A LOOK AT THE SCHRAMM-LOEWNER EVOLUTION (SLE) CURVE

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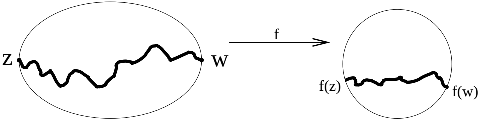
- The Schramm-Loewner evolution (SLE_κ) is a one-parameter family of paths invented by Oded Schramm in the late 1990s as a candidate for the limit of critical two-dimensional lattice models that exhibit conformal invariance in the scaling limit.
- It been shown to be the scaling limit of a number of models (percolation interfaces loop-erased random walk, uniform spanning tree, harmonic explorer, level lines of Gaussian free field, Ising interfaces), has been a tool in proving facts about Brownian motion, and is conjectured to be the limit of other models (self-avoiding walk).
- This talk will concentrate on SLE itself and will not discuss convergence of the lattice models.

ASSUMPTIONS ON SCALING LIMIT

Finite measure $\mu_D(z,w)$ and probability measure $\mu_D^{\#}(z,w)$ on curves connecting boundary points of a domain D.

$$\mu_D(z,w) = C(D;z,w)\,\mu_D^{\#}(z,w).$$



- **Conformal invariance**: If f is a conformal transformation

$$f \circ \mu_D^{\#}(z,w) = \mu_{f(D)}^{\#}(f(z),f(w)).$$

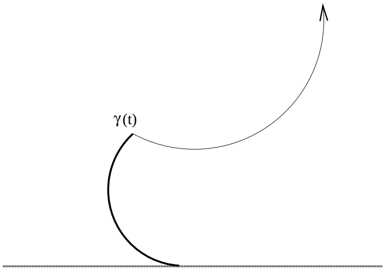
- **Scaling rule**

$$C(D;z,w) = |f'(z)|^b\,|f'(w)|^b\,C(f(D);f(z),f(w)).$$

- The constant $C(D;z,w)$ can be considered a (normalized) partition function.

- **Domain Markov property** Given $\gamma[0, t]$, the conditional distribution on $\gamma[t, \infty)$ is the same as

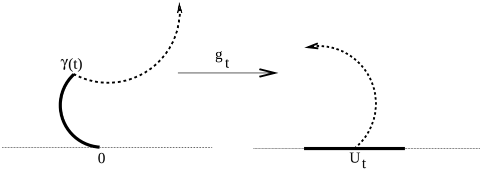
$$\mu^{\#}_{\mathbb{H}\setminus\gamma(0,t]}(\gamma(t), \infty).$$



- For simply connected D, $\mu^{\#}_{\mathbb{H}}(0, \infty)$ determines $\mu^{\#}_{D}(z, w)$ (Riemann mapping theorem).

LOEWNER EQUATION IN UPPER HALF PLANE

- Let $\gamma : (0, \infty) \to \mathbb{H}$ be a simple curve with $\gamma(0+) = 0$ and $\gamma(t) \to \infty$ as $t \to \infty$.
- $g_t : \mathbb{H} \setminus \gamma(0, t] \to \mathbb{H}$



- Can reparametrize if necessary so that

$$g_t(z) = z + \frac{2t}{z} + \cdots, \qquad z \to \infty$$

- g_t satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z.$$

Moreover, $U_t = g_t(\gamma(t))$ is continuous.

(Schramm) Suppose γ is a random curve satisfying conformal invariance and Domain Markov property. Then U_t must be a random continuous curve satisfying

- For every $s < t$, $U_t - U_s$ is independent of $U_r, 0 \leq r \leq s$ and has the same distribution as U_{t-s}.
- $c^{-1}\, U_{c^2 t}$ has the same distribution as U_t.

Therefore, $U_t = \sqrt{\kappa}\, B_t$ where B_t is a standard (one-dimensional) Brownian motion.

The *(chordal) Schramm-Loewner evolution with parameter* κ (SLE_κ) is the solution obtained by choosing $U_t = \sqrt{\kappa}\, B_t$.

To make the equations slightly simpler do linear change of variables

$$a = \frac{2}{\kappa}$$

$$g_t(z) = z + \frac{at}{z} + O(|z|^2)$$

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad U_t = -B_t.$$

If $Z_t = X_t + iY_t = Z_t(z) = g_t(z) - U_t$, then

$$dZ_t = \frac{a}{Z_t}\, dt + dB_t$$

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2}\, dt + dB_t, \qquad \partial_t Y_t = -\frac{aY_t}{X_t^2 + Y_t^2}.$$

EXISTENCE OF CURVE

- Deterministic estimates based on Hölder continuity properties of Brownian motion are insufficient to determine existence of curve (Marshall, Rohde, Lind,...)

