Random normal matrices

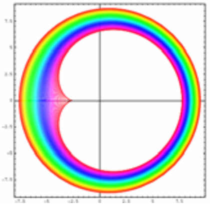
Nikolai Makarov

- based on joint work with Y. Ameur and H. Hedenmalm
- physical theory: P.Wiegmann and A. Zabrodin
- important references:

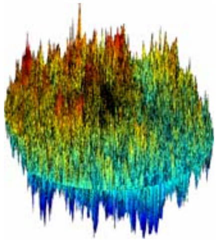
--- K. Johansson (Hermitian matrices)

--- B. Rider and B. Virag (Ginibre ensemble)

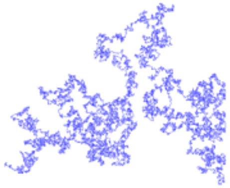
--- R. Berman, B. Berndtsson, J. Sjostrand
(asymptotics of Bergman kernels)



Potential theory

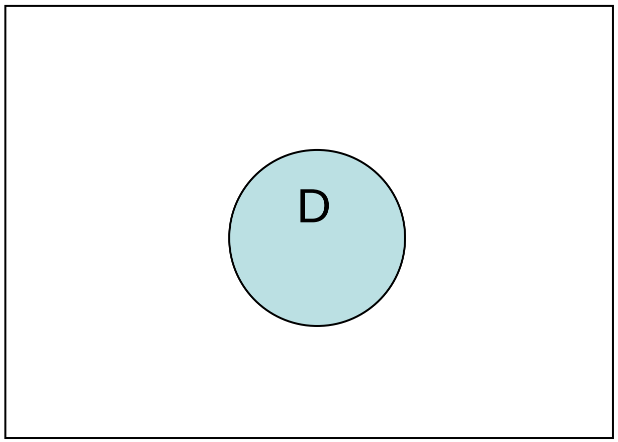


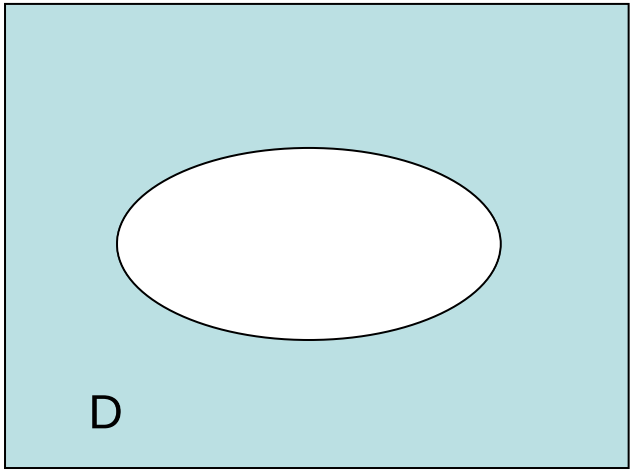
Statistical model



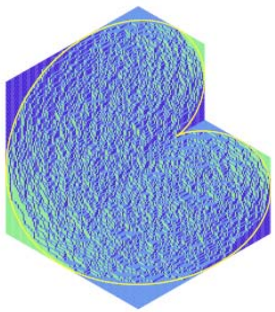
Field theory

Quadrature domains

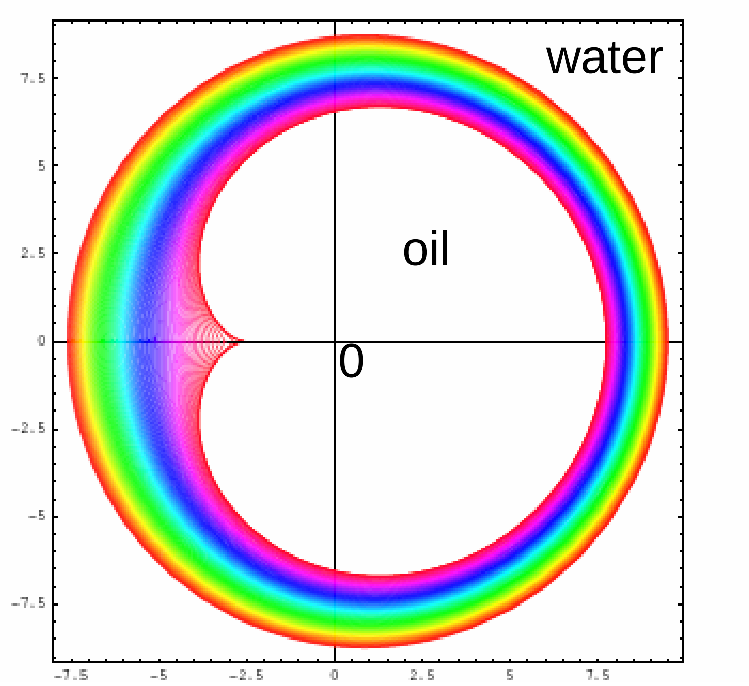




$$\int_D u = \sum_1^k c_j u(z_j)$$

similarity with 

Polubarinova-Kochina example

