The Brownian map

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Outline

Goal: To understand the continuous limit of large planar maps (planar maps are graphs drawn in the plane, or on the sphere) chosen uniformly at random in a certain class (p-angulations) viewed as metric spaces (for the graph distance)

- Expects universality of the limit
- Leads to an important continuous model (Brownian map)
- Gives insight into the properties of large planar maps.

Strong analogy with random paths and Brownian motion.

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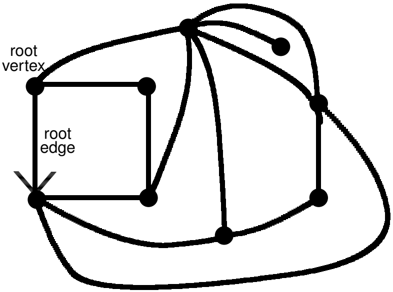
1. Introduction: Planar maps

Definition

A planar map is a proper embedding of a connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



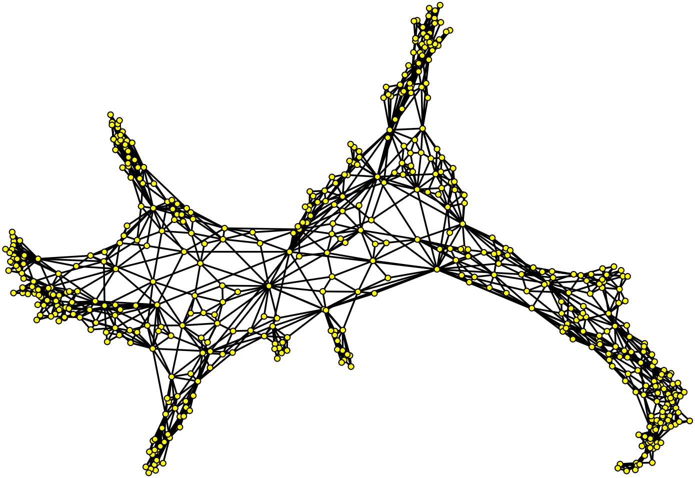
A rooted quadrangulation

p-angulation:

- each face has p adjacent edges

$p = 3$: triangulation
$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

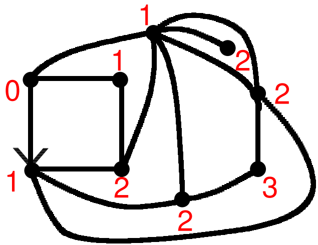


A large triangulation of the sphere (simulation by G. Schaeffer)
Can we get a continuous model out of this ?

What is meant by the continuous limit ?

M planar map

- $V(M)$ = set of vertices of M
- d_{gr} graph distance on $V(M)$
- $(V(M), d_{\mathrm{gr}})$ is a (finite) metric space



Goal

Let M_n be chosen uniformly at random in

$$\mathbb{M}_n^p = \{\text{rooted } p - \text{angulations with } n \text{ faces}\}$$

Then,

$$(V(M_n), n^{-1/4} d_{\mathrm{gr}}) \underset{n\to\infty}{\longrightarrow} \text{“continuous limiting space”}$$

in the sense of the Gromov-Hausdorff distance.

Remarks. (i) The limit should not depend on p (universality).
(ii) Rescaling by $n^{-1/4}$ because $\mathrm{diam}(V(M_n)) \approx n^{1/4}$.

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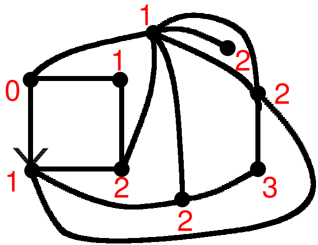

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The Gromov-Hausdorff distance

The Hausdorff distance. K_1, K_2 compact subsets of a metric space

$$d_{\mathrm{Haus}}(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}$$

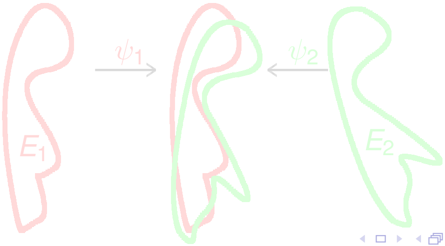
($U_\varepsilon(K_1)$ is the ε-enlargement of K_1)

Definition (Gromov-Hausdorff distance)

If (E_1, d_1) and (E_2, d_2) are two compact metric spaces,

$$d_{\mathrm{GH}}(E_1, E_2) = \inf\{d_{\mathrm{Haus}}(\psi_1(E_1), \psi_2(E_2))\}$$

the infimum is over all isometric embeddings $\psi_1 : E_1 \to E$ and $\psi_2 : E_2 \to E$ of E_1 and E_2 into the same metric space E.



