Universality for $\mathrm{SLE}(4)$

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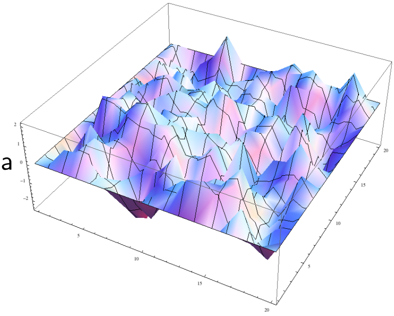
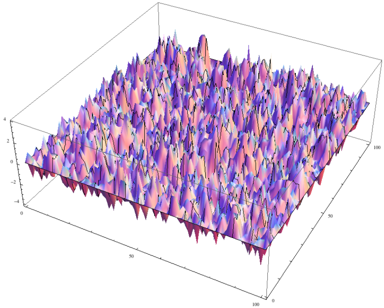
The DGFF





- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.
- Measure on functions $h \colon D \to \mathbf{R}$ with density

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$

for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$.

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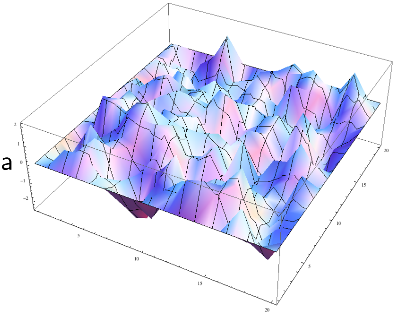

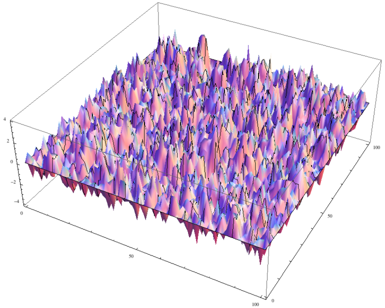

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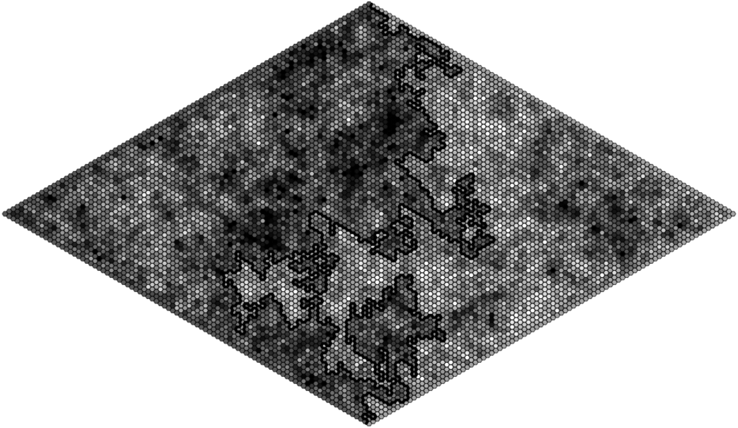
- **Covariance**: Green's function for SRW
- **Mean Height**: harmonic extension of ψ

A Single Contour of the DGFF



(Schramm-Sheffield, 2009)

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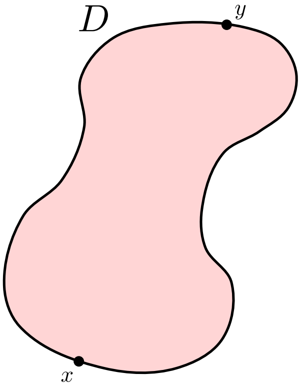

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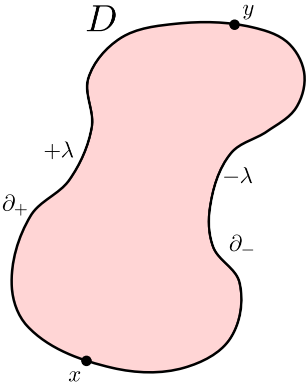

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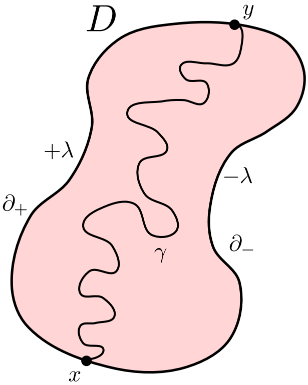

