Loops, trees and operators

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ENERGY and GREEN FUNCTION

Graph, with conductances and killing measure:

$$e(f,f) = \frac{1}{2}\sum_{x,y} C_{x,y}(f(x)-f(y))^2 + \sum_x \kappa_x f(x)^2$$
$$= \sum_x \lambda_x f(x)^2 - \sum_{x,y} C_{x,y} f(x) f(y)$$

with $\lambda_x = \kappa_x + \sum_y C_{x,y}$ and $C_{x,x} = 0$.
Under a transience assumption, the associated Green function is defined: $G(x,y) = G(y,x) = [(M_\lambda - C)^{-1}]_{x,y}$
with $M_\lambda :=$ multiplication by λ.

Also, given a "discrete one form": $\omega_{x,y} = -\omega_{y,x}$

$$e_{(\omega)}(f,\overline{f}) = \sum_x \lambda_x f(x)\overline{f}(x) - \sum_{x,y} C_{x,y} e^{i\omega_{x,y}} f(x)\overline{f}(y)$$

Green function: $G_{(\omega)}(x,y) = \overline{G_{(\omega)}(y,x)} = [(M_\lambda - Ce^{i\omega})^{-1}]_{x,y}$

LOOP MEASURE and loop functionals

Energy $e \to \lambda$-symmetric transition matrix $P^x_y = \frac{C_{x,y}}{\lambda_x} \to$ Markov chain $\to$ Bridge measure $\mathbb{P}^{x,y}_t$.
with mass $p_t(x,y) = p_t(y,x) = \frac{1}{\lambda_y}[\exp t(P-I)]_{x,y}$. There is also

$$\mu^{x,y} = \int_0^\infty \mathbb{P}^{x,y}_t dt$$

with mass $G(x,y)$ and the $\sigma-$finite loop measure

$$\mu(dl) = \sum_{x \in X} \int_0^\infty \frac{1}{t} \mathbb{P}^{x,x}_t (dl) \lambda_x dt$$

Occupation field of l

$$\widehat{l}^x = \frac{1}{\lambda_x} \int_0^{T(l)} 1_{\{l(s)=x\}} ds$$

LOOP MEASURE and loop functionals

Number of jumps from x to y in l: $N_{x,y}(l)$

$$N_{x,y}(l)\mu(dl) = C_{x,y}\mu^{x,y}(dl) \text{ and } \widehat{l}^x\mu(dl) = \mu^{x,x}(dl)$$

In particular,

$$\int N_{x,y}(l)\mu(dl) = C_{x,y}G(x,y) \text{ and } \int \widehat{l}^x\mu(dl) = G(x,x)$$

LOOP ENSEMBLE: $\mathcal{L}$ (or $\mathcal{L}_1$) = Poisson Point Process with intensity μ

$$\widehat{\mathcal{L}}^x = \sum_{l\in\mathcal{L}} \widehat{l}^x \quad N_{x,y}(\mathcal{L}) = \sum_{l\in\mathcal{L}} N_{x,y}(l)$$

$$E(\prod_1^n \widehat{\mathcal{L}}^{x_i}) = \sum_{\sigma\in\mathcal{S}_n} \prod_1^n G(x_i, x_{\sigma(i)})$$

RANDOM SPANNING TREE

Finite graph with $\kappa \neq 0$: Add a "cemetery point" δ.
Spanning trees are rooted in δ.
Generalized Cayley Theorem:

$$\sum_{spanning\ trees\ \tau} \ \prod_{(x,y)\ edge\ of\ \tau} C_{x,y} = \det(G)$$

with the convention $C_{x,\delta} = \kappa_x$.

$\rightarrow$ P_{ST} probability on spanning trees.

Sampling by Wilson algorithm, based on loop erasure.

$$\text{Erased loops } = \text{Loop ensemble } \mathcal{L}$$

FREE FIELD

The complex free field $\phi(x)$ is defined as the complex Gaussian field with covariance $G(x, y)$

$$E(\phi(x_1)...\phi(x_m)\overline{\phi}(y_1)...\overline{\phi}(y_n)) = \delta_{nm}Per(G(x_i, y_j))$$

Bosonic Fock space structure:
$L^2(\sigma(\phi))$ is isomorphic to the Hilbert space $\mathcal{F}_B$ generated by a "vacuum" vector 1 and creation/anihilation operators a_x, a_x^*, b_x, b_x^* with $[a_x, a_y^*] = [b_x, b_y^*] = G(x, y)$ and all others commutators vanishing.
Then $\phi(x)$ and $\overline{\phi}(x)$ are represented by two dual commuting operators $a_x + b_x^*$ and $a_x^* + b_x$:

$$\text{for any polynomial } P,\ \left\langle 1, P(\phi, \overline{\phi})1 \right\rangle_{\mathcal{F}_B} = \mathbb{E}(P(\phi, \overline{\phi}))$$

