Tilings of irregular hexagons

Large Conformal maps from probability to physics

May 23-28, 2010

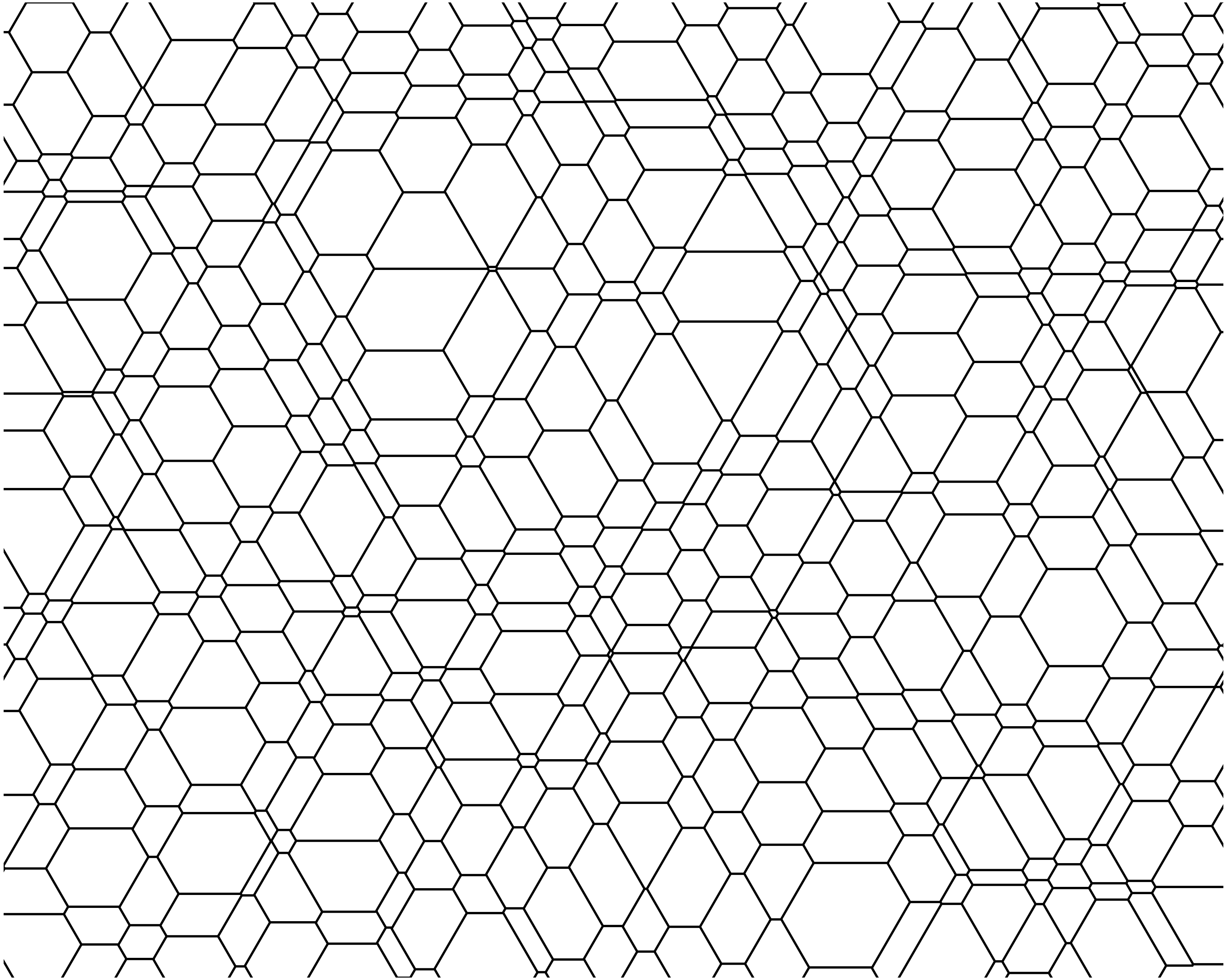
Monte Verità, Ascona, Ticino GGI

Bernard Nienhuis

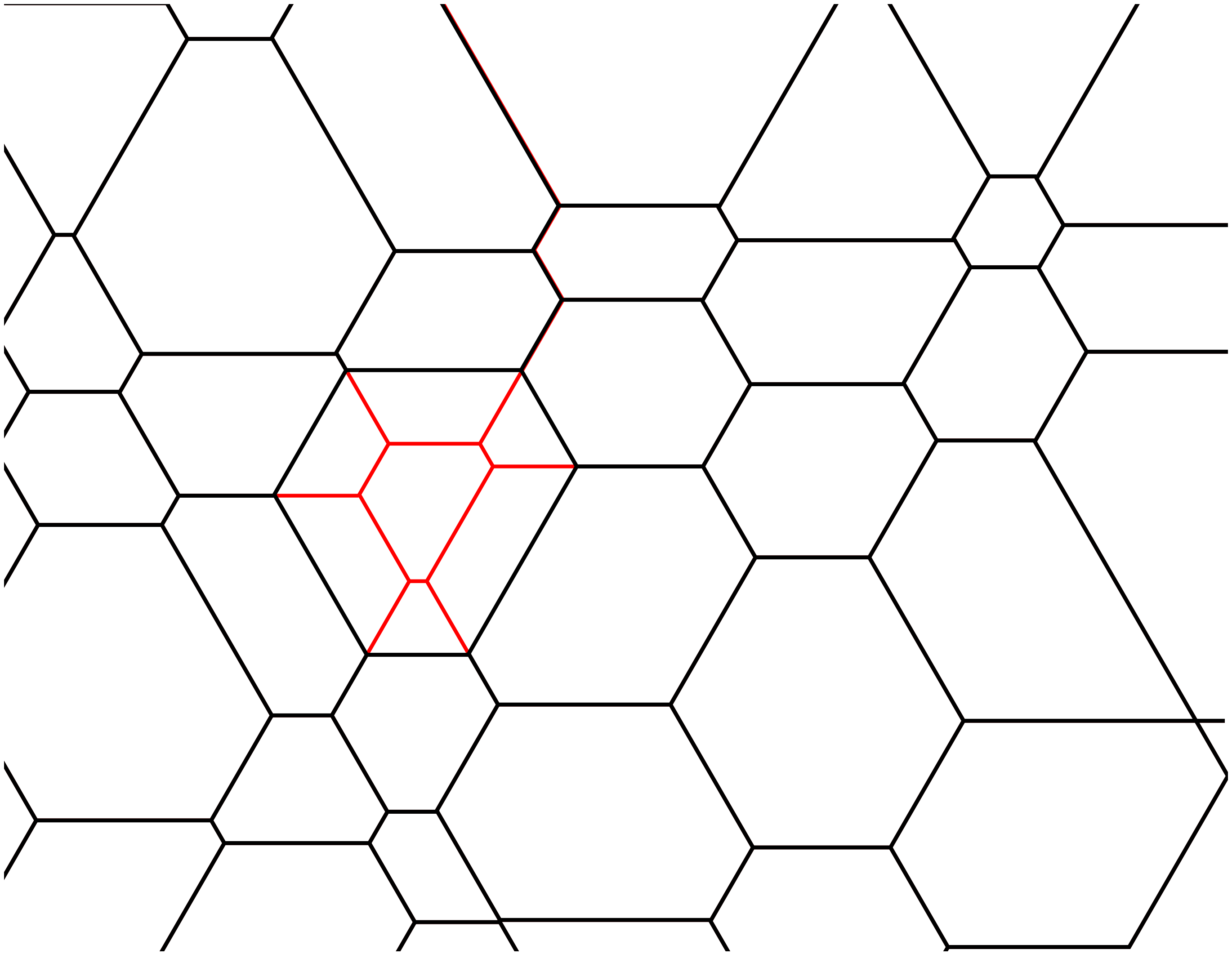
University of Amsterdam

Outline

- The model
- Why do physicists look at this model
- Why do mathematician look at this model
- Bethe Ansatz
- Yield transition
- Conformal behavior



The model: hexagonal net



The hexagons can grow at the expense of their neighbors, with the constraint that all sides are non-negative.