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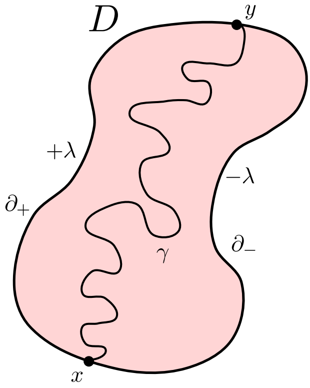
Theorem (Schramm, Sheffield 2009)

There exists a choice of $\lambda > 0$ such that $\gamma^n \to SLE(4)$.



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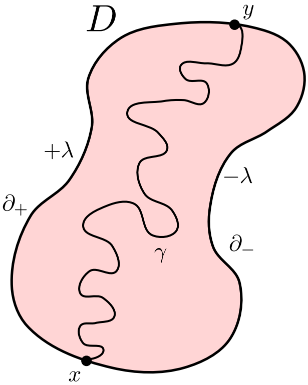

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Conjecture (Sheffield)

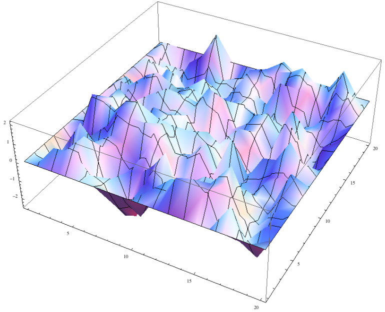
Zero level lines for many random surface models are described by variants of $\mathrm{SLE}(4)$. **Important examples***: dimer models, Ginzburg-Landau* $\nabla\varphi$ *interface model*

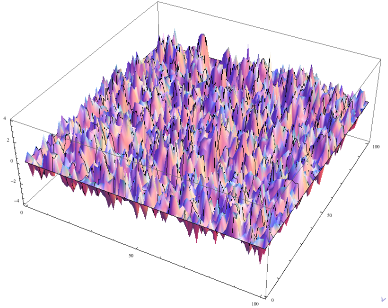
The Ginzburg-Landau $\nabla\varphi$ Model

- **Non-Gaussian** random surface model
- Measure on functions $h\colon D \to \mathbf{R}$ with density

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} \mathcal{V}(h(x) - h(y)) \right)$$

for $D \subseteq \mathbf{Z}^2$, $h|_{\partial D} = \psi$, $\mathcal{V} \in C^2$ **uniformly convex**.





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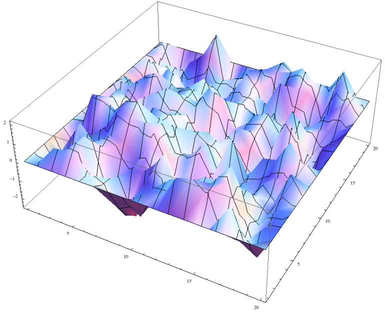
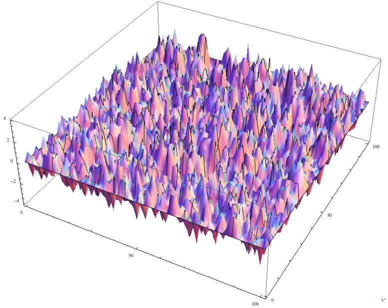
- $\mathcal{V}(x) = x^2$ (DGFF),
  $\mathcal{V}(x) = 4x^2 + \cos(x) + e^{-x^2}$.





