Laplacian Growth:

DLA and Algebraic Geometry

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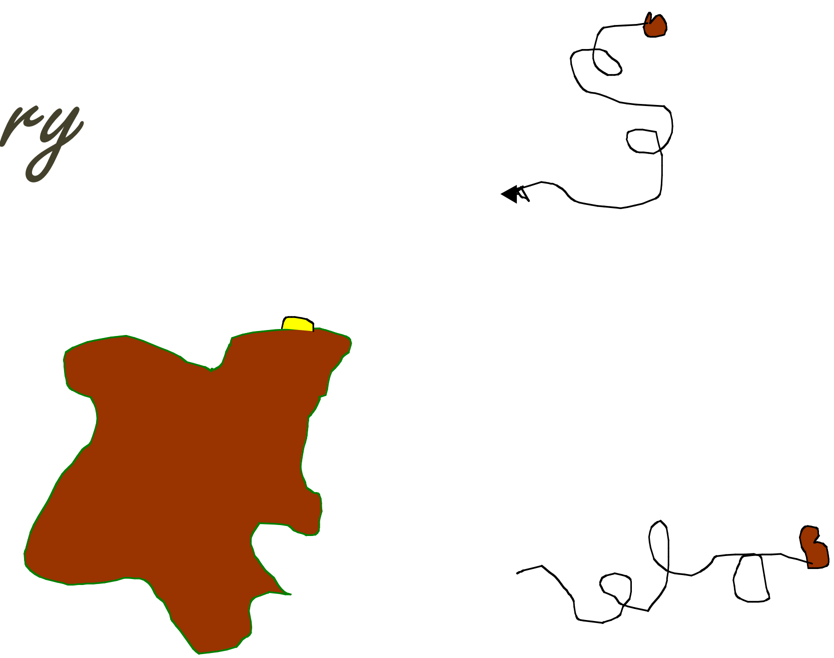
Ascona, 2010

Laplacian growth -

Moving planar interface which velocity is a gradient of a harmonic field

Brownian excursion of particles of a non-zero size

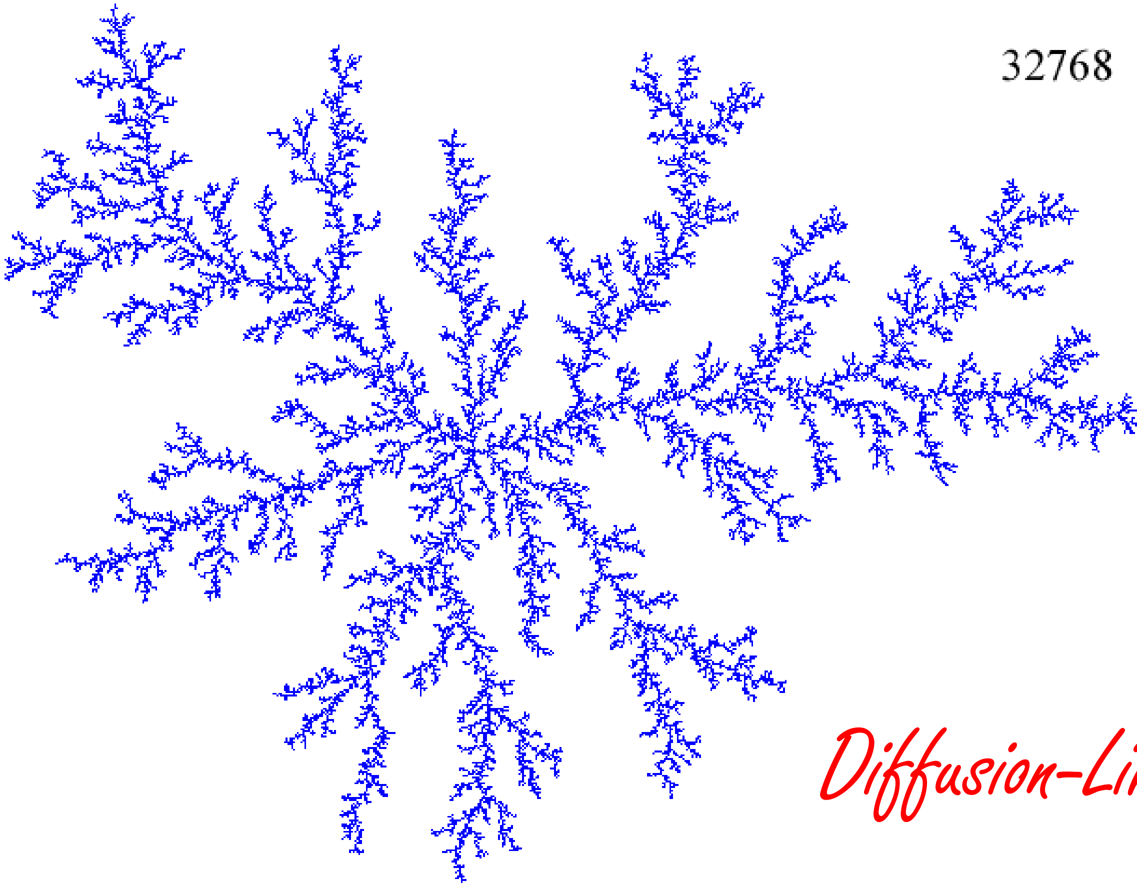
A probability of a Brownian particle to arrive is a harmonic measure of the boundary



$$|\nabla p| - Harmonic\, measure$$

$$\Delta p = 0, \quad p|_{\partial D} = 0,$$

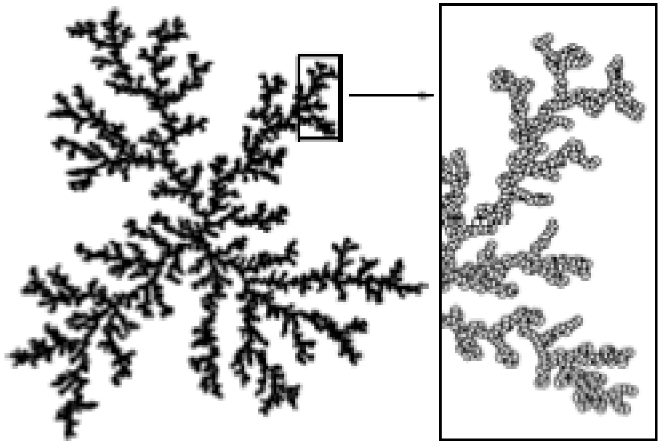
$$p|_{z\to\infty} \to -\log|z|$$



Diffusion-Limit of Aggregation, or DLA,

is a simple computer simulation of the formation of clusters by particles diffusing through a medium that jostles the particles as they move.

