Random curves, scaling limits and Loewner evolutions

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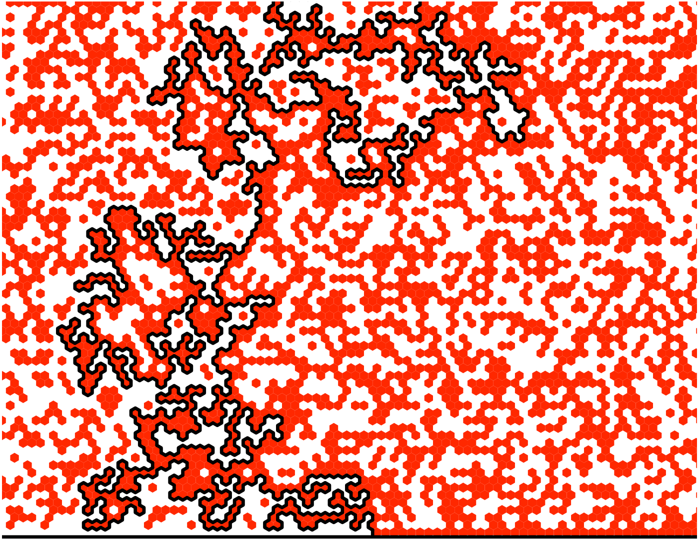
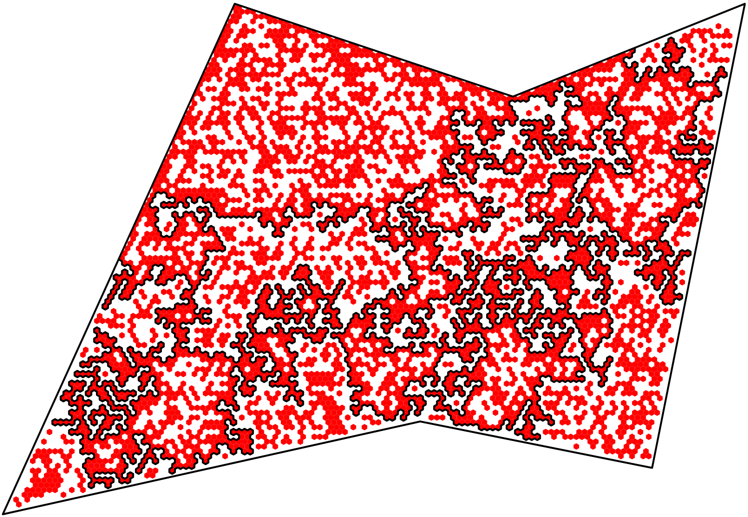
Interface separating two phases



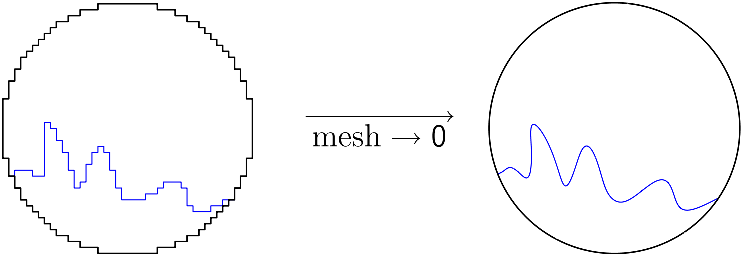
- 2D lattice models: percolation, Ising model, random cluster models

Scaling limit of an interface



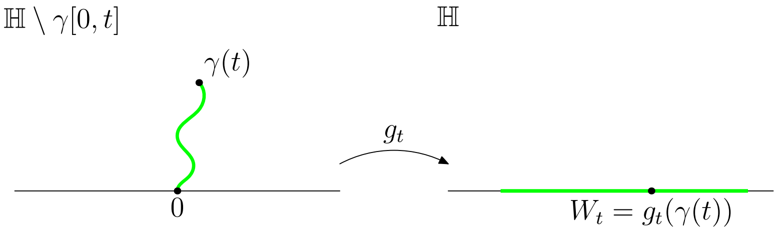
- $\mathbb{P}$ is the law of a random simple curve in a simply connected domain U and connecting $a, b \in \partial U$
- Scaling limit: mesh $\to 0$.

