Towards conformal invariance of 2D lattice models

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Abstract. Many 2D lattice models of physical phenomena are conjectured to have conformally invariant scaling limits: percolation, Ising model, self-avoiding polymers, ... This has led to numerous exact (but non-rigorous) predictions of their scaling exponents and dimensions. We will discuss how to prove the conformal invariance conjectures, especially in relation to Schramm-Loewner Evolution.

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1. Introduction

For several 2D lattice models physicists were able to make a number of spectacular predictions (non-rigorous, but very convincing) about exact values of various scaling exponents and dimensions. Many methods were employed (Coulomb Gas, Conformal Field Theory, Quantum Gravity) with one underlying idea: that the model at criticality has a continuum scaling limit (as mesh of the lattice goes to zero) and the latter is conformally invariant. Moreover, it is expected that there is only a one-parameter family of possible conformally invariant scaling limits, so universality follows: if the same model on different lattices (and sometimes at different temperatures) has a conformally invariant scaling limit, it is necessarily the same. Indeed, the two limits belong to the same one-parameter family, and usually it directly follows that the corresponding parameter values coincide.

Recently mathematicians were able to offer different, perhaps better, and certainly more rigorous understanding of those predictions, in many cases providing proofs. The point which is perhaps still less understood both from mathematics and physics points of view is why there exists a universal conformally invariant scaling limit. However such behavior is supposed to be typical in 2D models at criticality: Ising, percolation, self-avoiding polymers; with universal conformally invariant curves arising as scaling limits of the interfaces.

Until recently this was established only for the scaling limit of the 2D random walk, the 2D Brownian motion. This case is easier and somehow exceptional because of the Markov property. Indeed, Brownian motion was originally constructed by Wiener [46], and its conformal invariance (which holds in dimension 2 only) was shown by Paul Lévy [24] without appealing to random walk. Note also that unlike interfaces (which are often simple, or at most "touch" themselves), Brownian trajectory has many "transversal" self-intersections.

For other lattice models even a rigorous formulation of conformal invariance conjecture seemed elusive. Considering percolation (a model where vertices of a graph are declared open independently with equal probability p – see the discussion below) at criticality as an example, Robert Langlands, Philippe Pouliot and Yvan Saint-Aubin in [20] studied numerically crossing probabilities (of events that there is an open crossing of a given rectangular shape). Based on experiments they concluded that crossing probabilities should have a universal (independent of lattice) scaling limit, which is conformally invariant (a conjecture they attributed to Michael Aizenman). Thus the limit of crossing probability for a rectangular domain should depend on its conformal modulus only. Moreover an exact formula (5) using hypergeometric function was proposed by John Cardy in [9] based on Conformal Field Theory arguments. Later Lennart Carleson found that the formula has a particularly nice form for equilateral triangles, see [38]. These developments got many researchers interested in the subject and stimulated much of the subsequent progress.

Rick Kenyon [16, 17] established conformal invariance of many observables related to dimer models (domino tilings), in particular to uniform spanning tree and loop erased random walk, but stopped short of constructing the limiting curves.

In [32], Oded Schramm suggested to study the scaling limit of a single interface and classified all possible curves which can occur as conformally invariant scaling limits. Those turned out to be a universal one-parameter family of SLE(κ) curves, which are now called *Schramm-Loewner Evolutions*. The word "evolution" is used since the curves are constructed dynamically, by running classical Loewner Evolution with Brownian motion as a driving term. We will discuss one possible setup, chordal SLE(κ) with parameter $\kappa \in [0, \infty)$, which provides for each simply-connected domain Ω and boundary points a, b a measure μ on curves from a to b inside Ω. The measures $\mu(\Omega, a, b)$ are conformally invariant, in particular they are all images of one measure on a reference domain, say a half-plane $\mathbb{C}_+$. An exact definition appears below.

In [37, 38] the conformal invariance was established for critical percolation on triangular lattice. Conformally invariant limit of the interface was identified with SLE(6), though its construction does not use SLE machinery. See also Federico Camia and Charles Newman's paper [8] for the details on subsequent construction of the full scaling limit.

In [23] Greg Lawler, Oded Schramm and Wendelin Werner have shown that a perimeter curve of the uniform spanning tree converges to SLE(8) (and the related loop erased random walk – to SLE(2)) on a general class of lattices. Unlike the proof for percolation, theirs utilizes SLE in a substantial way. In [35] Oded Schramm and Scott Sheffield introduced a new model, Harmonic Explorer, where properties needed for convergence to SLE(4) are built in.

Despite the results for percolation and uniform spanning tree, the problem remained open for all other classical (spin and random cluster) 2D models, including percolation on other lattices. This was surprising given the abundance of the physics literature on conformal invariance. Perhaps most surprising was that the problem of a conformally invariant scaling limit remained open for the Ising model, since for the latter there are many exact and often rigorous results – see the books [27, 6].