T. Witten, L. Sandler 1981



Geometrical Growth

Believed to be self-similar

Capacity of a set

$$C(t) \sim t^{1/D} \quad t - \text{Number of particles}$$

Kersten estimate (a theorem) (1987): D >3/2

Numerical value D=1.710..-1.714.. (many authors, different methods)

Iterative Conformal maps (Hastings-Levitov, 1998)

Mathematical aspect of models of iterative maps:

Carleson, Makarov 2001

Rohde and Zinsmeister 2005

Numerical studies of iterative maps Procaccia et al , 2001-2005

D=1.710..-1.714.. Same as direct DLA simulation

Alternative view (2001-2010):

S. Y. Lee (CalTech), A. Zabrodin (Moscow)

E. Bettelheim (Jerusalem), I. Krichever (Columbia)

R. Teodorescu (Florida), P. W.

Based on Integrable structures

Related Phenomena

1) Viscous shocks in Hele-Shaw flow;
2) Dyson Diffusion;
3) Distributions of zeros of Orthogonal Polynomials;
4) Non-linear Stokes Phenomena in Painleve Equations;

Real Boutroux Curves (or Krichever-Boutroux Curves)

Real Boutroux Curves

Hyperelliptic Curves

Real (hyperelliptic) Boutroux Curves

$$(Y, X): \quad Y^2 = R_{2g+1}(X) - \text{real polynomial}$$

$$d\Omega(X) = -iY\,dX$$

$$\mathrm{Re} \oint_B d\Omega = 0, \quad \text{any cycle on the curve.}$$

conditions - # parameters = g

There is no general proof that Boutroux curves exist

Level Lines of Boutroux Curves:

A Graph Γ : $\quad \mathrm{X} \in \Gamma$:

$$\mathrm{Re}\,\Omega(\mathrm{X})|_{\Gamma} = 0,$$
$$\mathrm{Re}\,\Omega(\mathrm{X})|_{\mathrm{X}\to\Gamma} > 0$$

$$\Omega(X) = -i\int_e^X Y\,dX$$

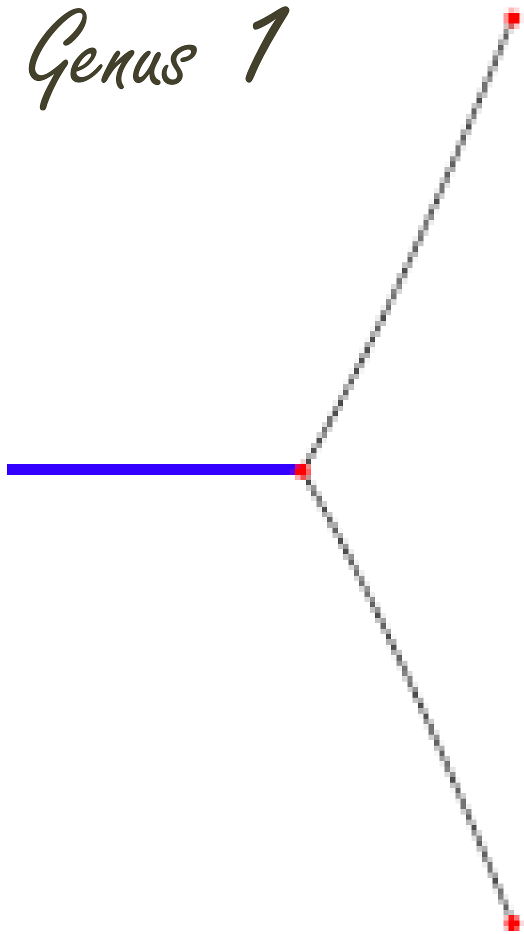
Level lines are Branch cuts drawn such that jump of Y *is Imaginary*

Alternative definition of Boutroux curves :

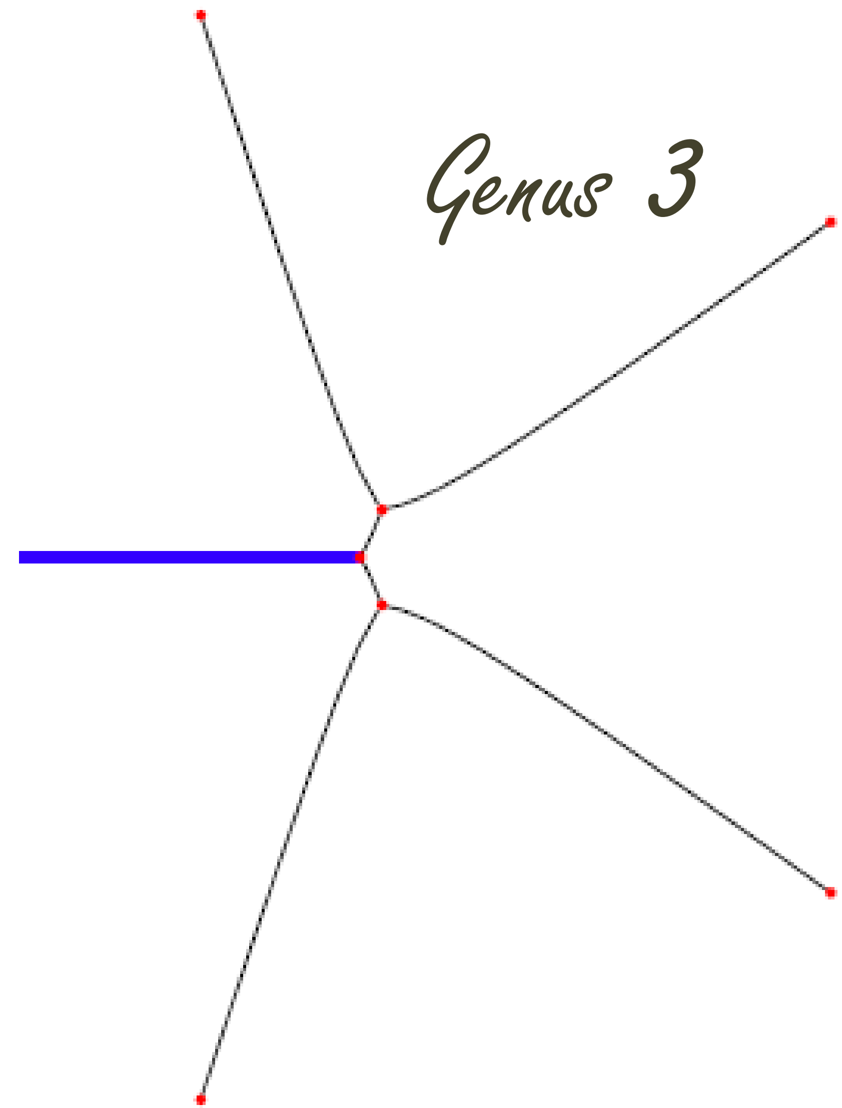
Branch cuts can be chosen such that jump of Y is Imaginary

Level Lines of Boutroux Curves

Genus 1



Genus 3



Growing branching graph (transcendental)

Deformation Parameters, Evolution, Capacity

Deformation Parameters, Evolution, Capacity

$$Y(X) := \sqrt{R_{2g+1}(X)} = \overbrace{X^{g+1/2} + t_{g-1}X^{g-1/2} + \ldots}^{deformation}$$

$$+ \underbrace{\frac{t}{X^{1/2}}}_{time} + \underbrace{\frac{C(t)}{X^{3/2}}}_{capacity} + Negative\ powers$$

g-2-deformation parameters and time t uniquely determine the curve

Evolve a curve in time,

keeping g-1 deformation parameters fixed,

follow the capacity C(t) and the graph

$$\text{time} = \text{res}_\infty(Y\, d\sqrt{X}),$$

$$\text{Capacity} = \text{res}_\infty(Y\sqrt{X} d\sqrt{X})$$

Attempt a limit $g \to \infty$

Marco Bertola presents......