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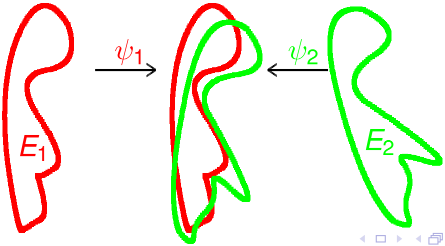


Gromov-Hausdorff convergence of rescaled maps

Fact

If $\mathbb{K} = \{$*isometry classes of compact metric spaces*$\}$, *then*

$(\mathbb{K}, d_{\mathrm{GH}})$ *is a separable complete metric space (Polish space)*

$\rightarrow$ It makes sense to study the convergence of

$$(V(M_n), n^{-1/4} d_{\mathrm{gr}})$$

as random variables with values in $\mathbb{K}$.

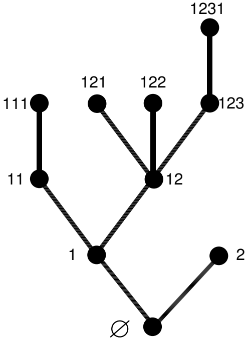
(Problem stated for triangulations by O. Schramm (ICM 06), also implicit in Chassaing-Schaeffer (04) for quadrangulations)

Why study continuous limits of random planar maps ?

- probability theory: models for a Brownian surface
  - universality of the limit (conjectured by physicists)
  - analogy with Brownian motion as continuous limit of discrete paths
- combinatorics
  - understanding the continuous limit should give information about the metric properties of large planar maps
- theoretical physics
  - large random planar maps as models of random geometry (cf Ambjørn-Durhuus-Jonsson 95)
  - connections with Liouville quantum gravity (Duplantier-Sheffield 08)

(Also geometric motivations, cf book by Lando-Zvonkin.)

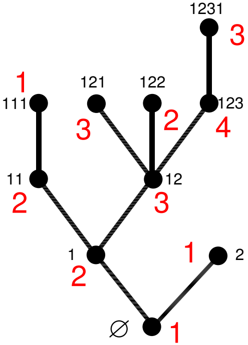
2. Bijections between maps and trees



Plane tree $\tau = \{\varnothing, 1, 2, 11, \ldots\}$

(rooted ordered tree)

the lexicographical order on vertices will play an important role in what follows



Well-labeled tree $(\tau, (\ell_v)_{v \in \tau})$

Properties of labels:

- $\ell_\varnothing = 1$
- $\ell_v \in \{1, 2, 3, \ldots\}, \forall v$
- $|\ell_v - \ell_{v'}| \leq 1$, if v, v' neighbors

Coding maps with trees, the case of quadrangulations

$$\mathbb{T}_n = \{\text{well-labeled trees with } n \text{ edges}\}$$
$$\mathbb{M}_n^4 = \{\text{rooted quadrangulations with } n \text{ faces}\}$$

Theorem (Cori-Vauquelin 81, Schaeffer 98)

There is a bijection $\Phi : \mathbb{T}_n \longrightarrow \mathbb{M}_n^4$ *such that, if* $M = \Phi(\tau, (\ell_v)_{v\in\tau})$, *then*

$$V(M) = \tau \cup \{\partial\} \qquad (\partial \text{ is the root vertex of } M)$$
$$d_{\rm gr}(\partial, v) = \ell_v \quad , \forall v \in \tau$$

Key facts.

- Vertices of τ become vertices of M
- The label in the tree becomes the distance from the root in the map.

Coding of more general maps: Bouttier, Di Francesco, Guitter (2004).

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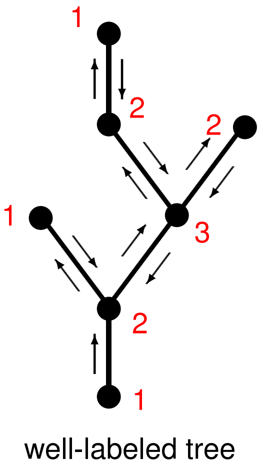
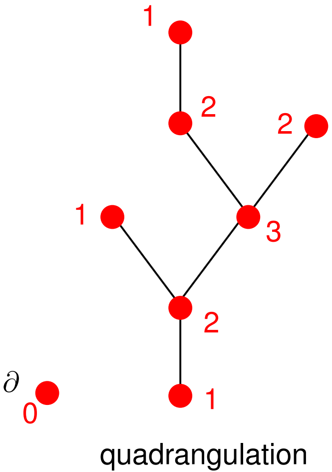
The bijection between quadrangulations and well-labeled trees





Rules.

- add extra vertex ∂ labeled 0
- follow the contour of the tree, connect each corner of the tree to the last visited corner with smaller label

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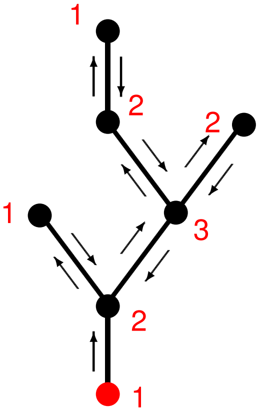
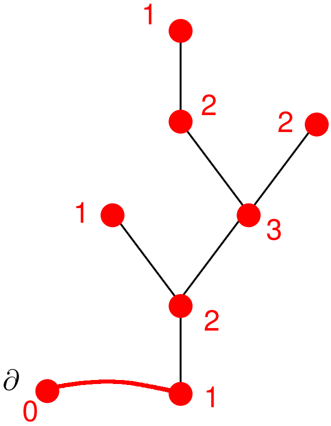


well-labeled tree



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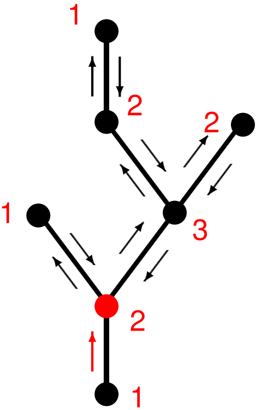

