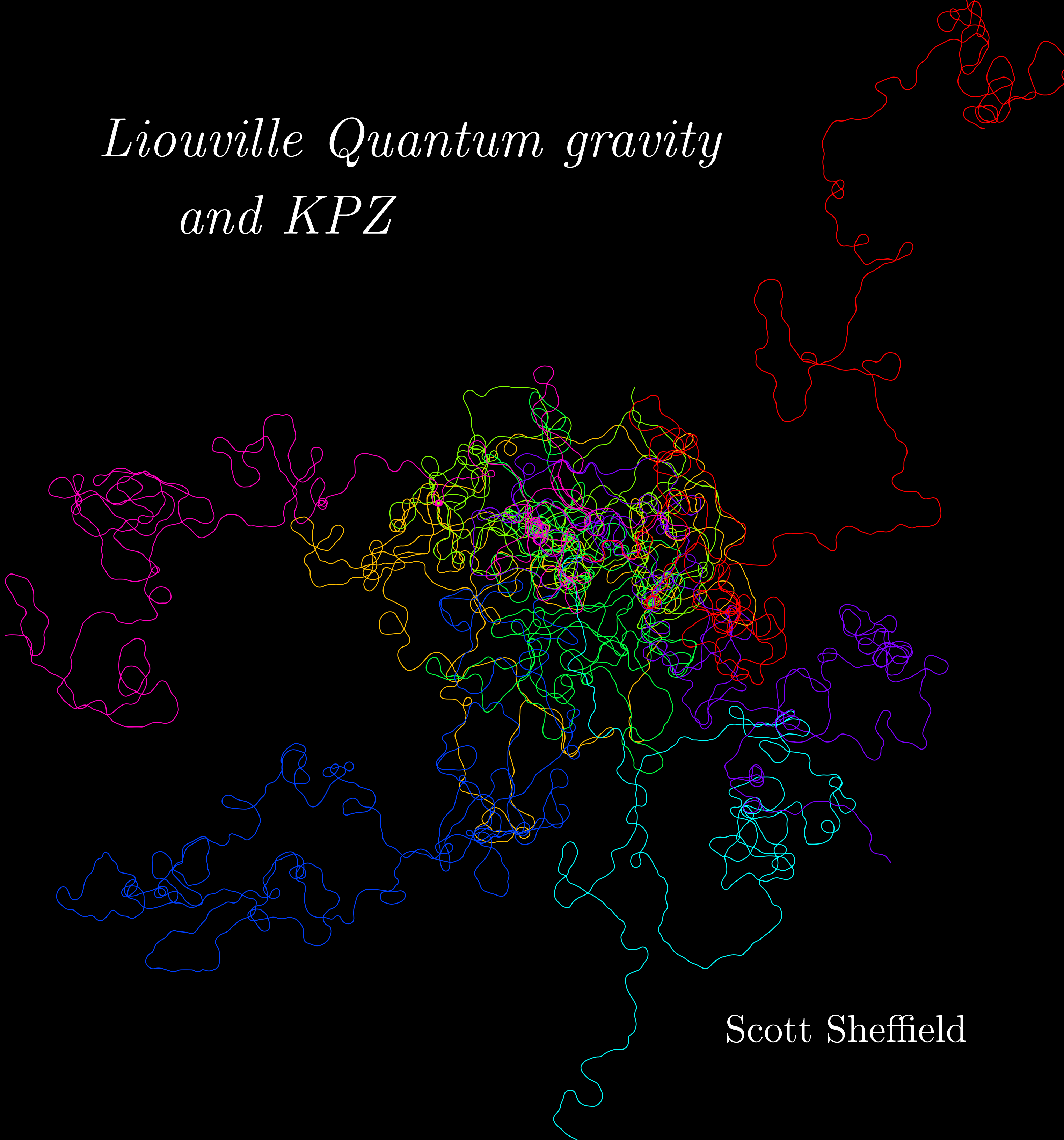
Liouville Quantum gravity and KPZ



Scott Sheffield

Scaling limits of random planar maps

Central mathematical puzzle: Show that the scaling limit of some kind of discrete quantum gravity (perhaps decorated by random loops, random trees, etc.) is some kind of continuum quantum gravity (perhaps decorated by continuum random loops, trees, etc.).

Problem: What is meant by “scaling limit”? What is meant by “continuum quantum gravity”? What is meant by “continuum random loops”?

Scaling limits of random planar maps

General idea: Let Γ be the union (over all lattice mesh sizes) of the discrete configuration spaces. Put some metric or topology on Γ. Let $\overline{\Gamma}$ be the completion of Γ w.r.t. that topology. Prove that random lattice configurations (viewed as random elements of Γ) converge in law to random elements of $\overline{\Gamma}$.

Analogy: scaling limits of random curves

Ising model configurations (see work of Smirnov plus Chelkak, Hongler, Kempainnen, etc.): How do we make sense of the term "scaling limit of the Ising model"?

Again, first consider some topology on discrete configuration space. Then prove convergence in law (w.r.t. that topology) to a a limiting law (a measure on the completion of configuration space w.r.t. that topology).

Topology 1: Two configurations are close if completed chordal interfaces are close in the Hausdorff topology. **Scaling limit:** A random closed set.

Topology 2: Two configurations are close if driving functions are close (e.g., in metric of L^∞ convergence on compact time intervals). **Scaling limit:** A random continuous driving function.

Topology 3: Two configurations are close if chordal interfaces are close in Fréchet distance. From wikipedia: "The Fréchet distance between two curves is the minimum length of a leash required to connect a dog and its owner as they walk without backtracking along their respective curves from one endpoint to the other." **Scaling limit:** A random continuous curve.

Topology 4: Two configurations are close if chordal interfaces with (normalized) graph-length parameterization are close as parameterized curves. **Scaling limit (conjectural):** A random parameterized curve. (Lawler, S.)

Topology 5: Two configurations are close if entire loop configurations are (in some sense) close. **Scaling limit:** A random collection loops.

Topology 6: Two configurations are close if normalized signed measures induced by plus minus spins are close. **Scaling limit:** A random distribution. (In case of percolation, this is white noise.)

