Nodal lines of random waves
Many questions and few answers

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Ascona, May 2010

Random 2D wave: random superposition of solutions to

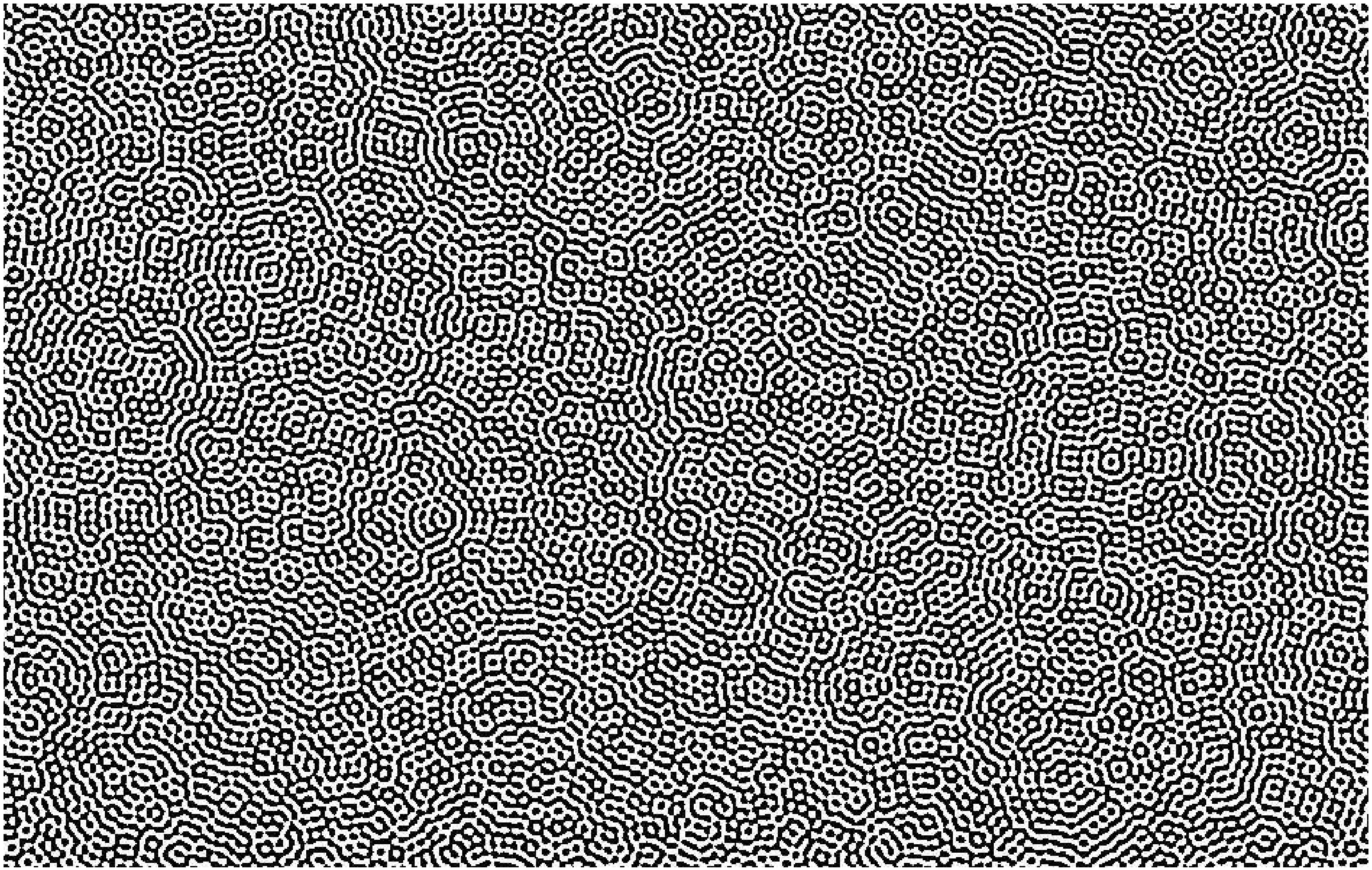
$$\Delta f + \kappa^2 f = 0$$

Examples:

- Random spherical harmonic of a given large degree
- Random plane monochromatic wave (in the large area limit)

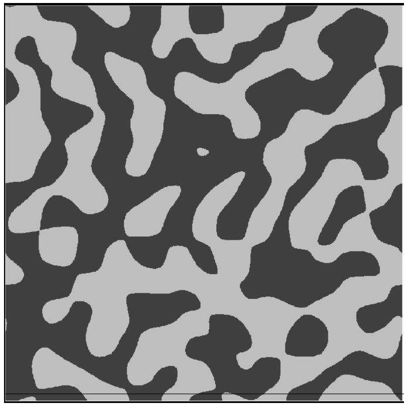
M.S.Longuet-Higgins: analysis of ocean waves (1950's)

M.Berry's conjecture: High-energy Laplace eigenfunctions on Riemannian surfaces with constant negative curvature are well modeled by random 2D waves (1970's)



Nodal portrait of a gaussian plane wave

Nodal portrait of random linear combinations of plane waves $e^{i\kappa \cdot x}$ with *different* wave numbers κ look dissimilar:



Spherical harmonic is a (real-valued) eigenfunction of the (minus) Laplacian on $\mathbb{S}^2$ with the eigenvalue $\lambda_n = n(n+1)$

equivalently,

a trace on $\mathbb{S}^2$ of a homogeneous harmonic polynomial in $\mathbb{R}^3$ of degree n

$\mathcal{H}_n$ is a $2n+1$-dim real Hilbert space of spherical harmonics of degree n with the $L^2(\mathbb{S}^2)$-norm

Gaussian spherical harmonic

$$f_n = \sum_{k=-n}^{n} \xi_k Y_k$$

ξ_k i.i.d. mean zero Gaussian (real) r.v. with $\mathcal{E}\xi_k^2 = \frac{1}{2n+1}$
$\{Y_k\}$ orthonormal basis in $\mathcal{H}_n$

- $\mathcal{E}\|f\|^2_{L^2(\mathbb{S}^2)} = 1$
- Distribution of f_n does not depend on the choice of the basis $\{Y_k\}$ in $\mathcal{H}_n$, and is rotation invariant on $\mathbb{S}^2$
- Covariance function: $\mathcal{E}\{f_n(x)f_n(y)\} = P_n(\cos\Theta(x,y))$
  $\Theta(x,y)$ angle between x and y
  P_n Legendre polynomial of degree n normalized by $P_n(1) = 1$.

Gaussian plane wave: 2D Fourier transform of the white noise on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. More formally,

$L^2_{\mathtt{sym}}(\mathbb{S}^1)$ the Hilbert space of complex valued L^2-functions with the symmetry $\phi(-\lambda) = \overline{\phi(\lambda)}$, $\lambda \in \mathbb{S}^1$;

$\mathcal{H} = \mathcal{F} L^2_{\mathtt{sym}}(\mathbb{S}^1)$ the Fourier image with the scalar product inherited from $L^2_{\mathtt{sym}}(\mathbb{S}^1)$; consists of real-analytic functions

$$\Phi(x) = \int_{\mathbb{S}^1} e^{\mathrm{i} x \cdot \lambda} \phi(\lambda) \, \mathrm{d}^1 \lambda, \qquad \Delta \Phi + \Phi = 0$$

The Gaussian plane wave: $F = \sum_k \eta_k \Phi_k$

η_k standard i.i.d. Gaussian r.v.'s, $\{\Phi_k\}$ orthonormal basis in $\mathcal{H}$.