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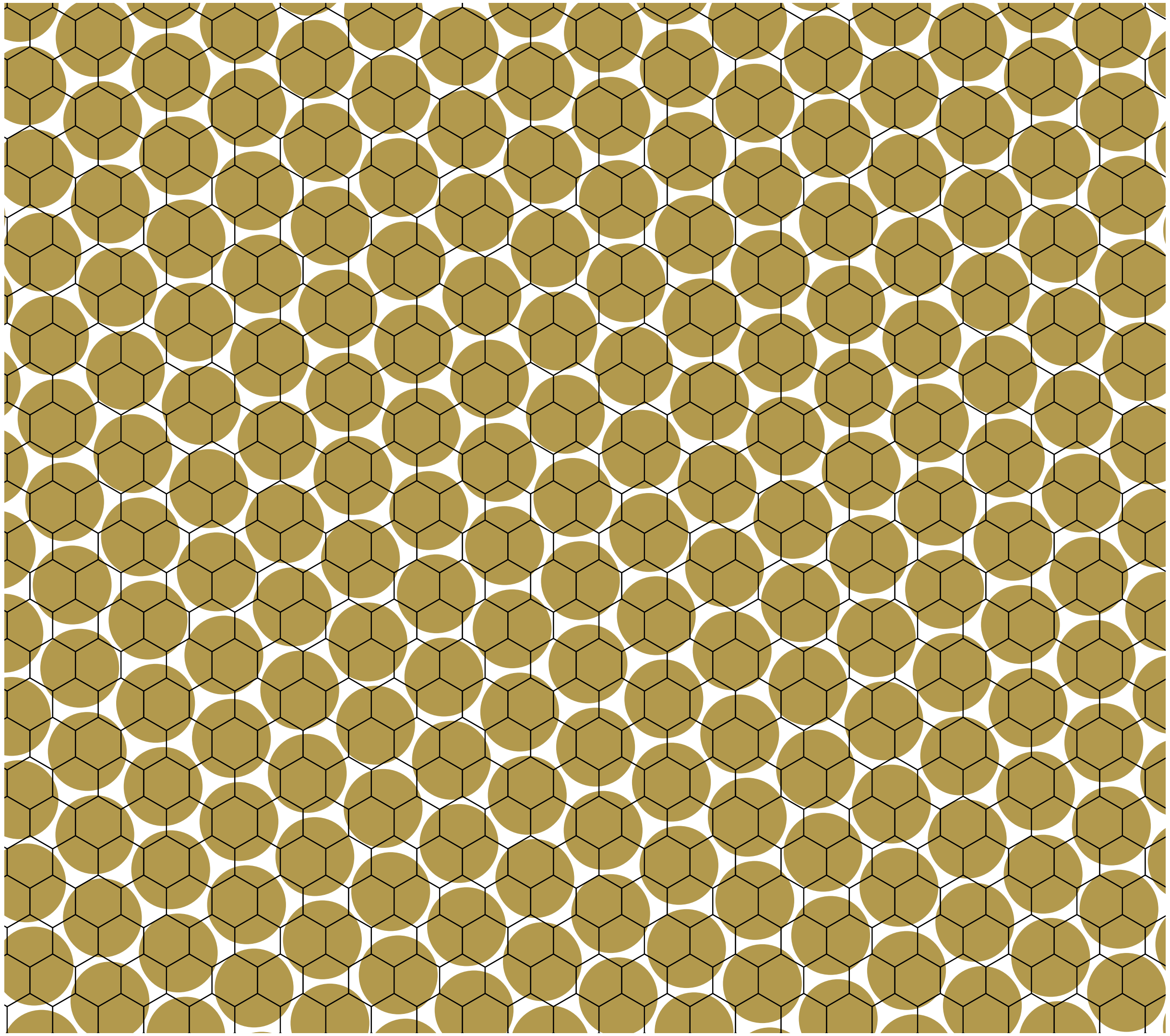
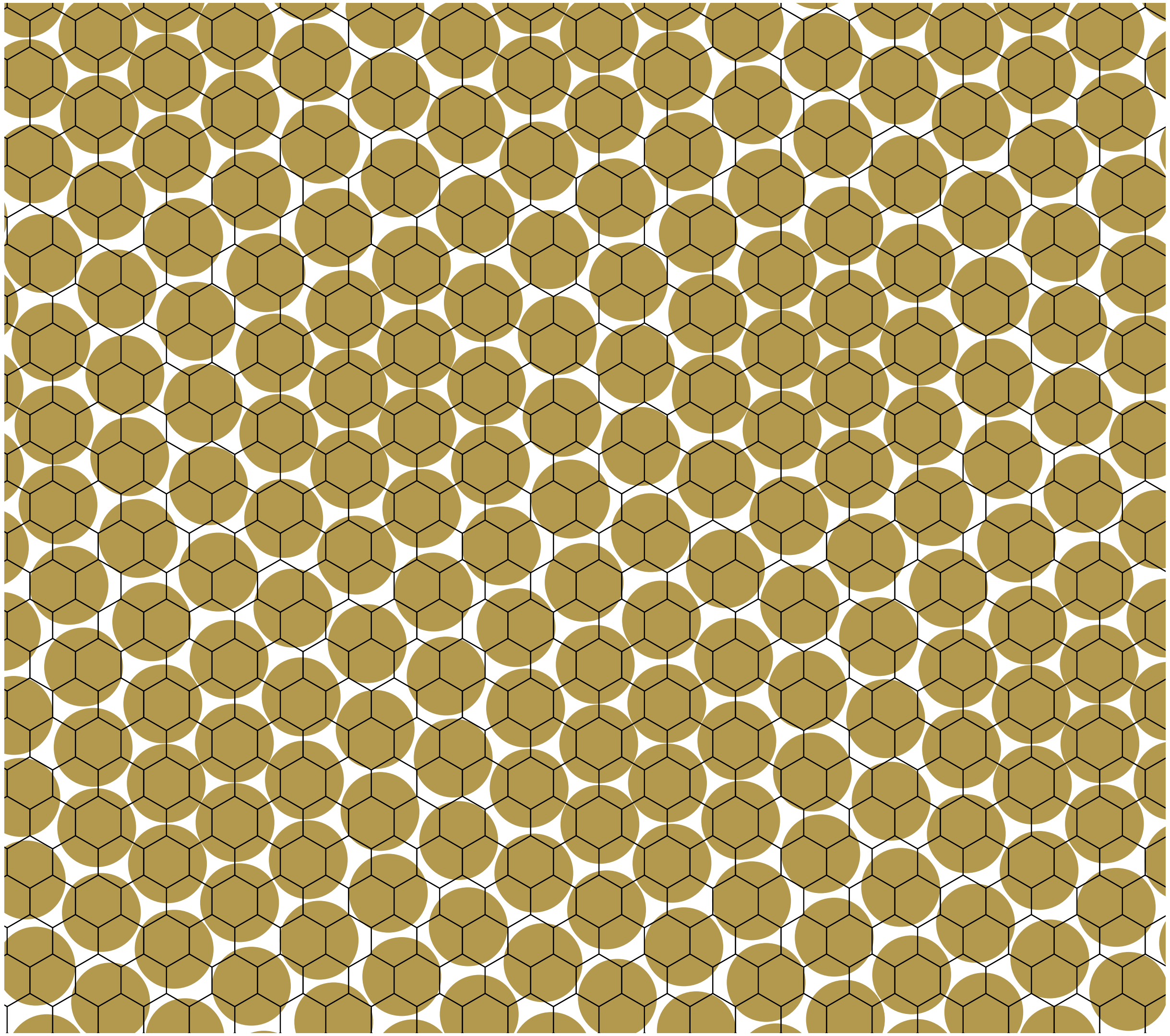


An monolayer of atoms adsorbed on a crystal surface, such that the lattices are incommensurable.



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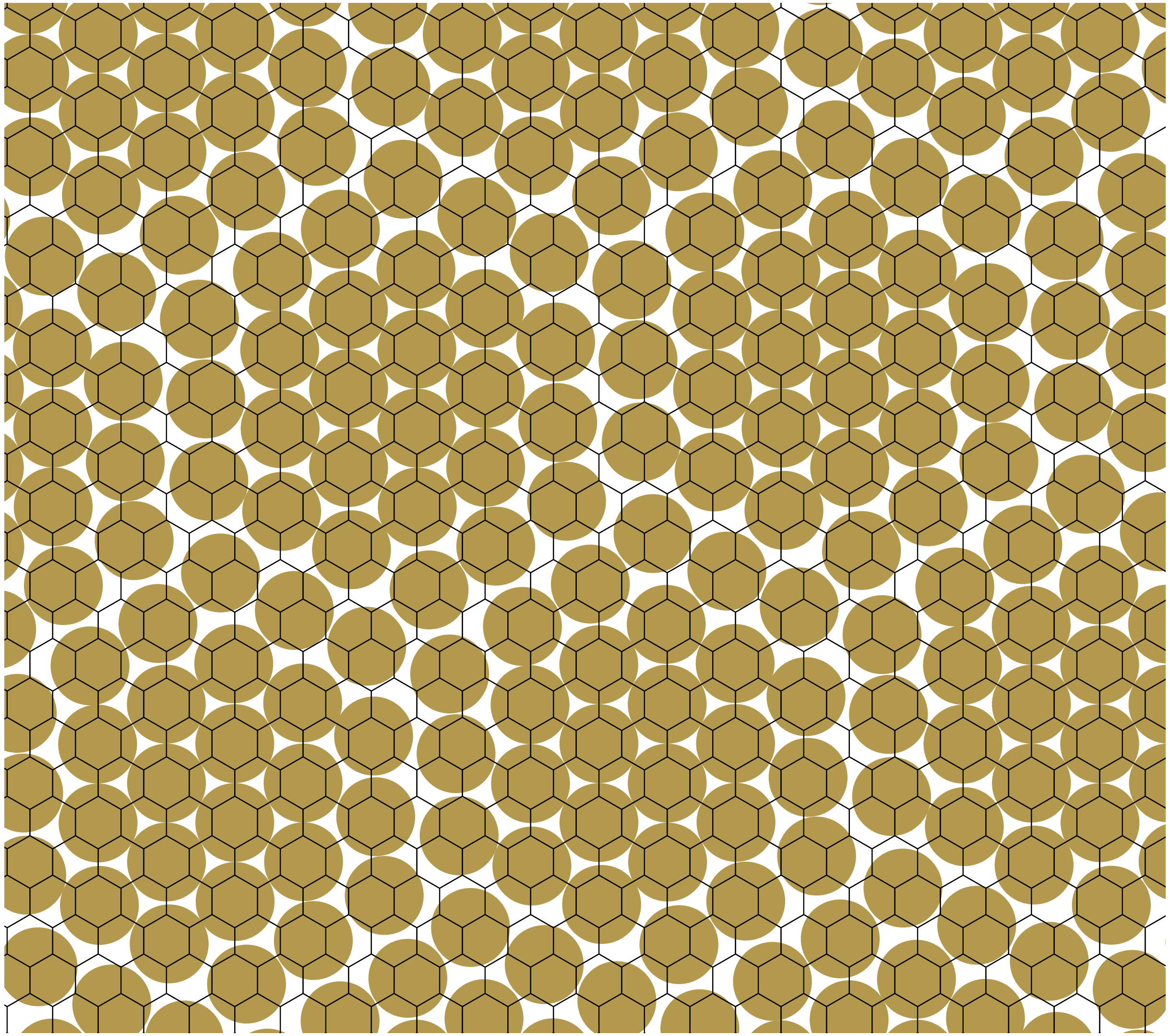