Let

$$f_t(z) = g_t^{-1}(z + U_t).$$

Intuitively, $\gamma(t) = g_t^{-1}(U_t) = f_t(0)$. Let

$$\gamma_n(t) = f_t(i/n).$$

- Goal: Try to show the limit

$$\gamma(t) = \lim_{n \to \infty} \gamma_n(t)$$

exists and gives a continuous function of t.

- Need to study distribution of $|f_t'(iy)|$ for small y.

- For $\kappa \neq 8$, Rohde and Schramm used moment estimates for $|f_t'(iy)|$ to show existence of curve.
- Finer estimates (RS, Lind, Johansson-L) show that the curve $\gamma(t), \epsilon \leq t \leq 1$, is α-Hölder continuous (with respect to capacity parametrization) if

$$\alpha < \alpha_* = \alpha_*(\kappa) = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}$$

  and not for $\alpha > \alpha_*$.
- $\alpha_* > 0$ if $\kappa \neq 8$.
- Existence of curve for $\kappa = 8$ known only through relation with uniform spanning tree (L-Schramm-Werner). It is not α-Hölder continuous for any $\alpha > 0$.
- Open problem: Find a lower bound for the modulus of continuity if $\kappa = 8$

PHASES FOR SLE_κ

- SLE_κ gives a simple curve iff $\kappa \le 4$.
- To prove, consider equivalent question: does SLE_κ hit $[x, \infty)$ for $x > 0$?
- Let $X_t = g_t(x) - U_t$. Does $X_t = 0$ for some t?
- X_t satisfies
$$dX_t = \frac{a}{X_t}\,dt + dB_t.$$
- Standard facts about Bessel equation show that this avoids origin iff $a = 2/\kappa \ge 1/2$.

- SLE_κ is plane-filling iff $\kappa \geq 8$.
- For $z \in \mathbb{H}$, let $\Theta_t = \arg[g_t(z) - U_t]$.
- After reparametrization, $\tilde{\Theta}_t = \Theta_{\sigma(t)}$ satisfies

$$d\tilde{\Theta}_t = (1 - 2a) \cot \tilde{\Theta}_t \, dt + dW_t.$$

- Θ_t is a martingale iff $\kappa = 4$ (related to harmonic explorer and GFF, Schramm-Sheffield)
- If $1 - 2a \geq 1/2$ ($\kappa \geq 8$) by comparison with Bessel, this never reaches zero (argument fluctuates as path approaches point z).
- For $\kappa < 8$ can determine probability that $\Theta_\infty = \pi$ (z is on left side of curve).

$$\int_0^\theta \frac{c \, dr}{\sin^{2-4a} r}, \quad \theta = \arg(z).$$

SLE_κ IN OTHER DOMAINS

- D simply connected domain, $z, w \in \partial D$.
- Schramm defined the probability measure $\mu_D^\#(z, w)$ as the conformal image of $\mu_{\mathbb{H}}^\#(0, \infty)$. This is defined modulo reparametrization.
- Consider $D \subset \mathbb{H}$ with $\mathbb{H} \setminus D$ bounded, $\mathrm{dist}(0, D) > 0$.
- Can we define SLE_κ from 0 to ∞ in D directly so that conformal invariance is a result? (Boundary perturbation)
- How about SLE_κ in $\mathbb{H}$ from 0 to $x \in \mathbb{R}$?
- We will consider the easier case $\kappa \leq 4$ with simple paths.

IMPORTANT PARAMETERS

- Central charge

$$\mathbf{c} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa} \in (-\infty, 1].$$

- The relationship $\kappa \leftrightarrow \mathbf{c}$ is two-to-one with a double root at $\kappa = 4, \mathbf{c} = 1$. The dual value of κ is $\tilde{\kappa} = 16/\kappa$.
- Boundary scaling exponent (dimension)

$$b = \frac{3a-1}{2} = \frac{6-\kappa}{2\kappa} \in \left(-\frac{1}{2}, \infty\right).$$

- b is strictly decreasing in κ.