A unique Elliptic Boutroux curve (Krichever, Ragnisco et al, 1991)

Degenerate curve

$$y^2 = -(x - e(t))\left(x + \frac{e(t)}{2}\right)^2$$

$$e(t) = -\sqrt{t_c - t}$$

Non-degenerate curve

$$y^2 = (x - e_1)(x - e_2)(x - e_3)$$

$$(e_{1,2}, e_3) = \sqrt{\frac{12}{4h^2 - 3}}(\tfrac{1}{2} \pm ih, -1)\sqrt{t}$$

Krichever constant $h \approx 3.246382253744278875676$

Appearance of Boutroux curves

Boutroux 1912:

semiclassical solution of Painleve I equations:

Adiabatic Invariant of a particle escaping to infinity E=P^2+V(x)

$$P^2 = -4X^3 + g_2 X + g_3, \quad I = \int P dX = \text{const}$$

2D-Dyson's Diffusion

Large Normal Matrix $\mathrm{M_{nm}} : \mathrm{MM^\dagger = M^\dagger M}$

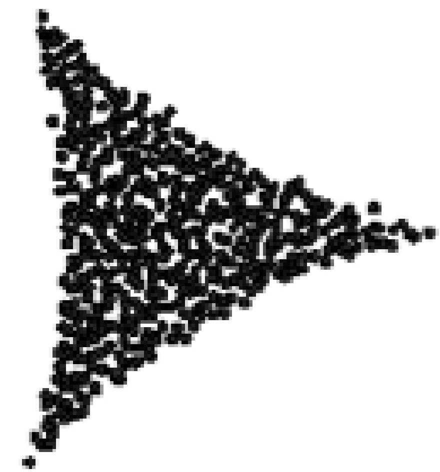
$$\dot{M} = M^\dagger + \underbrace{V'(M)}_{\mathrm{Pol_{g+2}}} + BM$$

Eigenvalues (complex)

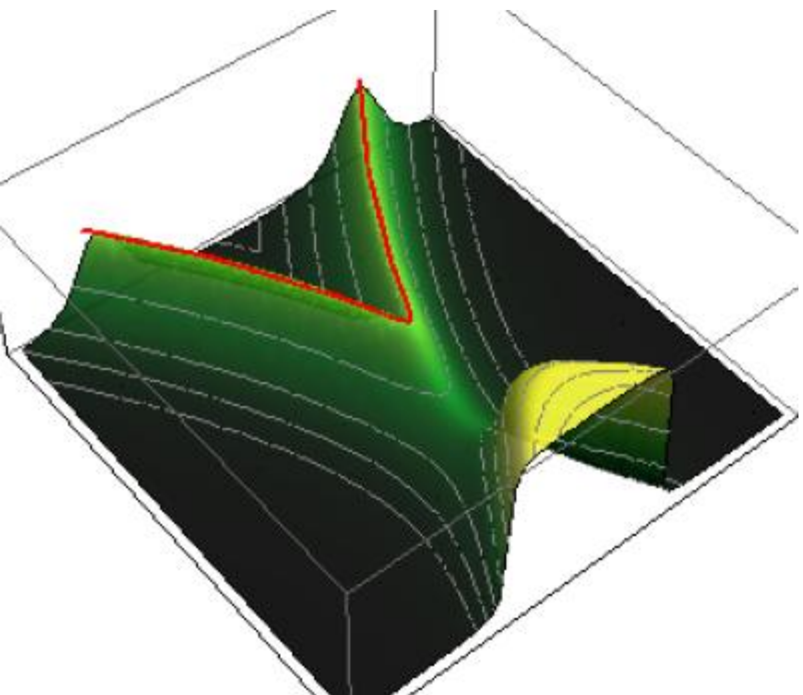
$$\dot{z}_i = \sum_j \frac{1}{\bar{z}_i - \bar{z}_j} + \bar{z}_i + V'(z_i) + \dot{\xi}_i,$$

$$\mathbf{E}[\xi_i \bar{\xi}_j] = \kappa \delta_{ij}, \quad \kappa = 4$$

$$\dot{z}_i = \sum_j^N \frac{1}{\bar{z}_i - \bar{z}_j} + \bar{z}_i + \underbrace{V'(z_i)}_{Pol_{g+2}} + \dot{\xi}_i,$$



Unstable directions: $V = z^3$

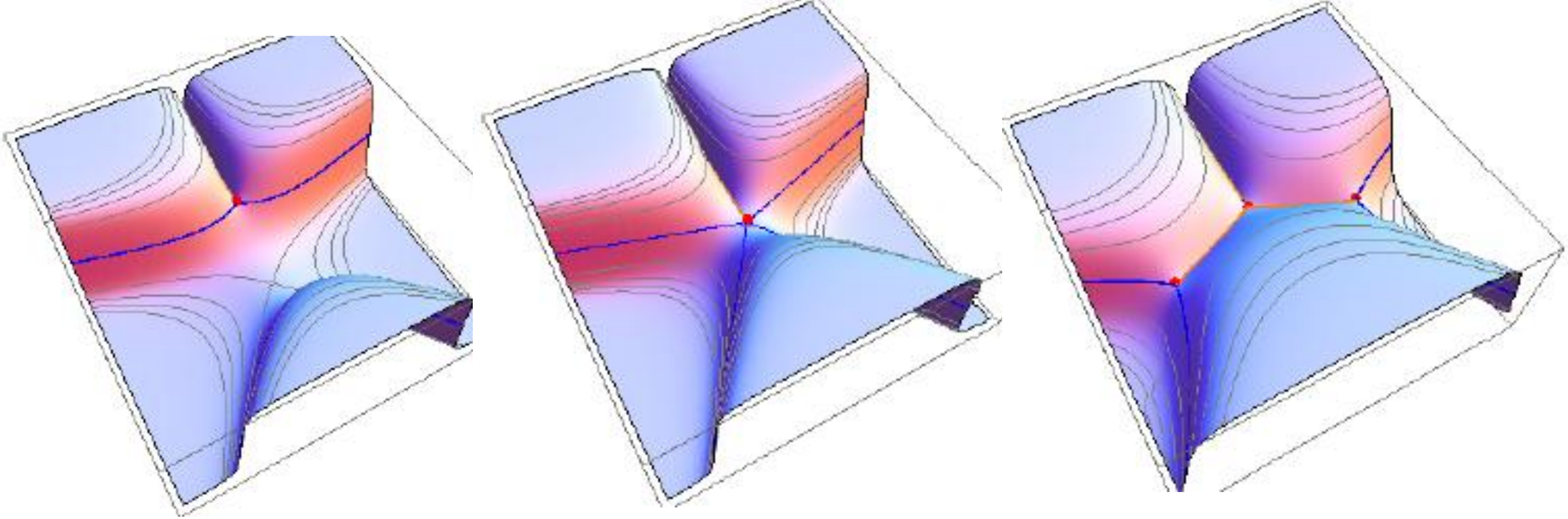


No Gibbs equilibrium:

One keeps pump particles to compensate escaping particles.