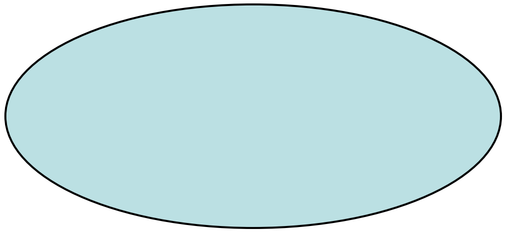


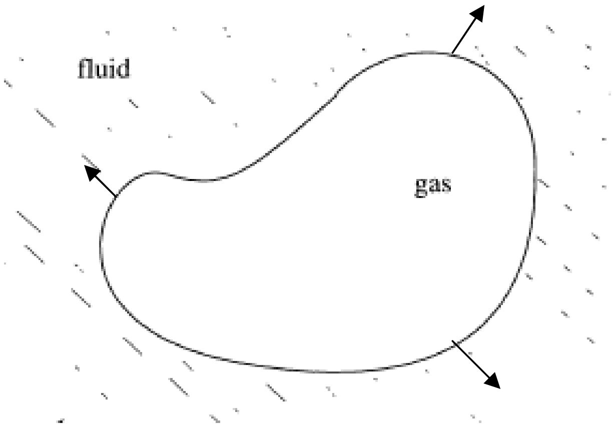
- inverted ellipse
- Varchenko, Etingof: Why the coundary of a round drop becomes a curve of order 4?

$$V_n = 2\pi \nabla G(\cdot, 0)$$

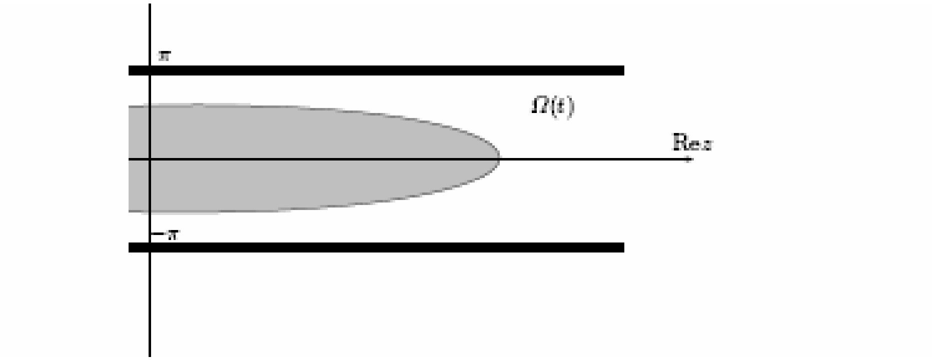
Hele-Shaw flow

[Gauss]





$$V_n = 2\pi \nabla G(\cdot, \infty)$$



[Saffman-Taylor]

Gauss variational problem

- σ electric charge (a positive measure) in $\mathbb{C}$
- $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ external field potential

$$\boxed{Q(z) \gg \log |z|, \quad z \to \infty} \qquad (*)$$

- Q-energy of the charge:

$$\int\int \log \frac{1}{|z-w|} d\sigma(z) d\sigma(w) + 2\sigma(Q)$$

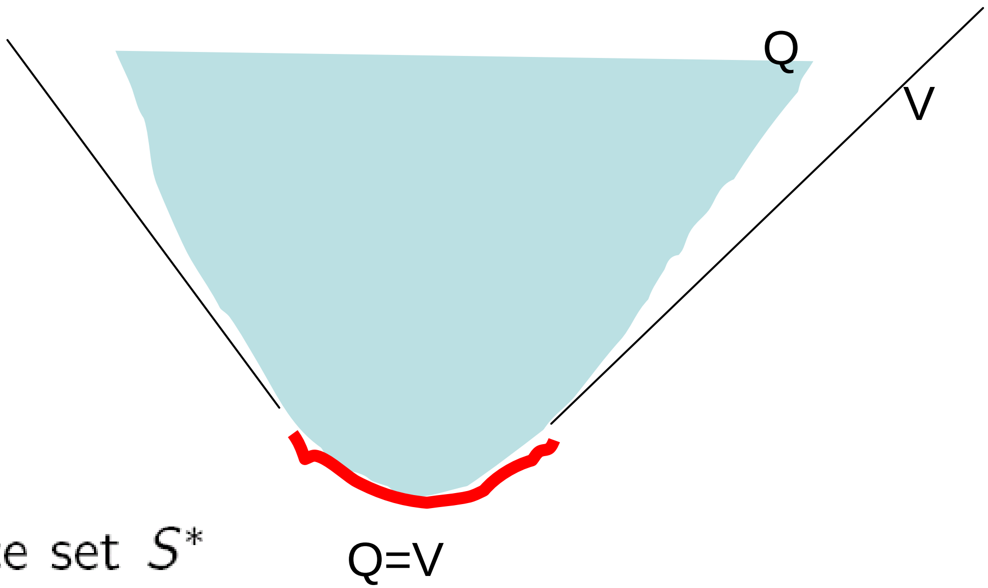
- Find σ of total charge one with minimal Q-energy