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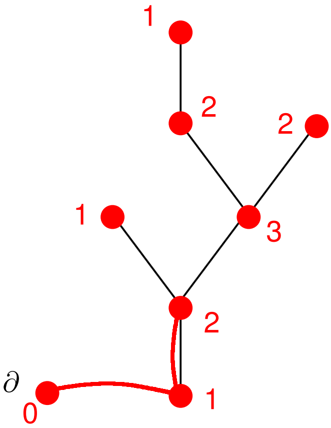

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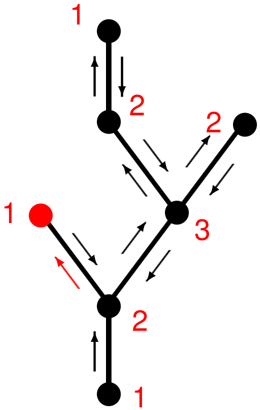

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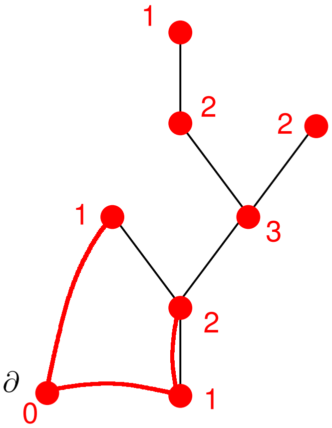

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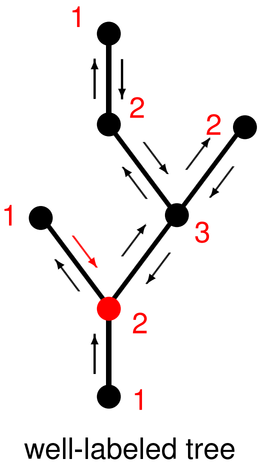

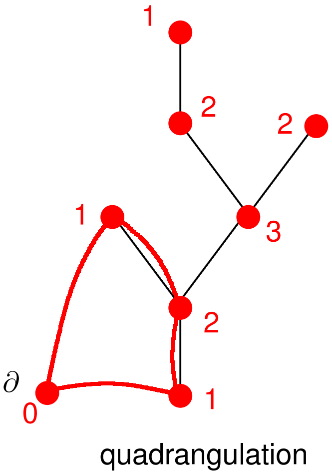

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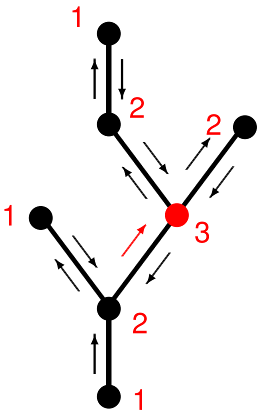

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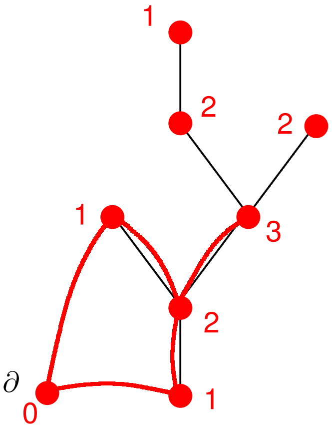

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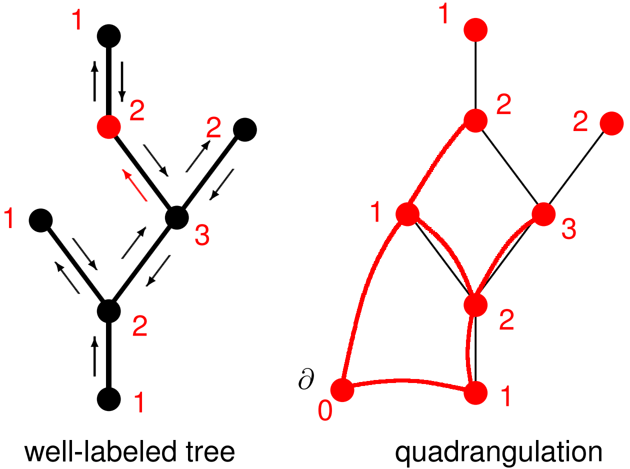

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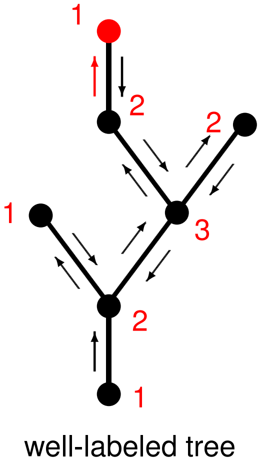