Discretized random surface conjectures

Topology 1: Two manifolds are close if they are close metric spaces in Gromov-Hausdorff sense. **Scaling limit:** A random metric space (recall talks of Le Gall, Miermont).

Topology 2: Two (triply-marked, topologically spherical) manifolds are close if conformal pull back of area measures to the sphere are close (e.g., in topology of weak convergence of measures on the sphere). **Scaling limit:** A random measure on the sphere.

Topology 3: Two (triply-marked, topologically spherical) loop-decorated manifolds are close if conformal images of loops in sphere are close. **Scaling limit:** A random loop ensemble on the sphere.

Topology 4: Two discretized loop-decorated surfaces are close if their "discrete driving functions" are close. **Scaling limit:** A random continuous path in the plane (or a random pair of continuum trees).

Hamburgers and cheeseburgers

Driving function topology: An actual convergence result can be obtained using a bijection with a model for inventory accumulation at a LIFO retailer with two products. (In case of a fixed graph, this is equivalent to the Bernardi bijection, which can be viewed as assigning a kind of exploration tree to a loop decorated surface.)

Conformal welding and independence

It is often the case in discrete quantum gravity that once you condition on the length of an interface, the random surfaces on the two sides of the interface are independent of one another. This has an analog in Liouville quantum gravity. If you conformally weld together two Liouville quantum surfaces (with free boundary conditions) the interface becomes a form of SLE.

Riemann uniformization theorem

Uniformization Every smooth simply connected Riemannian manifold $\mathcal{M}$ can be conformally mapped to either the unit disc $\mathbb{D}$, the complex plane $\mathbb{C}$, or the complex sphere $\mathbb{C} \cup \{\infty\}$.

Isothermal coordinates: $\mathcal{M}$ can be parameterized by points $z = x + iy$ in one of these spaces in such a way that the metric takes the form $e^{\lambda(z)}(dx^2 + dy^2)$ for some real-valued function λ. The (x, y) are called *isothermal coordinates* or *isothermal parameters* for $\mathcal{M}$.

Write D for the parameter space and suppose D is a simply connected bounded subdomain of $\mathbb{C}$ (which is conformally equivalent to $\mathbb{D}$ by the Riemann mapping theorem).

Isothermal coordinates

LENGTH of path in $\mathcal{M}$ parameterized by a smooth path P in D is $\int_P e^{\lambda(s)/2} ds$, where ds is the Euclidean length measure on D.

AREA of subset of $\mathcal{M}$ parameterized by a measurable subset A of D is $\int_A e^{\lambda(z)} dz$, where dz is Lebesgue measure on D.

GAUSSIAN CURVATURE DENSITY in D is $-\Delta\lambda$, i.e., if A is a measurable subset of the D, then the integral of the Gaussian curvature with respect to the portion of $\mathcal{M}$ parameterized by A is $\int_A -\Delta\lambda(z) dz$.

“There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics. Elementary excitations in gauge theories are formed by the flux lines (closed in the absence of charges) and the time development of these lines forms the world surfaces. All transition amplitude are given by the sums over all possible surfaces with fixed boundary.”

A.M. Polyakov, Moscow 1981

The *standard Gaussian* on n-dimensional Hilbert space

has density function $e^{-(v,v)/2}$ (times an appropriate constant). We can write a sample from this distribution as

$$\sum_{i=1}^{n} \alpha_i v_i$$

where the v_i are an orthonormal basis for $\mathbb{R}^n$ under the given inner product, and the α_i are mean zero, unit variance Gaussians.

