_0(z) = z,$$

where $k(t) = e^{-i\sqrt{\kappa}B_t}$. Set $w_t(z) = k(t) g_t(z)$.

- B_t : a 1-D standard Brownian motion on $\mathbb{R}$, $B_0 = 0$.
- SLE hulls: $K_t := \{z \in \overline{\mathbb{D}} : \tau(z) \leq t\}$.
- SLE path: $\gamma_t = \gamma[0, t]$, where $\gamma(t) = g_t^{-1}(e^{i\sqrt{\kappa}B_t})$.

- When $\kappa = 4$, we consider Makarov-Zhan's martingale observable

$$\varphi_t(z) = 2a \arg \frac{1 + \sqrt{w_t(z)}}{1 - \sqrt{w_t(z)}},$$

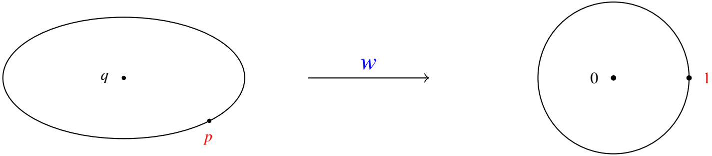
where $a = \pm 1/\sqrt{2}$.

Boundary conditions

Insertion of a chiral vertex

Denote $\widehat{\mathbb{E}}_p[\mathcal{X}] = \mathbb{E}[e^{\odot ia\Phi^\dagger(q,p)}\mathcal{X}]$ and let $\widehat{\mathcal{X}}_p$ denote the string $\mathcal{X}$ of F-fields under the boundary condition with

$$u = -2a \arg \frac{1+\sqrt{w}}{1-\sqrt{w}} \qquad w : (D, q, p) \to (\mathbb{D}, 0, 1).$$



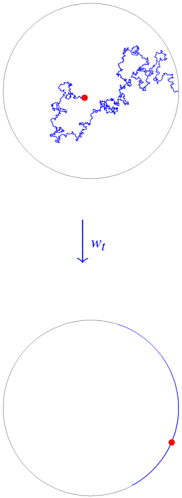
Lemma $$\widehat{\mathbb{E}}_p[\mathcal{X}] = \mathbb{E}[\widehat{\mathcal{X}}_p].$$

Main idea. Recall that $\mathbb{E}[\Phi(\zeta)\Phi(z)] = 2\log\left|\dfrac{1-\sqrt{\zeta\bar{z}}}{1+\sqrt{\zeta\bar{z}}}\dfrac{1+\sqrt{\zeta/z}}{1-\sqrt{\zeta/z}}\right|$ in $\mathbb{D}$. Thus

$$\mathbb{E}[\Phi^\dagger(q,p)\Phi(z)] = G^\dagger_{tw}(p,z) - G^\dagger_{tw}(q,z) = -2i \arg \frac{1+\sqrt{w(z)}}{1-\sqrt{w(z)}}.$$

Wick meets Itô and Schramm

Seoul National University



Suppose A_j's are conformally invariant (holomorphic differentials) in $\mathcal{F}(W)$. For every (D, q) consider

$$R_p(z_1, \cdots, z_n) = \widehat{\mathbb{E}}_p[A_1(z_1) \cdots A_n(z_n)], \; z_j \in D, \; p \in \partial D.$$

Denote

$$M_t := \widehat{\mathbb{E}}_{\gamma(t)}[A^1_{D_t}(z_1) \cdots]$$
$$(= w_t'(z_1)^{d_1} \cdots \widehat{\mathbb{E}}_1[A_{\mathbb{D}}(w_t(z_1)), \cdots])$$

Then

$$M_t \text{ is a (local) martingale,}$$

or the CFT $\mathcal{F}(W)$ does not change under SLE_κ evolution.

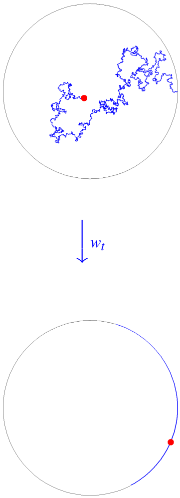
Main idea: conformal invariance + degeneracy at level two

$$\partial_\theta^2 R_{e^{i\theta}} = -\frac{1}{2} L_v R_{e^{i\theta}}, \quad v = v_{e^{i\theta}}.$$

Wick meets Itô and Schramm

Hadamard’s formula

Seoul National University



- A 1-pt function

$$\widehat{\varphi}(z) := \widehat{\mathbb{E}}[\Phi(z)]$$

is a martingale-observable.

- A 2-pt function

$$\widehat{\mathbb{E}}[\Phi(z_1)\Phi(z_2)] = \widehat{\varphi}(z_1)\widehat{\varphi}(z_2) + 2G(z_1, z_2)$$

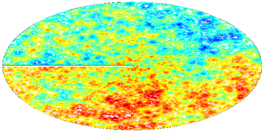
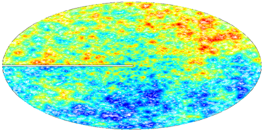
is a martingale-observable.

- Equating the drifts,

$$\begin{aligned} 2dG_t(z_1, z_2) &= -d\langle \varphi(z_1), \varphi(z_2)\rangle_t \\ &= -8\Re\frac{\sqrt{w_t(z_1)}}{1 - w_t(z_1)}\Re\frac{\sqrt{w_t(z_2)}}{1 - w_t(z_2)}\,dt. \end{aligned}$$

Summary

- Twisted CFT: $\Phi_n^{tw}(z) = \frac{1}{\sqrt{2}}(\Phi_n(\pm\sqrt{z}) - \Phi_n(\mp\sqrt{z}))$.

- Ward identity:

$$L_v^+ X(z) = \frac{1}{2\pi i}\oint_{(z)} vTX(z), \qquad T \in \mathcal{F}(W).$$

- Boundary condition modification: $\widehat{\Phi} = \Phi + \sqrt{2}\arg\frac{1+\sqrt{w(z)}}{1-\sqrt{w(z)}}$
- Boundary condition changing operator: $\widehat{\mathbb{E}}_p[\mathcal{X}](:= \mathbb{E}[e^{\odot ia\Phi^\dagger(p)}\mathcal{X}]) = \mathbb{E}[\widehat{\mathcal{X}}_p]$.
- Correlations of conformally invariant fields in $\mathcal{F}(W)$ are martingale-observables for SLE(4).