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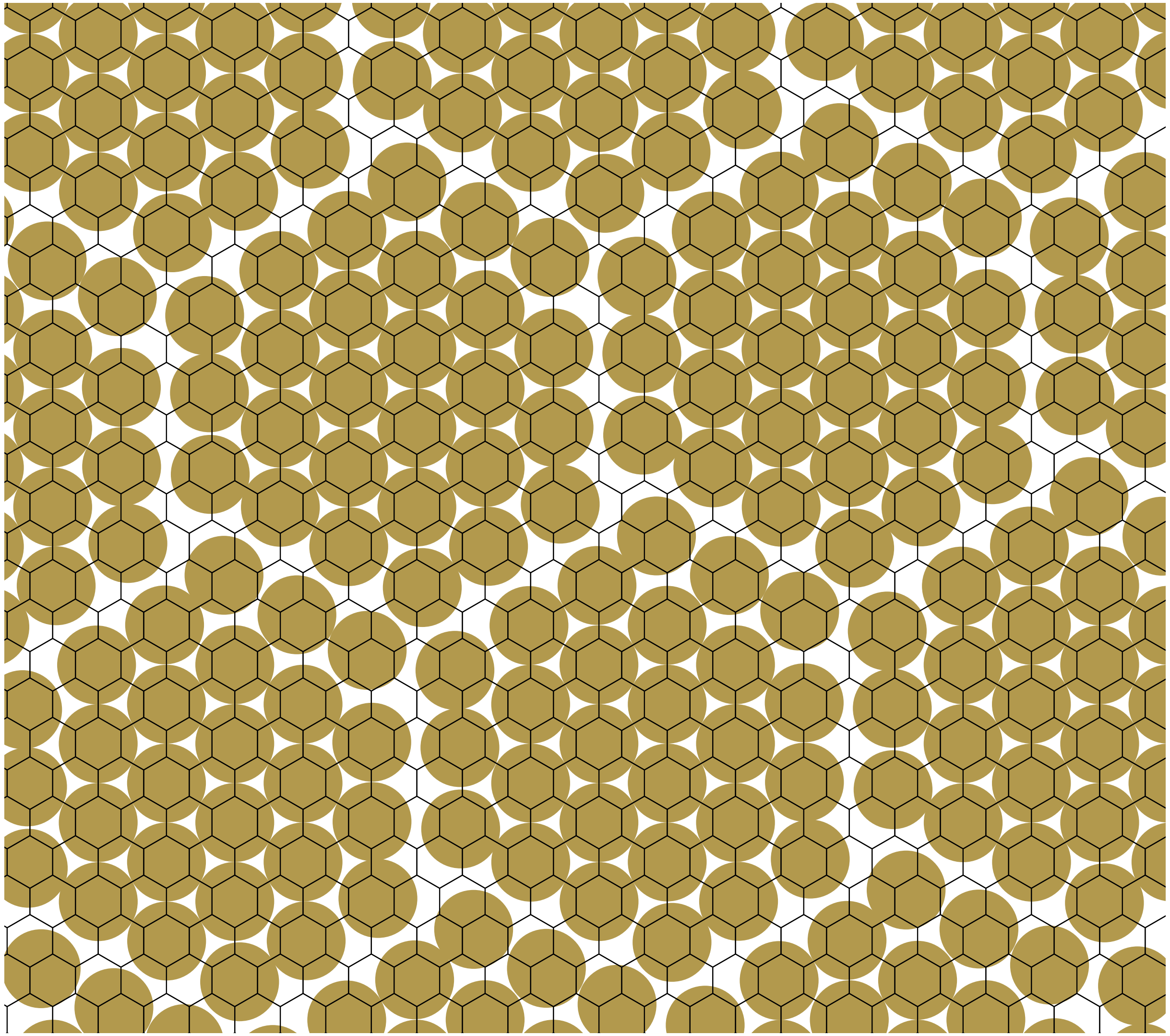