Anihilation operators can be interpreted in terms of functional derivatives: $a_x = \frac{\partial}{\partial \overline{\phi}(x)}$, $b_x = \frac{\partial}{\partial \phi(x)}$

GRASSMANN FIELD

Anticommuting variables $\psi_x, \overline{\psi}_x$ are defined as operators on the Fermionic Fock space $\mathcal{F}_F$ generated by a vector 1 and creation/anihilation operators c_x, c_x^*, d_x, d_x^* with $[c_x, c_y^*]^+ = [d_x, d_y^*]^+ = G(x,y)$ and with all others anticommutators vanishing. Then

$$\psi_x = d_x + c_x^* \quad \overline{\psi}_x = -c_x + d_x^*$$

Note that $\overline{\psi}_x$ is not the dual of ψ_x, but there is an involution $\mathfrak{I}$ on $\mathcal{F}_F$ such that $\overline{\psi} = \mathfrak{I}\psi^*\mathfrak{I}$

$$\left\langle 1, \psi(x_1)...\psi(x_m)\overline{\psi}(y_1)...\overline{\psi}(y_n)1\right\rangle = \delta_{nm}\det(G(x_i, y_j))$$

On a finite graph, $\psi_x, \overline{\psi}_x$ can also be defined in terms of differential forms.
"Supersymmetry" between ϕ and ψ: for any polynomial F

$$\left\langle 1, F(\phi\overline{\phi} - \psi\overline{\psi})1\right\rangle_{\mathcal{F}_B \otimes \mathcal{F}_F} = F(0)$$

(1 denotes $1_{(B)} \otimes 1_{(F)}$)

ISOMORPHISMS

$$\begin{array}{ccc} & \text{(Wilson Algorithm)} & \\ \text{Loop ensemble } \mathcal{L} & \longleftrightarrow & \text{Random Spanning Tree} \\ \updownarrow & & \updownarrow \\ \text{Free field } \phi, \overline{\phi} & \longleftrightarrow & \text{Grassmann field } \psi, \overline{\psi} \\ & \text{("Supersymmetry")} & \end{array}$$

The Bosonic isomorphism

Given two energy forms e, e' with $C' \leq C$, $\lambda' \geq \lambda$ and a "discrete one form" $\omega_{x,y} = -\omega_{y,x}$:

$$E(\prod_{x,y}[\frac{C'_{x,y}}{C_{x,y}}e^{i\omega_{x,y}}]^{N_{x,y}(\mathcal{L}_1)}e^{-\sum(\lambda'_x-\lambda_x)\widehat{\mathcal{L}}^x}) = \frac{\det(G'_{(\omega)})}{\det(G)}$$

$$= \left\langle 1, \exp(e(\phi,\overline{\phi}) - e'_{(\omega)}(\phi,\overline{\phi}))1 \right\rangle_{\mathcal{F}_B}$$

In particular for $C = C'$ and $\omega = 0$, we get that

$$E(e^{-\sum(\kappa'_x-\kappa_x)\widehat{\mathcal{L}}^x}) = \left\langle 1, \exp(\sum(\kappa'_x - \kappa_x)\phi_x\overline{\phi}_x))1 \right\rangle$$

Therefore the fields $\phi\overline{\phi}$ and $\widehat{\mathcal{L}}$ have the same joint distributions.

Variational Identities

Given n distinct vertices x_i

$$
\begin{aligned}
\frac{\partial^n}{\prod \partial \kappa_{x_i}} \mathbb{E}(F(\widehat{\mathcal{L}})) &= \frac{\partial^n}{\prod \partial \kappa_{x_i}} \mathbb{E}(F(\frac{1}{2}\phi\overline{\phi})) \\
&= \mathbb{E}(F(\widehat{\mathcal{L}}) \prod_i [\widehat{\mathcal{L}}^{x_i} - G(x_i, x_i)]) \\
&= \mathbb{E}(F(\phi\overline{\phi}) \prod_i [|\phi(x_i)|^2 - G(x_i, x_i)]) \\
&= \int E(F(\widehat{\mathcal{L}} + \sum \widehat{l_i}) - F(\widehat{\mathcal{L}})) \prod \mu^{x_i, x_i}(dl)
\end{aligned}
$$

In fact, more generally

$\frac{\partial^n}{\prod \partial \kappa_{x_i}} \mathbb{E}(F(\mathcal{L})) = \int E(F(\mathcal{L} \cup l_i) - F(\mathcal{L})) \prod \mu^{x_i, x_i}(dl)$

