# Factorization of density correlation functions for clusters touching the sides of a rectangle


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- Choice of co–ordinates
- Solving the PDEs
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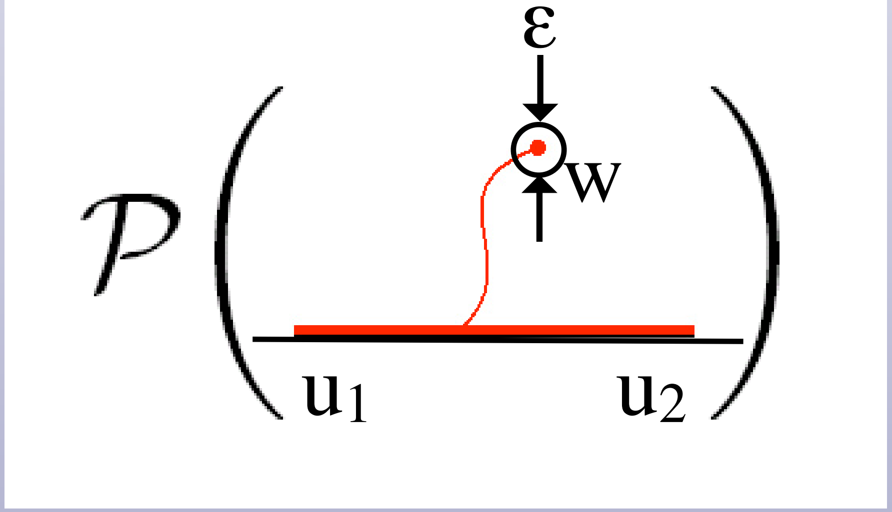
# A Guide for the (conformally) Perplexed:

Consider percolation in the upper half plane, in the continuum (field theory) limit. The probability that the interval $[u_1, u_2]$ on the real axis is connected to a small circle of size $\varepsilon$ around the point $w = u + i\,v$ is then



This is given by

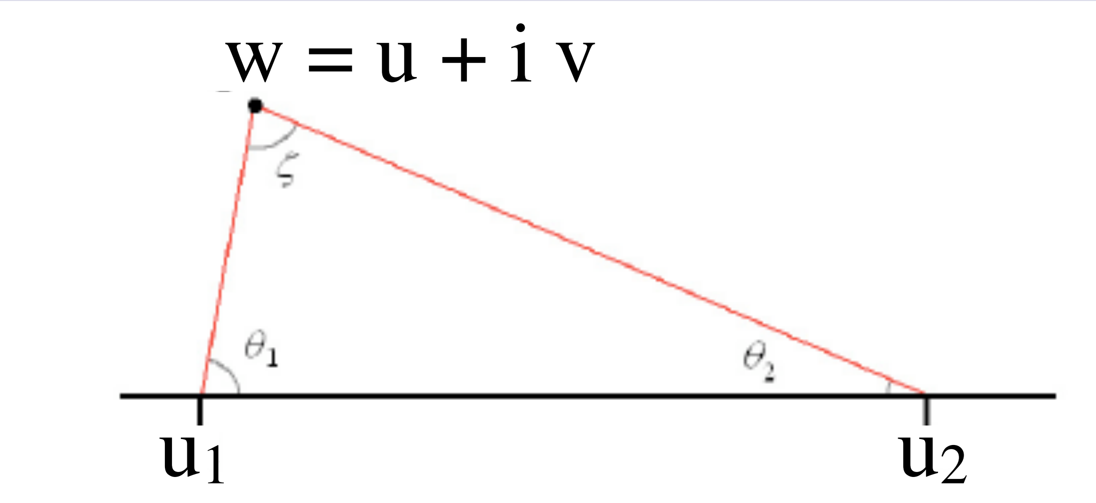
$$\mathcal{P}(u_1, u_2, w) = c_{1,2}^2\, c_{1/2,0}\, \epsilon^{5/48}\, f(u_1, u_2, w)$$

Here the (dimensionless) constants $c_{m,n}$ depend on the percolation model (lattice symmetry, type of percolation), and are thus “nonuniversal”. There is such a constant for each identified point (the subscripts will be explained below). We consider ratios in which the $c_{m,n}$ and $\varepsilon$ factors cancel out, so the ratios are “universal” (independent of the particular percolation model) and finite as $\varepsilon \to 0$.