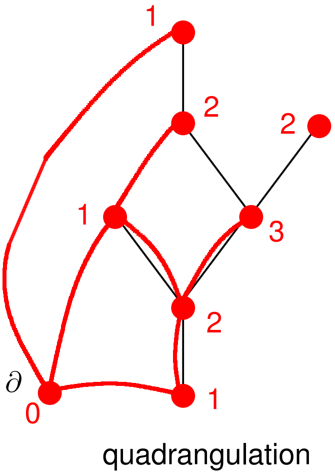

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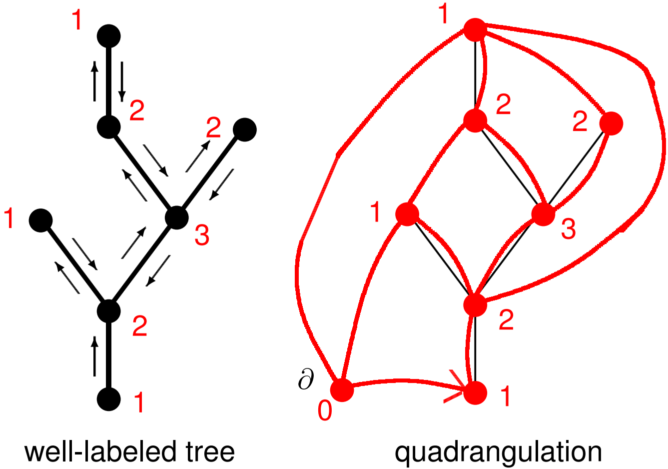

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General strategy

Understand continuous limits of trees ("easy")

in order to understand continuous limits of maps ("more difficult")

Key point. The bijections with trees allow us to handle distances from the root vertex, but **not** distances between two arbitrary vertices of the map (required if one wants to get Gromov-Hausdorff convergence)

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3. Asymptotics for trees

The case of plane trees

$$T_n^{\text{plane}} = \{\text{plane trees with } n \text{ edges}\}$$

Theorem (reformulation of Aldous 1993)

One can construct, for every n, a tree τ_n uniformly distributed over T_n^{plane}, in such a way that

$$(\tau_n, \frac{1}{\sqrt{2n}} d_{\text{gr}}) \longrightarrow (\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}}) \qquad \text{as } n \to \infty$$

almost surely, in the Gromov-Hausdorff sense.
Here $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the CRT (Continuum Random Tree)

The notation $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ comes from the fact that the CRT is the tree coded by a Brownian excursion **e**

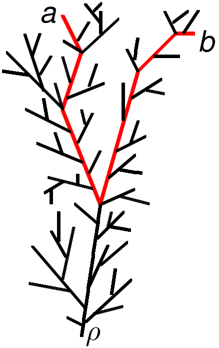
Definition of the CRT: notion of a real tree

Definition

A real tree is a (compact) metric space $\mathcal{T}$ such that:

- any two points $a, b \in \mathcal{T}$ are joined by a unique arc
- this arc is isometric to a line segment

It is a rooted real tree if there is a distinguished point ρ, called the root.



Remark. A real tree can have

- infinitely many branching points
- (uncountably) infinitely many leaves

Fact. The coding of discrete trees by Dyck paths can be extended to real trees.

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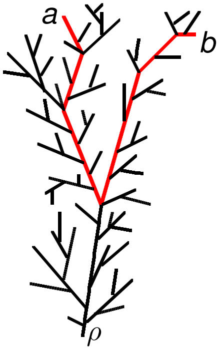

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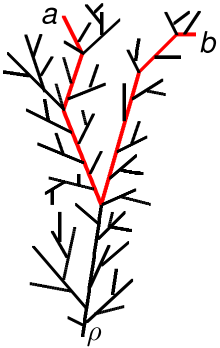

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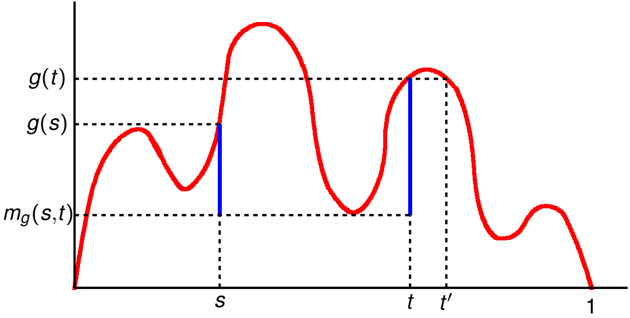
The real tree coded by a function g

$g : [0,1] \longrightarrow [0,\infty)$
continuous,
$g(0) = g(1) = 0$



$m_g(s,t) = m_g(t,s) = \min_{s \le r \le t} g(r)$

$d_g(s,t) = g(s) + g(t) - 2m_g(s,t)$ $\qquad t \sim t'$ iff $d_g(t,t') = 0$

Proposition

$\mathcal{T}_g := [0,1]/\sim$ equipped with d_g is a real tree, called the tree coded by g. It is rooted at $\rho = 0$.