- The construction does not depend on the choice of the basis $\{\Phi_k\}$ in $\mathcal{H}$
- The distribution of F is invariant with respect to translations and rotations of the plane

More explicit expression:

$$F(x) = \mathrm{Re} \sum_{m \in \mathbb{Z}} \zeta_m J_{|m|}(r) e^{\mathrm{i} m\theta}, \qquad x = (r, \theta),$$

ζ_m are i.i.d. complex Gaussian r.v.'s, $\mathcal{E}|\zeta_m|^2 = 2$

Covariance function of F: $\mathcal{E}\{F(x)F(y)\} = J_0(|x-y|)$

The Gaussian plane wave is a large n limit of the Gaussian spherical harmonic: the restrictions of the Gaussian functions f_n on spherical disks of radius R/n converge as random processes to the restriction of F on the euclidean disk of radius R

More formally, fix $x_0 \in \mathbb{S}^2$ and let $F_n(u) \stackrel{\text{def}}{=} \left(f_n \circ \exp_{x_0}\right)\left(\frac{u}{n}\right)$, $u \in T_{x_0}\mathbb{S}^2$.

Then

$$\mathcal{E}\left\{F_n(u)F_n(v)\right\} = P_n\left(\cos \underbrace{\Theta\left(\exp_{x_0}\left(\frac{u}{n}\right), \exp_{x_0}\left(\frac{v}{n}\right)\right)}_{\sim \frac{|u-v|}{n},\ n\to\infty}\right)$$

whence

$$\lim_{n\to\infty} \mathcal{E}\left\{F_n(u)F_n(v)\right\} = J_0(|u-v|) = \mathcal{E}\left\{F(u)F(v)\right\} \quad \text{loc unif in } u \text{ and } v$$

Nodal portrait of spherical harmonics

$g \in \mathcal{H}_n$ spherical harmonic of degree n

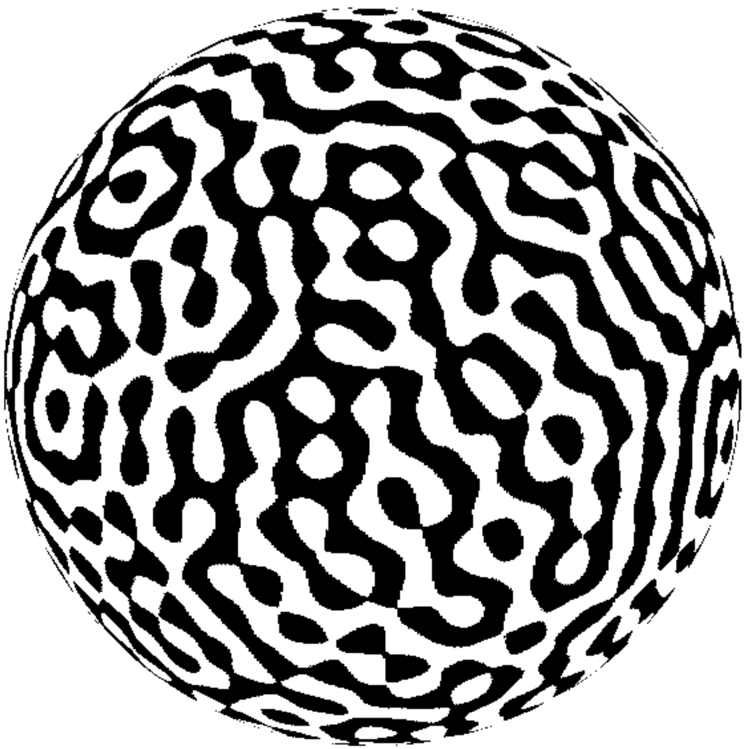
nodal set: $Z(g) = \{x \in \mathbb{S}^2 \colon g(x) = 0\}$

nodal domains: connected components of the set $\{x \in \mathbb{S}^2 \colon g(x) \neq 0\}$.