Brownian loop measure (BLM) (L.-Werner)

- Infinite (σ-finite) Conformally invariant measure on unrooted loops satisfying restriction property.
- Specify rooted loop $\omega : [0, t_\omega] \to \mathbb{H}$ as a triple $(z_0, t_\omega, \hat{\omega})$ then rooted loop measure is

$$\text{area} \times \left[\frac{1}{2\pi t^2} \, dt \right] \times \text{Brownian bridge}$$

- BLM in $\mathbb{C}$ obtained by forgetting root. BLM in $D \subset \mathbb{C}$ obtained by restriction.
- $\Lambda_D(V_1, V_2)$ denotes BLM of loops in D that intersect both V_1 and V_2.
- Well-defined for non-simply connected D.

- Define a measure $\mu_D(0,\infty)$ by

$$\frac{d\mu_D(0,\infty)}{d\mu_{\mathbb{H}}(0,\infty)}(\gamma) = 1\{\gamma \subset D\} \exp\left\{\frac{\mathbf{c}}{2}\Lambda_{\mathbb{H}}(\gamma, \mathbb{H}\setminus D)\right\}.$$

- Write

$$\mu_D(0,\infty) = C(D;0,\infty)\,\mu_D^{\#}(0,\infty).$$

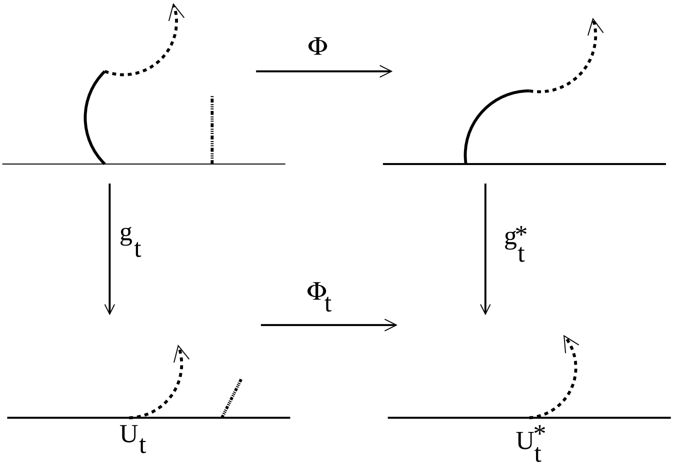
  where $\mu_D^{\#}(0,\infty)$ is a probability measure.

- Theorem: For $\kappa \le 4$,

$$C(D;0,\infty) = \Phi'(0)^b$$

  where $\Phi : D \to \mathbb{H}$ with $\Phi(\infty) = \infty, \Phi'(\infty) = 1$. Moreover, $\mu_D^{\#}(0,\infty)$ is SLE_κ in D as defined by Schramm.

- $\mathbf{c} = 0 (\kappa = 8/3)$ *restriction property*.



$$M_t = \Phi_t'(U_t)^b \, \exp\left\{ \frac{\mathbf{c}}{2} \, \Lambda_{\mathbb{H}}(\gamma_t, D) \right\}.$$

- $dM_t = b\,[\log \Phi_t'(U_t)]'\, dU_t$
- If one uses Girsanov theorem, to weight by the local martingale M_t, then one obtains a drift of $b\,[\log \Phi_t'(U_t)]$.
- This is the same as that from conformal image of SLE_κ in $\mathbb{H}$.
- Locally this holds for all κ; for $\kappa \leq 4$, M_t is actually a martingale and we can let $t \to \infty$.
- This analysis shows why SLE_κ is conjectured to be related to the b-Laplacian random walk. (This is rigorous for $\kappa = 2, b = 1$.)

- If D is bounded, simply connected domain and z, w smooth boundary points, define

$$C(D; z, w) = H_D(z, w)^b,$$

where $H_D(z, w)$ denotes (multiple of) Poisson kernel.

- $C(\mathbb{H}; 0, x) = x^{-2b}$.
- $\mu_D(z, w) = C(D; z, w)\, \mu^{\#}(z, w)$,
- $f \circ \mu_D(z, w) = |f'(z)|^b \, |f'(w)|^b \, \mu_{f(D)}(f(z), f(w))$.
- The function $C(D; z, w)$ can be called the (normalized) *partition function* for chordal SLE_κ.
- Although defined only for smooth boundaries, if $D_1 \subset D$, the ratio

$$\frac{C(D_1; z, w)}{C(D; z, w)}$$

is a conformal *invariant* and is defined for nonsmooth boundaries.