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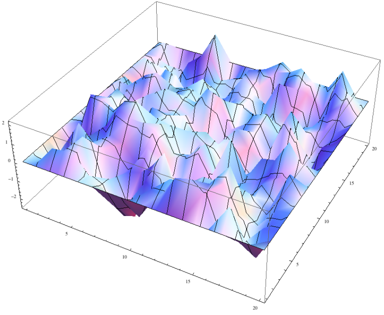
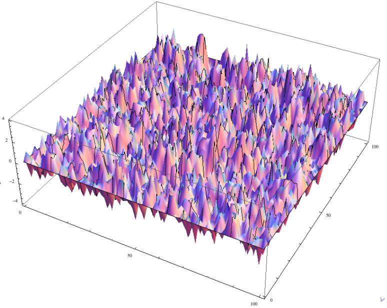
- $\mathcal{V}(x) = x^2$ (DGFF), $\mathcal{V}(x) = 4x^2 + \cos(x) + e^{-x^2}$.
- **Covariance**: Annealed Green's function of a RWRE (HS representation)
- **Mean Height**: Annealed harmonic measure of a RWRE (HS representation)





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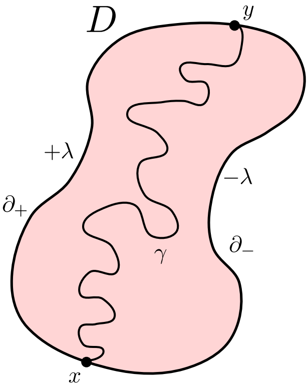
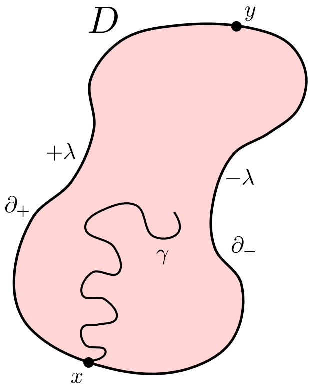


Theorem (M.)

There exists a choice of $\lambda > 0$ depending only on $\mathcal{V}$ such that $\gamma^n \to SLE(4)$.

Why SLE(4)?

- Let γ_t be an $\mathrm{SLE}(4)$ path connecting x to y at time t.



Why $\mathrm{SLE}(4)$?

- Let γ_t be an $\mathrm{SLE}(4)$ path connecting x to y at time t.
- Let $g_t(z)$ be **harmonic** in $D \setminus \gamma[0, t]$ with boundary values
  - $+\lambda$ on ∂_+, $-\lambda$ on ∂_-
  - $+\lambda$ to the left of γ, and
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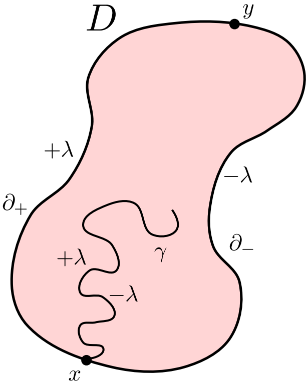

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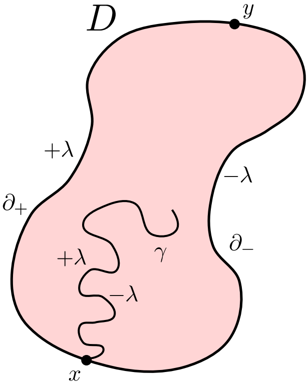

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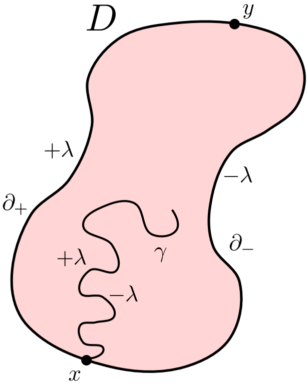
- $\mathrm{SLE}(4)$ is characterized by the property that (★) $g_t(z)$ **evolves as a martingale in time**.
- Proof idea: show (★) asymptotically holds for the GL contour by comparing $g_t(z)$ to the conditional mean $M_t(z)$.

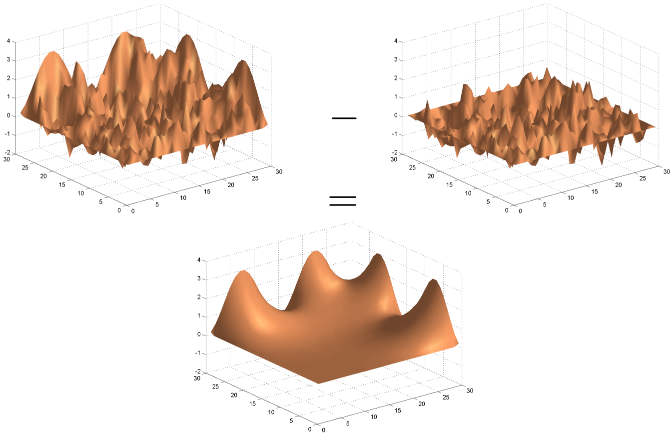
Step 1 Prove approximate harmonicity of $M_t(z)$,
Step 2 Handle the boundary behavior of $M_t(z)$.