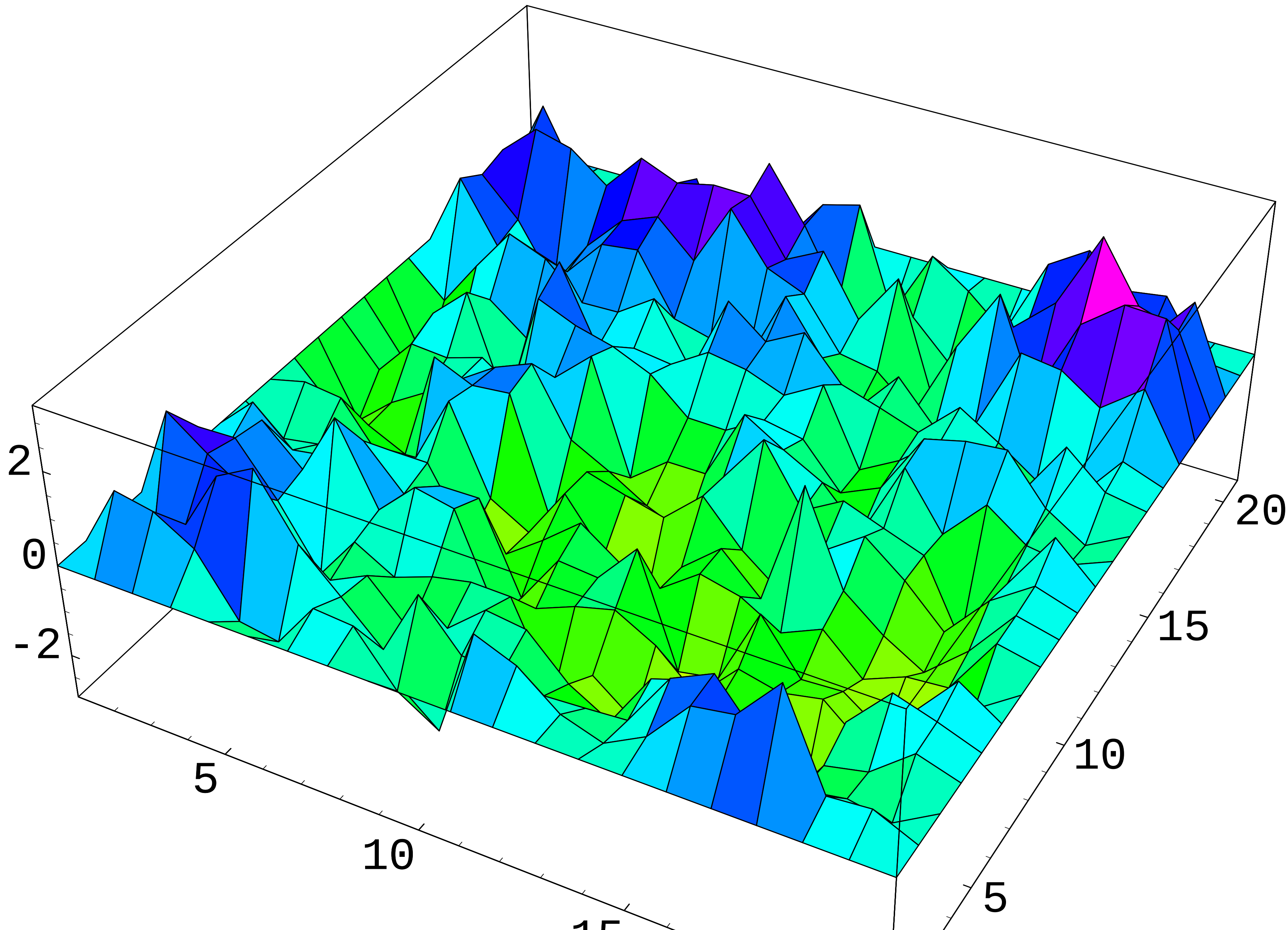
The discrete Gaussian free field

Let f and g be real functions defined on the vertices of a planar graph Λ. The **Dirichlet inner product** of f and g is given by

$$(f,g)_\nabla = \sum_{x \sim y} (f(x) - f(y))\,(g(x) - g(y))\,.$$

The value $H(f) = (f,f)_\nabla$ is called the **Dirichlet energy of** f**.** Fix a function f_0 on boundary vertices of Λ. The set of functions f that agree with f_0 is isomorphic to $\mathbb{R}^n$, where n is the number of interior vertices. The **discrete Gaussian free field** is a random element of this space with probability density proportional to $e^{-H(f)/2}$.

Discrete GFF on 20×20 grid, zero boundary



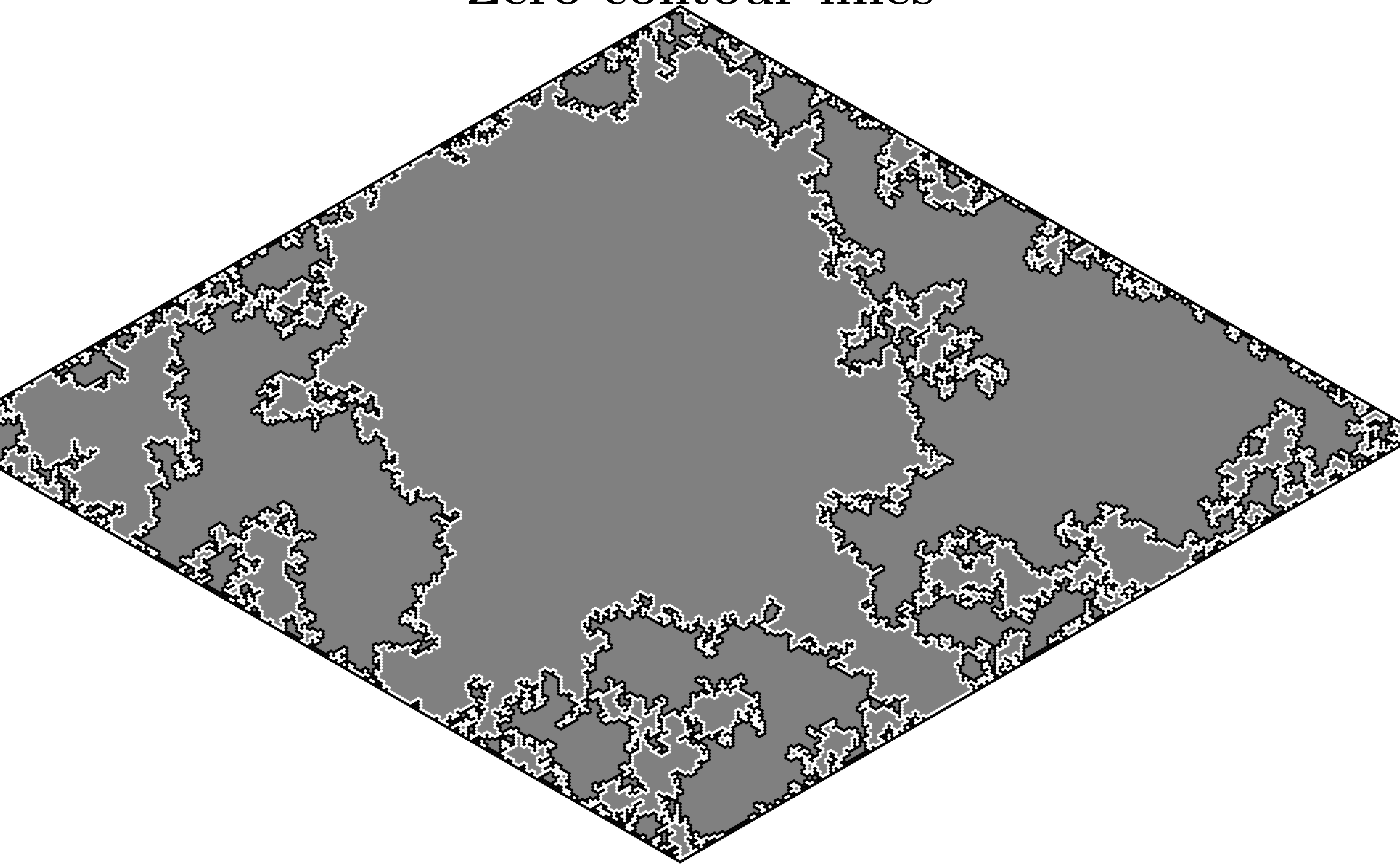
The continuum Gaussian free field

is a "standard Gaussian" on an *infinite* dimensional Hilbert space. Given a planar domain D, let $H(D)$ be the Hilbert space closure of the set of smooth, compactly supported functions on D under the conformally invariant *Dirichlet inner product*