Well-known (deterministic) facts:

- *for each $g \in \mathcal{H}_n$, the nodal set $Z(g)$ is a Cn^{-1}-net on $\mathbb{S}^2$*
- *for each $g \in \mathcal{H}_n$, every nodal domain of g contains a disk of radius cn^{-1}*

are valid for Laplace eigenfunctions on smooth Riemannian surfaces



Nodal portrait of a gaussian spherical harmonic of degree 40

Basic characteristics of the nodal set $Z(g)$:

- the length $L(g)$
- the number of connected components $N(g)$

No rigorous results about 'morphology' of the nodal portrait (distribution of shapes of nodal domains):

- area-to-perimeter ratio
- avoided intersections
- ...

though physicists have developed some heuristics Monastra-Smilansky-Gnutzmann, Foltin-Smilansky-Gnutzmann, Elon-Gnutzmann-Joas-Smilansky

The length of the nodal set: integral geometry is helpful:

- for each $g \in \mathcal{H}_n$, $C^{-1}n \le L(g) \le Cn$

The lower bound (with n replaced by $\sqrt{\lambda}$) is valid for any smooth Riemannian surface (Brüning), the upper bound is a celebrated conjecture by S.T.Yau proven by Donnelly and Fefferman for real-analytic surfaces.

For Gaussian spherical harmonic,

- $\mathcal{E}L(f_n) = \pi\sqrt{\lambda_n} = \sqrt{2}\pi n + O(1)$ Bérard (1985)
- variance of $L(f_n) = \frac{65}{32}\log n + O(1)$ I.Wigman (2009) (predicted by M.Berry)

The number of the nodal components

The celebrated Courant nodal domain theorem yields

- For every $g \in \mathcal{H}_n$, $N(g) \leq n^2$.

Pleijel: $\leq (0.69 + o(1))n^2$

The sharp asymptotic upper bound is not known yet (likely, $\frac{1}{2}n^2$)

H.Lewy: construction of spherical harmonics of any degree n whose nodal sets have one component for odd n and two components for even n, that is, no non-trivial lower bound for $N(g)$ is possible

Till recently, nothing had been known about the asymptotics of the r.v. $N(f_n)$ when $n \to \infty$. The main difficulty is *non-locality*: observing the nodal curves only locally, one cannot make any conclusion about the number of connected components

Blum-Gnutzmann-Smilansky: Nodal domains statistics: a criterion for quantum chaos (2002)

Bogomolny-Schmit: Percolation model for nodal domains of chaotic wave functions (2002)

Bogomolny-Schmit percolation-like model:

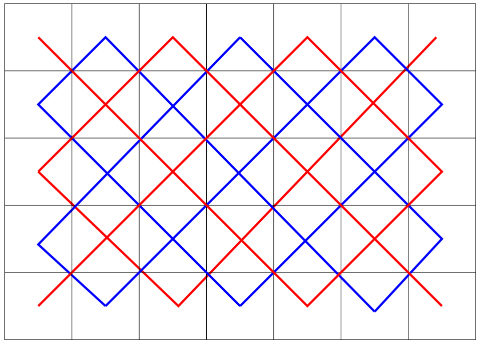
$\Delta f_n + n(n+1) f_n = 0$ hence local maxima are > 0, local minima are < 0

Checkerboard nodal picture: the square lattice with the total number of sites equal to $\left(\mathcal{E}L(f_n)\right)^2$, that is proportional to n^2. The sites are the saddle points; the saddle heights equal 0

Two dual square lattices:

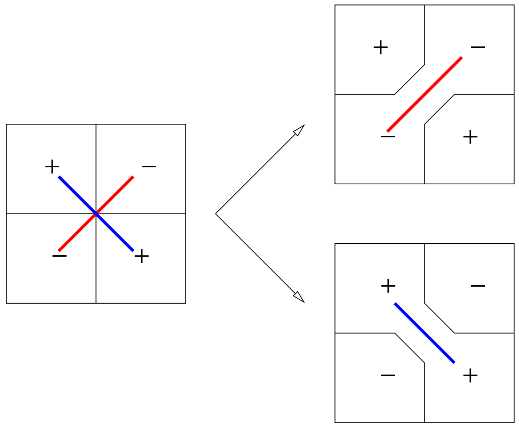
blue one (local maxima) vertices at the cells of the grid where $f_n > 0$

