Random Conformal Welding

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Random Planar Curves

2d Statistical Mechanics: phase boundaries

- Closed curves or curves joining boundary points of a domain
- Critical temperature: **Conformally invariant curves**

SLE:

- Describes curve growing in fictitious time
- Concrete stochastic process given in terms of Brownian motion
- Closed curves?

Welding

Conformal welding gives a correspondence between:

Closed curves in $\hat{\mathbb{C}}$ $\leftrightarrow$ **Homeomorphisms** $\phi : S^1 \to S^1$

Get **random curves** from **random homeomorphisms of circle**

- Möbius invariant construction
- Parametrized in terms of gaussian free field

Welding Closed Curves

From **Closed curves** in $\hat{\mathbb{C}}$ to **Homeomorphisms** of S^1:

Jordan curve $\Gamma \subset \hat{\mathbb{C}}$ splits $\hat{\mathbb{C}} \setminus \Gamma = \Omega_+ \cup \Omega_-$

Riemann mappings

$$f_+ : \mathbb{D} \to \Omega_+ \quad \text{and} \quad f_- : \mathbb{D}_\infty \to \Omega_-$$

f_- and f_+ extend continuously to $S^1 = \partial\mathbb{D} = \partial\mathbb{D}_\infty \implies$

$$\phi = (f_+)^{-1} \circ f_- : S^1 \to S^1 \ \text{Homeomorphism}$$

Welding problem: invert this:

Given $\phi : S^1 \to S^1$, **find** Γ and $f_\pm$.

QC homeomorphisms

Idea:

- Extend $\phi : S^1 \to S^1$ to a **quasiconformal** homeomorphism of the plane $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$
- Solve a **Beltrami equation** to get the conformal map f_-

Recall: a homeo $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is quasiconformal if

- ∇f is locally integrable
- **Complex dilation** of f

$$\mu(z) := \frac{\partial_{\bar{z}} f}{\partial_z f}$$

satisfies $|\mu(z)| < 1$ a.e.

This means f solves the **elliptic** Beltrami equation

$$\partial_{\bar{z}} f = \mu(z) \partial_z f$$

Beltrami equation

Suppose now

$$\phi = f|_{S^1}$$

for a quasiconformal f with dilation μ. Try to solve

$$\partial_{\bar{z}} F = \begin{cases} \mu(z)\partial_z F & \text{if } x \in \mathbb{D} \\ 0 & \text{if } x \in \mathbb{D}_\infty \end{cases}$$

Since $\partial_{\bar{z}} F = 0$ for $|z| > 1$

$$F|_{\mathbb{D}_\infty} := f_- : \mathbb{D}_\infty \to \Omega_-$$

is **conformal** and $\Gamma = F(S^1)$ is a Jordan curve.

Uniqueness

For $z \in \mathbb{D}$ we have two solutions of the **Beltrami equation**:

$$\begin{aligned}\partial_{\bar{z}} f &= \mu(z)\partial_z f \\ \partial_{\bar{z}} F &= \mu(z)\partial_z F\end{aligned}$$

How are they related?

- If f solves Beltrami, $g \circ f$ solves too for g **conformal**
- **Uniqueness of solutions:** all solutions of this form

If uniqueness holds $\exists$ **conformal** f_+ on $\mathbb{D}$ s.t.

$$F(z) = f_+ \circ f(z), \quad z \in \mathbb{D},$$

Solution

We found two **conformal** maps $f_{\pm}$:

$$f_{-} = F|_{\mathbb{D}_\infty}, \quad f_{+} \circ f|_{\mathbb{D}} = F|_{\mathbb{D}}$$

with $f_{-} : \mathbb{D}_\infty \to \Omega_{-}$ and $f_{+} : \mathbb{D} \to F(\mathbb{D}) := \Omega_{+}$.