Recently we were able to work out the Ising case [39, 40, 41, 42]:

Theorem 1. *As lattice step goes to zero, interfaces in Ising and Ising random cluster models on the square lattice at critical temperature converge to* SLE (3) *and* SLE (16/3) *correspondingly.*

Computer simulations of these interfaces (Figures 2, 4) as well as the definition of the Ising models can be found below. Similarly to mentioned experiments for percolation, Robert Langlands, Marc André Lewis and Yvan Saint-Aubin conducted in [21] numerical studies of crossing probabilities for the Ising model at critical temperature. A modification of the theorem above relating interfaces to SLE's (with drifts) in domains with five marked boundary points allows a rigorous setup for establishing their conjectures.

The proof is based on showing that a certain Fermionic lattice observable (or rather two similar ones for spin and random cluster models) is discrete analytic and solves a particular covariant Riemann Boundary Value Problem. Hence its limit is conformally covariant and can be calculated exactly. The statement is interesting in its own right, and can be used to study spin correlations. The observable studied has more manifest physics meaning than one in our percolation paper [38].

The methods lead to some progress in fairly general families of random cluster and $O(n)$ models, and not just on square lattices. In particular, besides Ising cases, they seem to suggest new proofs for all other known cases (i.e. site percolation on triangular lattice and uniform spanning tree).

In this note we will discuss this proof and general approach to scaling limits and conformal invariance of interfaces in the SLE context. We will also state some of the open questions and speculate on how one should approach other models.

We omit many aspects of this rich subject. We do not discuss the general mathematical theory of SLE curves or their connections to physics, for which interested reader can consult expository works [5, 10, 15, 44] and a book [22]. We do not mention the question of how to deduce the values of scaling exponents for lattice models with SLE help once convergence is known. It was explored in some detail only for percolation [43], where convergence is known and the required (difficult) estimates were already in place thanks to Harry Kesten [19]. We also restrict ourselves to one interface, whereas one can study the collection of all loops (cf. exposition [45]), and many of our considerations transfer to the loop soup observables. Finally, there are many other open questions related to conformal invariance, some of which are discussed in Oded Schramm's paper [34] in these proceedings.

2. Lattice models

We focus on two families of lattice models which have nice "loop representations". Those families include or are closely related to most of the "important" models, including percolation, Ising, Potts, spherical (or $O(n)$), Fortuin-Kasteleyn (or random cluster), self-avoiding random walk, and uniform spanning tree models. For their interrelations and for the discussion of many other relevant models one can consult the books [6, 12, 26, 27]. We also omit many references which can be found there.

There are various ways to understand the existence of the scaling limit and its conformal invariance. One can ask for the full picture, which can be represented as a loop collection (representing all cluster interfaces), random height function (changing by ± 1 whenever we cross a loop), or some other object. It however seems desirable to start with a simpler problem.

One can start with observables (like correlation functions, crossing probabilities), for which it is easier to make sense of the limit: there should exist a limit of a number sequence which is a conformal invariant. Though a priori it might seem to be a weaker goal than constructing a full scaling limit, there are indications that to obtain the full result it might be sufficient to analyze just one observable.

We will discuss an intermediate goal to analyze the law of just one interface, explain why working out just one observable would be sufficient, and give details on how to find an observable with a conformally invariant limit. To single out one interface, we consider a model on a simply connected domain with Dobrushin boundary conditions (which besides many loop interfaces enforce existence of an interface joining two boundary points a and b). We omit the discussion of the full scaling limit, as well as models on Riemann surfaces and with different boundary conditions.

2.1. Percolation. Perhaps the simplest model (to state) is Bernoulli percolation on the triangular lattice. Vertices are declared open or closed (grey or white in Figure 1) independently with probabilities p and $(1-p)$ correspondingly. The critical value is $p = p_c = 1/2$ – see [18, 11], in which case all colorings are equally probable.

Then each configuration can be represented by a collection of interfaces – loops which go along the edges of the dual hexagonal lattice and separate open and closed vertices.

We want to distinguish one particular interface, and to this effect we introduce Dobrushin boundary conditions: we take two boundary points a and b in a simply connected Ω (or rather its lattice approximation), asking the counterclockwise arc ab to be grey and the counterclockwise arc ba to be white. This enforces existence of a single non-loop interface which runs from a to b. The "loop gas" formulation of our model is that we consider all collections of disjoint loops plus a curve from a to b on hexagonal lattice with equal probability.