Existence, uniqueness, free boundary problem

- If Q is LSC, then there is a unique probability measure $\hat{\sigma} = \hat{\sigma}[Q]$ of minimal Q-energy, [Frostman '35]

  (e.g., $Q \in (*)$ continuous on a closed set and $+\infty$ elsewhere)

- Find maximal subharmonic $V \leq Q$ such that $V \sim \log|z|$ at ∞



coincidence set S^*

Smoothness, localization

- If $Q \in C^2(\mathrm{nbh}\ S^*)$, then $\Delta Q \geq 0$ on S^* and

$$2\pi\hat{\sigma} = \Delta Q \cdot 1_{S^*} = \Delta Q \cdot 1_S$$

- Conclusion: equilibrium measure is determined
  - by the support S ("droplet"), and
  - by the (conformal) metric ΔQ
- Examples of zero curvature metrics:

$$Q \;=\; |z|^2,\ |z|^{-2},\ \log|z|^2 \quad \text{plus harmonic } H$$

If ∂H is rational, then the components of $\mathbb{C} \setminus S$ are quadrature domains (after conformal transformation)

Laplacian growth

- What happens to the droplet if we slightly change the potential?

- Special case: $S_t = S(Q/t)$

  [S_t is support of $\sigma_t = t\hat{\sigma}(Q/t)$, the Q-extremal measure of mass t]

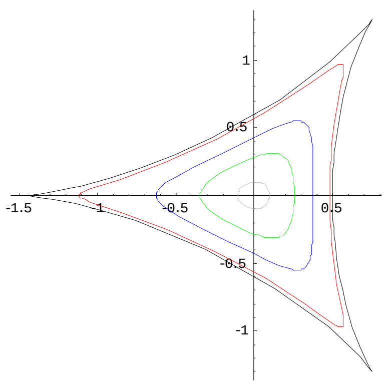
- Formally [Richardson]:

$$\dot{\sigma}_t = \omega_t^\infty,$$

or symbolically

$$\frac{1}{2\pi} V_n = \nabla G(\cdot, \infty)$$

[gradient wrt metric ΔQ)]



C^ω potentials

- Formal Hele-Shaw equation can be justified for (locally) C^ω potentials [Sakai]

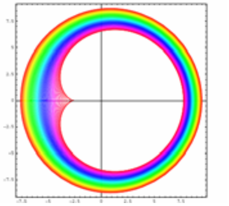
- Generalization: consider $Q_\varepsilon = Q - \varepsilon g$

  [$Q \in C^\omega$, $\partial S \in C^\omega$, g is smooth]

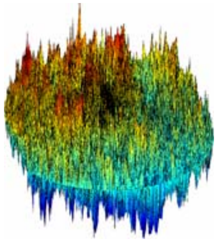
  Then

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int f \cdot \sigma(Q_\varepsilon) = \left(f^S, g^S\right)_\nabla$$

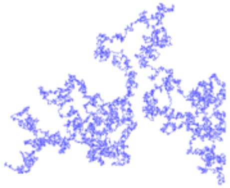
  [f^S is an extension of $f|_S$ harmonic in $\hat{\mathbb{C}} \setminus S$]