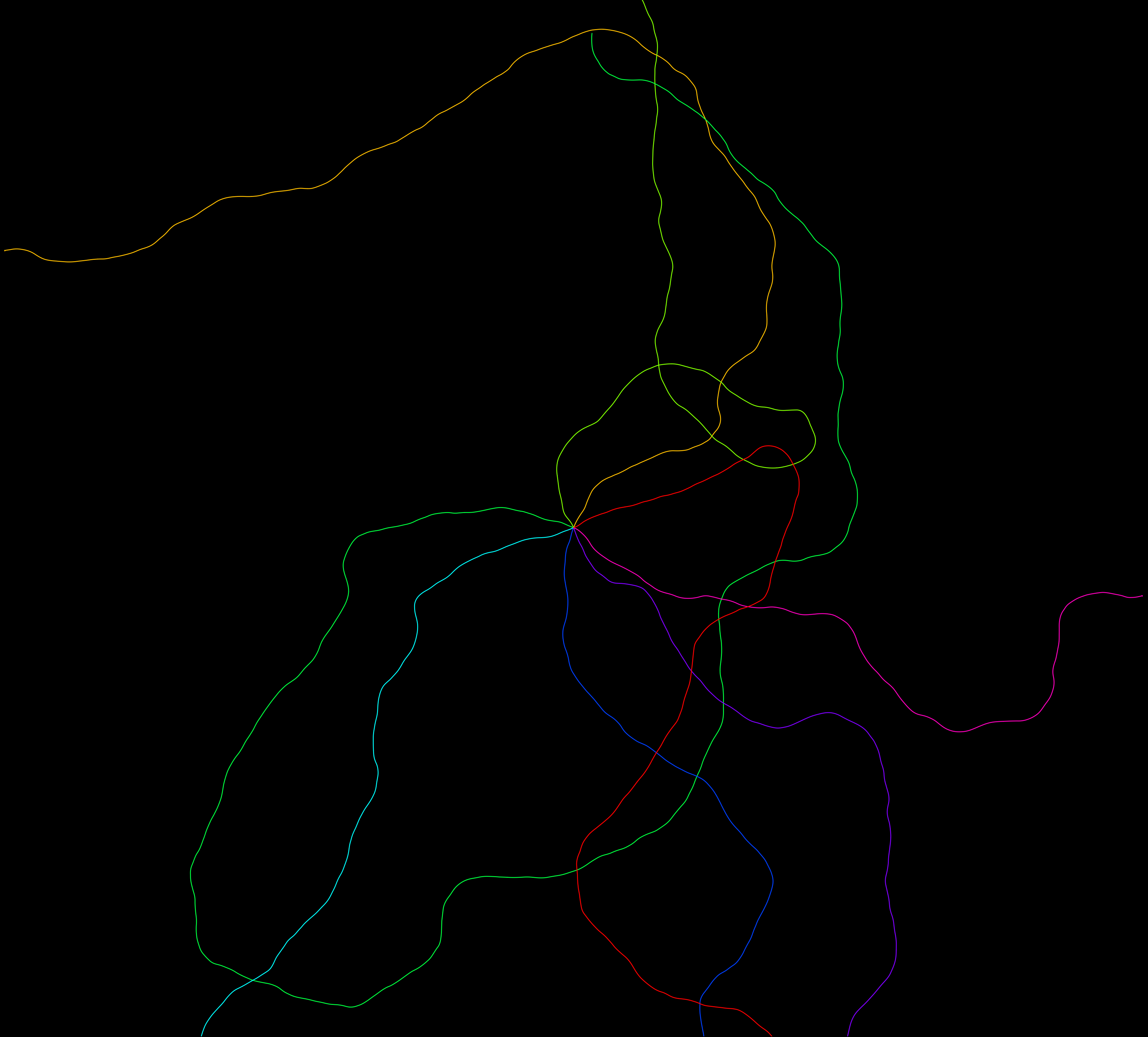
$$(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2) dx dy.$$

The GFF is the formal sum $h = \sum \alpha_i f_i$, where the f_i are an orthonormal basis for H and the α_i are i.i.d. Gaussians. The sum does not converge point-wise, but h can be defined as a *random distribution*—inner products (h, ϕ) are well defined whenever ϕ is sufficiently smooth.