Harmonic Coupling for the DGFF

Proposition

Let $h^{\psi}, h^{\widetilde{\psi}}$ have the law of the DGFF with boundary conditions $\psi, \widetilde{\psi}$, respectively. There exists a coupling $(h^{\psi}, h^{\widetilde{\psi}})$ such that $h^{\psi} - h^{\widetilde{\psi}}$ is harmonic.



Step 1: Approximate Harmonic Coupling for GL Model

Suppose $D \subseteq \mathbf{Z}^2$ is bounded with $R = \text{diam}(D)$ and $\psi, \widetilde{\psi} \colon \partial D \to \mathbf{R}$ boundary conditions with $\|\psi\|_\infty + \|\widetilde{\psi}\|_\infty \leq \overline{\Lambda}(\log R)^{\overline{\Lambda}}$ for some $\overline{\Lambda} > 0$.

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Let $h^{\psi}, h^{\widetilde{\psi}}$ have the law of the GL model with boundary conditions $\psi, \widetilde{\psi}$, respectively. There exists a coupling $(h^{\psi}, h^{\widetilde{\psi}})$ such that

$$\mathbf{P}[h^{\psi} - h^{\widetilde{\psi}} \text{ is harmonic in } D(R^{1-\epsilon})] = 1 - O_{\overline{\Lambda}}(R^{-\delta})$$

for $\epsilon, \delta > 0$ depending only on $\mathcal{V}$ where $D(r) = \{x \in D : \text{dist}(x, \partial D) \geq r\}$.

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Taking $\widetilde{\psi} = 0$ gives the **approximate harmonicity of the mean height** since $\mathbf{E}[h^\psi(x)] = \mathbf{E}[h^\psi(x) - h^0(x)]$.

We make no hypotheses on the regularity of ∂D nor $\psi, \widetilde{\psi}$.

Step 1: Harmonicity Up to the Boundary

- $D_t(s) = \{z \in D : \text{dist}(z, \partial D \cup \gamma[0,t]) \geq s\}$
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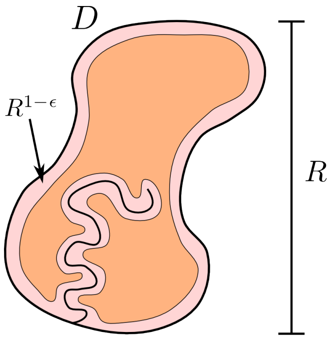

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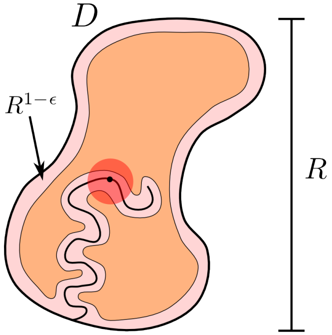

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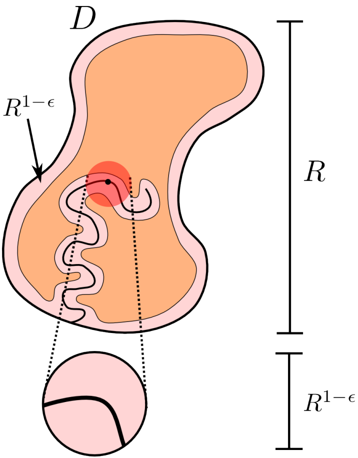

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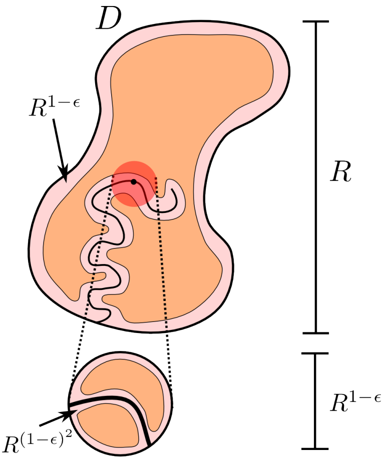