For each value of the lattice step $\epsilon > 0$ we approximate a given domain Ω by a lattice domain, which leads to a random interface, that is a probability measure μ_ϵ on curves (broken lines) running from a to b. The question is whether there is a

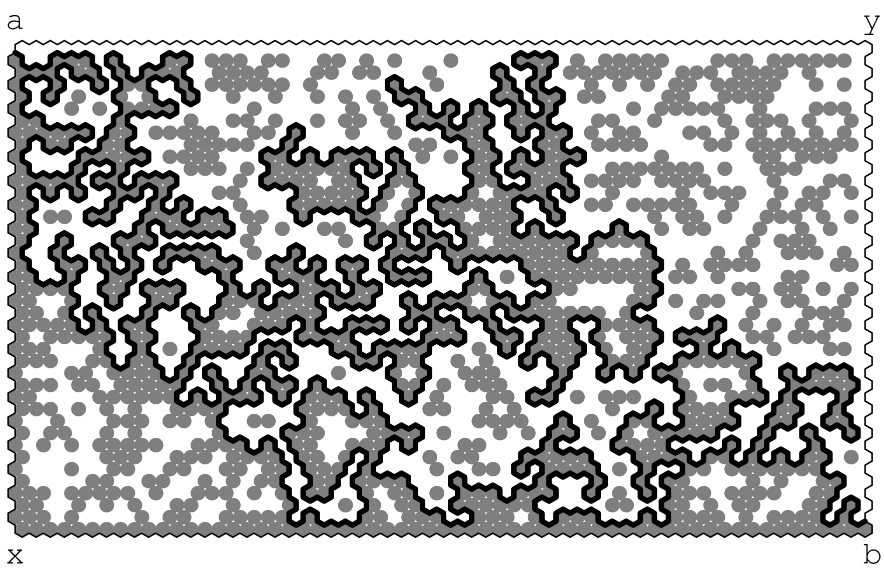

Figure 1. Critical site percolation on triangular lattice superimposed over a rectangle. Every site is grey or white independently with equal probability 1/2. Dobrushin boundary conditions (grey on lower and left sides, white on upper and right sides) produce an interface from the upper left corner a to the lower right corner b. The law of the interface converges to SLE (6) when lattice step goes to zero, while the rectangle is fixed.

limit measure $\mu = \mu(\Omega, a, b)$ on curves and whether it is conformally invariant. To make sense of the limit we consider the curves with uniform topology generated by parameterizations (with distance between γ_1 and γ_2 being $\inf \|f_1 - f_2\|_\infty$ where the infimum is taken over all parameterizations f_1, f_2 of γ_1, γ_2), and ask for weak-$*$ convergence of the measures μ_ϵ.

2.2. $O(n)$ and loop models.

Percolation turns out to be a particular case of the *loop gas* model which is closely related (via high-temperature expansion) to $O(n)$ (spherical) model. We consider configurations of non-intersecting simple loops and a curve running from a to b on *hexagonal lattice* inside domain Ω as for percolation in Figure 1. But instead of asking all configurations to be equally likely, we introduce two parameters: loop-weight $n \geq 0$ and edge-weight $x > 0$, and ask that probability of a configuration is proportional to

$$n^{\#\text{ loops}}\, x^{\text{length of loops}}\ .$$

The vertices not visited by loops are called monomers. Instead of weighting edges by x one can equivalently weight monomers by $1/x$.

We are interested in the range $n \in [0, 2]$ (after certain modifications $n \in [-2, 2]$ would work), where conformal invariance is expected (other values of n have different behavior). It turns out that there is a critical value $x_c(n)$, such that the model exhibits one critical behavior at $x_c(n)$ and another on the interval $(x_c(n), +\infty)$, corresponding to "dilute" and "dense" phases (when in the limit the loops are simple and non-simple correspondingly).

Bernard Nienhuis [28, 29] proposed the following conjecture, supported by physics arguments:

Conjecture 2. *The critical value is given by*

$$x_c(n) = \frac{1}{\sqrt{2+\sqrt{2-n}}} \ .$$

Note that though for all $x \in (x_c(n), \infty)$ the critical behavior (and the scaling limit) are conjecturally the same, the related value $\tilde{x}_c(n) = 1/\sqrt{2-\sqrt{2-n}}$ turns out to be distinguished in some ways.

The criticality was rigorously established for $n = 1$ only, but we still may discuss the scaling limits at those values of x. It is widely believed that at the critical values the model has a conformally invariant scaling limit. Moreover, the corresponding criticalities under renormalization are supposed to be unstable and stable correspondingly, so for $x = x_c$ there should be one conformally invariant scaling limit, whereas for the interval $x \in (x_c, \infty)$ another, corresponding to $\tilde{x}_c$. The scaling limit for low temperatures $x \in (0, x_c)$, a straight segment, is not conformally invariant.

Plugging in $n = 1$ we obtain weight

$$x^{\text{length of loops}} \ .$$

Assigning the spins ± 1 (represented by grey and white colors in Figure 1) to sites of triangular lattice, we rewrite the weight as

$$x^{\#\text{ pairs of neighbors of opposite spins}} \ , \tag{1}$$

obtaining the Ising model (where the usual parameterization is $\exp(-2\beta) = x$). The critical value is known to be $\beta_c = \log 3/4$, so one gets the Ising model at critical temperature for $n = 1$, $x = 1/\sqrt{3}$. A computer simulation of the Ising model on the square lattice at critical temperature, when the probability of configuration is proportional to (1), is shown in Figure 2.