$f_{\pm}$ **solve weding:**

Since $f|_{S^1} = \phi$ then on the circle

$$\phi = f_{+}^{-1} \circ f_{-}$$

and the curve corresponding to ϕ is

$$\Gamma = f_{\pm}(S^1)$$

Existence and Uniqueness

When does this work?

Reduction to Beltrami equation

- When can we extend $\phi : S^1 \to S^1$ to a QC map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$?

Existence and uniqueness for Beltrami

- Given μ, when is there a solution to $\partial_{\bar{z}} f = \mu(z) \partial_z f$, unique up to $f \to g \circ f$, g conformal?

Uniqueness of the curve Γ

- If $\phi = f_+^{-1} \circ f_-$ when are $f_\pm$ unique up to

$$f_\pm \to M \circ f_\pm, \quad M \;\; \text{Mobius}$$

Ellipticity

Extension

If ϕ is **quasisymmetric** i.e. if it has **bounded distortion**

$$\sup_{s,t\in S^1} \frac{|\phi(s+t)-\phi(s)|}{|\phi(s-t)-\phi(s)|} < \infty$$

then it can be extended to a QC homeo of $\hat{\mathbb{C}}$ with $\|\mu\|_\infty < 1$

$\exists$! of Beltrami

If $\|\mu\|_\infty < 1$ the Beltrami eqn is uniformly elliptic and:

- Solutions exist and are unique (up to conformal maps)
- Curve Γ is unique (up to Möbius)

Our ϕ are **not quasisymmetric** and our Beltrami is **not uniformly elliptic**.

Circle homeomorphisms

Homeomorphisms of S^1

- Identify $S^1 = \mathbb{R}/\mathbb{Z}$ by $t \in [0,1] \to e^{2\pi it} \in S^1$
- Homeo ϕ is a continuous increasing function on $[0,1]$ with $\phi(0) = 0$, $\phi(1) = 1$
- If ϕ were a **diffeomorphism** then $\phi'(t) > 0$ so $\phi'(t) = e^{X(t)}$, X real, and

$$\phi(t) = \int_0^t e^{X(s)} ds / \int_0^1 e^{X(s)} ds$$

Proposal of P. Jones: Take X a **random field**, the restriction of **2d free field** on the unit circle. The result is **not** differentiable.

Random measure

Let $X(s)$ be the **Gaussian random field** with covariance

$$\mathbb{E}\, X(s)X(t) = -\log|e^{2\pi is} - e^{2\pi it}|$$

- X is not a function: $\mathbb{E}\, X(t)^2 = \infty$
- Smeared field $\int_0^1 f(t)X(t)dt$ is a random variable

Define $e^{\beta X(s)}$:

- Regularize: $X \to X_\epsilon$
- Normal order : $e^{\beta X_\epsilon(s)} := e^{\beta X_\epsilon(s)}/\mathbb{E}\, e^{\beta X_\epsilon(s)} \quad \beta \in \mathbb{R}$

Then, almost surely

$$\mathrm{w} - \lim_{\epsilon \to 0} : e^{\beta X_\epsilon(s)} : ds = \tau_\beta(ds)$$

$\tau_\beta(ds)$ is a **random Borel measure** on S^1 ("quantum length").

Random homeomorphisms

Properties of τ:

- $\tau_\beta = 0$ if $\beta \geq \sqrt{2}$
- For $0 \leq \beta < \sqrt{2}$, τ_β has no atoms
- $\mathbb{E}\,\tau_\beta(B)^p < \infty$ for $-\infty < p < 2/\beta^2$

Let, for $\beta < \sqrt{2}$

$$\phi(t) := \tau_\beta([0,t])/\tau_\beta([0,1]).$$

ϕ is **almost surely Hölder continuous homeo**

By Hölder inequality the distortion

$$\frac{|\phi(s+t)-\phi(s)|}{|\phi(s-t)-\phi(s)|} = \frac{\tau([s,s+t])}{\tau([s-t,s])} \in L^p(\omega), \quad p < 2/\beta^2$$

but ϕ is a.s. **not quasisymmetric**.