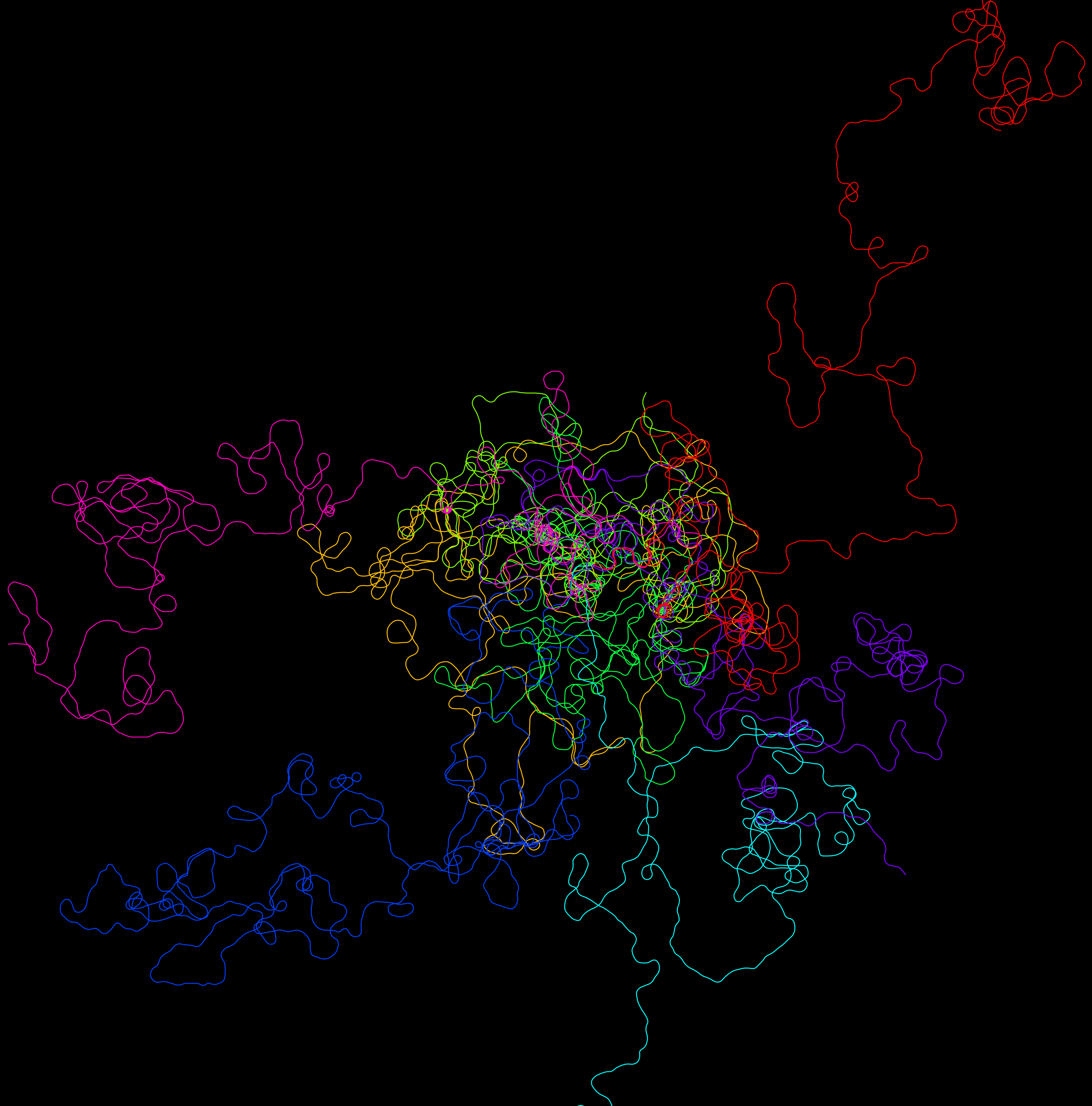
Zero contour lines



Geodesics flows of metric $e^h dL$ where h is .05 times the GFF.



Geodesics flows of metric $e^h dL$ where h is .2 times the GFF.



Geodesics flows of metric $e^h dL$ where h is 1 times the GFF.

“The gravitational action we are going to discuss has the form

$$S = \frac{d}{96\pi} \int_M (R \frac{1}{\Delta} R).$$

Here d^{-1} will play the role of coupling constant, Δ is a Laplacian in the metric g_{ab}, R is a scalar curvature and M is a manifold in consideration. This action is naturally induced by massless particles and appears in the string functional integral.
The most simple form this formula takes is in the conformal gauge, where $g_{ab} = e^{\phi}\delta_{ab}$ where it becomes a free field action. Unfortunately this simplicity is an illusion. We have to set a cut-off in quantizing this theory, such that it is compatible with general covariance. Generally, it is not clear how to do this. For that reason, we take a different approach...”