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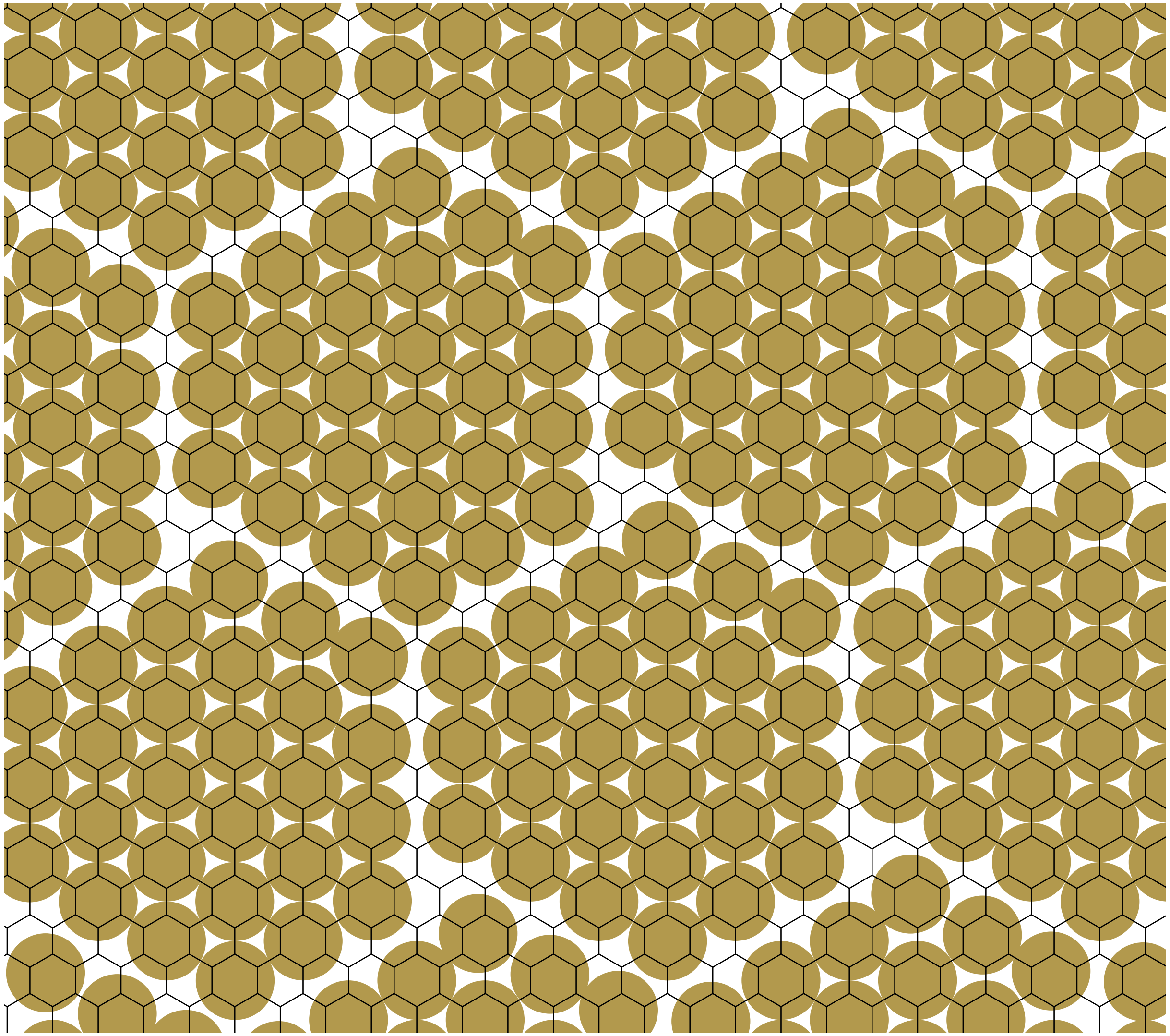
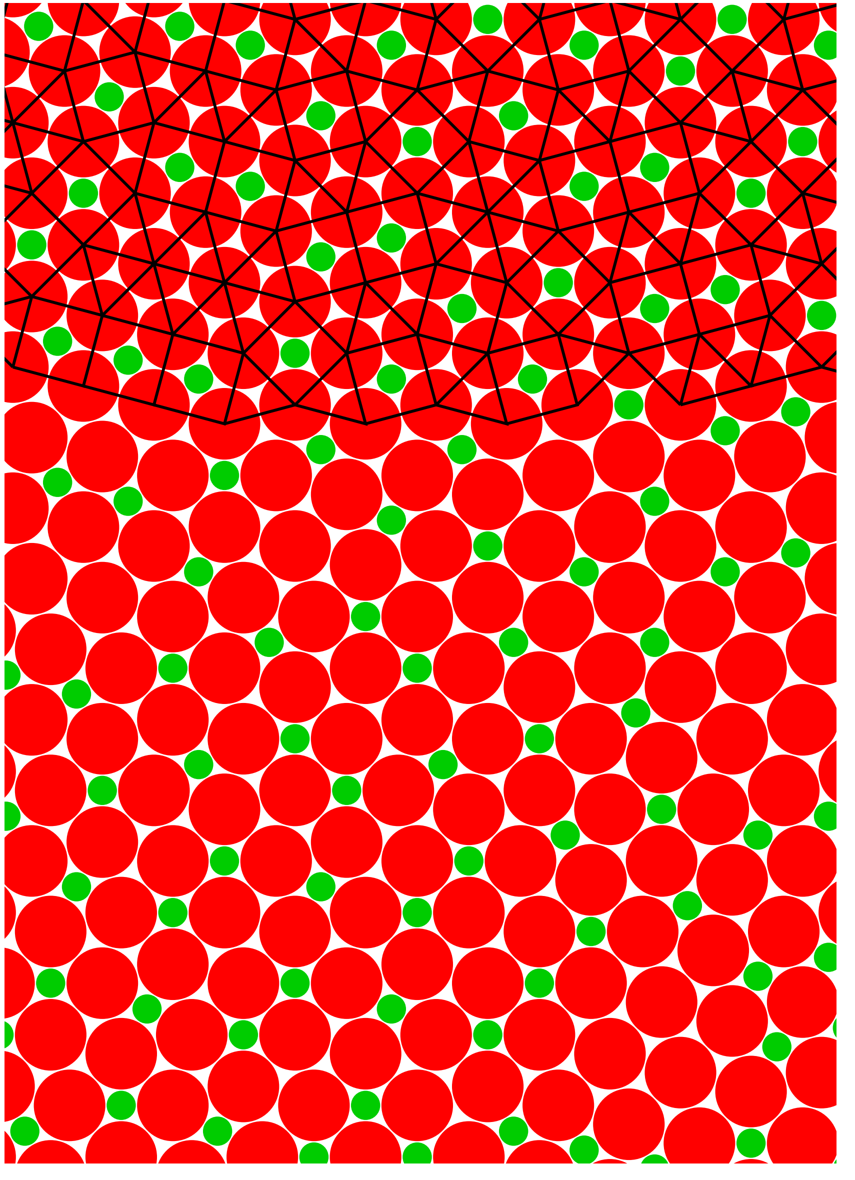
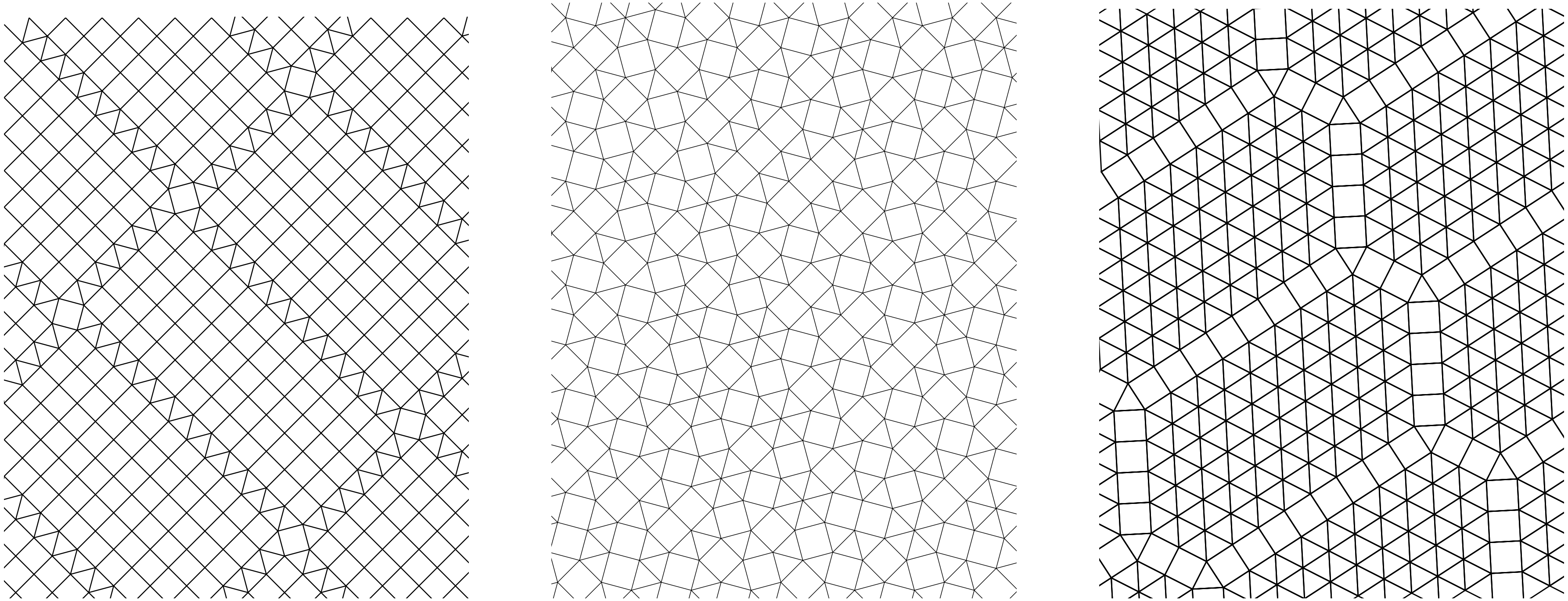


An monolayer of atoms adsorbed on a crystal surface, such that the lattices are incommensurable.
Hexagonal network of domain walls (Villain 1980)

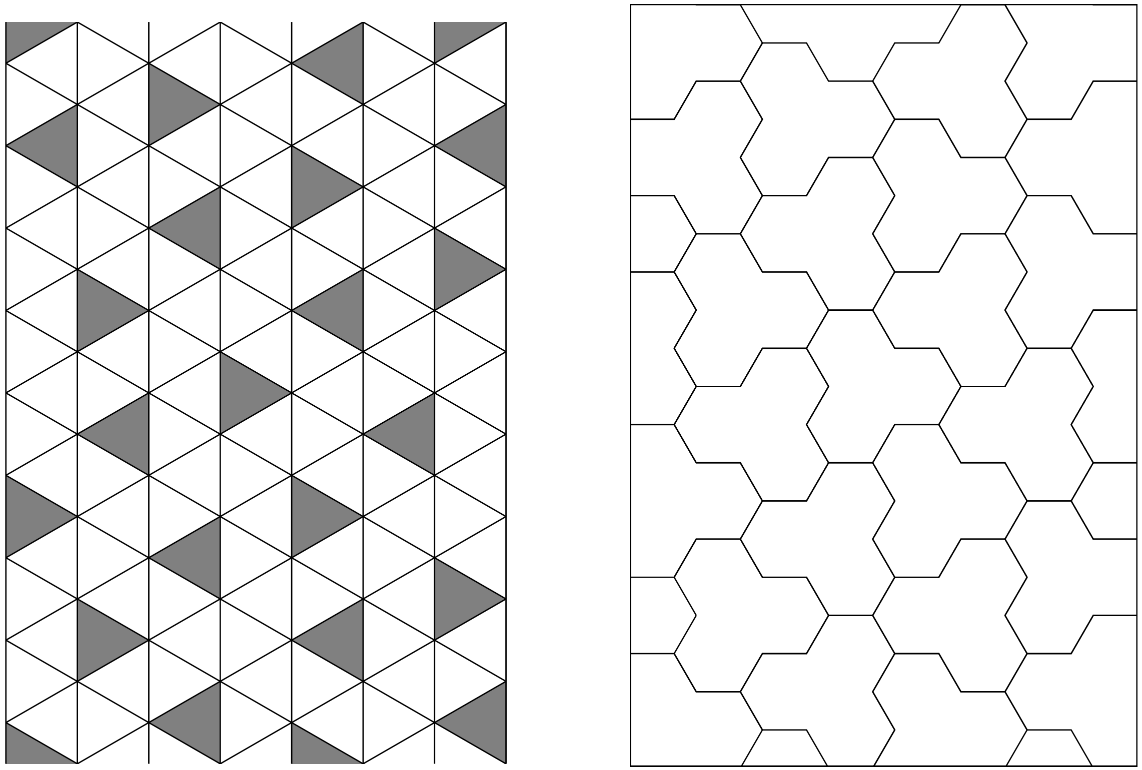
This random packing of disks is equivalent to a tiling of squares and triangles.