Remark. $\mathcal{T}_g$ inherits a "lexicographical order" from the coding.

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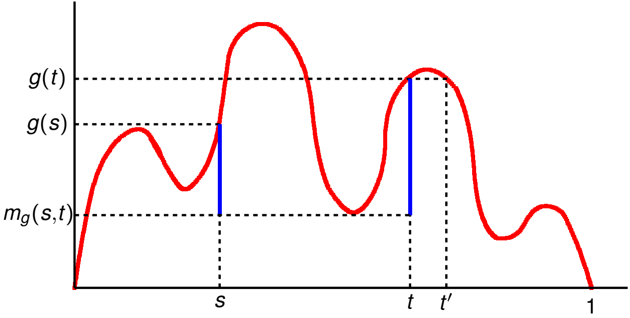

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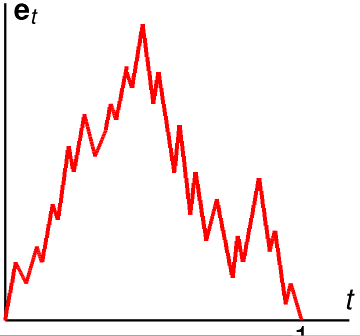
Back to Aldous' theorem and the CRT

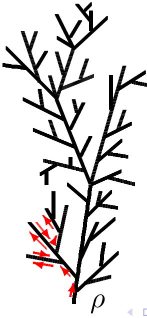
Aldous' theorem: τ_n uniformly distributed over T_n^{plane}

$$(\tau_n, \frac{1}{\sqrt{2n}} d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{\mathrm{a.s.}} (\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$$

in the Gromov-Hausdorff sense.

The limit $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the (random) real tree coded by a Brownian excursion $\mathbf{e} = (\mathbf{e}_t)_{0\leq t\leq 1}$ = Brownian motion starting from 0, conditioned to be at 0 at time 1 and to stay nonnegative over $[0, 1]$





Assigning labels to a real tree

Need to assign (random) labels to the vertices of a real tree $(\mathcal{T}, d)$

$(Z_a)_{a \in \mathcal{T}}$: Brownian motion indexed by $(\mathcal{T}, d)$
= centered Gaussian process such that

- $Z_\rho = 0 \qquad (\rho$ root of $\mathcal{T})$
- $E[(Z_a - Z_b)^2] = d(a,b), \qquad a, b \in \mathcal{T}$

Labels evolve like Brownian motion along the branches of the tree:

- The label Z_a is the value at time $d(\rho, a)$ of a standard Brownian motion
- Similar property for Z_b, but one uses
  - the same BM between 0 and $d(\rho, a \wedge b)$
  - an independent BM between $d(\rho, a \wedge b)$ and $d(\rho, b)$

Problem. The positivity constraint is not satisfied !

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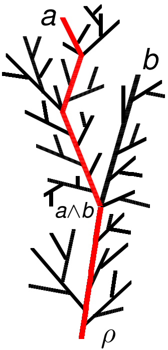

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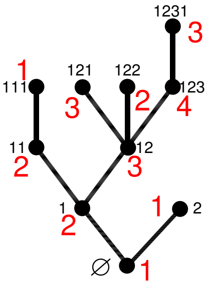
The scaling limit of well-labeled trees

Recall $\mathbb{T}_n = \{\text{well-labeled trees with } n \text{ edges}\}$

$(\theta_n, (\ell^n_v)_{v\in\theta_n})$ uniformly distributed over $\mathbb{T}_n$

Rescaling:

- Distances on θ_n are rescaled by $\frac{1}{\sqrt{n}}$ (Aldous' theorem)
- Labels ℓ^n_v are rescaled by $\frac{1}{\sqrt{\sqrt{n}}} = \frac{1}{n^{1/4}}$ ("central limit theorem")



Fact

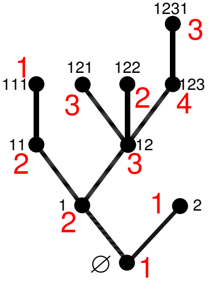
The scaling limit of $(\theta_n, (\ell^n_v)_{v\in\theta_n})$ is $(\mathcal{T}_{\mathbf{e}}, (\overline{Z}_a)_{a\in\mathcal{T}_{\mathbf{e}}})$, where

- *$\mathcal{T}_{\mathbf{e}}$ is the CRT*
- *$(Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$ is Brownian motion indexed by the CRT*
- *$\overline{Z}_a = Z_a - Z_*$, where $Z_* = \min\{Z_a, a \in \mathcal{T}_{\mathbf{e}}\}$*
- *$\mathcal{T}_{\mathbf{e}}$ is re-rooted at vertex minimizing Z*

The scaling limit of well-labeled trees

Recall $\mathbb{T}_n = \{\text{well-labeled trees with } n \text{ edges}\}$

$(\theta_n, (\ell_v^n)_{v\in\theta_n})$ uniformly distributed over $\mathbb{T}_n$