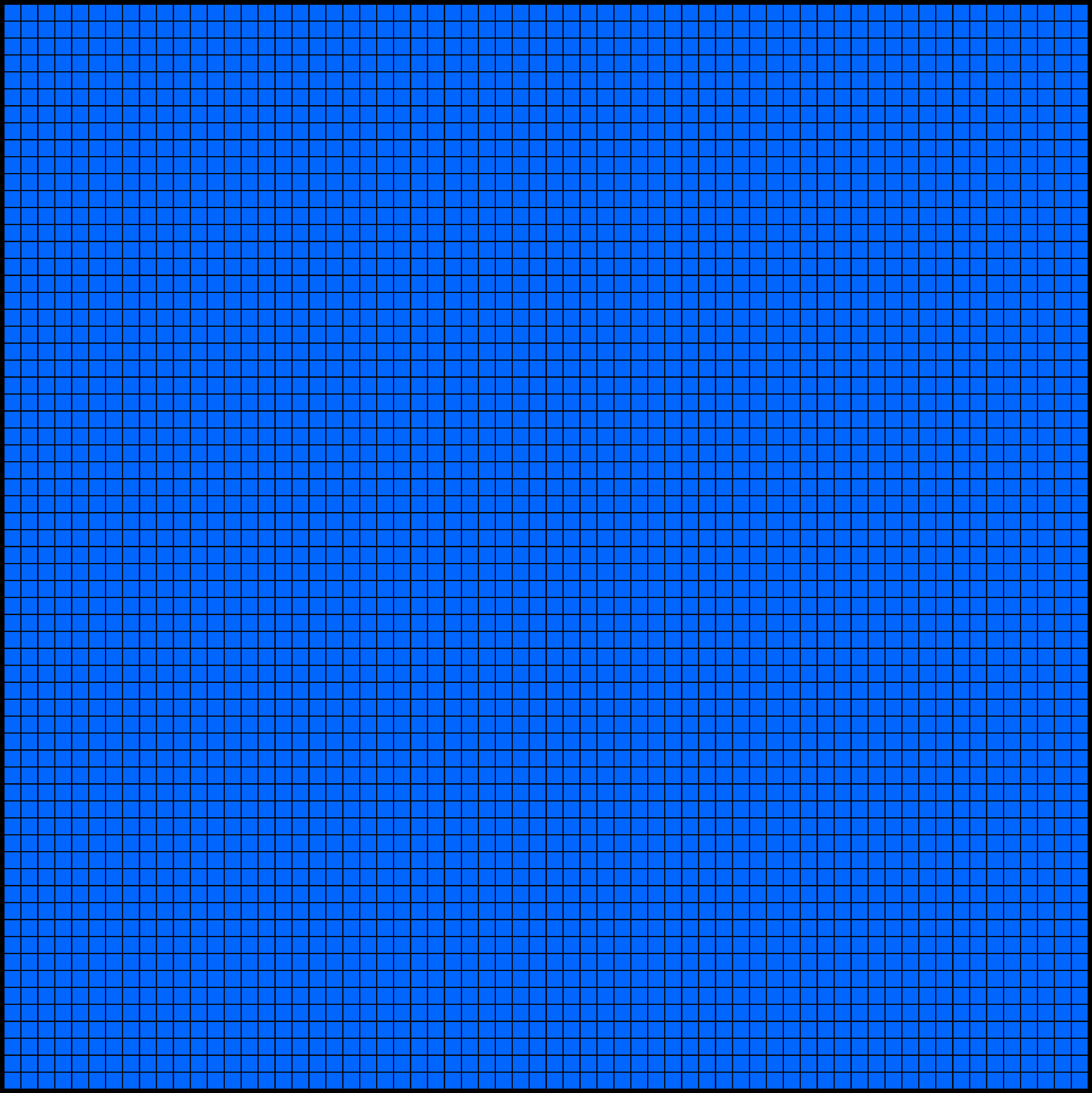
A.M. Polyakov, Moscow 1987

Constructing the random metric

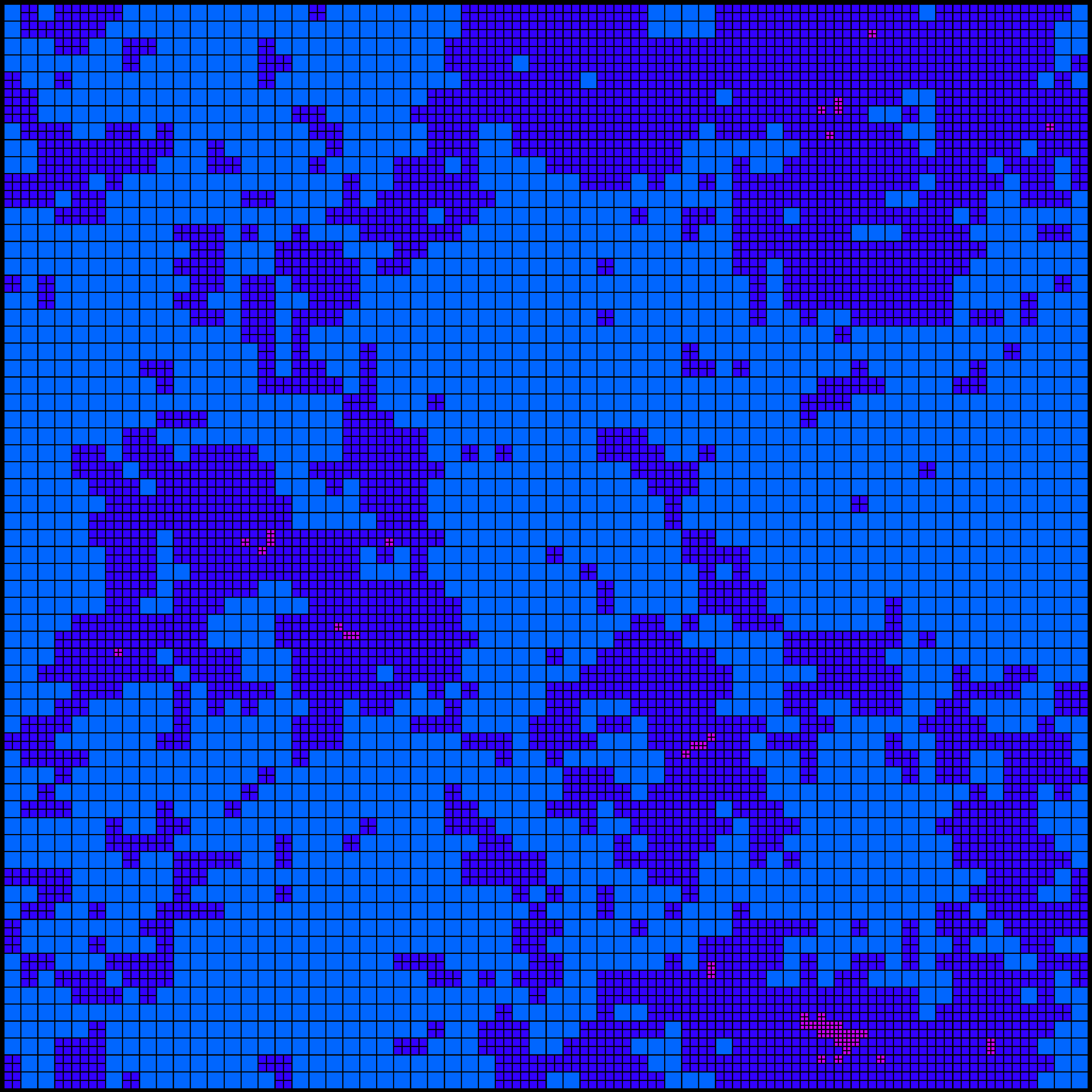
Let $h_\epsilon(z)$ denote the mean value of h on the circle of radius ϵ centered at z. This is almost surely a locally Hölder continuous function of (ϵ, z) on $(0, \infty) \times D$. For each fixed ϵ, consider the surface $\mathcal{M}_\epsilon$ parameterized by D with metric $e^{\gamma h_\epsilon(z)}(dx^2 + dy^2)$.

We define $\mathcal{M} = \lim_{\epsilon \to 0} \mathcal{M}_\epsilon$, but what does that mean?