Conformal field theory (CFT) calculates the function $f(u_1, u_2, w)$. This is a “correlation function”. In the case just mentioned we showed previously that

$$f(u_1, u_2, w) = v^{-5/48} \sin^{1/3}(\zeta/2),$$

where the angle ζ is as illustrated:



In CFT notation, the present correlation function (cf) is written

$$f(u_1, u_2, w) = \langle \phi_{1,2}(u_1)\, \phi_{1,2}(u_2)\, \phi_{1/2,0}(w) \rangle_H$$

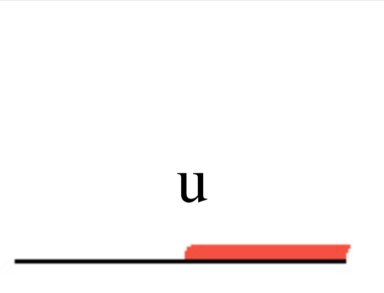
where the H denotes the upper half-plane. Here the $\phi_{m,n}$ are conformal "operators". One usually computes such a cf in the full plane by introducing an "image" operator at the reflected point $\bar{w} = u - i\, v$, following Cardy, ie

$$f(u_1, u_2, w) = \langle \phi_{1,2}(u_1)\, \phi_{1,2}(u_2)\, \phi_{1/2,0}(w)\, \phi_{1/2,0}(\bar{w}) \rangle$$

When the $\phi_{m,n}$ have integer indices, the cfs that they appear in satisfy DEs. That allows us to determine the cfs.

The conformal "operators" of use here are

1.) $\phi_{(1,2)}(u)$, which implements a change from fixed ("wired") to free boundaries at u.

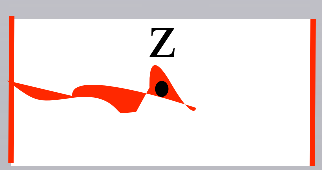



2.) $\phi_{(3/2,3/2)}(w) = \phi_{(1/2,0)}(w)$, the "magnetization" operator. This measures the density of clusters at the point w.

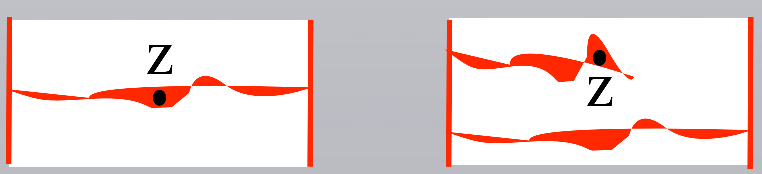
Conformal dimensions (@ c=0): $h_{(1,2)} = 0$, $h_{(3/2,3/2)} = 5/96$.

# Clusters touching the sides of a rectangle:

We consider percolation in a rectangle with the vertical edges "wired" (fixed bcs there). Take z = x + i y as the co–ordinate. Then let $\mathcal{P}_{L}(z)$ (resp. $\mathcal{P}_{R}(z)$, $\mathcal{P}_{LR}(z)$) be the probability of a cluster touching the point z and the left (resp. right, left & right) sides of the rectangle. Some examples:




$\mathcal{P}_{L}(z)$




$\mathcal{P}_{L}(z)$, $\mathcal{P}_{R}(z)$, $\mathcal{P}_{LR}(z)$

Then consider the ratio

$$C(z) = \frac{\mathcal{P}_{LR}(z)}{\sqrt{\mathcal{P}_L(z)\mathcal{P}_R(z)\Pi_h}}$$

Here $\pi_h$ is Cardy's (horizontal) crossing probability. The ratio C(z) is independent of the $c_{m,n}$ and ε factors mentioned, and therefore is universal and can be calculated from CFT.

The calculation is not so simple, since six-point cfs must be determined–four points for the corners of the rectangle, and two for the operator at z and its image.

# Why C(z)?

a. Previous results on a related ratio (with points rather than intervals) which factorizes exactly (i.e. C is independent of z). $\pi_h$ "improves" things, removing most aspect ratio dependence (it also make the ratio more homogeneous).

b. Bob Ziff's numerical results. He found that C(z) is

1. constant to within a few % everywhere in the rectangle– i.e. $P_{LR}(z)$ "almost" factorizes

2. a function of x only (i.e. it is independent of the vertical coordinate).