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Application to the radius of a planar map

Recall

- Bijection : quadrangulations $\leftrightarrow$ well-labeled trees
- labels on the tree correspond to distances from the root in the map

Theorem (Chassaing-Schaeffer 2004)

Let R_n be the maximal distance from the root in a quadrangulation with n faces chosen at random. Then,

$$n^{-1/4} R_n \xrightarrow[n\to\infty]{(d)} \left(\frac{8}{9}\right)^{1/4} (\max Z - \min Z)$$

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Extensions to much more general planar maps (including triangulations, etc.) by

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4. The scaling limit of planar maps

$\mathbb{M}_n^{2p} = \{$rooted $2p-$ angulations with n faces$\}$ (bipartite case)
M_n uniform over $\mathbb{M}_n^{2p}$, $V(M_n)$ vertex set of M_n, $d_{\rm gr}$ graph distance

Theorem (The scaling limit of $2p$-angulations)

At least along a sequence $n_k \uparrow \infty$, one can construct the random maps M_n so that

$$(V(M_n), c_p \frac{1}{n^{1/4}} d_{\rm gr}) \xrightarrow[n\to\infty]{\text{a.s.}} (\mathbf{m}_\infty, D)$$

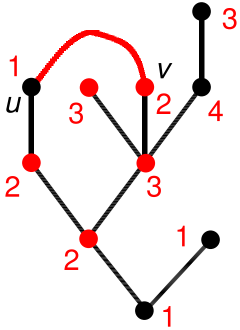
in the sense of the Gromov-Hausdorff distance.
Furthermore, $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\approx$ where

- *$\mathcal{T}_{\mathbf{e}}$ is the CRT (re-rooted at vertex minimizing Z)*
- *$(Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$ is Brownian motion indexed by $\mathcal{T}_{\mathbf{e}}$, and $\overline{Z}_a = Z_a - \min Z$*
- *$\approx$ equivalence relation on $\mathcal{T}_{\mathbf{e}}$: $a \approx b \Leftrightarrow \overline{Z}_a = \overline{Z}_b = \min_{c\in[a,b]} \overline{Z}_c$ ($[a,b]$ lexicographical interval between a and b in the tree)*
- *D distance on $\mathbf{m}_\infty$ such that $D(\rho, a) = \overline{Z}_a$*
  D induces the quotient topology on $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\approx$

Interpretation of the equivalence relation $\approx$

Recall the Cori-Vauquelin-Schaeffer bijection:
$\exists$ edge between u and v if

- $\ell_u = \ell_v - 1$
- $\ell_w \geq \ell_v$, $\forall w \in]u, v]$



Explains why in the continuous limit

$$a \approx b \quad \Rightarrow \quad \overline{Z}_a = \overline{Z}_b = \min_{c \in [a,b]} \overline{Z}_c$$

$$\Rightarrow \quad a \text{ and } b \text{ are identified}$$

Key point: Prove the converse (no other pair of points are identified)

Remark: Equivalence classes for $\approx$ contain 1, 2 or 3 points.

Consequence and open problems

Corollary

The topological type of any Gromov-Hausdorff sequential limit of $(V(M_n), n^{-1/4} d_{\rm gr})$ is determined:

$$\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}} / \approx \quad \textit{with the quotient topology.}$$

Open problems

- Identify the distance D on $\mathbf{m}_\infty$
  (would imply that there is no need for taking a subsequence)
  - Recent progress: Bouttier-Guitter (08) three-point function
- Show that D does not depend on p
  (universality property, expect same limit for triangulations, etc.)

STILL MUCH CAN BE PROVED ABOUT THE LIMIT !

Any possible limiting space $(\mathbf{m}_\infty, D)$ is called a Brownian map
[Marckert, Mokkadem 06, with a different approach]

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Two theorems about the Brownian map

Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D) = 4 \qquad a.s.$$

(Already "known" in the physics literature.)

Theorem (topological type, LG-Paulin 07)

Almost surely, $(\mathbf{m}_\infty, D)$ is homeomorphic to the 2-sphere $\mathbb{S}^2$.

Consequence: for n large, no separating cycle of size $o(n^{1/4})$ in M_n, such that both sides have diameter $\geq \varepsilon n^{1/4}$

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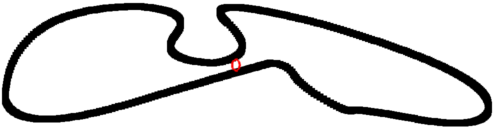

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5. Geodesics in the Brownian map

Brownian map: $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\approx$, root ρ

$\prec$ lexicographical order on $\mathcal{T}_{\mathbf{e}}$

Recall $D(\rho, a) = \overline{Z}_a$ (labels on $\mathcal{T}_{\mathbf{e}}$)

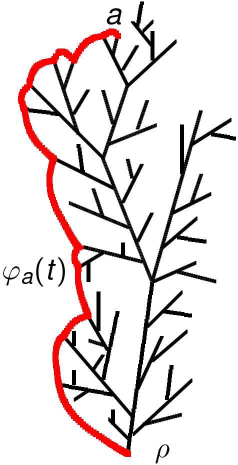
Fix $a \in \mathcal{T}_{\mathbf{e}}$ and for $t \in [0, \overline{Z}_a]$, set

$$\varphi_a(t) = \sup\{b \prec a : \overline{Z}_b = t\}$$

(same formula as in the discrete case !)