Potential theory



Statistical model



Field theory of CG

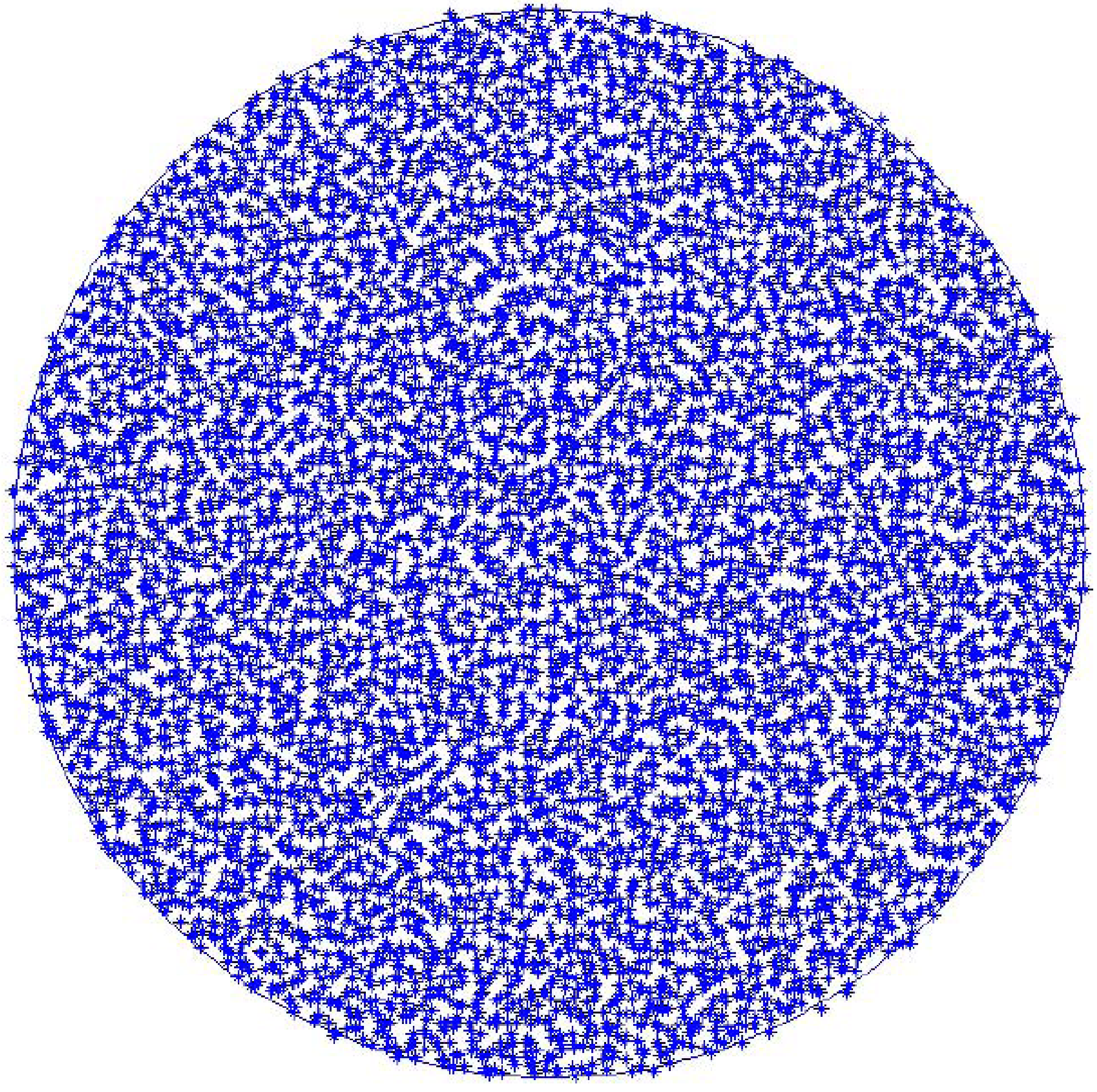
Statistical model

- n electrons at random positions λ_j in external field nQ

$$H = \sum \sum_{j \neq k} \log \frac{1}{|\lambda_j - \lambda_k|} + 2n \sum Q(\lambda_j).$$

$$Z_n(Q) = \int_{\mathbb{C}^n} e^{-\frac{\beta}{2}\ H},$$

- 2D Dyson ensembles; one-component plasma; [Wiegmann-Zabrodin]



Density of states

Define $\boldsymbol{\sigma_n} \in \mathrm{Prob}(\mathbf{C})$ as

$$\boldsymbol{\sigma_n(e) = E\, \frac{\#\{z_j \in e\}}{n}}, \qquad e \subset \mathbf{C},$$

[$\boldsymbol{E}$ is expectation wrt $\boldsymbol{P_n}$]

Theorem [Johansson, Elbau-Felder, Hedenmalm-M.]

$$\sigma_n \to \sigma := \hat{\sigma}[Q] \qquad \text{in} \quad (C \cap L^\infty)^* .$$

In particular, if $Q \in C^2$ in some neighborhood of the "droplet" $S := \operatorname{supp} \sigma$, then

$$\sigma = \frac{1}{2\pi} \Delta Q \cdot 1_S .$$

Random normal matrices

Special case $\beta = 2$:

$$Z_n(Q) = \int_{\mathcal{M}_n} e^{-2n \text{ trace } Q(M)} \, dM,$$

where $\mathcal{M}_n \subset \mathrm{C}^{(n^2)}$ is the set of normal $n \times n$ matrices.

P_n describes the distribution of eigenvalues.

Example: $Q = +\infty$ on $\mathrm{C} \setminus \Sigma$

- $\Sigma = \mathbb{R}$ Hermitian ensembles.
- $\Sigma = \mathbb{T}$ unitary ensembles.

Fermionic description of $\mathbb{P}_n$ for $\beta = 2$

Lemma

$\mathbb{P}_n$ is a determinantal process. The kernel $K_n(z, w)$ is equal to the reproducing kernel in weighted polynomial Bergman-Fock space