Our calculation verifies 1. and shows that 2. holds exactly.

The cf that must be determined is

$$<\phi_{(1,2)}(u_1)\ \phi_{(1,2)}(u_2)\ \phi_{(1,2)}(u_3)\ \phi_{(1,2)}(u_4)\phi_{(1/2,0)}(w)\phi_{(1/2,0)}(\bar{w})>$$

$w = u + i\,v$ is the half–plane co–ordinate.
This is a complicated function. Aside from an algebraic pre-factor, it depends on a function F of three independent cross-ratios.
We write equations for arbitrary central charge (equivalently, arbitrary $\kappa$ ) but for brevity present conclusions for percolation only ($c = 0$, $\kappa = 6$)

Because of the $\phi_{(1,2)}$ operators, the cf is annihilated by certain second–order operators. For ex., the operator associated with $u_1$ is

$$\frac{2h_{1/2,0}\,\mathrm{Re}\left[(w-u_1)^2\right]}{|w-u_1|^4}-\frac{2\,\mathrm{Re}\left[(\bar{w}-u_1)\partial_w\right]}{|w-u_1|^2}+\sum_{j=2}^{4}\left[\frac{h_{1,2}}{(u_j-u_1)^2}-\frac{\partial_{u_j}}{u_j-u_1}\right]-\frac{3}{2(1+2h_{1,2})}\partial_{u_1}{}^2$$

Because of this, the factor F satisfies second–order PDEs. We can choose the cross-ratios so that

$$F\left(\frac{(w-u_1)(u_4-u_3)}{(u_3-u_1)(u_4-w)}, \frac{(\bar{w}-u_1)(u_4-u_3)}{(u_3-u_1)(u_4-\bar{w})}, \frac{(u_2-u_1)(u_4-u_3)}{(u_3-u_1)(u_4-u_2)}\right)$$

Letting $\{u_1, u_2, u_3, u_4\} \rightarrow \{0, \lambda, 1, \infty\}$, gives $F \rightarrow F(w, \bar{w}, \lambda)$. Note that $\lambda$ determines the aspect ratio r of the rectangle.

Proper choice of co-ordinates greatly simplifies the PDEs, as the numerical result suggests.

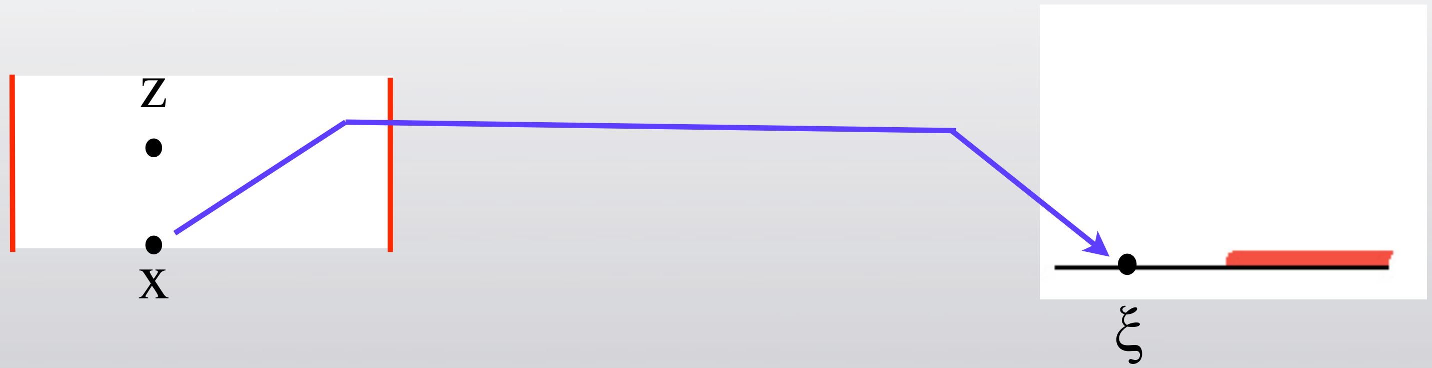
# Choice of co–ordinates:

We next transform from rectangle coordinates $z = x + i\, y$ to the upper half–plane via a Schwarz–Christoffel map

$$w(z) = m\, \mathrm{sn}(z|K'(m))^2$$

Here

1. $K'(m) = K(1\text{-}m)$, with K the complete elliptic integral of the first kind
2. sn( |m) is the Jacobi elliptic function. The elliptic parameter m is defined by $r = K(m)/K'(m)$, r being the aspect ratio of the rectangle.