Then $(\varphi_a(t))_{0 \leq t \leq \overline{Z}_a}$ is a geodesic from ρ to a

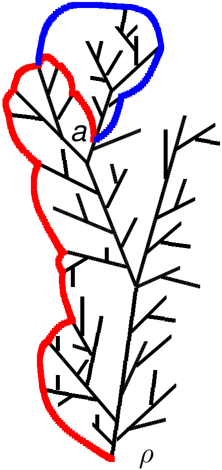
(called a simple geodesic)



How many simple geodesics from a given point ?

- If *a* is a leaf of $\mathcal{T}_{\mathbf{e}}$,
  there is a unique simple geodesic from ρ to *a*
- If $a \in \mathrm{Sk}(\mathcal{T}_{\mathbf{e}}) = \mathcal{T}_{\mathbf{e}} \setminus \{\text{leaves of } \mathcal{T}_{\mathbf{e}}\}$
  (skeleton of $\mathcal{T}_{\mathbf{e}}$)
  - 2 distinct simple geodesics if *a* is a simple point of the skeleton
  - 3 distinct simple geodesics if *a* is a branching point

(3 is the maximal multiplicity in $\mathcal{T}_{\mathbf{e}}$)



Proposition (key result)

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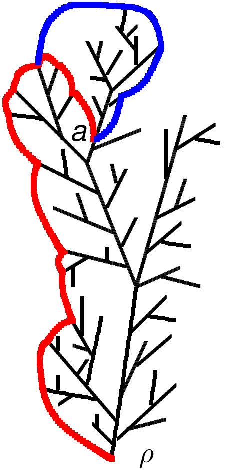

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The main result about geodesics

Recall $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})$ = skeleton of $\mathcal{T}_{\mathbf{e}}$, and set

$$\mathrm{Skel} = \pi(\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})) \qquad (\pi : \mathcal{T}_{\mathbf{e}} \to \mathcal{T}_{\mathbf{e}}/\approx = \mathbf{m}_\infty \text{ canonical projection})$$

Then

- the restriction of π to $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})$ is a homeomorphism onto Skel
- $\dim(\mathrm{Skel}) = 2$ (recall $\dim(\mathbf{m}_\infty) = 4$)

Theorem (Geodesics from the root)

Let $x \in \mathbf{m}_\infty$. Then,

- *if $x \notin \mathrm{Skel}$, there is a unique geodesic from ρ to x*
- *if $x \in \mathrm{Skel}$, the number of distinct geodesics from ρ to x is the multiplicity $m(x)$ of x in Skel (note: $m(x) = 2$ or 3).*

Remarks

- Skel is the cut-locus of $\mathbf{m}_\infty$ relative to ρ: cf classical Riemannian geometry [Poincaré, Myers, ...], where the cut-locus is a tree.
- same results if ρ replaced by a point chosen "at random" in $\mathbf{m}_\infty$.
- other approach to the uniqueness of geodesics: Miermont (2007)

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Confluence property of geodesics

Fact: Two simple geodesics coincide near the root.
(easy from the definition)

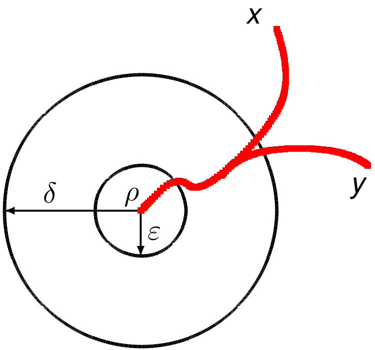
Corollary

Given $\delta > 0$, there exists $\varepsilon > 0$ s.t.

- *if $D(\rho, x) \geq \delta$, $D(\rho, y) \geq \delta$*
- *if γ is any geodesic from ρ to x*
- *if γ' is any geodesic from ρ to y*

then

$$\gamma(t) = \gamma'(t) \quad \text{for all } t \leq \varepsilon$$



"Only one way" of leaving ρ along a geodesic.
(also true if ρ is replaced by a typical point of $\mathbf{m}_\infty$)

Other results about geodesics in large quadrangulations:
Bouttier-Guitter (07,08)

6. Canonical embeddings: Open problems

Recall that a planar map is defined up to (orientation-preserving) homeomorphisms of the sphere.