red one (local minima) vertices at the cells of the grid where $f_n < 0$



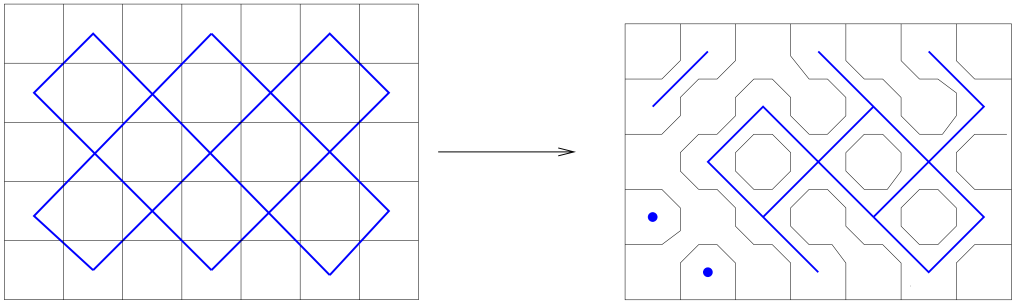
If the saddle height is positive then the bond between two neighboring maxima is open, if it is negative, then the bond is closed.

Bogomolny-Schmit assumptions: saddle heights are uncorrelated and have equal probability being positive or negative:



Each realization generates two graphs: the blue one whose vertices are the blue lattice points and the red one whose vertices are the red lattice points.

Each of these graphs uniquely determines the whole picture, and each of them represents the critical bond percolation on the corresponding square lattice:



Using heuristics from statistical mechanics, Bogomolny and Schmit predicted that for $n \to \infty$,

$$\mathcal{E}N(f_n) = (a + o(1))n^2, \qquad \text{variance of } N(f_n) = (b + o(1))n^2$$

with explicitly computed positive constants a and b.

They argued that *the fluctuations of the random variable $N(f_n)$ are asymptotically Gaussian* when $n \to \infty$, and concluded with a prediction of the power distribution law for the areas of nodal domains.

They also checked consistency of all their predictions with numerics

Major problem with the B-S approach: it ignores correlations (which decay only as $\text{dist}^{-1/2}$)

Minor problem: so far, no rigorous mathematical treatment of the critical bond percolation on the square lattice ...

Obvious question: Convergence of the percolation cluster to SLE_6? Bogomolny-Dubertrand-Schmit checked it numerically

Maybe, it is possible to prove it avoiding the B-S model?

Good news:

THEOREM: (F.Nazarov-M.S., 2008) *There exists a constant* $a > 0$ *such that, for every* $\epsilon > 0$*, we have*

$$\mathcal{P}\left\{\left|\frac{N(f_n)}{n^2} - a\right| > \epsilon\right\} \leq C(\epsilon)e^{-c(\epsilon)n} \qquad (*)$$

where $c(\epsilon)$ *and* $C(\epsilon)$ *are some positive constants depending on* ϵ *only.*

Since $L(f_n) \leq Cn$, this yields that for a typical spherical harmonic, most of its nodal domains have diameters $\simeq 1/n$.

QUESTION: *estimate the mean number of large components of the nodal set whose diameter is much bigger than* $1/n$*. E.g., of those whose diameter is* $\simeq n^{-\alpha}$ *with* $0 < \alpha < 1$.

Sharpness of $(*)$: for small $\kappa > 0$, $\mathcal{P}\left\{N(f_n) < \kappa n^2\right\} \geq e^{-C(\kappa)n}$.

On the other hand ...

- our proof does not give us the value of $a = \lim \mathcal{E} N(f_n)/n^2$
- it gives us the exponential bound $e^{-c(\epsilon)n}$ only with $c(\epsilon) \simeq \epsilon^{15}$
- we cannot prove that the variance of $N(f_n)$ tends to infinity

Other news: (good or bad?)