Result

Theorem. Let ϕ_β be the random homeomorphism

$$\phi_\beta(t) = \tau_\beta([0,t])/\tau_\beta([0,1])$$

with $\beta < \sqrt{2}$. Then a.s. ϕ_β admits a conformal welding

$$(\Gamma_\beta, f_{\beta+}, f_{\beta-}).$$

The Jordan curve Γ_β is unique, up to a Möbius transformation and almost surely continous in β.

Connection to SLE

- Welding homeo looses continuity as $\beta \uparrow \sqrt{2}$
- $\nu(ds) := \tau_\beta(ds)/\tau_\beta([0,1])$ is a **Gibbs measure** of a **Random Energy Model** with logarithmically correlated energies, β^{-1} **temperature**
- Conjecture: $\lim_{\epsilon \to 0} \nu_\epsilon$ nontrivial also for $\beta \geq \sqrt{2}$
- $\beta > \sqrt{2}$ is a **spin glass phase** and ν is believed to be atomic.

Questions. Γ vs SLE_κ, $\kappa = 2\beta^2$? Duplantier, Sheffield (need to compose our maps)

Does welding exist for $\beta = \sqrt{2}$?

What is the right framework for $\beta > \sqrt{2}$?

Outline of proof

1.**Extension** of ϕ to $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by **Beurling-Ahlfors**
$\implies$ bound for $\mu = \bar{\partial} f / \partial f$ in terms of the measure τ.

2. **Existence** for Beltrami equation by a method of Lehto to control moduli of annuli

3. Probabilistic **large deviation estimate** for the **Lehto integral** which controls moduli of annuli

4. **Uniqueness** of welding: theorem of Jones-Smirnov on removability of Hölder curves

Extension

Beurling-Ahlfors extension: $\phi : \mathbb{R} \to \mathbb{R}$ extends to $F_\phi : \mathbb{H} \to \mathbb{H}$

$$F_\phi(x+iy) = \frac{1}{2}\int_0^1 (\phi(x+ty)+\phi(x-ty)+i(\phi(x+ty)-\phi(x-ty))dt.$$

Solve Beltrami

$$\partial_{\bar{z}} F = \chi_{\mathbb{D}}(z)\mu(z)\partial_z F$$

with

$$\mu = \partial_{\bar{z}} F_\phi / \partial_z F_\phi$$

to get

$$\Gamma = F(\partial\mathbb{D})$$

Existence and Hölder

Existence by equicontinuity of regularized solutions:

$\mu \to \mu_\epsilon := (1-\epsilon)\mu$ **elliptic**: $\|\mu_\epsilon\|_\infty \leq 1-\epsilon$.

Show for balls $B_r(w)$

$$\mathrm{diam}(F_\epsilon(B_r(w))) \leq Cr^a$$

uniformly in ϵ. Then F_ϵ uniformly Hölder continuous.

Bonus: F **Hölder** (use for uniqueness of welding)

Moduli

Idea by Lehto: **control images of annuli** under F:

$$\mathrm{diam}(F_\epsilon(B_r(w))) \leq 80e^{-\pi \mathrm{mod} \mathcal{A}_r}.$$

- Annular region $\mathcal{A}_r := F(B_1(w) \setminus B_r(w))$
- $\mathrm{mod}\mathcal{A}_r$ **modulus** of $\mathcal{A}_r$

Hölder continuity follows if can show

$$\mathrm{mod}\mathcal{A}_r \geq c \log(1/r), \quad c > 0$$

Lehto integral

Lower bound for moduli of images of annuli:

$$\mathrm{mod} F(B_R(w) \setminus B_r(w)) \geq 2\pi L(w, r, R)$$

$L(w, r, R)$ is the **Lehto integral**:

$$L(w, r, R) = \int_r^R \frac{1}{\int_0^{2\pi} K\left(w + \rho e^{i\theta}\right) d\theta} \frac{d\rho}{\rho}$$

K is the **distortion** of F_ϕ

$$K(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

Need to show

$$L(w, r, 1) \geq a \log(1/r)$$

Localization

Let $w \in \partial\mathbb{D}$ and decompose in scales:

$$L(w, 2^{-n}, 1) = \sum_{k=1}^{n} L(w, 2^{-k}, 2^{-k+1}) := \sum_{k=1}^{n} L_k$$

Point:

- L_k are **i.i.d.** and **weakly correlated**
- $P(L_k < \epsilon) \to 0$ as $\epsilon \to 0$.