For $n = 1$, $x = 1$ we obtain critical site percolation on triangular lattice. Taking $n = 0$ (which amounts to considering configurations with no loops, just a curve running from a to b), one obtains for $x_c = 1/\sqrt{2+\sqrt{2}}$ a version of the self-avoiding random walk.

The following conjecture (see e.g. [15]) is a direct consequence of physics predictions and SLE calculations:

Conjecture 3. *For $n \in [0, 2]$ and $x = x_c(n)$, as lattice step goes to zero, the law of the interface converges to Schramm-Loewner Evolution with*

$$\kappa = 4\pi/(2\pi - \arccos(-n/2)) \ .$$

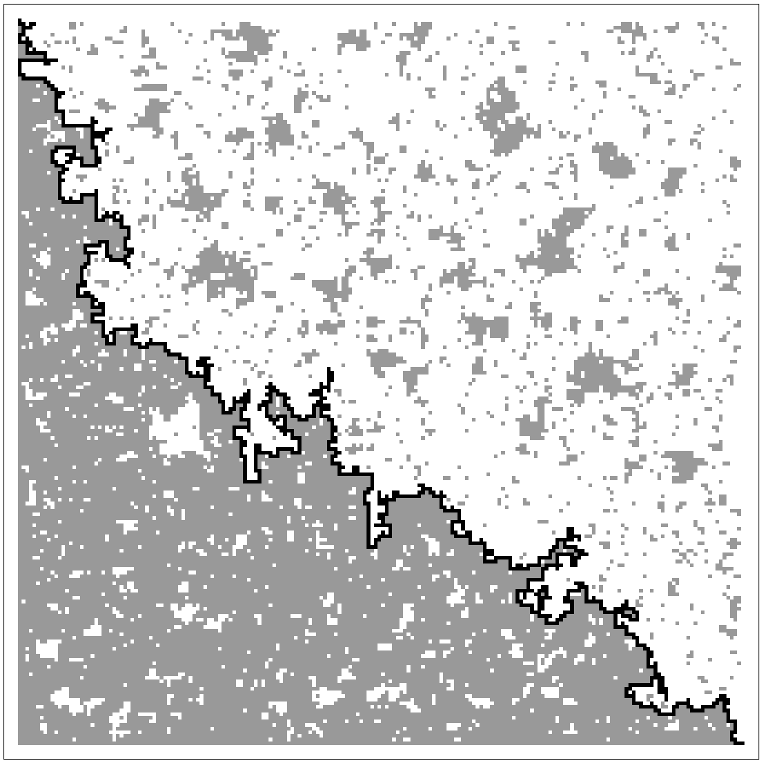

Figure 2. The Ising model at critical temperature on a square. White and grey sites represent ± 1 spins. Dobrushin boundary conditions (grey on lower and left sides, white on upper and right sides) produce, besides loop interfaces, an interface from the upper left to the lower right corner, pictured in black. When lattice step goes to zero, the law of the interface converges to SLE (3), which is a conformally invariant random curve, almost surely simple and of Hausdorff dimension 11/8.

For $n \in [0, 2]$ and $x \in (x_c, \infty)$ (in particular for $x = \tilde{x}_c$), as lattice step goes to zero, the law of the interface converges to Schramm-Loewner Evolution with

$$\kappa = 4\pi / \arccos(-n/2) \ .$$

Note that to address this question one does not need to prove that the Nienhuis temperature is indeed critical (Conjecture 2).

We discussed loops on the hexagonal lattice, since it is a trivalent graph and so at most one interface can pass through a vertex. One can engage in similar considerations on the square lattice with special regard to a possibility of two interfaces passing through the same vertex, in which case they can be split into loops in two different ways (with different configuration weights). In the case of Ising ($n = 1$) this poses less of a problem, since number of loops is not important. For $n = 1$ and $x = 1$ we get percolation model with $p = 1/2$, but for a general lattice this p need not be critical, so e.g. critical site percolation on the square lattice does not fit directly into this framework.

2.3. Fortuin-Kasteleyn random cluster models. Another interesting class is Fortuin-Kasteleyn models, which are random cluster representations of q-state Potts model. The random cluster measure on a graph (a piece of the *square lattice* in our case) is a probability measure on edge configurations (each edge is declared either open or closed), such that the probability of a configuration is proportional to

$$p^{\#\text{ open edges}}\,(1-p)^{\#\text{ closed edges}}\,q^{\#\text{ clusters}}\,,$$

where clusters are maximal subgraphs connected by open edges. The two parameters are edge-weight $p \in [0,1]$ and cluster-weight $q \in (0,\infty)$, with $q \in [0,4]$ being interesting in our framework (similarly to the previous model, $q > 4$ exhibits different behavior). For a square lattice (or in general any planar graph) to every configuration one can prescribe a cluster configuration on the dual graph, such that every open edge is intersected by a dual closed edge and vice versa. See Figure 3 for a picture of two dual configurations with respective open edges. It turns out that the probability of a dual configuration becomes proportional to