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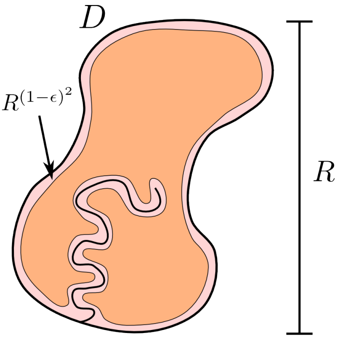

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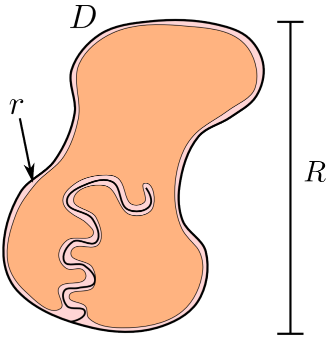
- Compare $M_t|_B$ to $\mathbf{E}\widetilde{h}$, get approx. harmonicity of $M_t(z)$ in $D_t(R^{(1-\epsilon)^2})$.
- Iterate. Arrive at

$$\sup_{z \in D_t(r)} |M_t(z) - f_t(z)| = O(r^{-\delta})$$

where $f_t(z)$ is harmonic in $D_t(r)$ with $f_t|_{\partial D_t(r)} = M_t$; δ depends only on $\mathcal{V}$.



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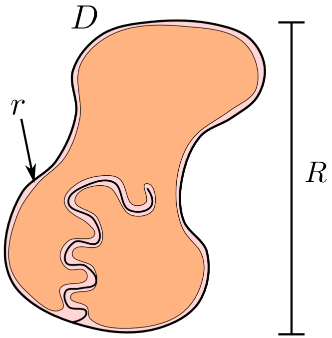

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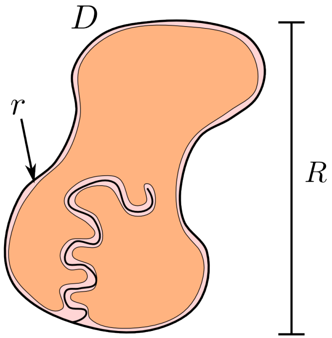
- Hard: γ is **rough**, M_t is **very irregular** near γ



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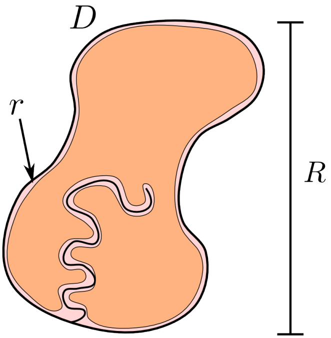

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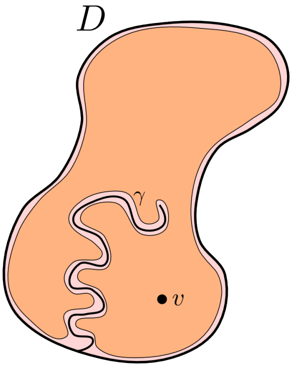
  - Fix v far from γ.



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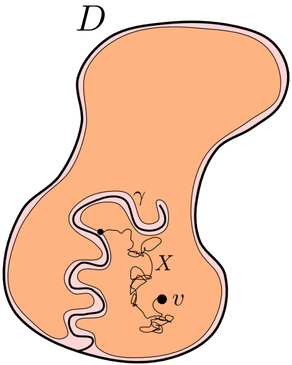

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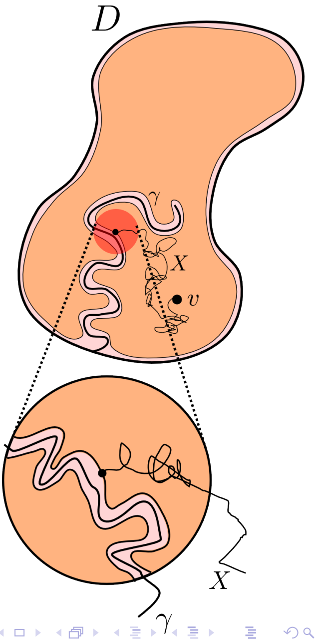