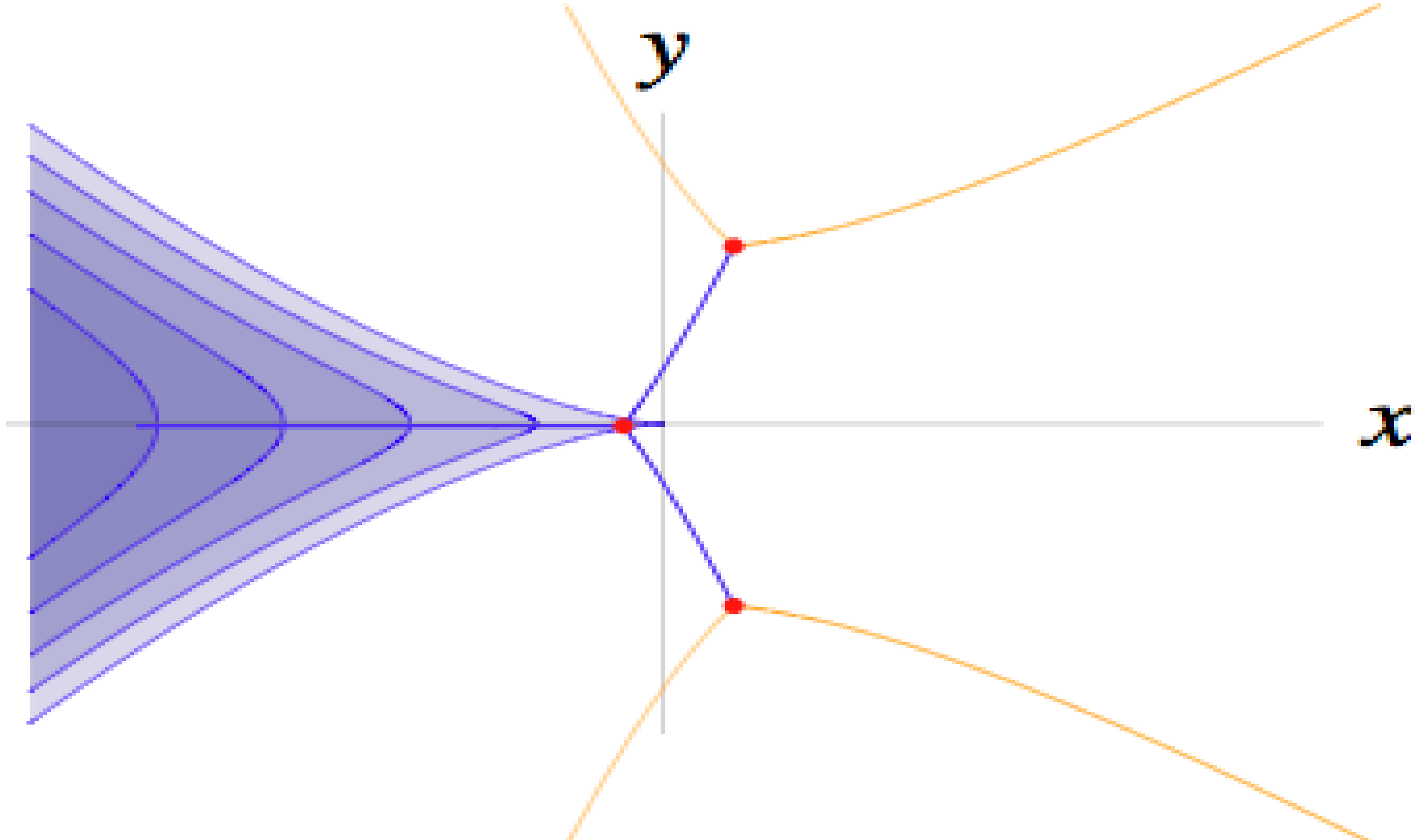
Evolution $N \longrightarrow N+1$

Particles escaping through cusps



Support for a non-equlibrium distribution of eigenvalues is the Boutroux-level graph: Red dot is a position of a new particle in a steady state

David (1991), Marinari-Parisi (1991), P. W.



2D⟶1D

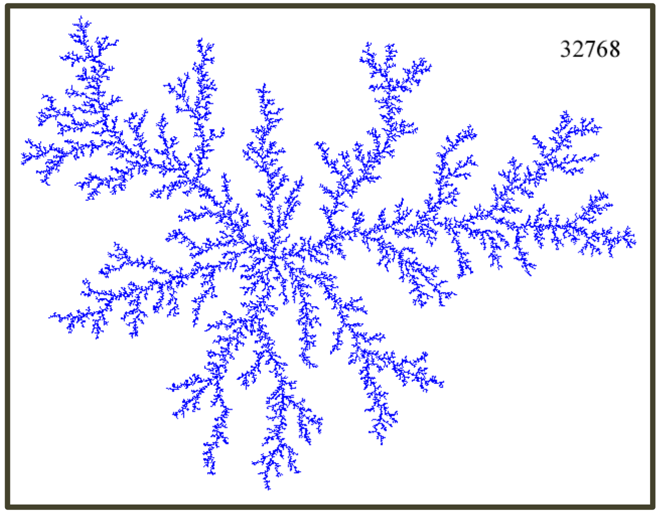
support of eigenvalues changes: it becomes level lines of Boutroux curve

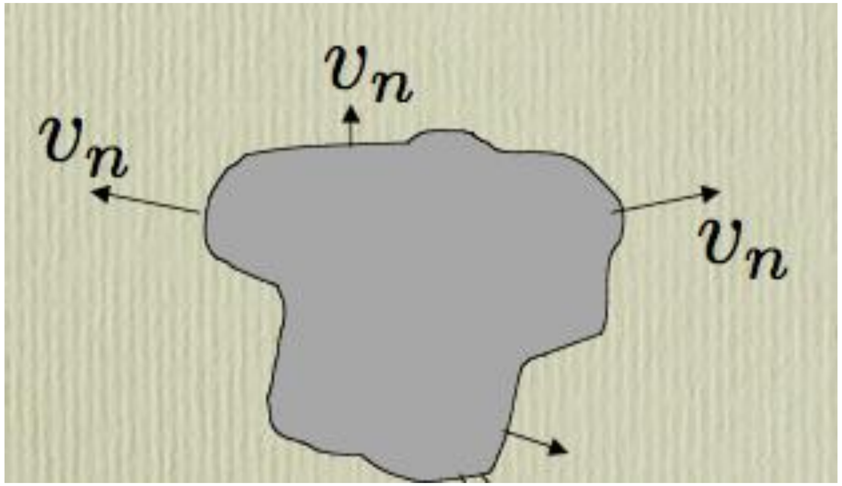
Hele-Shaw problem;

Fingering instability;

Finite time singularities;