- SLE_κ from 0 to x in $\mathbb{H}$ can be obtained by weighting SLE_κ from 0 to ∞ by the partition function:

$$C(\mathbb{H}; U_t, g_t(x)) = X_t^{-2b}, \quad X_t = g_t(x) - U_t.$$

$$M_t = g_t'(x)^\lambda \, X_t^{-2b}$$

$$dM_t = \frac{2b}{X_t} \, M_t \, dU_t.$$

- Girsanov theorem states that there is a BM W_t in new measure such that

$$dU_t = \frac{2b}{X_t} \, dt + dW_t.$$

- This gives an example of a $SLE(\kappa, \rho)$ process. The probability measure $\mu_{\mathbb{H}}^{\#}(0, x)$ can be described in terms of $SLE(\kappa, \rho)$ processes only.

OPEN PROBLEM: NON-SIMPLY CONNECTED DOMAINS

- Conformal invariance and domain Markov property insufficient to define SLE_κ in general domains.
- Two possible approaches: find partition function or find "drift term" to process. In each case expect locally absolutely continuous with respect to SLE_κ.
- For $\kappa = 2$ (loop-erased random walk, $b = 1$), one can choose the partition function to be the Poisson kernel (which makes sense in general domain). However, this is not correct for other κ.
- One can define process using Radon-Nikodym derivative and Brownian loop measure, but a number of technical issues are open (as well as the question — is this what we want?)

MULTIPLE *SLE* PATHS

- Consider two *SLE* paths γ^1, γ^2 growing from $0, x$ in $\mathbb{H}$; $\gamma_s^j = \gamma^j[0, s]$
- If paths are interacting, give Radon-Nikodym derivative at (γ_s^1, γ_t^2) with respect to independent *SLE*s.
- Parametrization can be tricky, but the R-N derivative should be independent of the choice of parametrization.

NON-INTERSECTING PATHS ($\kappa \leq 4$)

(L-Kozdron, Dubédat, Cardy, L-Lind, Bauer-Bernard, Kenyon-Wilson...)

- Consider simply connected D with z_1, z_2, w_1, w_2, smooth boundary points.
- Measure on pairs (γ^1, γ^2) where γ^j connects z_j to w_j in D.
- Choose γ^1 according to $\mu_D(z_1, w_1)$ weighted by $C(\tilde{D}; z_2, w_2)$ where $\tilde{D} = \tilde{D}(\gamma^1)$ is the appropriate component of $D \setminus \gamma^1$. Then choose γ^2 from $\mu^{\#}_{\tilde{D}}(z_2, w_2)$
- R-N derivative with respect to product measure is

$$1\{\gamma^1 \cap \gamma^2 = \emptyset\} \exp\left\{\frac{\mathbf{c}}{2}\Lambda_D(\gamma^1, \gamma^2)\right\}.$$

- Much easier to describe using μ_D (nonprobability measure) rather than $\mu^{\#}_D$.
- Can let $z_1 \to z_2, w_1 \to w_2$.

EXAMPLE: REVERSIBILITY

- Take $0 < x$. Grow γ_s^1 using SLE_κ from 0 to x in $\mathbb{H}$ (with some stopping time s before path reaches x)
- Given γ_s^1, grow γ_t^2 using SLE_κ from x to $\gamma^1(s)$ in $\mathbb{H} \setminus \gamma_s^1$.
- Can give R-N derivative in terms of BLM and partition function for SLE_κ. This formulation shows that the process above is symmetric in the two paths.
- (Zhan) In fact for $\kappa \leq 4$. one can grow the paths in any order that one wants and they will eventually meet. The distribution of the final path does not depend on the order. This shows that $SLE_\kappa, \kappa \leq 4$ is reversible.

RADIAL SLE_κ

- Describes evolution of curve from boundary point z to interior point w in domain D.
- For $\kappa \leq 4$, write as

$$\tilde{\mu}_D(z,w) = \tilde{C}(D;z,w)\,\tilde{\mu}_D^{\#}(z,w).$$

$$f \circ \tilde{\mu}_D^{\#}(z,w) = \tilde{\mu}_{f(D)}^{\#}(f(z),f(w)),$$

$$\tilde{C}(D;z,w) = |f'(z)|^b\,|f'(w)|^{\tilde{b}}\,\tilde{C}(f(D),f(z),f(w))$$

$$\tilde{b} = \frac{\kappa-2}{4}\,b.$$

- Usually described with $D = \mathbb{D}$, $w = 0$ using radial Loewner equation.