The next step is key. We chose real coordinates which reflect the rectangular symmetry, namely

$$\xi = \mathrm{sn}(x|K'(m))^2$$

$$\psi = \mathrm{sn}(y|K'(1\text{-}m))^2.$$

$\xi$ is the half–plane image of x. $\psi$ is more complicated. Exchanging $x \leftrightarrow y$ (i.e. $z \leftrightarrow i\bar{z}$ ), and rescaling the rectangle to preserve its aspect ratio r, makes $\psi$ the half–plane image of y.

# The transformation to the upper–plane then becomes

$$w = m\frac{\xi(1-(1-m)\psi)-(1-\xi)(1-m\xi)\psi(1-\psi)}{(1-\psi+m\xi\psi)^2} + i\,m\frac{2\sqrt{\xi(1-\xi)(1-m\xi)\psi(1-\psi)(1-(1-m)\psi)}}{(1-\psi+m\xi\psi)^2}\,.$$

# In these co–ordinates, the PDE (with λ = m) is (thank you, Mathematica!)

$$\begin{aligned}
0 \;=\; & \big(8(6-\kappa)(\xi+\psi-\xi\psi)^2+(8-\kappa)^2(1-m\xi-\psi+m\psi)^2 \qquad \big( \\
& \qquad +8(8-\kappa)\left((1-\xi)(1-\psi)-\xi\psi(1-m\xi)(1-\psi+m\psi)\right)\big)\,F \\
& +2\kappa m(1-m)\left(((8-\kappa)(\xi-\psi)-8\xi\psi(1-2m))(2-\xi-\psi+\xi\psi)-(1-2m)\kappa\left(\xi^2-\xi^2\psi+\psi^2-\xi\psi^2\right)\right)\partial_m F \\
& +4\kappa\xi(1-\xi)\left(4\left(1-\psi+m^2\xi\psi\right)-(\kappa-2\kappa\xi+4\xi)(1-\psi+m\psi)^2-(\kappa-4)\xi m(1-\psi+m\psi)\right)\partial_\xi F \\
& +4\kappa\psi(1-\psi)\left(4\left(1-\xi+(1-m)^2\xi\psi\right)-(\kappa-2\kappa\psi+4\psi)(1-m\xi)^2-(\kappa-4)\psi(1-m)(1-m\xi)\right)\partial_\psi F \\
& -4\kappa^2 m^2(1-m)^2(\xi+\psi-\xi\psi)^2\partial_m^2 F-8\kappa^2 m(1-m)\xi(1-\xi)(1-(1-m)\psi)(\xi+\psi-\xi\psi)\partial_m\partial_\xi F \\
& +8\kappa^2 m(1-m)(1-m\xi)\psi(1-\psi)(\xi+\psi-\xi\psi)\partial_m\partial_\psi F-4\kappa^2\xi^2(1-\xi)^2(1-(1-m)\psi)^2\partial_\xi^2 F \\
& +8\kappa^2\xi(1-\xi)(1-m\xi)\psi(1-\psi)(1-(1-m)\psi)\partial_\psi\partial_\xi F-4\kappa^2(1-m\xi)^2\psi^2(1-\psi)^2\partial_\psi^2 F\,.
\end{aligned}$$

# Solving the PDEs, origin of the y–independence, and identifying cluster configurations:

How do we handle this?

1.) Using the symmetries of the rectangle: mirror about $x = r/2$, mirror about $y = 1/2$, and rotation by $90^0$ (with change of aspect ratio $r \to 1/r$) gives three additional PDEs. Because of the symmetry of our choice of $\xi$ and $\psi$, these are the same equations that arise from applying CFT at the other $\phi_{(1,2)}$ operators.