Scaling limit of an interface



- Goal: present a setting for proving the convergence.
- Often: a sequence of $(U, a, b, \mathbb{P})$ so that (U, a, b) approximates $(\hat{U}, \hat{a}, \hat{b})$, $\mathbb{P}$ from a chosen lattice model.
- Useful to choose a reference domain, e.g. the unit-disc $\mathbb{D}$, and map conformally $\phi : U \to \mathbb{D}$, $\phi(a) = -1$, $\phi(b) = 1$.
- We will consider a set Σ, whose elements are triplets $(U, \phi, \mathbb{P})$.

Schramm-Loewner evolution

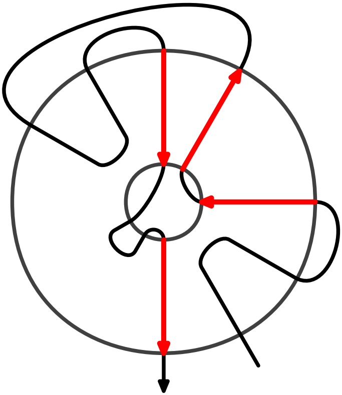


- g_t conformal mapping with a specific normalization: $g_t(z) = z + \frac{C(t)}{z} + \dots$ capacity
- Loewner equation

$$\frac{\partial g_t}{\partial t}(z) = \frac{C'(t)}{g_t(z) - W_t}$$

where $t \mapsto W_t$ continuous. Capacity param. $C(t) = 2t$.
- SLE_κ, $\kappa > 0$, a random curve s.t. $W_t = \sqrt{\kappa} B_t$. $(B_t)_{t \geq 0}$ standard, one-dimesional Brownian motion.

Crossings of an annulus

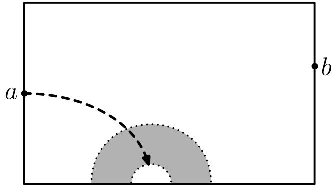


A *crossing* of an annulus $A(z_0, r, R) = \{z : r < |z - z_0| < R\}$ is a subcurve such that its end points are in the different components of $\mathbb{C} \setminus A(z_0, r, R)$.

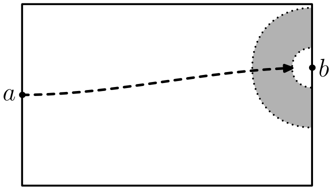
Crossings of an annulus

There are three types of crossings of an annulus $A = A(z_0, r, R)$.

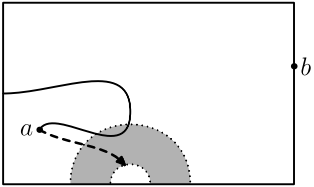
Unforced



Forced



Ambiguous



Condition assumed to hold for the random curve

Choose a parametrization for all curves, $\gamma : [0,1] \to \mathbb{C}$.

Condition

$\exists C > 1$ s.t. for any $(U, \phi, \mathbb{P}) \in \Sigma$, for any stopping time τ and for any annulus $A = A(z_0, r, R)$, $0 < Cr \leq R$,

$$\mathbb{P}\big(\gamma[\tau, 1] \text{ crosses any } V \in \mathcal{U}(U_\tau, A) \,\big|\, \gamma[0,\tau]\big) < \frac{1}{2}$$

for almost every $\gamma[0, \tau]$.

where $U_t = U \setminus \gamma(0, t]$ and

$$\mathcal{U}(U_t, A) = \begin{cases} \emptyset & \text{if } A \text{ is not on } \partial U_t \\ \text{those components of } U_t \cap A \text{ that don't separate } \gamma(t) \text{ and } b & \text{otherwise} \end{cases}$$

Condition assumed to hold for the random curve

Choose a parametrization for all curves, $\gamma : [0,1] \to \mathbb{C}$.

Condition

$\exists K > 0, \Delta > 0$ s.t. for any $(U, \phi, \mathbb{P}) \in \Sigma$, for any stopping time τ and for any annulus $A = A(z_0, r, R)$, $0 < r < R$,

$$\mathbb{P}\big(\gamma[\tau, 1] \text{ crosses any } V \in \mathcal{U}(U_\tau, A) \,\big|\, \gamma[0,\tau]\big) < K \left(\frac{r}{R}\right)^{\Delta}$$

for almost every $\gamma[0,\tau]$.

where $U_t = U \setminus \gamma(0, t]$ and

$$\mathcal{U}(U_t, A) = \begin{cases} \emptyset & \text{if } A \text{ is not on } \partial U_t \\ \text{those components of } U_t \cap A \text{ that don't separate } \gamma(t) \text{ and } b & \text{otherwise} \end{cases}$$