Illegitimate limit: vanishing size of a particle $\hbar \to 0$

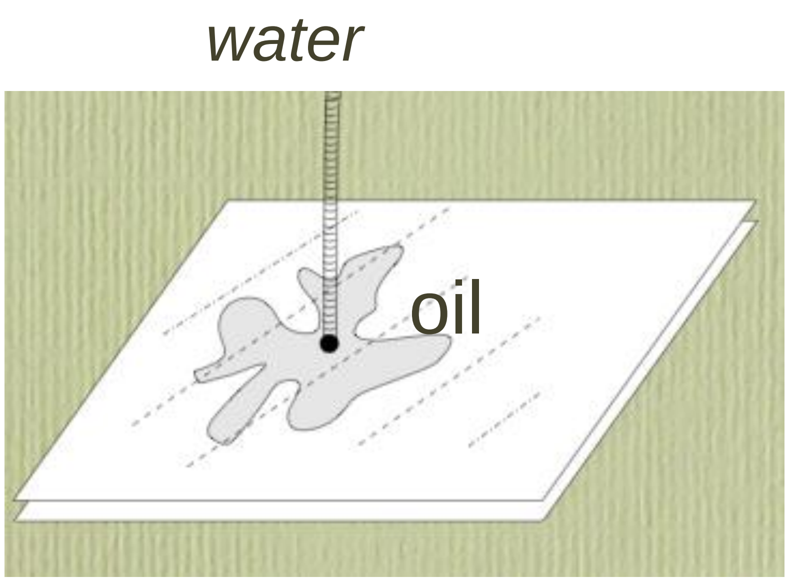




Darcy Law : $\quad \mathrm{C}\backslash\mathrm{D} : v = -\nabla p, \quad \Delta p = 0,$

$$p|_{\partial D} = 0, \quad p|_{z\to\infty} \to -\log|z|$$

Hele-Shaw Cell (1894)

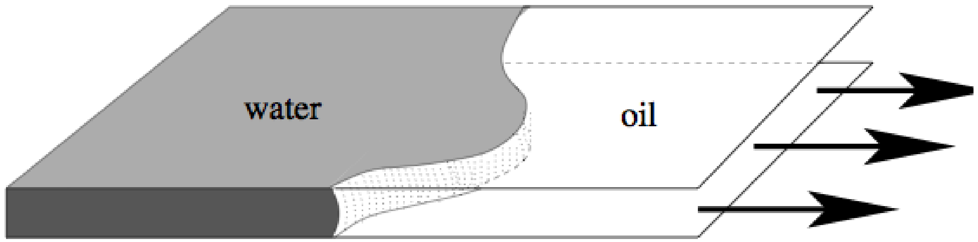


HS Hele-Shaw, inventor of the Hele-Shaw cell (and the variable-pitch propeller)

Oil (exterior) - incompressible viscous fluid

Water (interior) - incompressible inviscid liquid

Interface between Incompressible fluids with different viscosities



Darcy Law

$$\mathbf{v} = -\nabla p$$

Incompressibility

$$\nabla \cdot \mathbf{v} = 0 \qquad \Delta p = 0$$

Drain

$$Q = \oint_{\infty} \mathbf{v} \times d\ell$$

No surface tension

$$p|_{\text{boundary}} = 0$$

Velocity of a boundary=Harmonic measure of the boundary

Fingering Instabilities in fluid dynamics

Any but plane front is unstable - an arbitrary small deviation from a plane front causes a complex set of fingers growing out of control

Hele-Shaw cell fingers

Flame with no convection

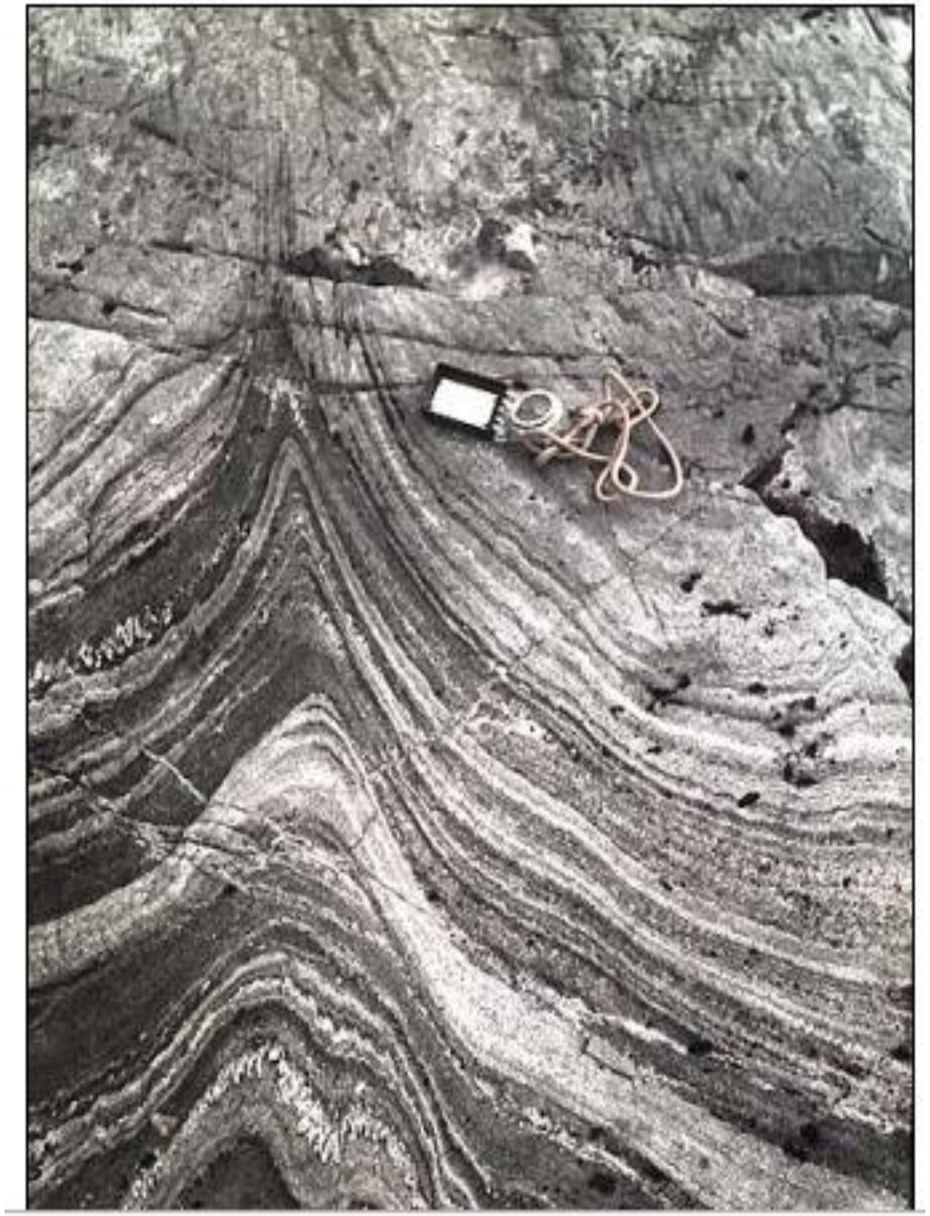
Saffman-Taylor, 1958 (linear analysis)