Reason:

- L_k can be bounded in terms of $\frac{\tau(I)}{\tau(J)}$, I, J dyadic intervals of size $\mathcal{O}(2^{-n})$ near w
- Random variables $\tau(I) = \int_I e^{\beta X(s)} ds$ depend mostly on the scale 2^{-n} part of the free field $X(s)$ for $s \in I$ and these are almost independent.

Probabilistic estimate

Prove a **large deviation estimate**:

$$\mathrm{Prob}\big(L(w, 2^{-n}, 1) < \delta n\big) \leq 2^{-(1+\epsilon)n}$$

For some $\delta > 0$, $\epsilon > 0$ and all $n > 0$

Rest is **Borel-Cantelli**:

- Pick a grid, spacing $2^{-(1+\frac{1}{2}\epsilon)n}$, points w_i, $i = 1, \ldots 2^{(1+\frac{1}{2}\epsilon)n}$.
- Then for almost all ω: for $n > n(\omega)$ and all w_i

$$L(w_i, 2^{-n}, 1) > \delta n$$

Then for all balls

$$\mathrm{diam}(F_\epsilon(B_r)) < Cr^a$$

$\implies$ Hölder continuity a.e. in ω

Uniqueness

Uniqueness for welding follows from Hölder continuity:

Suppose $f_\pm$ and $f'_\pm$ are two solutions, mapping $\mathbb{D}, \mathbb{D}_\infty$ onto $\Omega_\pm$ and $\Omega'_\pm$. Show:

$$f'_\pm = \Phi \circ f_\pm, \qquad \Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \text{ Möbius.}$$

Now

$$\Psi(z) := \begin{cases} f'_+ \circ (f_+)^{-1}(z) & \text{if } z \in \Omega_+ \\ f'_- \circ (f_-)^{-1}(z) & \text{if } z \in \Omega_- \end{cases}$$

is **continuous** on $\hat{\mathbb{C}}$ and **conformal** outside $\Gamma = \partial\Omega_\pm$.

Result of **Jones-Smirnov**: Hölder curves are **conformally removable** i.e. Ψ extends conformally to $\widehat{\mathbb{C}}$ i.e.it is Möbius.

Decomposition to scales

Decompose the free field X into scales:

$$X = \sum_{n=0}^{\infty} X_n$$

X_n are **i.i.d.** modulo scaling:

- $X_n \sim x(2^n \cdot)$ in law
- x **smooth field correlated on unit scale:** $x(s)$ and $x(t)$ are **independent** if $|s - t| > \mathcal{O}(1)$
- $\implies X_n(s)$ and $X_n(t)$ are independent if $|s - t| > \mathcal{O}(2^{-n})$.

Decomposing free field

Nice representation of free field in terms of white noise (Kahane, Bacry, Muzy):

$$X(s) = \int_{\mathbb{H}} \frac{W(dxdy)}{y} \chi(|x - s| \leq y)$$

W is white noise in $\mathbb{H}$.

$$X_n(s) = \int_{\mathbb{H}} \frac{W(dxdy)}{y} \chi(|x - s| \leq y) \chi(y \in [2^{-n}, 2^{-n+1}])$$

Questions

- Is Γ_β "locally like $\mathrm{SLE}_{2\beta^2}$" (or, if compose such weldings)?
- What happens at $\beta^2 \geq 2$?
- Easier: understand the Gibbs measure at $\beta^2 \geq 2$.