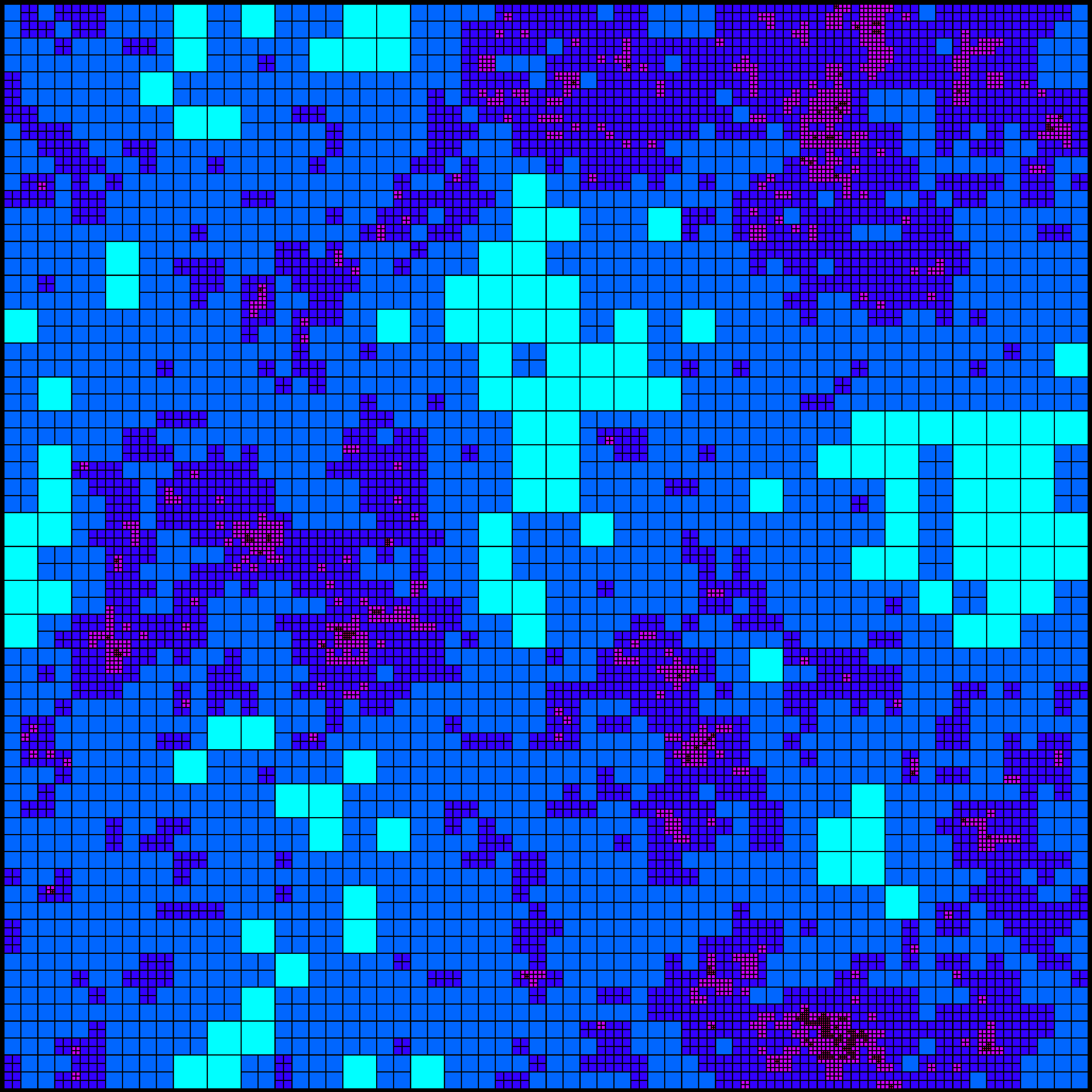
PROPOSITION: Fix $\gamma \in [0, 2)$ and define h, D, and μ_ϵ as above. Then it is almost surely the case that as $\epsilon \to 0$ along powers of two, the measures $\mu_\epsilon := \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} dz$ converge weakly to a non-trivial limiting measure, which we denote by $\mu = \mu_h = e^{\gamma h(z)} dz$.



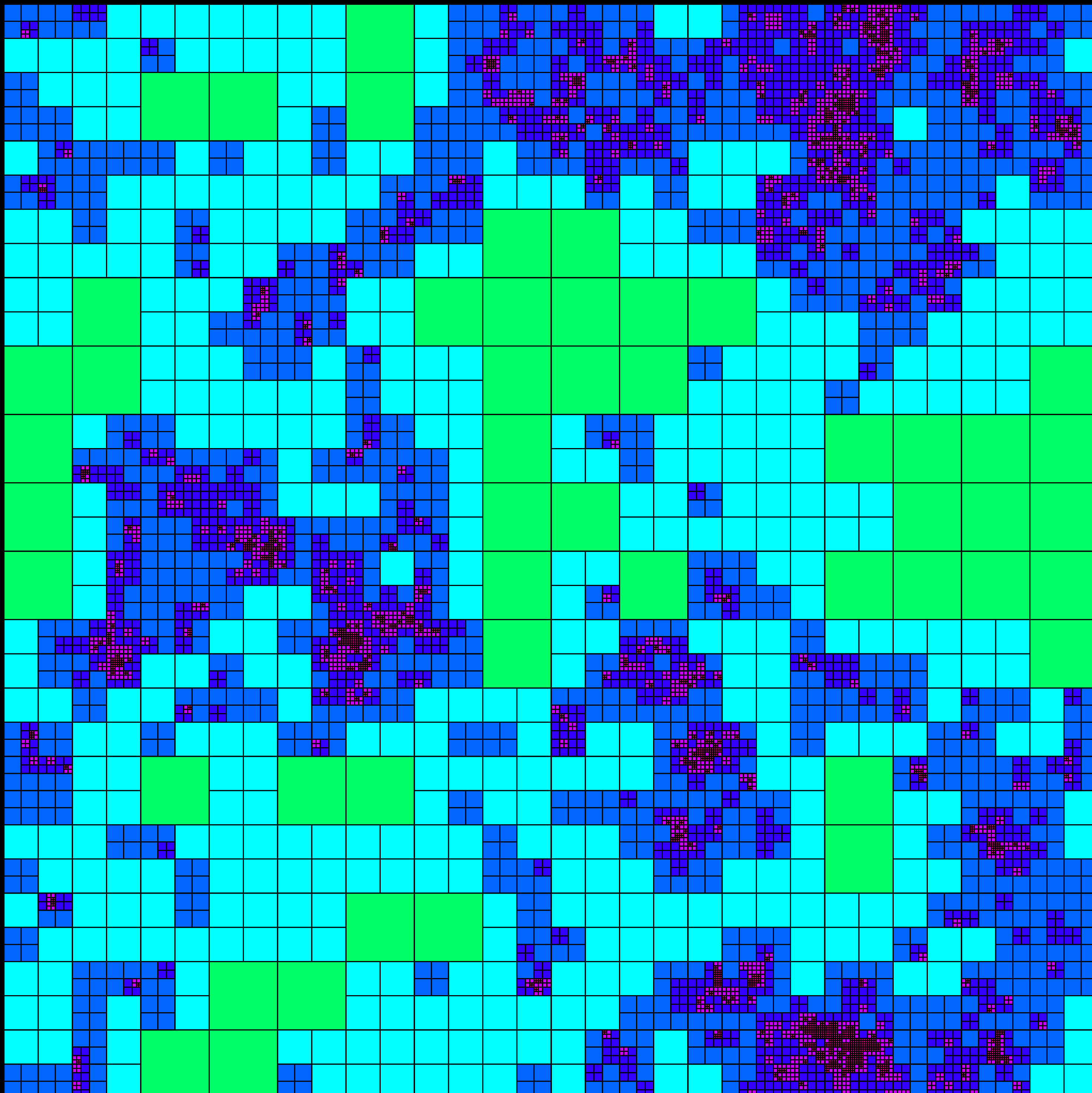
Area/4096 square decomposition of $e^{\gamma h} d^2 z$ for $\gamma = 0$