Finite Time Cusp Singularities

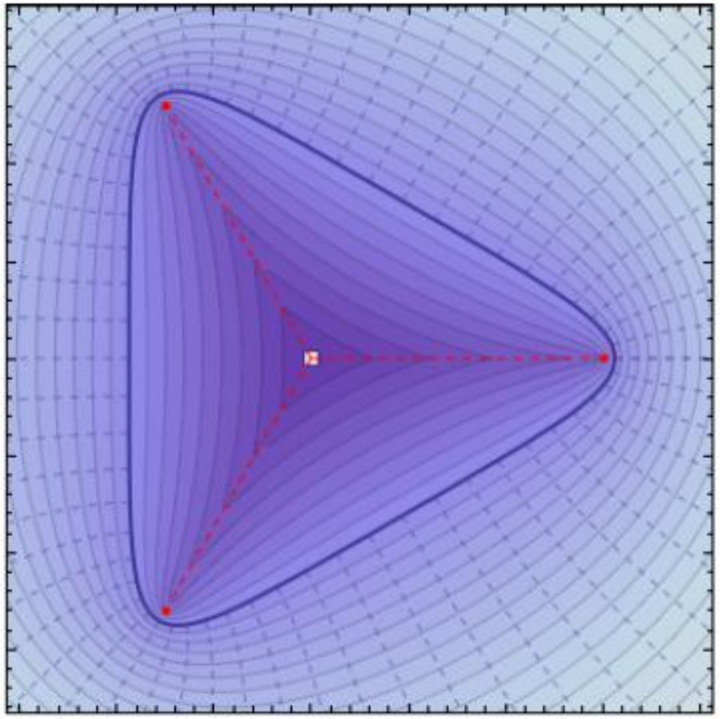
Finite time singularities:

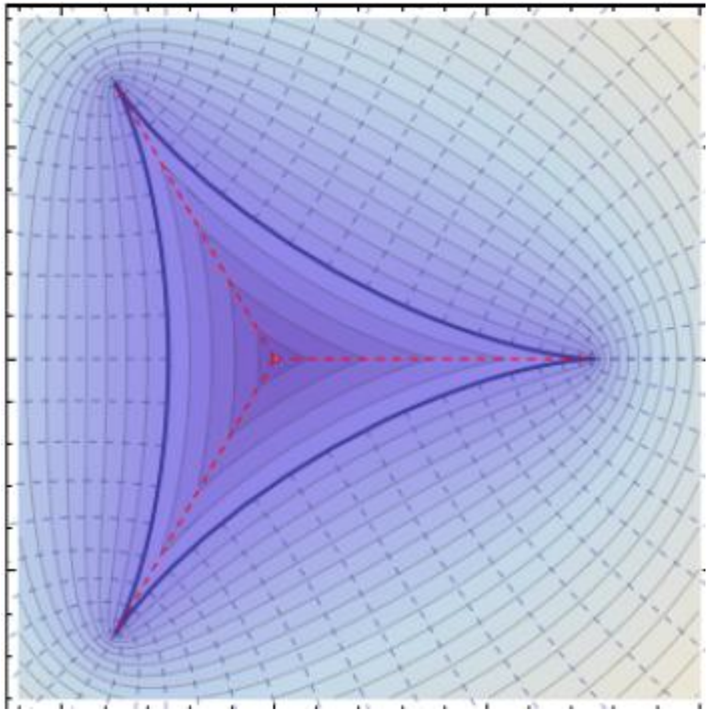
any but plain algebraic domain develops cusp singularities occurred at a finite time (the area of the domain)

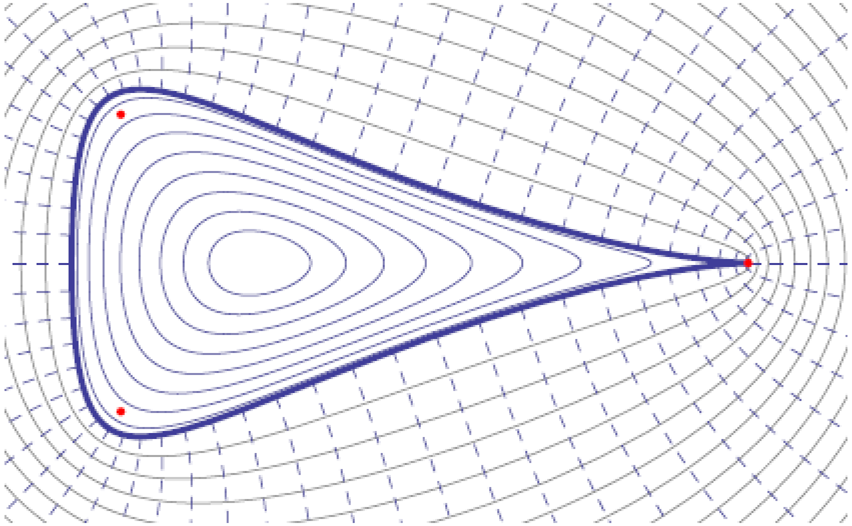
$$y^2 \sim x^3$$



Saffman-Taylor, Howison, Shraiman,.....







Evolution of a hypertrocoid

Zabrodin, Teodorescu, Lee, P. W:

Cusps: A graph of an evolving finger is:

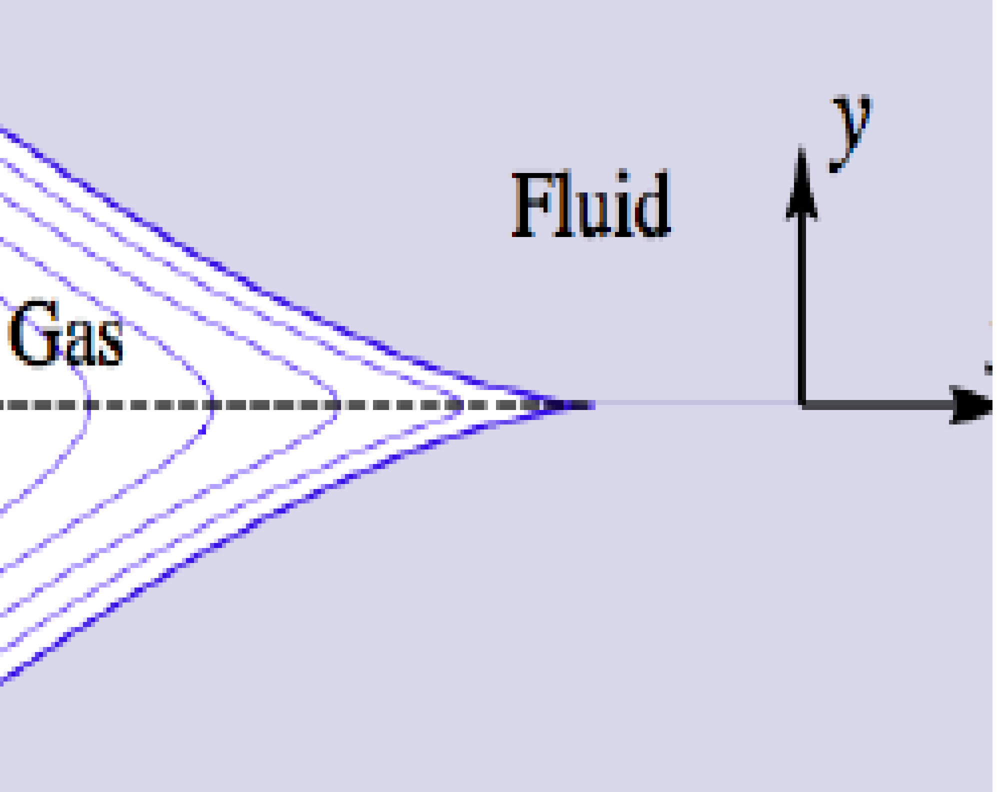
1) a degenerate hyperelliptic real Boutroux curve;