Variational Identities

If we set
$$\begin{aligned} T^{x,y}(\mathcal{L}) &= \widehat{\mathcal{L}}^x + \widehat{\mathcal{L}}^y - N^{x,y}(\mathcal{L}) - N^{y,x}(\mathcal{L}) \\ K^{(x,y),(u,v)} &= G^{x,u} + G^{y,v} - G^{x,v} - G^{y,u}, \end{aligned}$$

then
$$\mathbb{E}(T^{x,y}(\mathcal{L})) = \mathbb{E}(|\phi(x) - \phi(y)|^2) = K^{(x,y),(x,y)}$$

Given n DISTINCT edges (x_i, y_i), we get a 2nd variational formula:

$$\begin{aligned} \frac{\partial^n}{\prod \partial C_{x_i,y_i}} \mathbb{E}(F(\widehat{\mathcal{L}})) &= \frac{\partial^n}{\prod \partial C_{x_i,y_i}} \mathbb{E}(F(\frac{1}{2}\phi\overline{\phi})) \\ &= \mathbb{E}(F(\widehat{\mathcal{L}}) \prod_i [T^{x_i,y_i}(\mathcal{L}) - K^{(x_i,y_i),(x_i,y_i)}]) \\ &= \mathbb{E}(F(\frac{1}{2}\phi\overline{\phi}) \prod_i [|\phi(x_i) - \phi(y_i)|^2 - K^{(x_i,y_i),(x_i,y_i)}]) \end{aligned}$$

$$= \int E(F(\widehat{\mathcal{L}} + \sum \widehat{l_i}) - F(\widehat{\mathcal{L}})) \prod [\mu^{x_i,x_i}(dl) - 2\mu^{x_i,y_i}(dl) + \mu^{y_i,y_i}(dl)]$$

The fermionic isomorphism

$$E_{ST}\Big(\prod_{(x,y)\in\tau}\frac{C'_{x,y}}{C_{x,y}}e^{i\omega_{x,y}}\prod_{x,(x,\delta)\in\tau}\frac{\kappa'_x}{\kappa_x}\Big)=\frac{\det(G)}{\det(G'_{(\omega)})}$$
$$=\Big\langle 1,\exp(e(\psi,\overline{\psi})-e'_{(\omega)}(\psi,\overline{\psi}))1\Big\rangle_{\mathcal{F}_F}$$

The Transfer Current Theorem follows directly

$$P_{ST}((x_i,y_i)\in\tau)=\det(K^{(x_i,y_i),(x_j,y_j)})\prod C_{x_i,y_i}$$

In particular, $P_{ST}((x_i,\delta)\in\tau)=\det(G(x_i,x_j))\prod\kappa_{x_i}$.
NB: ϕ and ψ can also be used jointly to represent bridge functionals: In particular

$$\int F(\widehat{l})\mu_{x,y}(dl)=\langle 1,\phi_x\overline{\phi}_yF(\phi\overline{\phi}-\psi\overline{\psi})1\rangle_{\mathcal{F}_B\otimes\mathcal{F}_F}$$
$$=\langle 1,\psi_x\overline{\psi}_yF(\phi\overline{\phi}-\psi\overline{\psi})1\rangle_{\mathcal{F}_B\otimes\mathcal{F}_F}$$

IN THE CONTINUUM

Domain $D \subset R^d$, or Riemannian manifold with metric and killing rate

$$e(f,f) = \frac{1}{2}\int a_{i,j}(x)\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\det(a)^{-\frac{1}{2}}dx + \int k(x)f(x)^2\det(a)^{-\frac{1}{2}}dx$$

$$(-\frac{1}{2}\Delta_x + k(x))G(x,y) = \delta_y(x)$$

with $\Delta_x = \sum a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}$

Bridge measures and $\sigma-$finite measure μ on Brownian loops are defined in the same way (Lawler and Werner "loop soup").

IN THE CONTINUUM

Occupation field $\widehat{l}$: for $d = 1$, local times. For $d \geq 2$ random measure, defined on test functions.
For $d = 1$ the fields ϕ and ψ are defined in the same way as on graphs. For $d \geq 2$ they are generalized field i.e. defined only on test functions).

For $d = 2$ and 3, "$\widehat{\mathcal{L}}^x - G(x,x)$" is well defined on test functions as the compensated sum of the $\widehat{l_i}^x$, $l_i \in \mathcal{L}$.

Associated by a version of the Bosonic isomorphism with the Wick square of the free field "$|\phi(x)|^2 - G(x,x)$" (square for the tensor product structure).

For $d = 2$, n-th renormalized powers of $\widehat{\mathcal{L}}^x$ can be defined and are associated with Wick $2n$-th powers of ϕ.

IN THE CONTINUUM

Problems of definitions in higher dimensions, or for the field T, even in $d = 1$.

$\rightarrow$ They should be interpreted as unbounded operators, on adequate domains:

e.g., in any dimension, for any point x inside a domain D, $|\phi(x)|^2 - G(x, x)$ operates on polynomials in the Fock space associated to the boundary ∂D.