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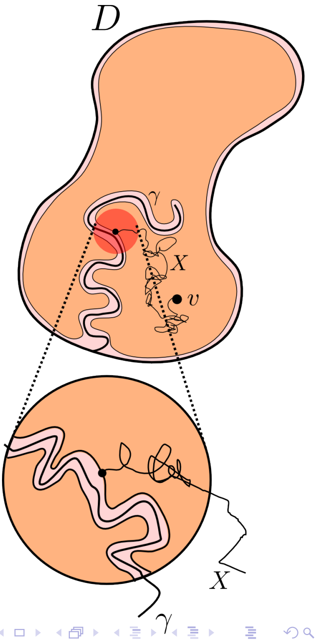

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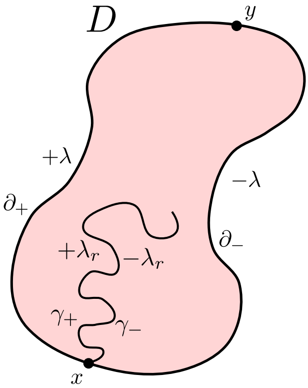

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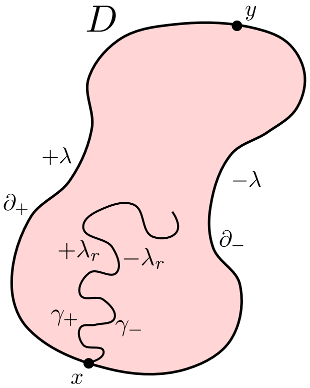

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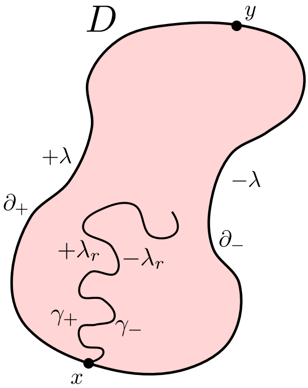

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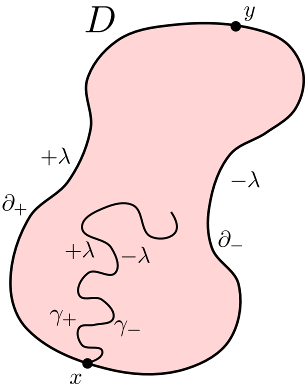

CLT for Linear Functionals on Bounded Domains

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- G.O.S. proof based on the Kipnis-Varadhan approach; shift-ergodicity is an important ingredient.

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where $(h,\cdot)_\nabla$ is a whole-plane GFF.

Other new results for GL models

- Gibbs states as limits of finite volume models
- CLT on bounded domains in the tilted regimes
  - Limit is a linear transformation of a GFF
- Explicit representation of the limiting covariance
  - G.O.S. express it in terms of a complicated variational problem.

Works in Progress [with S. Sheffield]

Full scaling limit of all of the level sets exists and is conformally invariant.

- Odd integer multiple of λ level sets converge to $\mathrm{CLE}(4)$
- Construction of the GFF as a functional of $\mathrm{CLE}(4)$
- SLE based proof of the CLT for GL

Future Directions

- Precise asymptotics of the maximum height
  - DGFF: exact results (Bolthausen, Deuschel, Giacomin '01 and Daviaud '06)
  - GL: known to be between $c_{\mathcal{V}} \log R$, $C_{\mathcal{V}} \log R$ (Deuschel, Giacomin '00). Should have $c_{\mathcal{V}} = C_{\mathcal{V}}$; proof by iterating harmonic coupling. **Exact value should come from CLT.**

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- Non-convex models
  - Perturbations of quadratic (Cotar, Deuschel '10 and Cotar, Deuschel, Müller '08). Harmonic coupling, CLT, and SLE convergence results should hold.
  - Gaussian mixture models: non-uniqueness of Gibbs states (Biskup, Kotecky '07) but still have a CLT (Biskup, Spohn '09) for zero-tilt Gibbs states. **Finite domains?**

Thanks!