The proof uses tools from the classical analysis, which should also work in a more general setting of Gaussian non-monochromatic waves in any dimension (and for higher Betti numbers)

Steps in the proof:

1. The lower bound $\mathcal{E}N(f_n) \geq cn^2$ with some $c > 0$
2. Uniform lower semicontinuity of the functional $f \mapsto N(f)/n^2$ in $L^2(\mathbb{S}^2)$ outside of an exceptional set $E \subset \mathcal{H}_n$
3. Existence of the limit $\mathcal{E}N(f_n)/n^2$

Uniform lower semicontinuity of $f \mapsto N(f)/n^2$:

LEMMA: *For every* $\epsilon > 0$, *there exist* $\rho > 0$ *and an exceptional set* $E \subset \mathcal{H}_n$ *of probability* $\mathcal{P}(E) \le C(\epsilon) e^{-c(\epsilon) n}$ *s.t. for all* $f \in \mathcal{H}_n \setminus E$ *and for all* $g \in \mathcal{H}_n$ *satisfying* $\|g\|_{L^2(\mathbb{S}^2)} \le \rho$,

$$N(f+g) \ge N(f) - \epsilon n^2 \, .$$

Together with the concentration of Gaussian measure principle (='Gaussian isoperimetry', P.Levy-Sudakov-Tsirelson-Borell), this yields the exponential concentration of the r.v. $N(f_n)/n^2$ around its median.

Exceptional harmonics with unstable nodal portraits:

instability is caused by points where f and ∇f are simultaneously small.

$0 < \alpha, \delta \ll 1$, $R \gg 1$ parameters (depending on ϵ)

Cover $\mathbb{S}^2$ by $\simeq R^{-2}n^2$ disks D_j of radius R/n s.t. the concentric disks $4D_j$ cover $\mathbb{S}^2$ with a bounded multiplicity.

Unstable disks D_j: there is $x \in 3D_j$ s.t. $|f(x)| < \alpha$ and $|\nabla f(x)| < \alpha n$.

$f \in \mathcal{H}_n$ is *exceptional* if the number of unstable disks is at least δn^2, E is the set of all exceptional spherical harmonics of degree n.

- $\mathcal{P}(E) \le C(\delta) e^{-c(\delta)n}$ provided that α is sufficiently small
- for all $f \in \mathcal{H}_n \setminus E$ and all $g \in \mathcal{H}_n$ with $\|g\|_{L^2(\mathbb{S}^2)} \le \rho$, we have $N(f+g) \ge N(f) - \epsilon n^2$

Level sets:

Using non-critical bond percolation, Bogomolny and Schmit gave good predictions for the behaviour of the components of the level set. Nevertheless,

- *we cannot even prove* that for each $\epsilon > 0$ and each $\eta > 0$, the probability that the level set $\{x \in \mathbb{S}^2 : f_n(x) > \epsilon\}$ has a component of diameter larger than η tends to zero as $n \to \infty$.

Reasons for our ignorance:

- non-locality of the number of connected components
- a very slow decay of the correlations

More questions:

- Nothing is known about *the number of connected components of the nodal set for 'randomly chosen' high-energy Laplace eigenfunction* f_λ *on an arbitrary compact surface* M *without boundary endowed with a smooth Riemannian metric* g.

It's tempting to expect that our theorem models what is happening when M is the two-dimensional sphere $\mathbb{S}^2$ endowed with a generic Riemannian metric g that is sufficiently close to the constant one.

Instead of perturbing the 'round metric' on $\mathbb{S}^2$, one can add a small (random) potential to the spherical Laplacian. The question remains just as hard.

And yet the ear cannot right now part with the music and allow the tale to fade; the chords of fate itself continue to vibrate; and no obstruction for the sage exists where I have put The End: the shadows of my world extend beyond the skyline of the page, blue as tomorrow's morning haze - nor does this terminate the phrase.

Vladimir Nabokov, The Gift

THE END