(given the appropriate size ratio, and maximal packing density)

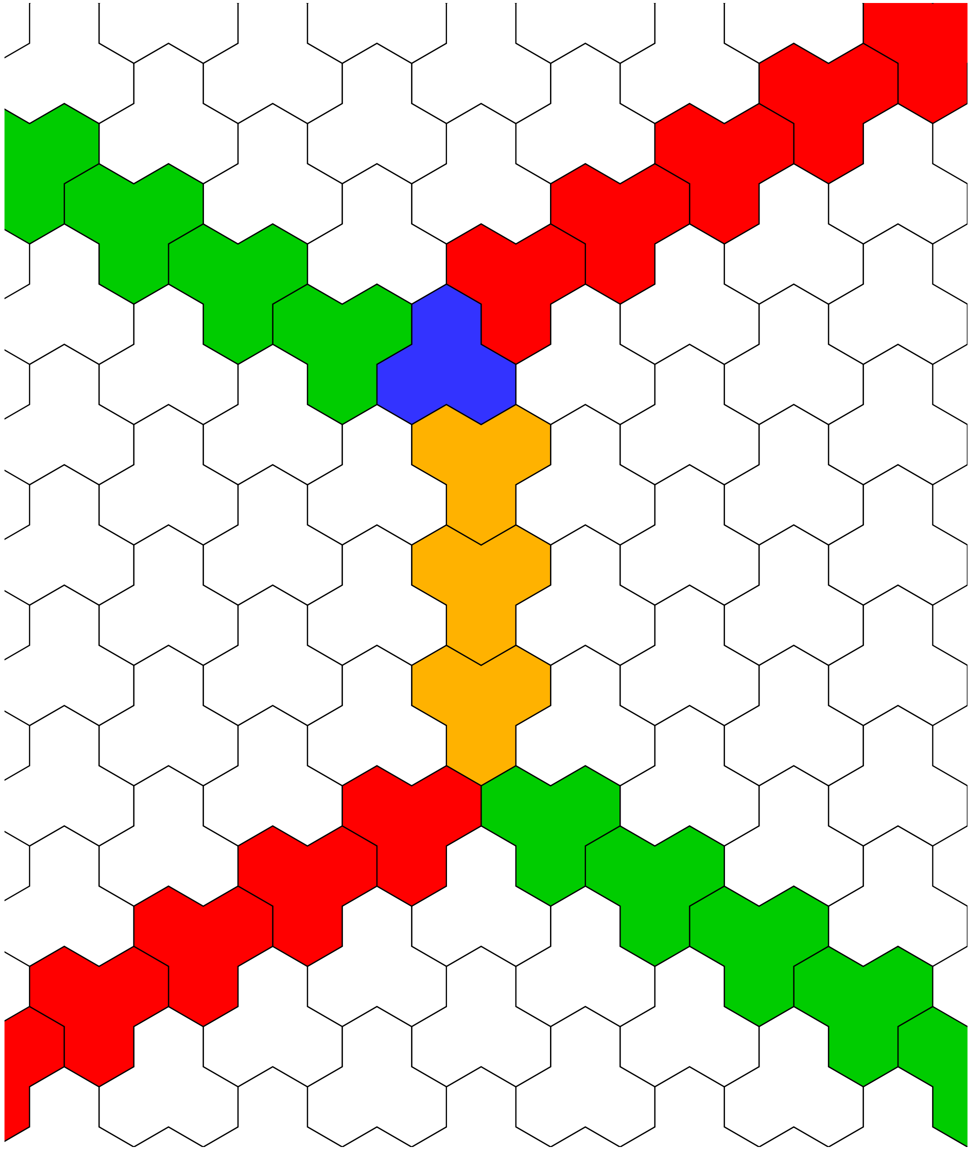




When there are far more triangles than squares, the tiling behaves like a hexagonal net. (Oxborrow, Henley 1993)



In the trimer model, analogous to the dimer model, every site of the triangular lattice part of one triangular trimer. (Verberkmoes, BN 1999)
Equivalently the hexagonal lattice is tiled with 'triominos'.

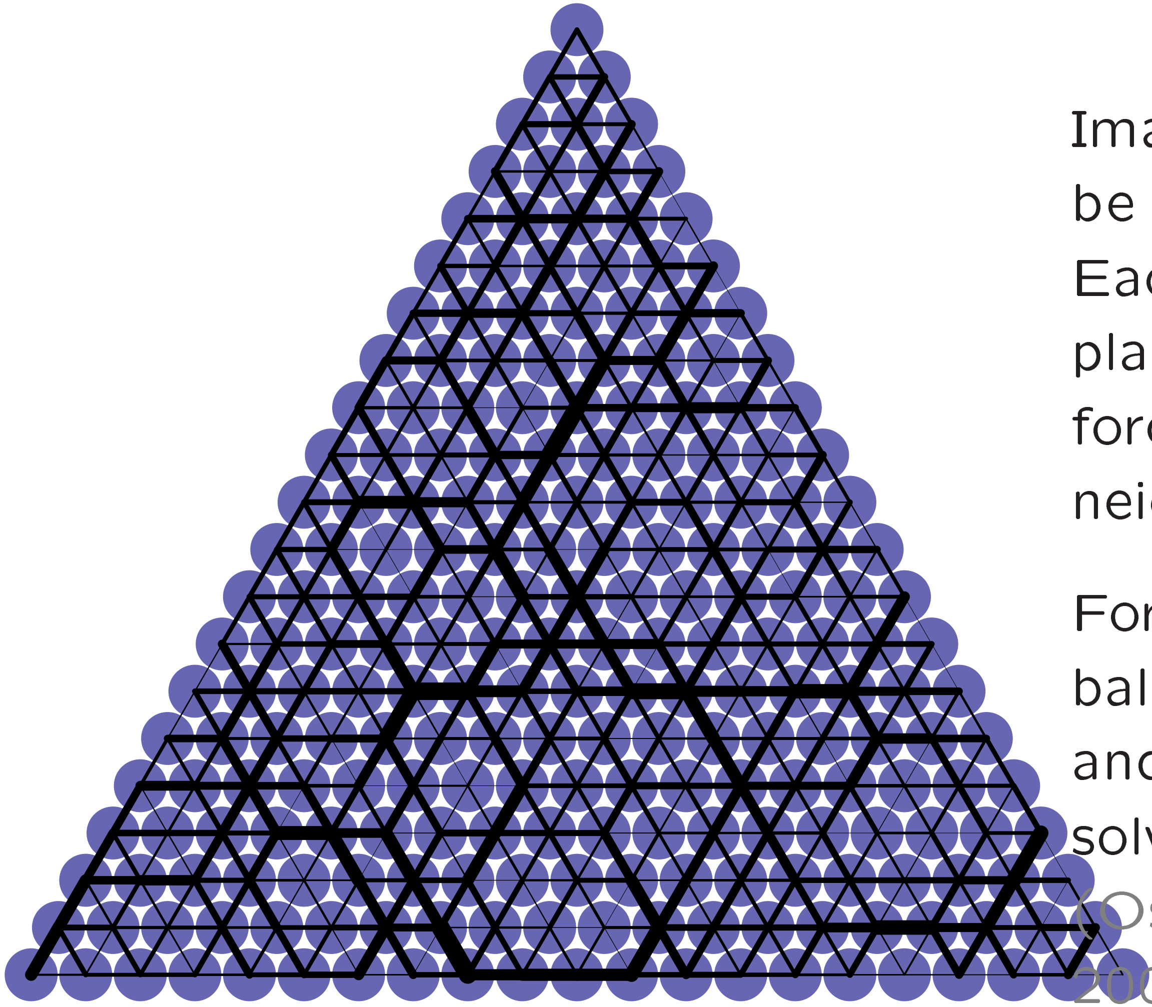


When the trimers predominantly point the same way, the trimers with minority orientation are sorted out along a hexagonal net.

The different colors represent different sublattices.

(The Y vertex comes in two chiralities)

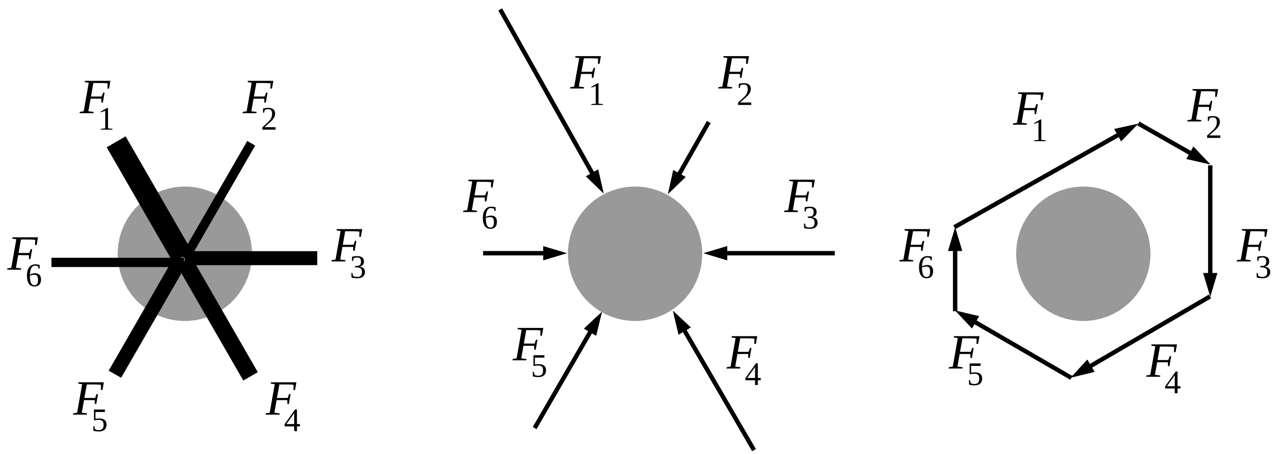
Forces in disk packings



Imagine the disks to be perfectly hard. Each disk is held in place by the contact forces from its six neighbors.

For each disk two balance equations, and three forces to solve them.