These symmetries translate, respectively, into

$$(\xi, \psi, m) \to \left(\frac{1-\xi}{1-m\,\xi}, \psi, m\right), \left(\xi, \frac{1-\psi}{1-(1-m)\psi}, m\right), (\psi, \xi, 1-m) \;.$$

2.) There is a linear combination of the four equations which gives

$$\partial_\psi \partial_\xi F(\xi, \psi, m) = 0 \, ,$$

(2. is the origin of the y–independence of C(z) discussed above.)

3.) Thus all solutions must be of the form

$$F(\xi, \psi, m) = G(\xi, m) + \widetilde{G}(\psi, m) \, .$$

(A solution of the form g(m) can't satisfy the original PDEs.)

4.) Further, the symmetry $\{\psi \leftrightarrow \xi, \, m \leftrightarrow 1\text{-}m\}$ implies

$$F(\xi, \psi, m) = G(\xi, m) + G(\psi, 1\text{-}m).$$

## 5.) Redefining G as

$$G(\xi,m) = \frac{(1-m)^{2/\kappa}}{m^{(6-\kappa)/\kappa}\,[\xi(1-\xi)(1-m\xi)]^{(8-\kappa)/(2\kappa)}} H(m, m\,\xi)$$

substituting into the DEs, and taking linear combinations that allow us to cancel out all factors of ψ we arrive at

$$\begin{aligned}
0 &= \frac{2(8-\kappa)(\kappa-4)}{\kappa^2} H(s,t) + \frac{2(\kappa-4)-4(\kappa-5)t}{\kappa}\partial_t H + t(1-t)\partial_t^2 H + \frac{2(8-\kappa)}{\kappa} s\partial_s H + s(1-t)\partial_t\partial_s H \;, \\
0 &= -\frac{4(\kappa-4)}{\kappa^2} H(s,t) - \frac{4}{\kappa} t\partial_t H + \frac{2(\kappa-4)-2\kappa\, s}{\kappa}\partial_s H + t(1-s)\partial_t\partial_s H + s(1-s)\partial_s^2 H \;, \quad \text{and} \\
0 &= -\frac{4}{\kappa}\partial_t H - \frac{2(8-\kappa)}{\kappa}\partial_s H + (t-s)\partial_t\partial_s H \;.
\end{aligned}$$

where s = m and t = m ξ, the standard form of Appell's hypergeometric DEs for the Appell function $F_1$. (Note we write this for arbitrary $\kappa$ .)

The Appell function $F_1$ is a two–variable generalization of the hypergeometric function:

$$F_1\left(\begin{matrix} a;\ b_1, b_2 \\ c \end{matrix}\middle|\, x,\ y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}\,(b_1)_n\,(b_2)_m}{(c)_{n+m}\ n!\ m!}\, x^n\, y^m$$

6.) There is a three–dimensional solution space, including the five convergent Frobenius series

$$H_{\mathrm{I}}(s,t) = s^{-4/\kappa}t^{12/\kappa-1}F_1\left(\frac{\kappa-4}{\kappa};\frac{4}{\kappa},\frac{4}{\kappa};\frac{12}{\kappa}\middle|\frac{t}{s},t\right),$$

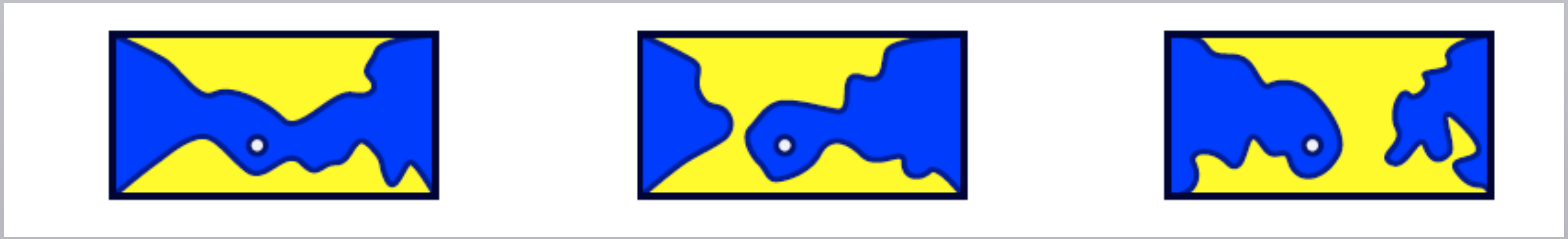
$$H_{\mathrm{II}}(s,t) = F_1\left(\frac{\kappa-4}{\kappa};\frac{4}{\kappa},-\frac{2(8-\kappa)}{\kappa};\frac{2(\kappa-4)}{\kappa}\middle|1-s,1-t\right),$$