Some variational identities are defined and valid for adequate functionals of $\mathcal{L}$ and for variations of the killing rate k or of the inverse metric $a_{i,j}$

IN THE CONTINUUM

For $d = 1$,

$$\text{Loop ensemble } \mathcal{L} = \text{ensemble of Brownian excursions}$$
$$\text{(from the minima, or the maxima of the loops)}$$

Describes the history of a (quadratic) continuous branching process with immigration.

For $d = 1$,

the determinantal process formed by the points x_i such that $(x_i, \delta) \in \tau$ exists	=	determinantal processes with independant spacings defined by Macchi.

For constant killing rate, $G(x, y) = \rho \exp(-|x - y|/\alpha)$.
Interpretation of the law of the spacings:

$$K \; e^{-\frac{x}{\alpha}} \sinh(\sqrt{1 - 2\rho\alpha}\frac{x}{\alpha}).$$

For $d = 2$. $SLE(2)$ trees linking any finite set of sites and SLE(8) contour (LSW).

Addendum: OTHER LOOP FUNCTIONALS

A loop l in $\mathcal{L}$ includes more information than the fields $\widehat{l}^x$ and $N^{x,y}(l)$.

Hitting distributions

If F_1 and F_2 are disjoint, if $D_i = F_i^c$ and G^D denotes the Green function of the chain killed outside D

The probability that no loop in $\mathcal{L}$ intersects F_1 and F_2 equals,

$$\exp(-\mu(\{\widehat{l}(F_1)\widehat{l}(F_2) > 0\})) = \frac{\det(G^{D_1})\det(G^{D_2})}{\det(G)\det(G^{D_1 \cap D_2})} \qquad (1)$$

>From that, by some matrix manipulations

$$\mu(\widehat{l}(F_1)\widehat{l}(F_2) > 0) = \sum_1^\infty \frac{1}{2k} Tr([H_{12}H_{21}]^k + [H_{12}H_{21}]^k) \qquad (2)$$

with $H_{12} = H^{F_2}|_{F_1}$(hitting distributions of F_2 from F_1) and $H_{21} = H^{F_1}|_{F_2}$.

The k-th term of the expansion can be interpreted as the measure of loops with exactly k-crossings between F_1 and F_2.

Hitting distributions

In the continuum: The right-hand side of equation (1)

$$\frac{\det(G^{D_1})\det(G^{D_2})}{\det(G)\det(G^{D_1\cap D_2})}$$

is well defined but determinants diverge. For Brownian motion killed at the exit of a bounded domain, Weyl asymptotics show that the divergences may cancel.
In fact, the right-hand side of equation (2)

$$\sum_1^\infty \frac{1}{2k} Tr([H_{12}H_{21}]^k + [H_{12}H_{21}]^k)$$

is well defined in terms of the densities of the hitting distributions of F_1 and F_2 with respect to their capacitary measures, which allow to take the trace.

Multiple local times $\widehat{l}^{x_1,x_2,\dots x_n}$

$$\widehat{l}^{x_1,\dots,x_n} = \sum_{j=0}^{n-1} \int_{0<t_1<\dots<t_n<T} 1_{\{l(t_1)=x_{1+j},\dots,l(t_{n-j})=x_n,\dots,l(t_n)=x_j\}} \prod \frac{1}{\lambda_{x_i}} dt_i$$

Note that in general $\widehat{l}^{x_1,\dots,x_k}$ cannot be expressed in terms of $\widehat{l}$.

$$\mu(\widehat{l}^{x_1,\dots,x_n}) = G^{x_1,x_2} G^{x_2,x_3} \dots G^{x_n,x_1}$$

In particular,

$$\mu(\widehat{l}^{x_1} \dots \widehat{l}^{x_n}) = \frac{1}{n} \sum_{\sigma \in \mathcal{S}_n} G^{x_{\sigma(1)},x_{\sigma(2)}} G^{x_{\sigma(2)},x_{\sigma(3)}} \dots G^{x_{\sigma(n)},x_{\sigma(1)}}$$

These variables generate a dense set of funtionals and form a **Shuffle algebra** by multiplication.

Holonomies

Attach to each oriented edge x, y a unitary matrix $U_{x,y}^{i,j}$, with $U_{y,x} = U_{x,y}^{-1}$. Then associate to a loop l the trace $h_U(l)$ of the corresponding matrix product.
Allow explicit computations under μ.

The variables $h_U(l)$ determine l up to tree-like components.

Proof follows from the fact that traces of unitary representations separate the conjugacy classes of finite groups and from the so-called CS-property satisfied by free groups: Given two elements belonging to different conjugacy classes, there exists a finite quotient of the group in which they are not conjugate.

The continuum analogue is not proved. Holonomies should characterize Brownian loops. Close to T. Lyons signature conjecture for Brownian paths.