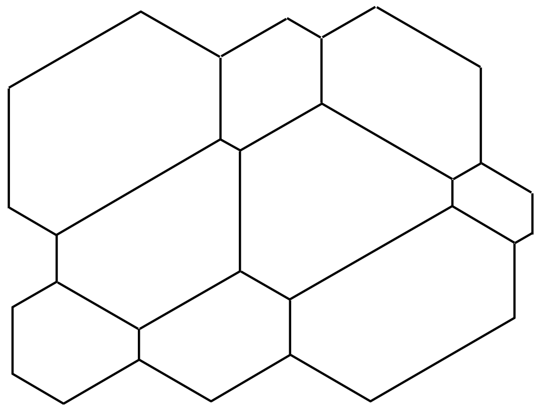
(Ostojic, BN 2005)[2cm]



Represent the forces by linesegments perpendicular to the direction of the force with length proportional to the strength.

Then these linesegments form a closed hexagon.(Maxwell)

Placing these in one figure gives a hexagonal net



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Sums of hermitian matrices

Let λ be a weakly decreasing sequence of real numbers, and $\mathcal{H}_\lambda$ be set of hermitian matrices with eigenvalues λ.

For what λ, μ and ν does

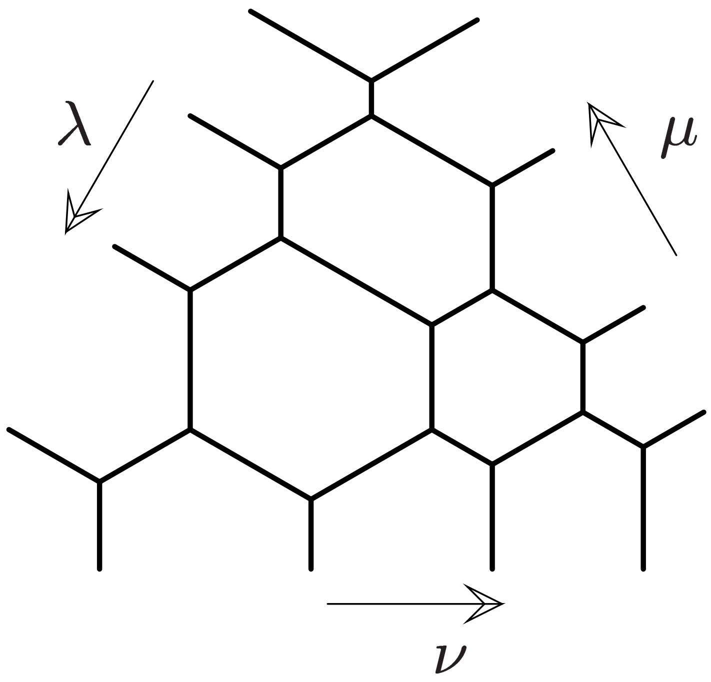
$$M_\lambda + M_\mu + M_\nu = 0 \tag{1}$$

have solutions in $\mathcal{H}_\lambda$, $\mathcal{H}_\mu$ and $\mathcal{H}_\nu$, respectively?

In other words: if we know (only) the spectrum of two hermitian matrices, what does that tell us of the spetrum of their sum?

Horn (1962) conjectured that a set of inequalities (together with the obvious condition on the trace) are sufficient conditions for (1) to have a solution.

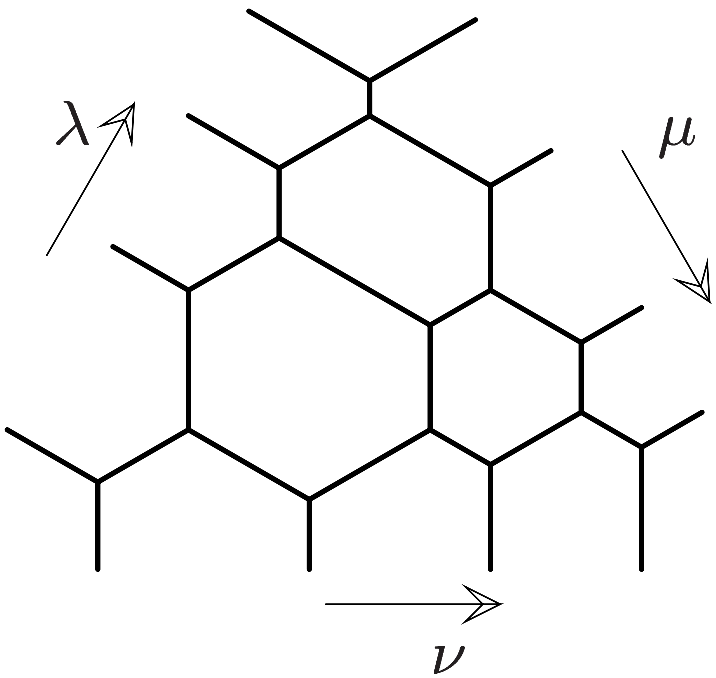
In 2000 Knutson and Tao proved this to be correct, and formulated it in terms of the hexagonal net:

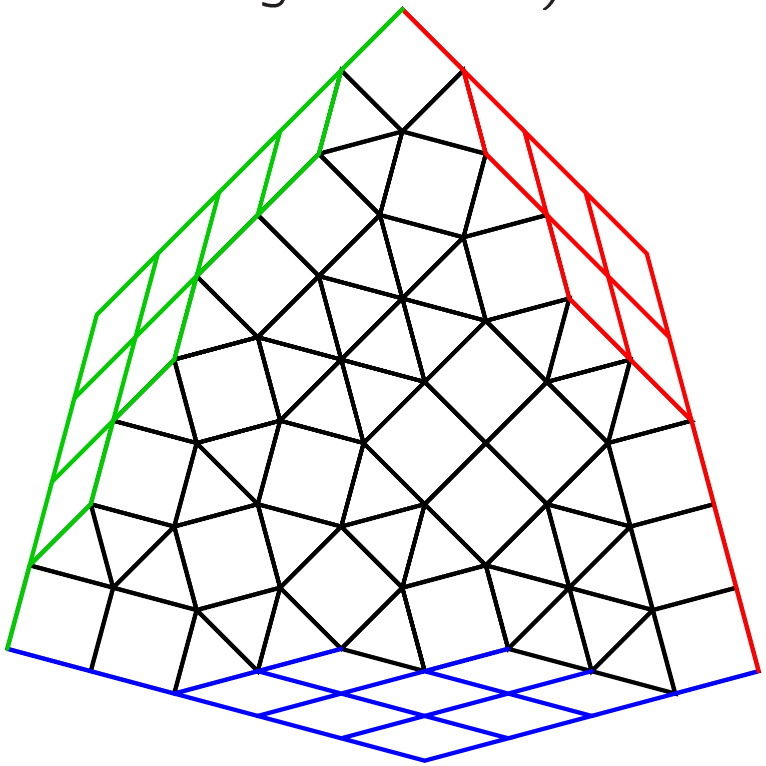


Knutson & Tao: The matrices with spectra λ, μ and ν can add to zero if the spectra can be connected with a hexagonal net.

Even stronger:
with uniform measure on the matrices and on the hexagonal net, the partition function of the hexagonal net measures the relative probability that $-\nu$ is the spectrum of $M_\lambda + M_\mu$.

For integer sequences λ (or equivalent Young tableaux):





Let $\mathcal{V}_\lambda$ be a representation of U(n). Then the number of times that $\mathcal{V}_\nu$ occurs in $\mathcal{V}_\lambda \times \mathcal{V}_\mu$ is given by the number of (integer) hexagonal nets that connect λ, μ and ν.

These Littlewood-Richardson coefficients are also naturally represented as tilings with squares and triangles.
(Zinn-Justin, 2009)

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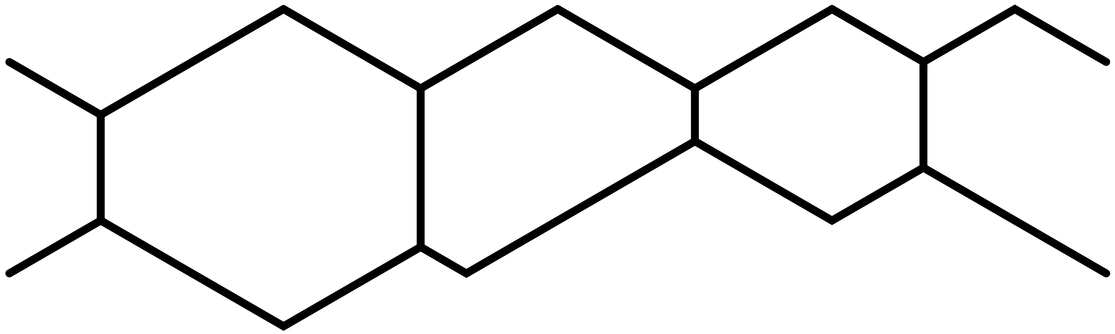
Some calculational tools

Transfer matrix: add one row of disks



Use the Bethe Ansatz to diagonalize the transfer matrix:

The positions of the diagonal lines proceed with plane waves as weight function, between collisions (i.e. events that two lines coincide). On collision the wave numbers may be interchanged.

This suggests that the eigenstates of the transfer matrix can be written as a linear combination of products of plane waves:

$$\Psi(x_1, x_2, \ldots; y_1, y_2, \ldots) = \sum_{\pi,\sigma} A_{\pi,\sigma} \exp\left(\sum_{j=1}^{N} \mathrm{i} x_j p_{\pi(j)} + \mathrm{i} y_j q_{\sigma(j)}\right)$$

This turns out to "work" (i.e. solve the eigenvalue equation). This implies that multiple collisions can be written as a sequence of binary collisions.