Question

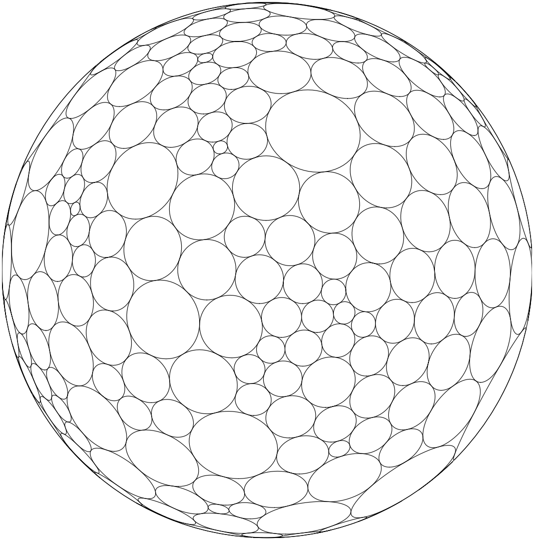
Can one choose a particular (canonical) embedding of the graph satisfying conformal invariance properties ?

The answer is yes (at least up to the Möbius transformations, which are the conformal transformations of the sphere $\mathbb{S}^2$).

Question

Applying this canonical embedding to M_n (uniform over p-angulations with n faces), can one let n tend to infinity and get a random metric on the sphere as the limit of the (scaled) graph distance on $V(M_n)$?

Canonical embeddings via circle packings 1

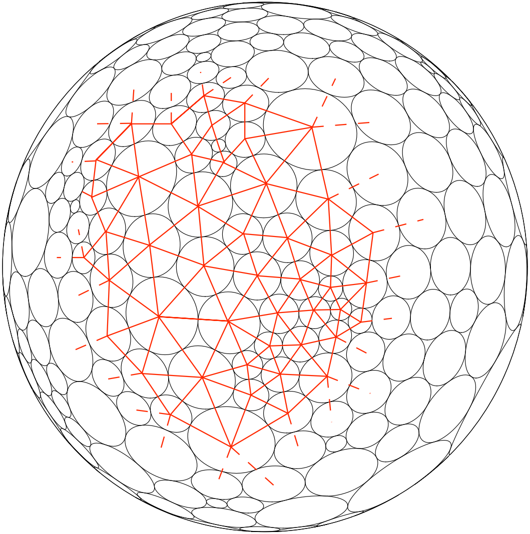


From such a circle packing, construct a graph M embedded in the sphere:

- $V(M) = \{\text{centers of circles}\}$
- draw an edge between a and b if the corresponding circles are tangent.

Figure by Nicolas Curien

Canonical embeddings via circle packings 2

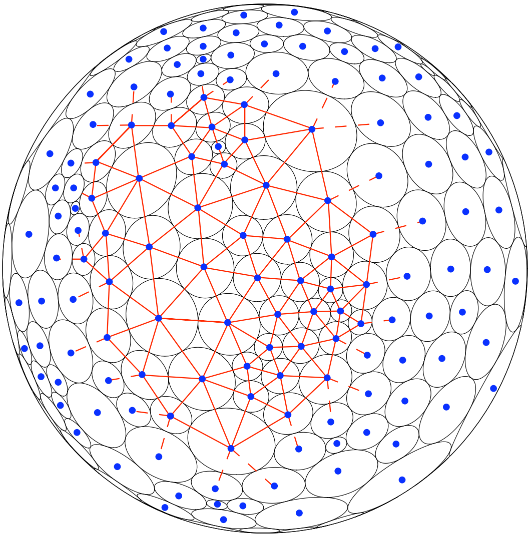


A triangulation (without loops or multiple edges) can always be represented in this way.

Representation unique up to Möbius transformations.

Figure by Nicolas Curien

Canonical embeddings via circle packings 3



Apply this to M_n uniform over {triangulations with n faces}.
Let $n \to \infty$. Expect to get

- Random metric on $\mathbb{S}^2$ (with conformal invariance properties)
- Random volume measure on $\mathbb{S}^2$

Connections with the Gaussian free field and Liouville quantum gravity ? (cf Duplantier-Sheffield).

Figure by Nicolas Curien

A few references

BENJAMINI: Random planar metrics. Preprint.
BOUTTIER, DI FRANCESCO, GUITTER: Planar maps as labeled mobiles. Electr. J. Combinatorics **11**, #R69 (2004)
BOUTTIER, GUITTER: The 3-point function... J. Stat. Mech. (2008)
BOUTTIER, GUITTER: Statistics of geodesics ... J. Physics A (2008)
DUPLANTIER, SHEFFIELD: Liouville quantum gravity and KPZ. Preprint.
LE GALL: The topological structure of scaling limits of large planar maps. Invent. Math. **169**, 621-670 (2007)
LE GALL: Geodesics in large planar maps ... Acta Math., to appear.
LE GALL, PAULIN: Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. GAFA **18**, 893-918 (2008)
MARCKERT, MIERMONT: Invariance principles for random bipartite planar maps. Ann. Probab. **35**, 1642-1705 (2007)
MARCKERT, MOKKADEM: Limit of normalized quadrangulations: The Brownian map. Ann. Probab. **34**, 2144-2102 (2006)
MIERMONT: Tesselations of random maps ... Ann. ENS (2009)
MIERMONT: On the sphericity of scaling limits ... ECP (2008)