$$p_*^{\#\text{ dual open edges}}\,(1-p_*)^{\#\text{ dual closed edges}}\,q^{\#\text{ dual clusters}}\,,$$

with the dual to p value $p_* = p_*(p)$ satisfying $p_*/(1-p_*) = q(1-p)/p$. For $p = p_{\mathrm{sd}} := \sqrt{q}/(\sqrt{q}+1)$ the dual value coincides with the original one: one gets $p_{\mathrm{sd}} = (p_{\mathrm{sd}})_*$ and so the model is self-dual. It is conjectured that this is also the critical value of p, which was only proved for $q = 1$ (percolation), $q = 2$ (Ising) and $q > 25.72$.

Again we introduce Dobrushin boundary conditions: wired on the counterclockwise arc ab (meaning that all edges along the arc are open) and dual-wired on the counterclockwise arc ba (meaning that all dual edges along the arc are open, or equivalently all primal edges orthogonal to the arc are closed) - see Figure 3. Then there is a unique interface running from a to b, which separates cluster containing the arc ab from the dual cluster containing the arc ba.

We will work with the loop representation, which is similar to that in 2.2. The cluster configurations can be represented as Hamiltonian (i.e. including all edges) non-intersecting (more precisely, there are no "transversal" intersections) loop configurations on the medial lattice. The latter is a square lattice which has edge centers of the original lattice as vertices. The loops represent interfaces between cluster and dual clusters and turn by $\pm\frac{\pi}{2}$ at every vertex – see Figure 3. It is well-known that probability of a configuration is proportional to

$$\left(\frac{p}{1-p}\frac{1}{\sqrt{q}}\right)^{\#\text{ open edges}} \cdot (\sqrt{q})^{\#\text{ loops}}\,,$$

which for the self-dual value $p = p_{sd}$ simplifies to

$$(\sqrt{q})^{\#\text{ loops}}\,. \tag{2}$$

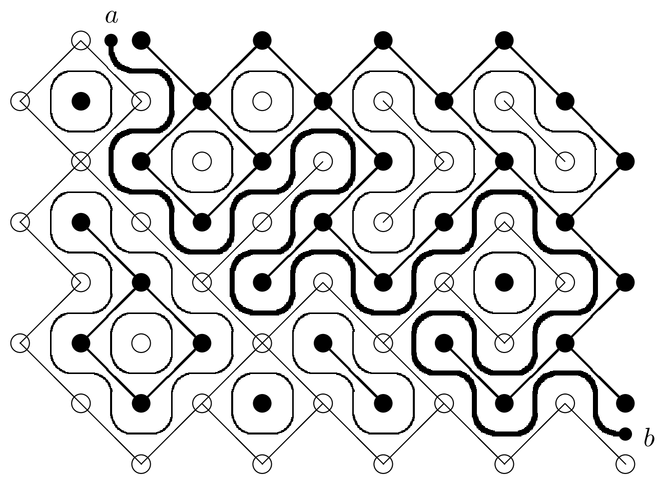

Figure 3. Loop representation of the random cluster model. The sites of the original lattice are colored in black, while the sites of the dual lattice are colored in white. Clusters and dual clusters and loops separating them are pictured. Under Dobrushin boundary conditions besides a number of loops there is an interface running from a to b, which is drawn in bold. Weight of the configuration is proportional to $(\sqrt{q})^{\#\ \text{loops}}$.

Dobrushin boundary conditions amount to introducing two vertices with odd number of edges: a source a and a sink b, which enforces a curve running form a to b (besides loops) – see Figure 3 for a typical configuration.

Conjecture 4. *For all $q \in [0,4]$, as the lattice step goes to zero, the law of the interface converges to Schramm-Loewner Evolution with $\kappa = 4\pi/\arccos(-\sqrt{q}/2)$.*

Conjecture was proved by Greg Lawler, Oded Schramm and Wendelin Werner [23] for the case of $q = 0$, when they showed that the perimeter curve of the uniform spanning tree converges to SLE (8). Note that with Dobrushin boundary conditions loop representation still makes sense for $q = 0$. In fact, the formula (2) means that we restrict ourselves to configurations with no loops, just a curve running from a to b (which then necessarily passes through all the edges), and all configurations are equally probable.

Below we will outline our proof [39, 40] that for the Ising parameter $q = 2$ the interface converges to SLE (16/3), see Figure 4. It almost directly translates into a proof that the interface of the spin cluster for the Ising model on the square lattice at the critical temperature (which can be rewritten as the loop model in 2.2 for $n = 1$, only on the square lattice) converges to SLE (3), as shown on Figure 2. It seems likely that it will work in the $n = 1$ case for the loop model on hexagonal lattice described above, providing convergence to SLE (3) for $x = x_c$ and (a new proof of) convergence to SLE (6) for $x = \tilde{x}_c$ (and possibly for all $x > x_c$).