The eigenvalue equation $T \cdot \Psi = \Lambda \ \Psi$ results in the following consistency conditions (B.A.eqns) on the wave numbers:

$$\prod_{j=1}^{N} (u_k - v_j \mathrm{e}^{-\mu}) = u_k^{F_\mathrm{u}} \qquad \prod_{k=1}^{N} (u_k - v_j \mathrm{e}^{-\mu}) = v_j^{-F_\mathrm{v}}$$

where $u_k = \exp(\mathrm{i}p_k)$ and $v_j = \exp(-\mathrm{i}q_j)$, and e^{μ} is the weight of one unit vertical line.

In these equations we want to take two (independent) limits:
(i) The discrete hexagonal to the continuum, and
(ii) the number N of hexagons in a row to ∞.

The second limit turns the (logarithm of the) products into integrals.

In the limit we get integral equations:

$$f(z) = -F_{\mathsf{u}} + \int_{\bar\eta}^{\eta} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \, \frac{g(y)}{(z+\mu-y)}$$

$$g(y) = F_{\mathsf{v}} + \int_{\zeta}^{\bar\zeta} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \, \frac{f(z)}{(y-\mu-z)}$$

with the restriction that

$$\int_{\bar\eta}^{\eta} \frac{g(y)\,\mathrm{d}y}{2\pi\mathrm{i}} = \int_{\zeta}^{\bar\zeta} \frac{f(z)\,\mathrm{d}z}{2\pi\mathrm{i}} = 1$$

The entropy per added hexagon is then given by

$$S = \int_{\bar\eta}^{\eta} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \int_{\zeta}^{\bar\zeta} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \log(y-z-\mu)$$

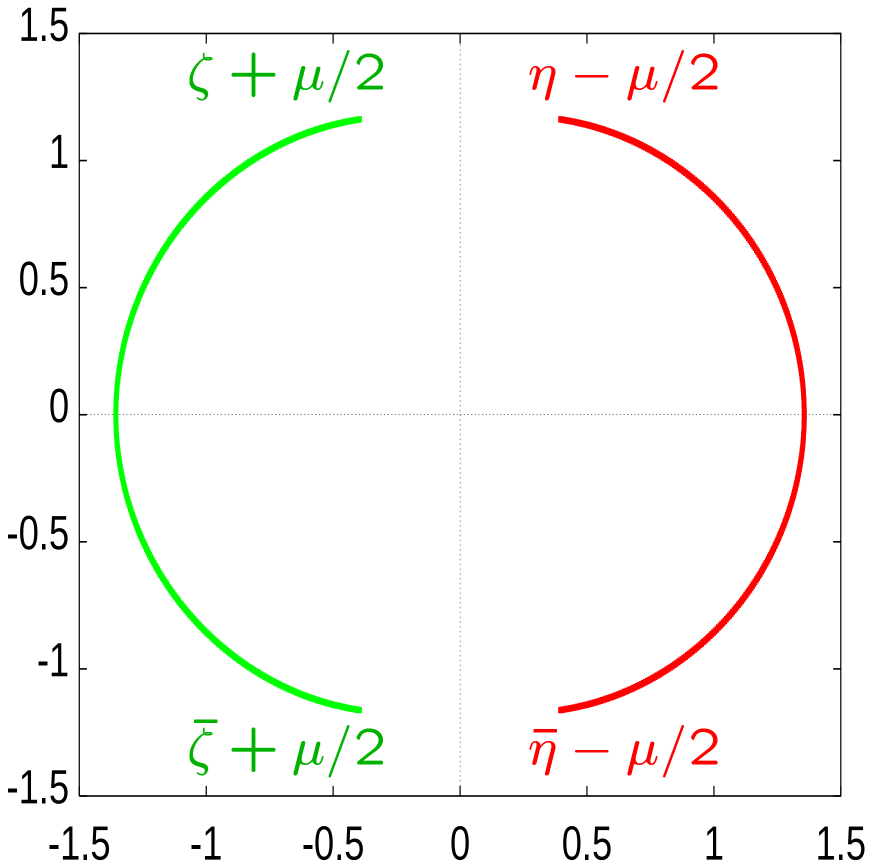
The function f is analytic except in a cut $\bar\eta - \mu \to \eta - \mu$,
and g is analytic except in a cut $\zeta + \mu \to \bar\zeta + \mu$.

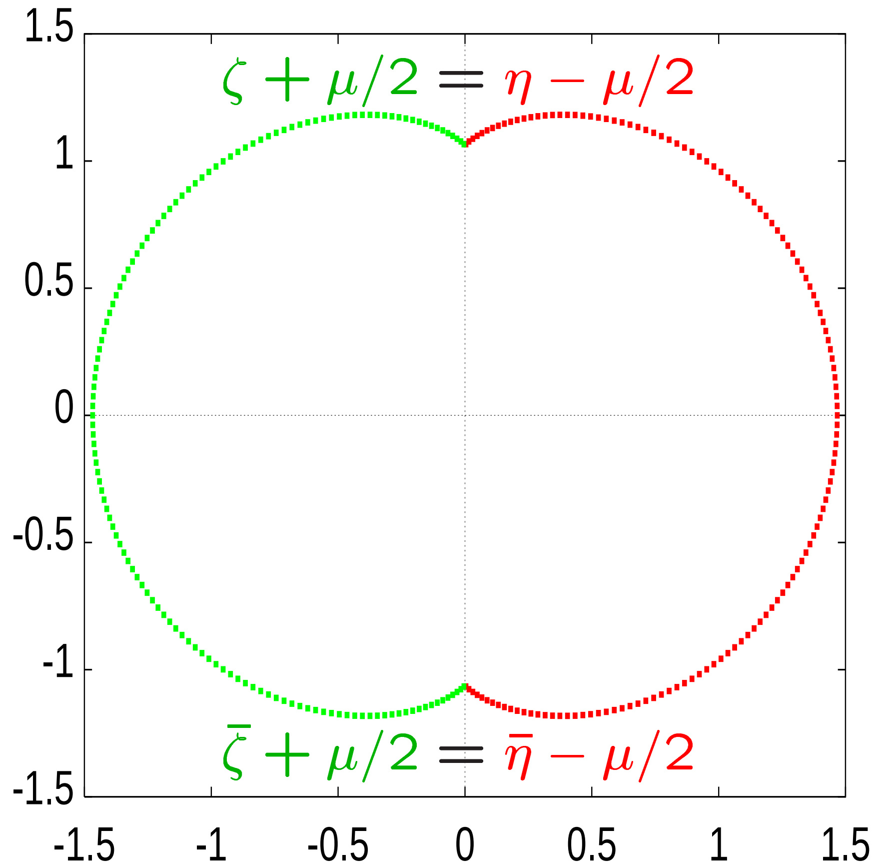
From the integral equations it follows that for ξ on the cut in f:
$f(\xi+\varepsilon) - f(\xi-\varepsilon) = g(\xi+\mu)$

In two limits we can make progress with these equations:

(i) In the limit that the cuts in f and g are very far apart.

(ii) In the special case that the end points of the cuts in $f(z)$ and in $g(z+\mu)$ coincide, the monodromy structure simplifies.





The coincidence of the endpoints of the cuts (one complex equation) requires the symmetry condition $F_{\mathsf{u}} = F_{\mathsf{v}}$: the ascending lines and descending lines are equal on average, and a special tuning of the parameter μ, which controls F_{w} the length of the vertical lines.

This case was solved as follows (Kalugin (1994)):

Passing through both cuts 6 times leaves both f and g invariant This immediately suggests an unfoliation of the Riemann sheets, to find new variables in which the solution is analytic.

On can assume the co-incidence to happen, leaving the special value of μ where this happens, and the position of the endpoint η as unknowns. The known properties of the functions leaves enough equations to solve for these unknowns.

The result for the partition sum per hexagon is

$$\mathrm{e}^{S} = 6F\,\mathrm{exp}(-3/2)$$

with $F = F_{\mathsf{u}} = F_{\mathsf{v}} = F_{\mathsf{w}}$ the mean side length in all three directions.