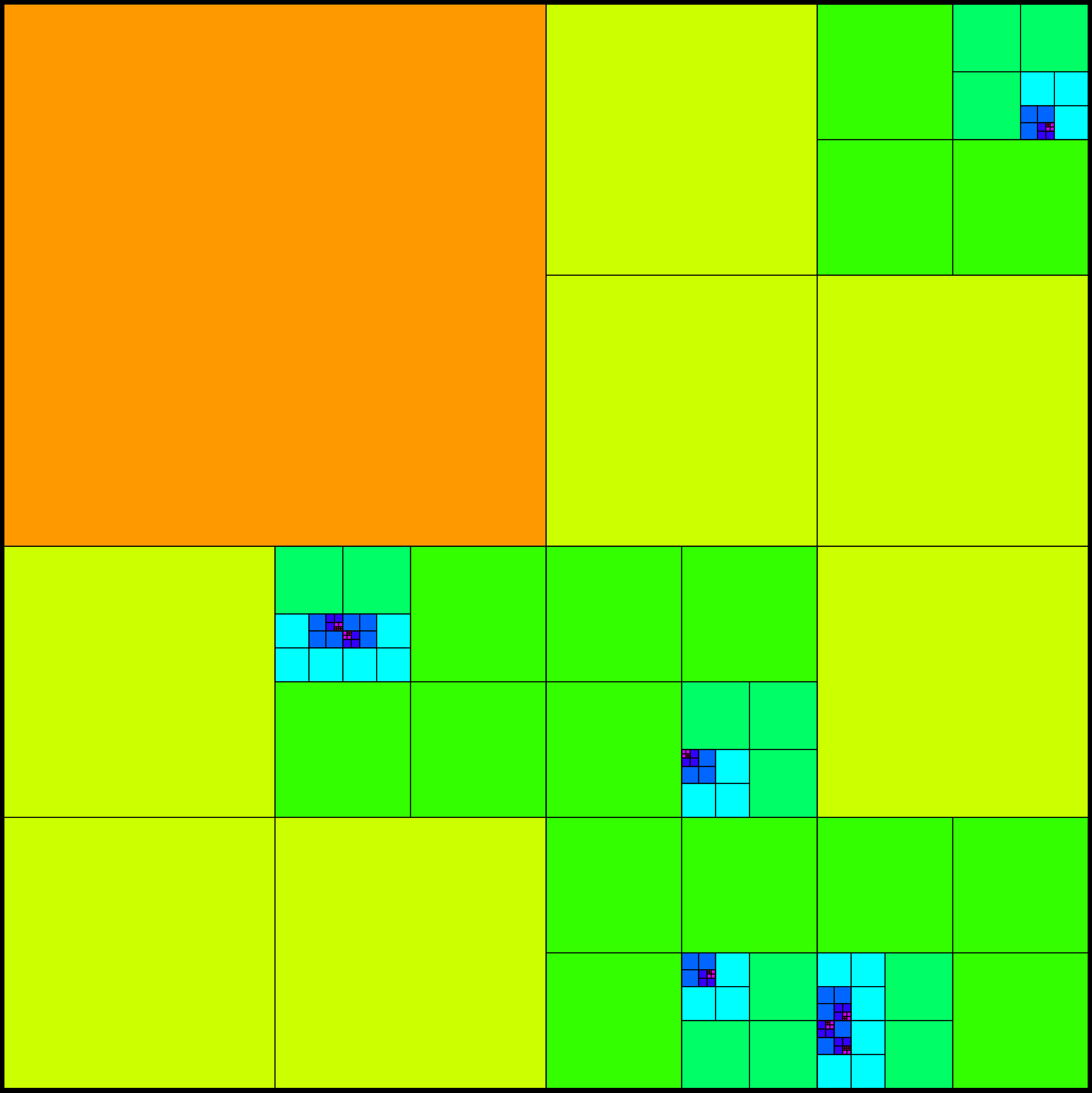
Area/4096 square decomposition of $e^{\gamma h} d^2 z$ for $\gamma = 1/2$



Area/4096 square decomposition of $e^{\gamma h} d^2 z$ for $\gamma = 1$



Area/4096 square decomposition of $e^{\gamma h} d^2 z$ for $\gamma = 2$



Area/4096 square decomposition of $e^{\gamma h} d^2 z$ for $\gamma = 10$

"A few years before this work Kazakov and David suggested that the discrete version of 2d gravity can be described by the various matrix models. It was hard to be certain that these models really have a continuous limit described by the Liouville theory, there were no proofs of this conjecture. To our surprise we found that the anomalous dimensions coming from our theory coincide with the ones computed from the matrix model. That left no doubts that in the case of the minimal models the Liouville description is equivalent to the matrix one. This relation received a lot of attention."

A.M. Polyakov, *From quarks to strings*, 2008

Knizhnik-Polyakov-Zamolodchikov (KPZ) Formula

THEOREM [Duplantier, S.]: Fix $\gamma \in [0,2)$ and let X be a compact subset of D. Let $N(\mu,\delta,X)$ be the number of (μ,δ) boxes intersected by X and $N(\epsilon,X)$ the number of diadic squares intersecting X that have edge length ϵ (a power of 2). Then if

$$\lim_{\epsilon\to 0} \frac{\log \mathbb{E}[\epsilon^2 N(\epsilon,X)]}{\log \epsilon^2} = x.$$

for some $x > 0$ then

$$\lim_{\delta\to 0} \frac{\log \mathbb{E}[\delta N(\mu,\delta,X)]}{\log \delta} = \Delta,$$

where Δ is the non-negative solution to

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta.$$