$$\mathcal{H}_n = \mathcal{P}_n \cdot e^{-nQ/2} \subset L^2(\mathbb{C}).$$

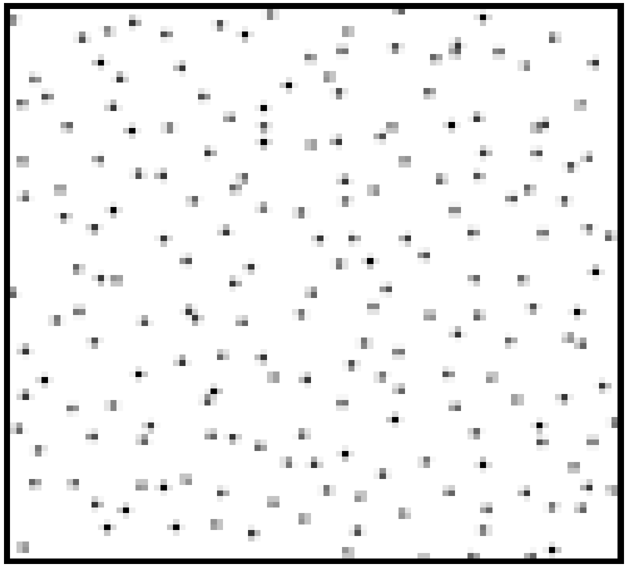
- k-point intensity is $\det\{K(z_i, z_j)\}$.
- Quantum mechanical interpretation: fermions in magnetic field at lowest Landau level
- k-point wave function:

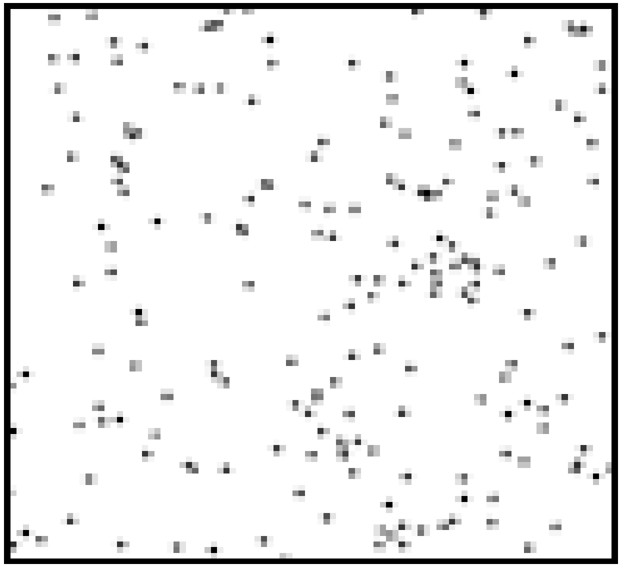
$$\psi_{k,n}(z) = \frac{1}{\sqrt{k!}} \det \psi_i(z_j),$$

$\{\psi_i\}$ are ON weighted polynomials.

from now on, beta=2!

Finer structure





our electrons

(lattice-type)

Poisson

Statistical correction

Theorem [Ameur-Hedenmalm-M]
If $Q \in C^\omega$ in a nbh of S and if f is a smooth test function, then

$$\mathbb{E}\sum f(\lambda_j) \;=\; n\sigma(f) + \nu(f) + o(1),$$

where

$$8\pi\nu(f) = \int_S f\,\Delta\log\Delta Q + \int_S \Delta f + \int_{\partial S}\left(f^S\partial_* L - L\partial_* f^S\right)$$

- $L = \log\Delta Q$
- ∂_* is normal derivative wrt $\mathbb{C}\setminus S$
- f^S is an extension of $f|_S$ harmonic in $\hat{\mathbb{C}}\setminus S$
- constant 8π depends on β

[Johansson] for Hermitian

[Rider-Virag] for Ginibre

Footnotes

- universality of the double layer (independence of Q)
- the jump in the potential depends on β
- Polyakov-Alvarez ($\det_\zeta \Delta$) & MacKean-Singer (heat kernel asymptotics)
- Partition function:

$$\frac{d}{ds} \log Z_n(sQ) \;=\; n\, \mathbb{E}_{sQ} \sum Q(\lambda_j),$$

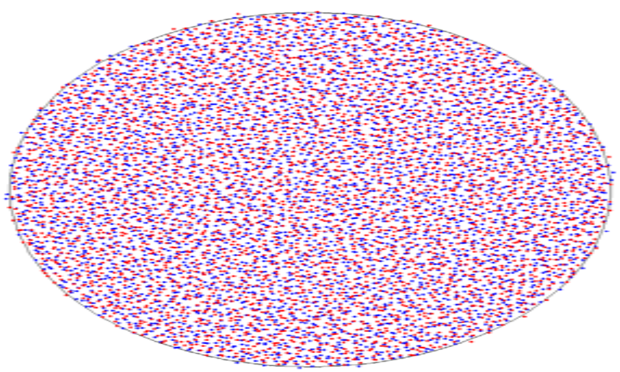
integrate over Hele-Shaw flow [explicit formulae by Wiegmann-Zabrodin]