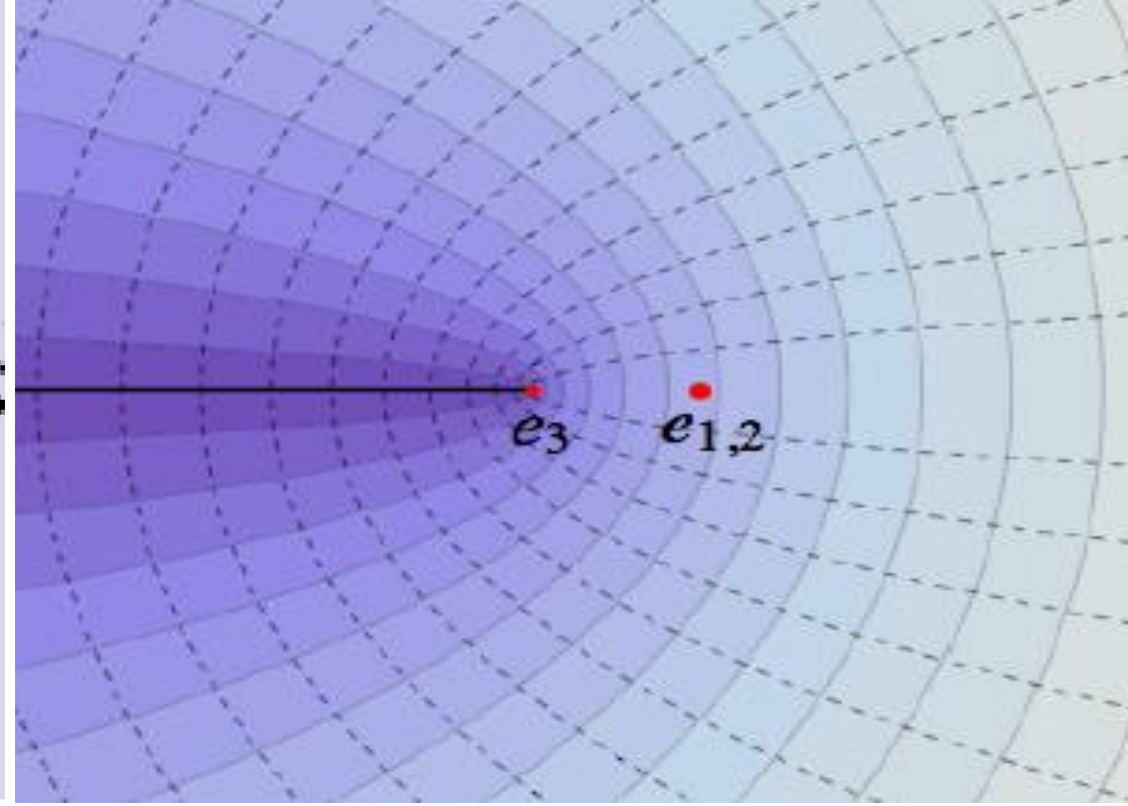
$$Y = \prod_{i=1}^{g} (X - d_i(t)) \sqrt{X - e(t)}$$

2) genus of the curve and a finite number of deformation parameters do not evolve

$$Y(X) := \sqrt{R_{2g+1}(X)} = \overbrace{X^{g+1/2} + t_{g-1} X^{g-1/2} + \dots}^{deformation}$$

$$+ \underbrace{\frac{t}{X^{1/2}}}_{time} + \underbrace{\frac{C(t)}{X^{3/2}}}_{capacity} + Negative\ powers$$





Evolution of a real elliptic degenerate curve

$$Y^2 = -4\,(X - e)\left(X + \frac{e}{2}\right)^2, \quad e = -2\sqrt{-T}, \quad T < 0$$

Darcy law is ill-defined – no physical solution beyond a cusp

$$\mathbf{v} = -\nabla p$$

Weak solution:

Allow discontinuities at some moving graph

Integral form of Darcy law

Lee, Teodorescu, P. W

Integral form of Darcy law → $\mathbf{v} = -\nabla p$

$$d\Omega(X) = -iY\,dX$$

$$i\frac{d}{dT}\oint_\infty d\Omega = flux$$

$$\frac{d}{dT}\mathrm{Re}\oint_B d\Omega = -\oint_B \mathbf{v}\cdot\mathbf{dl} = -\oint_B dp = circulation = 0.$$

$$\mathrm{Re}\oint_B d\Omega = 0$$

← Boutroux condition

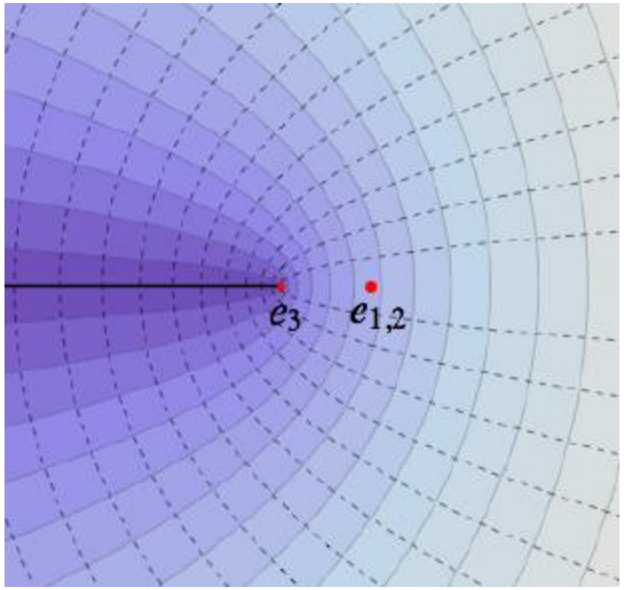
Boutroux condition uniquely define evolution and a graph of shocks (lines of discontinuities)