Summing it up, Conjecture 3 was proved earlier for $n = 1$, $x = \tilde{x}_c$, see [38, 37],

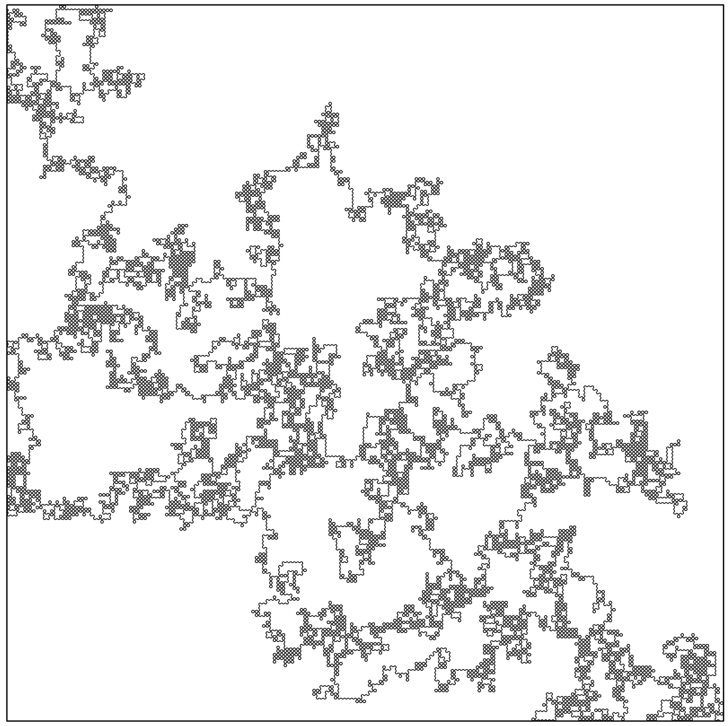

Figure 4. Interface in the random cluster Ising model at critical temperature with Dobrushin boundary conditions (loops not pictured). The law converges to SLE (16/3) when mesh goes to zero, so in the limit it has Hausdorff dimension 5/3 and touches itself almost surely. The random cluster is obtained by deleting some bonds from the spin cluster, so the interfaces are naturally different. Indeed, they converge to different SLE's and have different dimensions. However they are related: conjecturally, the outer boundary of the (non-simple) pictured curve and the (simple) spin interface in Figure 2 have the same limit after appropriate conditioning.

whereas Conjecture 4 was established for $q = 0$, see [23]. We outline a technique, which seems to prove conformal invariance in two new cases, and provide new proofs for the only cases known before, making Conjecture 4 solved for $q = 0$ and $q = 2$, and Conjecture 3 for $n = 1$. The method also contributes to our understanding of universality phenomenon.

Much of the method works for general values of n and q. The most interesting values of the parameters (where it does not yet work all the way) are $n = 0$, related to the self-avoiding random walk, and $q = 1$, equivalent to the critical bond percolation on the square lattice (in the latter case some progress was achieved by Vincent Beffara by a different method). Hopefully the lemma (essentially the discrete analyticity statement – see below) required to transfer our proof to other models will be worked out someday, leading to full resolution of these Conjectures.

3. Schramm-Loewner Evolution

3.1. Loewner Evolution. Loewner Evolution is a differential equation for a Riemann uniformization map for a domain with a growing slit. It was introduced by Charles Loewner in [25] in his work on Bieberbach's conjecture.

In the original work, Loewner considered slits growing towards interior point. Though such *radial* evolution (along with other possible setups) is also important in the context of lattice models and fits equally well into our framework, we will restrict ourselves to the *chordal* case, when the slit is growing towards a point on the boundary.

In both cases we choose a particular Riemann map by fixing its value and derivative at the target point. *Chordal Loewner Evolution* describes uniformization for the upper half-plane $\mathbb{C}_+$ with a slit growing from 0 to ∞ (one deals with a general domain Ω with boundary points a, b by mapping it to $\mathbb{C}_+$ so that $a \mapsto 0$, $b \mapsto \infty$).

Loewner only considered slits given by smooth simple curves, but more generally one allows any set which grows continuously in conformal metric when viewed from ∞. We will omit the precise definition of *allowed slits* (more extensive discussion in this context can be found in [22]), only noting that all simple curves are included. The random curves arising from lattice models (e.g. cluster perimeters or interfaces) are simple (or can be made simple by altering them on the local scale). Their scaling limits are not necessarily simple, but they have no "transversal" self-intersections. For such a curve to be an allowed slit it is sufficient if it touches itself to never venture into the created loop. This property would follow if e.g. a curve visits no point thrice.