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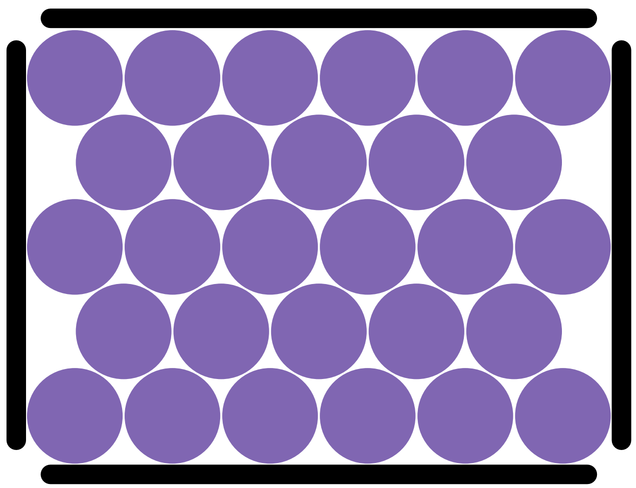
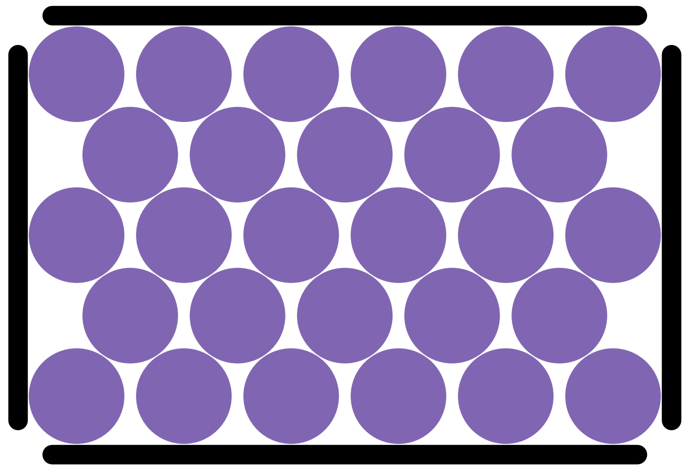
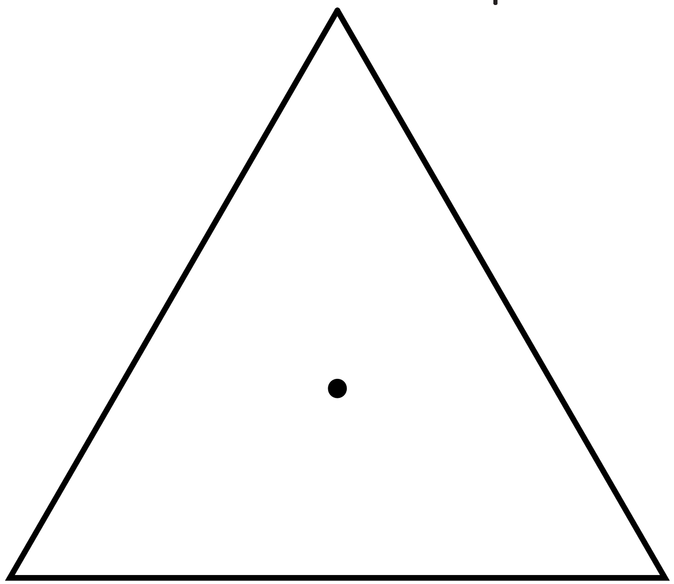
Questions:

- Is the distribution scale invariant
- ... conformally invariant
- partition function
- correlation functions
- nature of the yield transition:





In the continuous case only the ratios of F_{u}, F_{v} and F_{w} matter. There are two free parameters.



In the center we have an exact expression for the partition function. We know that it vanishes at the boundary of the triangle, (where one of the F's vanishes).

$$f(z) = -F_{\mathsf{u}} + \int_{\bar{\eta}}^{\eta} \frac{\mathrm{d}y}{2\pi \mathrm{i}} \frac{g(y)}{(z+\mu-y)}$$

$$g(y) = F_{\mathsf{v}} + \int_{\zeta}^{\bar{\zeta}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{f(z)}{(y-\mu-z)}$$

When $-\mu \gg 1$ the denominator of the integrand is always big, and can be expanded in powers of $1/\mu$. In this limit F_{w} is very small.

This is a tedious process, as the functi ons f and g as well as the integration boundaries η and ζ have to be calculated order by order. But it leads to a consistent series

For compact notation I introduce: $a = \pi F_{\mathsf{u}}^{-1}\mu^{-1}$ and $b = \pi F_{\mathsf{v}}^{-1}\mu^{-1}$.

In that notation:

$$S = -\log(-\mu) - \frac{1}{6}(a^2 + b^2) - \frac{1}{2\pi}(a^3 + b^3) + \frac{1}{20}(a^4 + b^4) + \frac{1}{6}a^2b^2 +$$

$$\frac{5}{12\pi}(a^5+b^5)+\frac{2}{3\pi}a^2b^2(a+b)+\frac{9}{4\pi^2}a^3b^3-\frac{1}{42}(a^6+b^6)-\frac{1}{6}a^2b^2(a^2+b^2)-$$

$$\frac{7}{20\pi}(a^7 + b^7) - \frac{17}{12\pi}a^2b^2(a^3 + b^3) - \frac{23}{30\pi}a^3b^3(a + b) + \cdots$$

so far up to 10 orders. Conversion from $S(F_{\mathsf{u}}, F_{\mathsf{v}}, \mu)$ to $S(F_{\mathsf{u}}, F_{\mathsf{v}}, F_{\mathsf{w}})$ is still needed, but does not seem to improve the esthetics.

What is surprising is that the partition sum can be expanded like this near an instability.

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Scale- and conformal invariance

In a conformally invariant system one finds

$$\Lambda_j = \exp(SN + v_{\mathsf{F}}\frac{c}{6N} - v_{\mathsf{F}}\frac{2\pi\Delta_j}{N} + o(1/N))$$

Here S is the bulk free energy and Δ_j gives the dominant size dependence, and v_{F} (Fermi velocity) is a geometric factor relating the scales in the 'time-like' and 'space-like' direction.

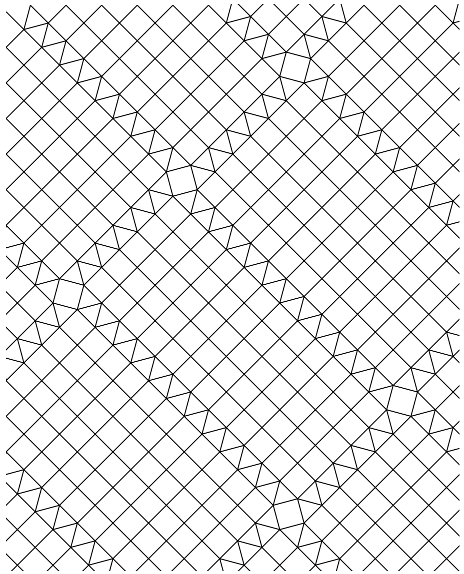
The fact that the gaps are inversely proportional to the (linear) system size clearly indicates scale invariance: these corrections play the role of inverse correlation lengths. If a correlation length is proportional to the linear system size there is apparently no intrinsic scale.

A typical character of conformally invariant systems is that there are many integer valued Δ_j or $\Delta_j - \Delta_k$, the signature of the Virasoro algebra.

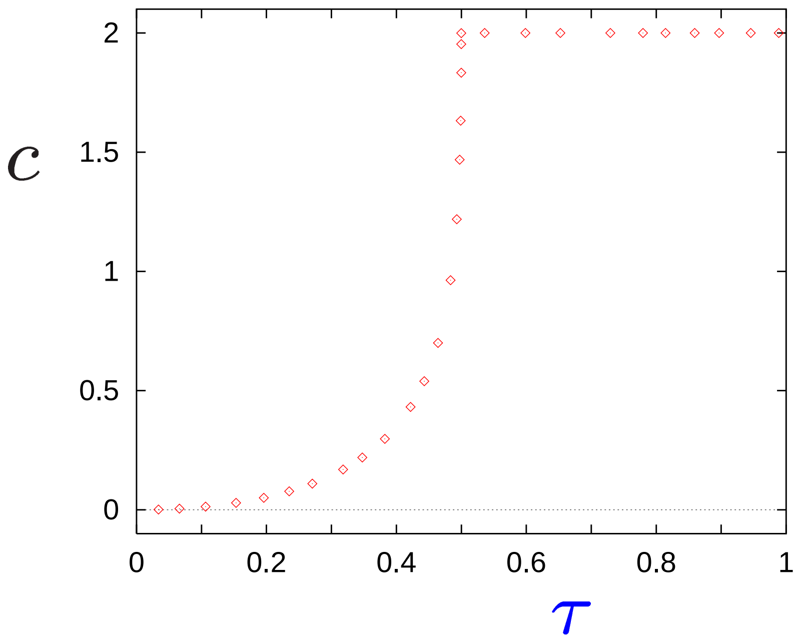
What do we find in the hexagonal net?

In the isotropic point the spectrum has all these characteristics, and a central charge $c = 2$.

In an anisotropic hexagonal net, one may need to find v_F empirically, (rather than by a priori geometric arguments). To avoid this complication we look at the square-triangle tiling in the square phase.



This is equivalent to a phase with stacks of elongated hexagons, but in the square triangle tiling the symmetry is more obvious.



The effective central charge as function of the triangle density. In the right half the triangles dominate and form an isotropic hexagonal net. In the left half the squares dominate and form a fluctuating square net of domain walls.

In the spectrum we find many instances with $|\Delta_j - \Delta_k| = 1$, but with complex values of Δ.

These findings can be explained as follows: The diagonally running domain walls form two families of fermions. The B.A. equations show that they interact only with the members of the other family.

The pictures suggest that the space-time of each species of fermions is elongated in the direction of the corresponding domain wall.

The characteristic space-like scale is the inverse density.
The characteristic time-like scale is the time between collisions.

Because the space-time stretching is different for the two families of fermions, it can not be corrected with a v_{F} or a tilt in the transfer matrix.

But the theory gives explains precisely the relation between the value of the effective central charge, and the imaginary part of Δ_j.

Also it predicts that the effective central charge is larger than 2, when the transfer matrix is oriented axially w.r.t. the domain walls.

- many hexagonal nets are seen in physics
- they are a useful tool in mathematics
- Satisfy Bethe Ansatz, but not always 'solvable'
- Yield transition accessible analytically
- Strange conformal behavior