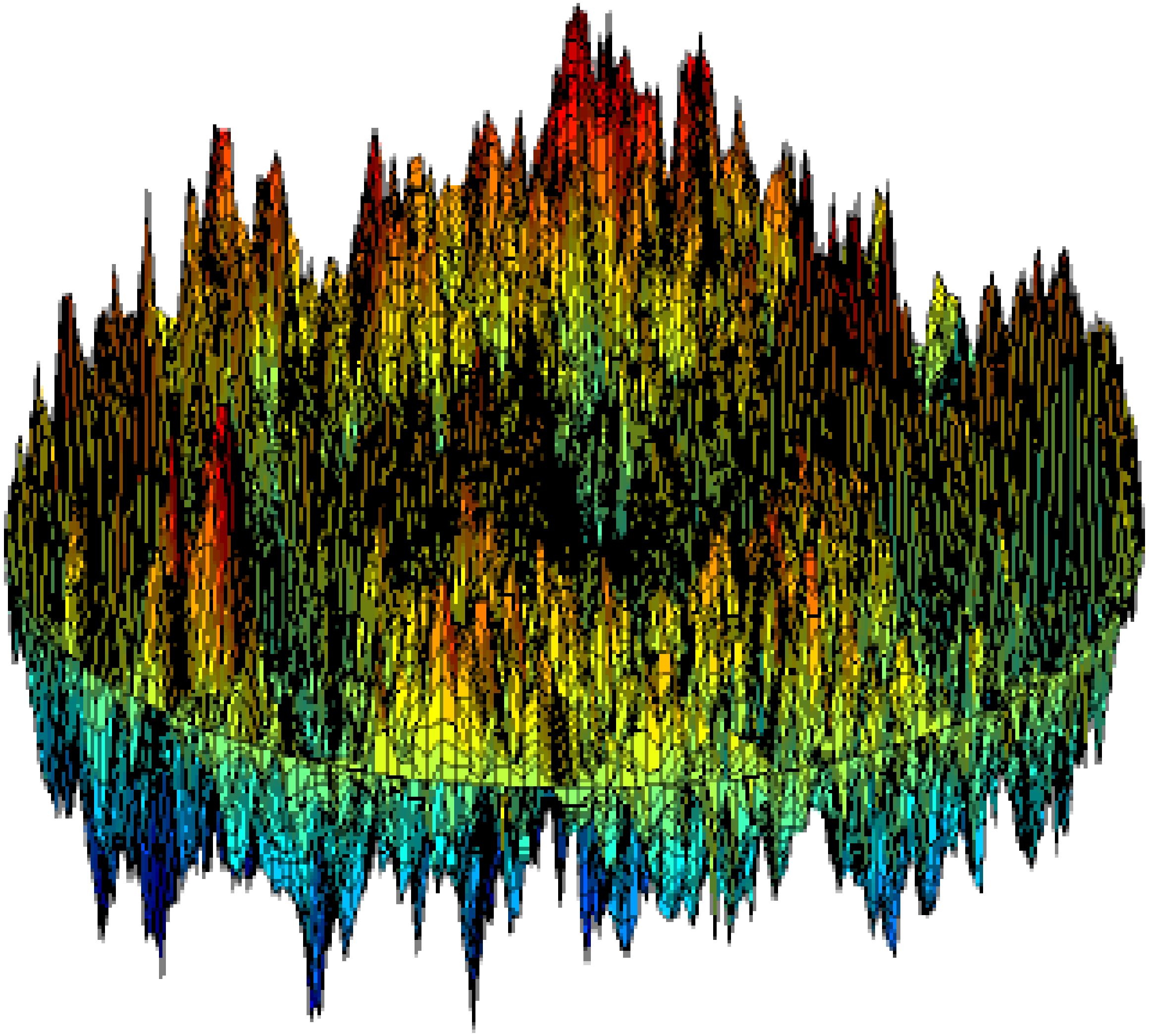
GFF convergence

Corollary

$$\sum f(\lambda_j) - \mathbb{E}\sum f(\lambda_j) \to N\left(0, \frac{1}{4\pi}\|f^S\|_\nabla^2\right)$$

- In other words, $\log \frac{p(z,\tilde{M}_n)}{p(z,M_n)}$ converges to GFF in S with free boundary ($\tilde{M}$ and M are independent matrices)
- no dependence on Q or β
- the derivation uses only the classical term $\sigma(f)$, not $\nu(f)$, (plus estimates)





Proof of Cor.

$\text{Fluct}_n f := \sum f(\lambda_j) - n\sigma(f)$. Define

$$F(\lambda) = \log \mathbb{E}\, e^{\lambda \text{Fluct}_n f}, \quad \lambda \in (0,1).$$

Then (following Kurt Johansson)

$$F'(\lambda) = \tilde{\mathbb{E}}\ \text{Fluct}_n f \quad \text{wrt} \quad \tilde{Q} = Q - \frac{\lambda f}{2n}$$

(and also $F'' > 0$), so by Laplacian growth

$$F'(\lambda) = n[\tilde{\sigma}(f) - \sigma(f)] + \tilde{\nu}(f) \ \to \ \frac{\lambda}{2\pi}\|f^S\|_\nabla^2 + \nu(f).$$

Integrate.