Parameterizing the slit γ in some way by time t, we denote by $g_t(z)$ the conformal map sending $\mathbb{C}_+ \setminus \gamma_t$ (or rather its component at ∞) to $\mathbb{C}_+$ normalized so that at infinity $g_t(z) = z + \alpha(t)/z + \mathcal{O}(1/|z|^2)$, the so called *hydrodynamic normalization*. It turns out that $\alpha(t)$ is a continuous strictly increasing function (it is a sort of capacity-type parameter for γ_t), so one can change the time so that

$$g_t(z) \;=\; z + \frac{2t}{z} + \mathcal{O}\left(\frac{1}{|z|^2}\right) \tag{3}$$

Denote by $w(t)$ the image of the tip $\gamma(t)$. The family of maps g_t (also called a *Loewner chain*) is uniquely determined by the real-valued *"driving term"* $w(t)$. The general Loewner theorem can be roughly stated as follows:

Loewner's theorem. *There is a bijection between allowed slits and continuous real valued functions $w(t)$ given by the ordinary differential equation*

$$\partial_t g_t(z) \;=\; \frac{2}{g_t(z) - w(t)}\;, \quad g_0(z) \;=\; z\;. \tag{4}$$

The original Loewner equation is different since he worked with smooth radial slits and evolved them in another (but related) way.

3.2. Schramm-Loewner Evolution. While a deterministic curve γ corresponds to a deterministic driving term $w(t)$, a random γ corresponds to a random $w(t)$. One obtains SLE (κ) by taking $w(t)$ to be a Brownian motion with speed κ:

Definition 5. *Schramm-Loewner Evolution*, or SLE (κ), is the Loewner chain one obtains by taking $w(t) = \sqrt{\kappa} B_t$, $\kappa \in [0, \infty)$. Here B_t denotes the standard (speed one) Brownian motion (Wiener process).

The resulting slit will be almost surely a continuous curve. So we will also use the term SLE for the resulting random curve, i.e. a probability measure on the space of curves (to be rigorous one can think of a Borel measure on the space of curves with uniform norm). Different speeds κ produce different curves: we grow the slit with constant speed (measured by capacity), while the driving term "wiggles" faster. Naturally, the curves become more "fractal" as κ increases: for $\kappa \leq 4$ the curve is almost surely simple, for $4 < \kappa < 8$ it almost surely touches itself, and for $\kappa \geq 8$ it is almost surely space-filling (i.e. visits every point in $\mathbb{C}_+$) – see [22, 30] for these and other properties. Moreover, Vincent Beffara [7] has proved that the Hausdorff dimension of the SLE (κ) curve is almost surely $\min(1 + \kappa/8, 2)$.

3.3. Conformal Markov property. Suppose we want to describe the scaling limits of cluster perimeters, or interfaces for lattice models assuming their existence and conformal invariance. We follow Oded Schramm [32] to show that Brownian motion as the driving force arises naturally. Consider a simply connected domain Ω with two boundary points, a and b. Superimpose a lattice with mesh ϵ and consider some lattice model, say critical percolation with the Dobrushin boundary conditions, leading to an interface running from a to b, which is illustrated by Fig. 1 for a rectangle with two opposite corners as a and b. So we end up with a random simple curve (a broken line) connecting a to b inside Ω. The law of the curve depends of course on the lattice superimposed. If we believe the physicists' predictions, as mesh tends to zero, this measure on broken lines converges (in an appropriate weak-$*$ topology) to some measure $\mu = \mu(\Omega, a, b)$ on continuous curves from a to b inside Ω.

In this setup the conformal invariance prediction can be formulated as follows:

(A) Conformal Invariance. *For a conformal map ϕ of the domain Ω one has*

$$\phi\left(\mu(\Omega, a, b)\right) \ = \ \mu\left(\phi(\Omega), \phi(a), \phi(b)\right) \ .$$

Here a bijective map $\phi : \Omega \to \phi(\Omega)$ induces a map acting on the curves in Ω, which in turn induces a map on the probability measures on the space of such curves, which we denote by the same letter. By a conformal map we understand a bijection which locally preserves angles.

Moreover, if we start drawing the interface from the point a, we will be walking around the grey cluster following the right-hand rule – see Figure 1. If we stop at some point a' after drawing the part γ' of the interface, we cannot distinguish the boundary of Ω from the part of the interface we have drawn: they both are colored

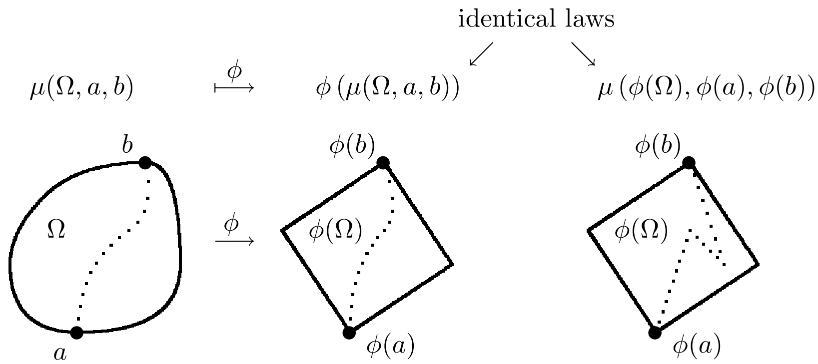

Figure 5. **(A) Conformal Invariance:** conformal image of the law of the curve γ (dotted) in Ω coincides with the law of the curve γ in the image domain $\phi(\Omega)$.