- Radial SLE_κ in $\mathbb{H}$ from 0 to $w \in \mathbb{H}$ is locally absolutely continuous w.r.t. chordal SLE_κ.
- Can obtain radial SLE_κ by weighting chordal SLE_κ by the partition function

$$\tilde{C}(\mathbb{H}, g_t(z), g_t(w)).$$

  (Equivalently, can weight by Poisson kernel although Poisson kernel is not a local martingale.)
- Valid for all κ until path disconnects w from infinity.
- The interior scaling exponent $\tilde{b}$ is related to certain critical exponents. For example, for $\kappa = 8/3$, the exponent $\tilde{b} = 5/48$ is related (by some algebra that we skip) to the exponent $43/32$ predicted by Nienhuis for the number of self-avoiding walks.

(CHORDAL) *SLE* GREEN'S FUNCTION $\kappa < 8$

- Let $\Upsilon_t(z)$ denote (two times) the conformal radius (comparable to distance) between z and $\gamma_t \cup \mathbb{R}$. Problem: find d, G such that if $\Upsilon = \Upsilon_\infty(z)$,

$$\mathbf{P}\{\Upsilon \leq \epsilon\} \sim c_* \, G(z) \, \epsilon^{2-d}.$$

- d is the fractal dimension, G is the Green's function.
- More generally, can define $G_D(z; w_1, w_2)$ for chordal SLE_κ from w_1 to w_2 in D. Scaling relation

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d} \, G_{f(D)}(f(z); f(w_1), f(w_2)).$$

- (Rohde-Schramm) Expect $G_{H_t}(z; \gamma(t), \infty)$ to be a local martingale.

$$d = 1 + \frac{\kappa}{8}, \qquad G(re^{i\theta}) = r^{d-2} \sin^{\frac{8}{\kappa}-1} \theta.$$

- Can consider SLE_κ weighted by $G_{H_t}(z;\gamma(t),\infty)$. Gives two-sided radial SLE_κ (chordal SLE_κ conditioned to go through z). By studying this process can show that

$$\mathbf{P}\{\Upsilon \le \epsilon\} \sim c_* \, G(z) \, \epsilon^{2-d}.$$

- (Beffara) Two-point estimate

$$\mathbf{P}\{\Upsilon(z) \le \epsilon, \Upsilon(w) \le \epsilon\} \asymp \epsilon^{2-d} \, \epsilon^{2-d} \, |z-w|^{d-2}.$$

  Using this, one can show that the Hausdorff dimension of the paths is $d = 1 + \frac{\kappa}{8}$.

- (L-Werness, in progress) Can define a multi-point Green's function such that

$$\mathbf{P}\{\Upsilon(z) \le \epsilon, \Upsilon(w) \le \delta\} \sim c_*^2 \, G(z,w) \, \epsilon^{2-d} \, \delta^{2-d} \, .$$

  Open problem: find closed form expression for $G(z,w)$.

NATURAL PARAMETRIZATION (LENGTH) ($\kappa < 8$)

- The capacity parametrization is very convenient (e.g., it makes the Loewner differential equation nice), but is not "natural".
- For discrete processes, expect a scaling limit for the length (number of steps) of paths. This length often appears in discrete Hamiltonians.
- Expect limit to be a d-dimensional parametrization.
- Should be conformally covariant. If γ is parametrized naturally, and f is a conformal transformation, the "length" of $f(\gamma[s,t])$ should be

$$\int_s^t |f'(\gamma(r))|^d \, dr.$$

- CONJECTURE: can give in terms of "Minkowski content": "length" of $\gamma[s,t]$ is

$$\lim_{\epsilon \to 0} \epsilon^{d-2} \mathrm{Area} \{z : \mathrm{dist}(z, \gamma[s,t]) \leq \epsilon\}$$

- This limit not established.
- Would imply that the expected amount of "time" spent in a domain D should be (up to multiplicative constant)

$$\int_D G(z) \, dA(z).$$

- Given γ_t amount of time spent in D after time t is

$$\Psi_t(D) = \int_D G_{\mathbb{H} \setminus \gamma_t}(z; \gamma(t), \infty) \, dA(z).$$

- (L.- Sheffield) Can define length $\Theta_t(D)$ so that

$$\Theta_t(D) + \Psi_t(D)$$

  is a martingale. ($\kappa < 5. \cdots$)
- (L.- Wang Zhou, in progress) can define for $\kappa < 8$.
- (Alberts - Sheffield) A similar measure can be given for amount of time $SLE_\kappa, 4 < \kappa < 8$, spends on real line.
- (L.- Rezaei, in progress) Can show that definition of length is independent of the domain it lies on.
- Still open to establish that one can define it with Minkowski content.