$$H_{\mathrm{III}}(s,t) = F_1\left(\frac{\kappa-4}{\kappa};\frac{4}{\kappa},-\frac{2(8-\kappa)}{\kappa};\frac{2(\kappa-4)}{\kappa}\middle|s,t\right),$$

$$H_{\mathrm{IV}}(s,t) = \frac{(1-t)^{2(8/\kappa-1)}}{(1-s)^{8/\kappa-1}}F_1\left(\frac{\kappa-4}{\kappa};\frac{4}{\kappa},-\frac{2(8-\kappa)}{\kappa};\frac{2(\kappa-4)}{\kappa}\middle|1-s,\frac{1-s}{1-t}\right),$$

$$H_{\mathrm{V}}(s,t) = \frac{(s-t)^{12/\kappa-1}(1-t)^{4/\kappa-1}}{s^{4/\kappa}(1-s)^{8/\kappa-1}}F_1\left(\frac{\kappa-4}{\kappa};\frac{4}{\kappa},\frac{4}{\kappa};\frac{12}{\kappa}\middle|\frac{s-t}{1-t},\frac{s-t}{s(1-t)}\right),$$

7.) Making use of this, we find five solutions for the original six–point correlation function. That is still not quite the whole story: since each of these is a Frobenius series, it corresponds to a conformal block (function associated with a single term in the CFT operator product expansion). But what we really want are the functions associated with each cluster configuration of interest:



LR, L, R R L

8.) Some analysis involving vertex operators (translation: integral representations) and various limits gives for our five solutions (here $\Pi$ is the weight of the indicated configuration)

$$
\begin{aligned}
G_I &\sim \Pi_R \\
G_{II} &\sim \Pi_{LR} + \Pi_R \\
G_{III} &\sim \Pi_{LR} + \Pi_L + \Pi_R \\
G_{IV} &\sim \Pi_{LR} + \Pi_L \\
G_V &\sim \Pi_L
\end{aligned}
$$

9.) Finally, transforming to a rectangle (and noting that corner operators enter) we find

$$\Pi_{LR} = f(x,y)\,(G_{II}(\xi) - G_I(\xi))$$
$$\Pi_R = f(x,y)\,G_I(\xi)$$
$$\Pi_L = f(x,y)\,G_V(\xi)$$

with the common factor

f(x,y) =
$$2^{h_{1,3}}(K')^{4h^c_{1,2}+2h_{1/2,0}}(m(1-m))^{2h_{1,2}}\left(\frac{\operatorname{Im}\left[\operatorname{sn}(zK'|m)^2\right]}{|\operatorname{sn}(zK'|m)\operatorname{cn}(zK'|m)\operatorname{dn}(zK'|m)|}\right)^{h_{1,3}-2h_{1/2,0}}.$$

(As above, K′=K(1-m), with K the complete elliptic integral.)

The common factor $f(x,y)$ cancels out of the ratio, so that $C(z)$ only depends on m (parameterizing the aspect ratio) and $\xi$, which is independent of y. Thus for a given rectangle

$$C = C(x),$$

demonstrating that the original numerical observation is exact.

Furthermore, the whole derivation can be generalized to arbitrary central charge (equivalently, arbitrary n or $\kappa$ ). The (approximate) factorization generalizes as well. In doing this, the cfs that enter C must generalized as well.

# Note added:

Dmitri Beliaev and Konstantin Izyurov have recently obtained a rigorous derivation of these PDEs for the case $SLE_6$ with one interval of infinitesimal length.

# Conclusion:

Results from conformal field theory show that a certain ratio of correlation functions in 2-D critical rectangles almost factorizes exactly, with the deviation from exact factorization depending on only one co–ordinate. This follows from a solution of the CFT PDEs in properly chosen co–ordinates.

# YAPPS

YAPPS

Yet
Another
Peculiar
Percolation
Symmetry