"Quantum Hele-Shaw"

Corollary

$$|P_n|^2 e^{-nQ} \equiv R^1_{n+1} - R^1_n \; \to \; \omega^\infty$$

- P_n is the n-th ON polynomial in $L^2\left(e^{-nQ}\right)$
- R^1_{n+1} and R^1_n are intensities wrt the same potential nQ
- Compare: σ_t is density of states of n electrons in potential nQ and $\dot{\sigma} = \omega^\infty$

Double scaling in the bulk

- Let $0 \in \mathrm{Int}(S)$, $Q \in C^2$ near 0, $\Delta Q(0) > 0$.
- Rescale so that the average spacing is one:

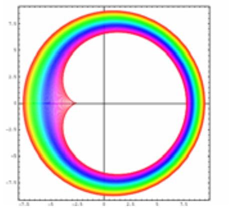
$$z \mapsto \sqrt{n}\,\sqrt{\Delta Q(0)}\,z$$

- $\mathbb{P}_{n,nQ} \mapsto \mu_n$, a new n-point process in $\mathbb{C}$

Theorem

$$\mu_n \longrightarrow Ginibre(\infty).$$

- Ginibre(∞) is a det-process with $K(z,w) = e^{-2z\bar{w}-|z|^2-|w|^2}$
- in C^ω case, hierarchies of universal laws at boundary singularities (?)



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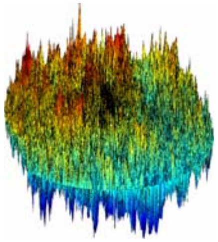
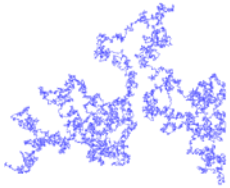


Statistical model



Field theory

CG approximation

- Consider $\Phi_n = \sqrt{2}\sum G(\cdot, \lambda_j) - \sqrt{2}\sum G(\cdot, \lambda'_j)$ as an approximation of GFF in S with Dirichlet boundary [eigenvalues of independent matrices]
- Definition of Φ_n^{*2} in terms of OPE:

$$\Phi_n(w)\Phi_n(z) = \log\frac{1}{|w-z|} + \Phi_n^{*2}(z) + o(1)$$

as $w \to z$ and $n^{-1/2} \ll |w-z|$ (in correlations)
- True with

$$\Phi_n^{*2} = \Phi_n^2 - \log\sqrt{n} - \frac{1}{2} - \frac{\gamma}{2} + 2c.$$

[Euler's constant and conformal log-radius]

cont'd

- Similarly, we define Φ_n^{*3}, Φ_n^{*4}, ... so that OPE exponentials $e^{*\sigma\Phi_n}$ are non-random modifications of $e^{\sigma\Phi_n}$.

- Claim:

$$e^{*\sigma\Phi_n} \to e^{*\sigma\Phi}$$

    - in correlations (for all σ's)
    - as random distributions (if $|\sigma| < 1$)

Ramified fields

- $\tilde{\Phi}_n$ (harmonic conjugation)

  [like $\sum \arg(z-\lambda_j)$]
- modifications of $e^{\sigma\tilde{\Phi}_n}$

  [like $\prod \sqrt{z-\lambda_j}$]
- Claim:

$$\tilde{\Phi}_n \to \int *d\Phi$$

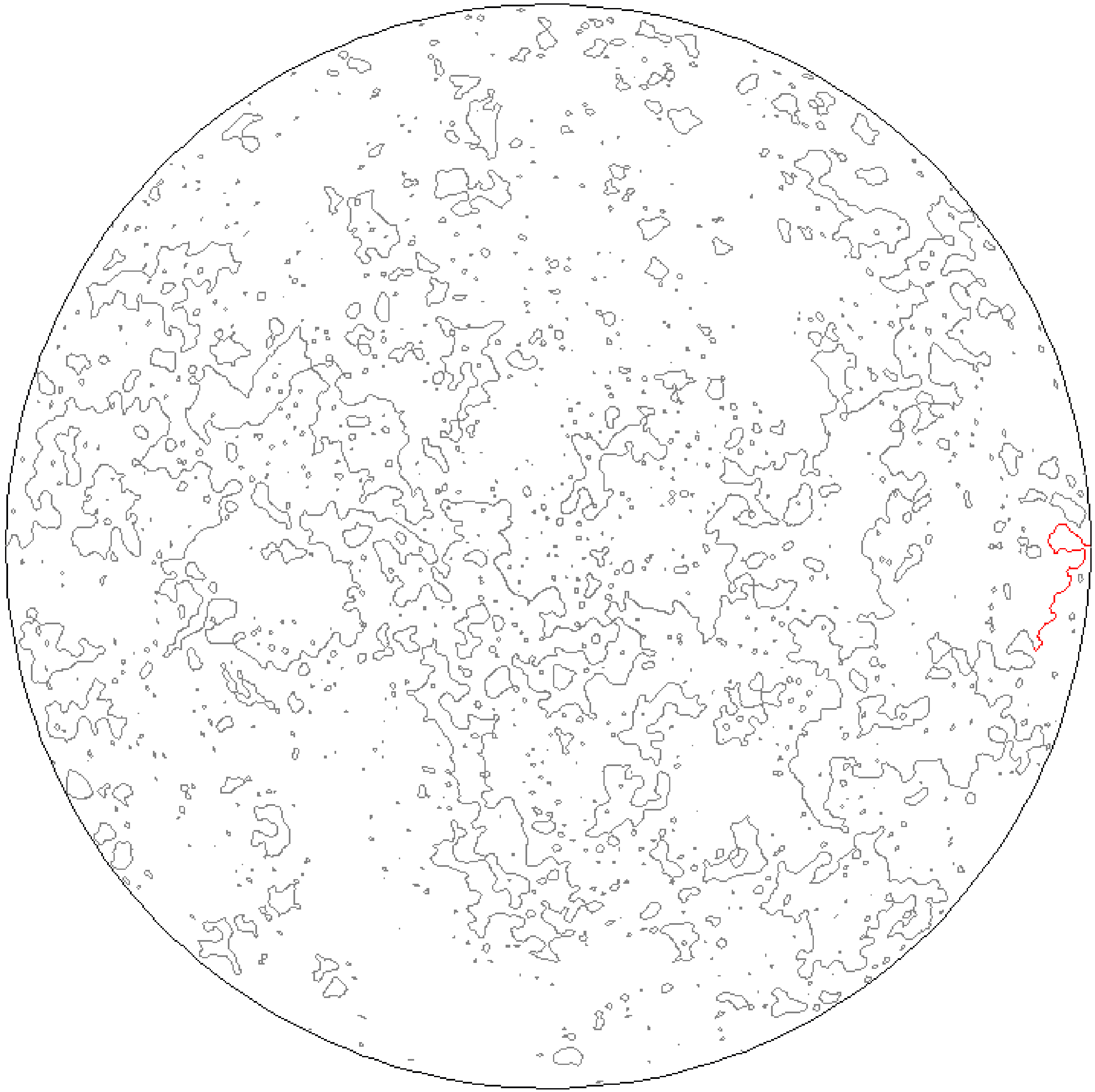
  in correlations with $\Phi_n(z_1)\ldots\Phi_n(z_m)$

  [monodromy group is $\pi_1(D\setminus\{z_1,\ldots,z_m\})$]
- etc (Ameur, Kang, M)

- Sheffield's interpretation of SLE: flow lines of $e^{*i\sigma\Phi}$ (with boundary conditions and central charge modifications)

  [$\sigma = \sigma(\kappa)$ from equation spin=-1]
- Q: Is this true (in the limit) for flow lines of $e^{*i\sigma\Phi_n}$?
- Equivalent reformulation: ... for geodesics of $e^{*\sigma\tilde{\Phi}_n}$ (with corresponding modifications)?

  [metric is not well-defined but the geodesics are]
- Classical limit ($\kappa = 0$): SLE is a hyperbolic geodesic



Stress tensor

- S.E.T. is a map

$$v \mapsto W_v$$

from vector fields in $\mathbb{C}$ to random variables

- in terms of Hamiltonian $H = H(\lambda_1, \ldots, \lambda_n)$,

$$W_v^+ = -\nabla_v H + Tr(\partial v)$$

  - $\nabla_v H := \sum v(\lambda_j)\partial_{\lambda_j} H$ and $Tr(\partial v) := \sum \partial v(\lambda_j)$
  - in RNM model

$$W_v^+ = \sum_{j<k} \frac{v(\lambda_j) - v(\lambda_k)}{\lambda_j - \lambda_k} - 2n Tr[v\partial Q] + Tr[\partial v].$$

Ward's identities

- for all $F = F(\lambda_1, \dots, \lambda_n)$,

$$\mathbb{E}[\nabla_v F] + \mathbb{E}[W_v F] = 0$$

- equivalently, W is stress tensor for the density field $\rho(z) = \sum \delta(z - \lambda_j)$:

$$\mathbb{E}[\mathcal{L}_v \rho(z_1) \dots \rho(z_m)] = \mathbb{E}[W_v \rho(z_1) \dots \rho(z_m)]$$

[Here $\mathcal{L}_v$ is Lie derivative: $\mathcal{L}_v \rho(z) = \frac{d}{dt}\Big|_{t=0} (\rho \| \psi_{-t})(z)$, where ψ_t is the flow of v, and $(\rho \| \psi_{-t})$ is the expression for ρ in local coordinates ψ_{-t}]

The magic of Ward's identities

An example of (infinitely many) exact computations:

$$Var(W_v^+) \;=\; 2n\,\mathbb{E}\;Tr(|v|^2\Delta Q) \;+\; \mathbb{E}\;Tr|\bar\partial v|^2$$

Cor:

$$Var\left(\frac{1}{n}W_v^+\right) \;\to\; 2\int_S |v|^2(\Delta Q)^2$$

E.g.,

$v = 1$ gives Main Thm for $f = \partial Q$,

$v(z) = z$ for $f = z\partial Q$, ...

Cauchy kernels, etc