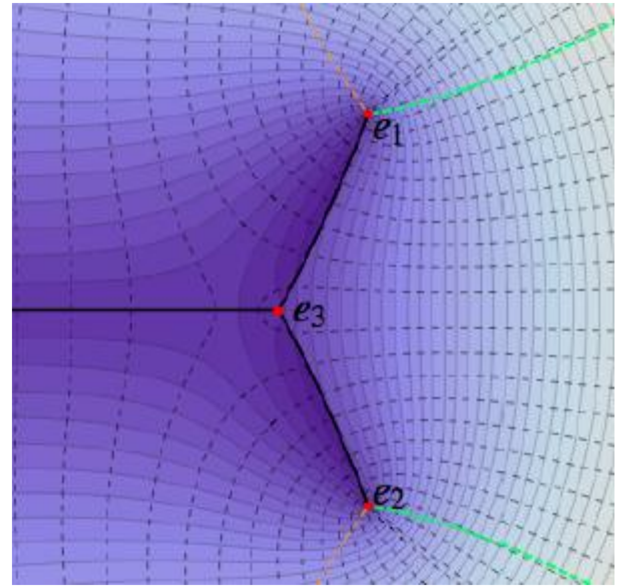
$$\mathrm{Re}\oint_B d\Omega = 0 \qquad d\Omega(X) = -iY\,dX$$

Shocks are Level lines of Boutroux curves:

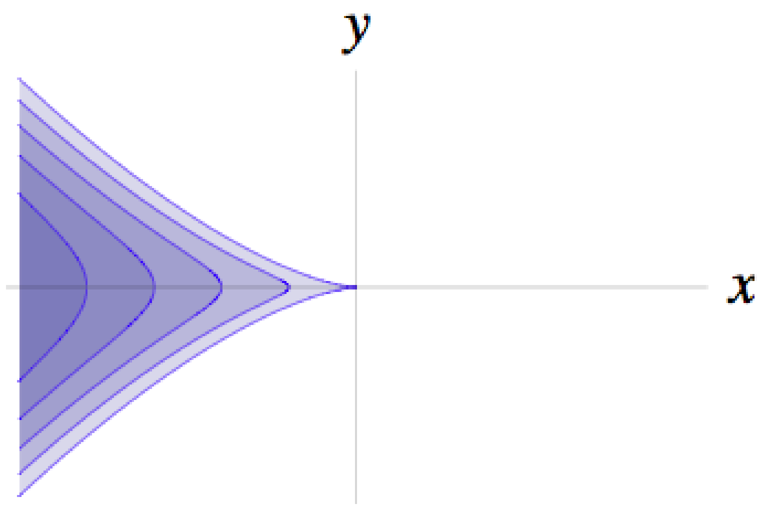
$$\mathrm{Re}\,\Omega|_{\mathrm{shocks}} = 0$$

$$\mathrm{Re}\,\Omega(\mathrm{X})|_{\mathrm{X}\to\mathrm{shocks}} > 0$$



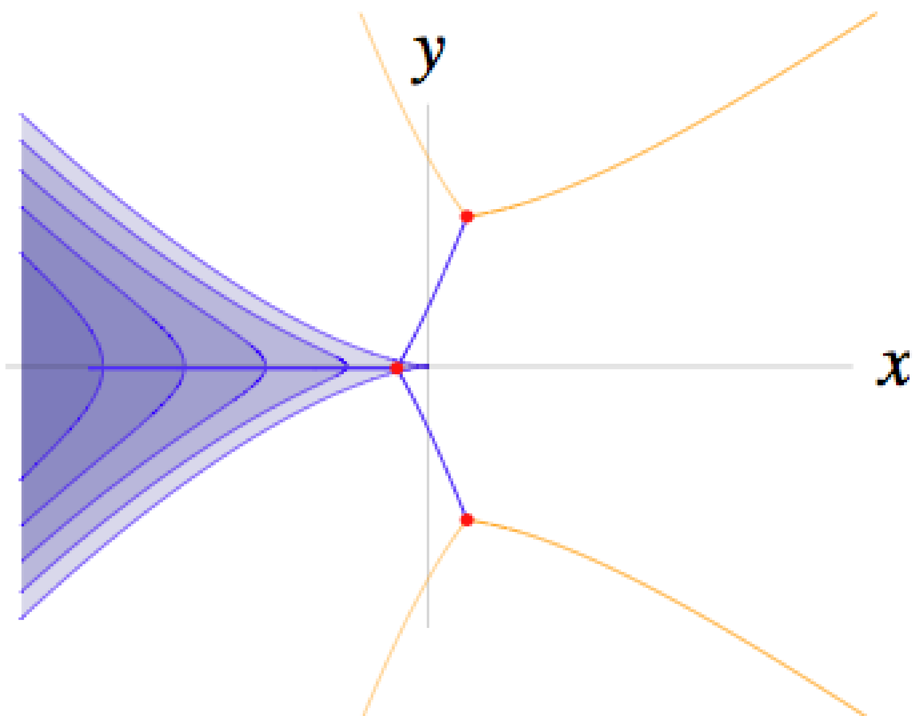


Level lines of elliptic Boutroux curves:
genus 0⟶1 transition



$$y^2 = -(x - e(t))\left(x + \frac{e(t)}{2}\right)^2$$

$$e(t) = -\sqrt{t_c - t}$$



$$y^2 = (x - e_1)(x - e_2)(x - e_3)$$

$$(e_{1,2}, e_3) = \sqrt{\frac{12}{4h^2 - 3}}\left(\tfrac{1}{2} \pm ih, -1\right)\sqrt{t}$$

$$h \approx 3.2463822537442788756\,76$$

Krichever constant

Universal jump of capacity at a branching

$$\eta = \frac{\dot{C}_{\text{after branching}}}{\dot{C}_{\text{before branching}}} \approx 0.91522030388$$

Computed through elliptic integrals

Evolution of Boutroux Curves
is equivalent to Laplacian Growth

Genus transition gives raise to branching of the level tree. Every branching produces a universal capacity "jump"

Manual for planting and growing trees

1. Fix a Polynomial of a large degree Pol_{2g+1};

2. Determine a degenerate Boutroux curve

$$Y = \prod_{i=1}^{l} (X - d_i(t)) \sqrt{(X - e(t))}$$

   which positive part is a Pol_{2g+1}

$$Y = \mathrm{Pol}_{2g+1}(\sqrt{X}) + \frac{t}{\sqrt{X}} + \frac{C(t)}{X^{3/2}} + \dots$$

3. Run t. Pinched cycles begin to open. Level graph branches. When all cycles are opened, the process stops.

4. Send genus $g \to \infty$. Follow capacity $C(t)$.

Questions:

- Does Capacity scale with (large) time (at large genus)

$$C(t) \sim t^{1/D}?$$

- Does the asymptote of capacity remember initial data - a chosen Pol_{g-1} ?

- Does the exponent is expressed solely through Krichever constant -capacity branching jump?

$$\eta = \frac{\dot{C}_{\text{after branching}}}{\dot{C}_{\text{before branching}}} \approx 0.91522030388$$

- Simple-minded (not even a conjecture) formula for dimension in terms of universal capacity jump

$$\frac{1}{D} - \frac{1}{2} = 1 - \eta = 0.08477969612, \quad D = \overbrace{\underbrace{1.71004}_{\textit{of DLA dimension}}}^{\textit{numerical value}} 56918$$