Main theorem, shortly

- Condition assumed is carried nicely under conformal mappings. Hence it holds in chosen reference domain(s).
- Under Condition the following holds for $\Sigma_{\mathbb{D}} = \{\phi\mathbb{P} : (U, \phi, \mathbb{P}) \in \Sigma\}$: $\exists F_n$ so that $\mathbb{P}(F_n) \to 1$ as $n \to \infty$ uniformly in $\mathbb{P} \in \Sigma_{\mathbb{D}}$ and so that each F_n is
  - precompact in the path topology: The space of curves X, the equivalence classes of $C([0,1], \mathbb{C})$ under increasing reparametrizations. A metric in X:

    $$\mathrm{d}_X(\gamma, \hat{\gamma}) = \inf \left\{ \|f - \hat{f}\|_\infty : f, \hat{f} \text{ parametr. of } \gamma, \hat{\gamma} \right\}.$$

  - precompact in the driving process topology: The capacity parametrization and the norm $\| \cdot \|_\infty$
  - On $\overline{F_n}$ the above descriptions are the same: the tip is uniformly visible from the target point.

Precompactness in the path convergence

Theorem

There exists $\alpha > 0$ s.t. each curve can be parametrized α-Hölder continuously with the Hölder norm being a tight random variable on $\Sigma_{\mathbb{D}}$.

- Tight r.v. Y on Σ_0: for each $\varepsilon > 0$, $\exists M > 0$ s.t. $\mathbb{P}(Y \in [-M, M]) > 1 - \varepsilon$, for any $\mathbb{P} \in \Sigma_0$.
- Aizenman&Burchard [1999]: the above holds for a collection of probability measures Σ_0 on X if

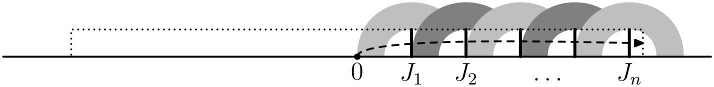
$$\mathbb{P}(\exists\ n \text{ crossings of } A(z_0, r, R)) \leq K_n \left(\frac{r}{R}\right)^{\Delta_n}$$

  for each $\mathbb{P} \in \Sigma_0$, where Δ_n large when n large.
- Half of the crossings are unforced. $\Delta_n \geq ((n-2)/12) \cdot \Delta$

Precompactness in the driving process convergence

Theorem

For any $0 < \beta < 1/2$, the driving processes of the curves are β-Hölder continuous with a tight Hölder norm on $\Sigma_{\mathbb{D}}$.

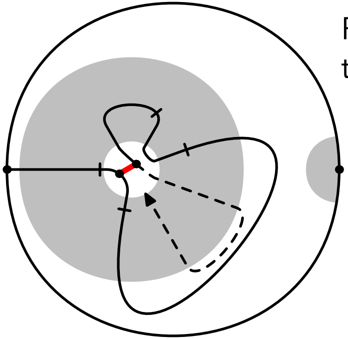


- The increments of the driving process have exponential tails:

$$\mathbb{P}\left[\, |W_t - W_s| \geq L \,\middle|\, W[0,s] \right] \leq K e^{-c \frac{L}{\sqrt{t-s}}}$$

(note: capacity parametrization)

Excluding a six-arms event



For fixed ρ, define $E(r, R)$ as the event that $\exists (s,t) \in [0,1]^2$, $s < t$, s.t.

1. diam $\left(\gamma[s,t]\right) \geq R$ and
2. $\exists$ a crosscut C, diam$(C) < r$, that separates $\gamma(s,t]$ from $B(1,\rho)$ in $\mathbb{D} \setminus \gamma(0,s]$.

Theorem

Uniformly for $\mathbb{P} \in \Sigma_{\mathbb{D}}$, $\mathbb{P}\Big(E(r,R)\Big) = o(1)$ *as* $r \to 0$.

Conclusion

- The condition assumed is uniform over the scales. Scale invariance is not assumed.
- Conformal maps were used (mostly for resolving a problem with non-smooth boundary near *a* and *b*), but conformal invariance is not assumed.
- Condition assumed is a natural property to check. Gives a conceptual way to prove the existence of subsequential limits and that the limits are well-described by Loewner equation.
- Should be applicable for any random curve converging to SLE_κ, $\kappa < 8$.
- Generalizes to several points, $\mathrm{SLE}_\kappa(\rho_1, \rho_2, \ldots)$ processes.

Thank you for your attention!