grey on the (counterclockwise) arc $a'b$ and white on the arc ba' of the domain $\Omega\setminus\gamma'$. So we can say that the conditional law of the interface (conditioned on it starting as γ') is the same as the law in a new domain with a slit. We expect the limit law μ to have the same property:

(B) Markov property. *The law conditioned on the interface already drawn is the same as the law in the slit domain:*

$$\mu\left(\Omega, a, b\right) | \gamma' \;=\; \mu\left(\Omega \setminus \gamma', a', b\right) \;.$$

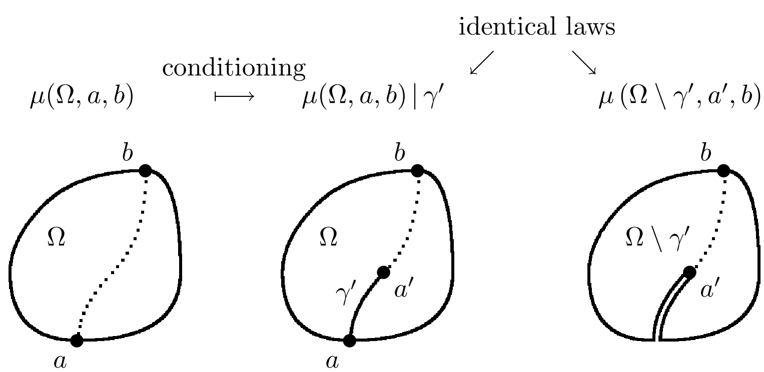

Figure 6. **(B) Markov property:** The law conditioned on the curve already drawn is the same as the law in the slit domain. In other words when drawing the curve we do not distinguish its past from the boundary.

If one wants to utilize these properties to characterize μ, by (A) it is sufficient to study some reference domain (to which all others can be conformally mapped), say the upper half-plane $\mathbb{C}_+$ with a curve running from 0 to ∞. Given (A), the second property (B) is easily seen to be equivalent to the following:

(B') Conformal Markov property. *The law conditioned on the interface already drawn is a conformal image of the original law. Namely, for any conformal map $G = G_{\gamma'}$ from $\mathbb{C}_+ \setminus \gamma'$ to $\mathbb{C}_+$ preserving ∞ and sending the tip of γ' to 0, we have*

$$\mu\left(\mathbb{C}_+, 0, \infty\right) | \gamma' \;=\; G^{-1}\left(\mu\left(\mathbb{C}_+, 0, \infty\right)\right) \;.$$

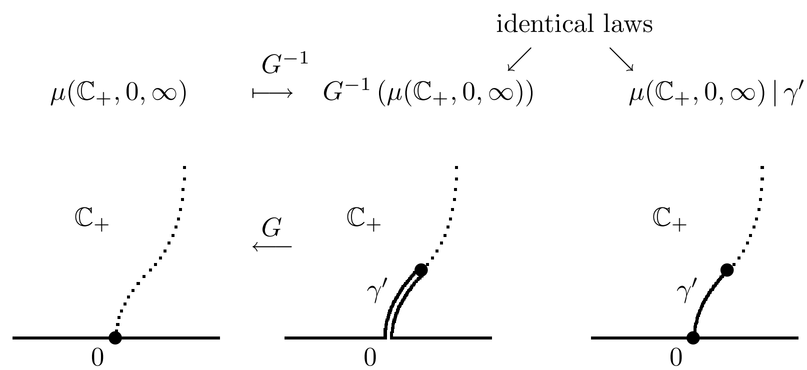

Figure 7. **(B') Conformal Markov property:** The law conditioned on the curve already drawn is a conformal image of the original law. In other words the curve has "Markov property in conformal coordinates".

Remark 6. Note that property (B') is formulated for the law μ for one domain only, say $\mathbb{C}_+$ as above. If we extend μ to other domains by conformal maps, it turns out that (A) and (B) are equivalent to (B') together with scale invariance (under maps $z \mapsto kz$, $k > 0$).

To use the property (B'), we describe the random curve by the Loewner Evolution with a certain random driving force $w(t)$ (we assume that the curve is almost surely an allowed slit). If we fix the time t, the property (B') with the slit $\gamma[0, t]$ and the map $G_t(z) = g_t(z) - w(t)$ can be rewritten for random conformal map $G_{t+\delta}$ conditioned on G_t (which is the same as conditioning on $\gamma[0, t]$) as

$$G_{t+\delta} | G_t \;=\; G_t\left(G_\delta\right) \;.$$

Expanding G's near infinity we obtain

$$z - w(t+\delta) + \ldots | G_t \;=\; (z - w(t) + \ldots) \circ (z - w(\delta) + \ldots) \;=\; z - (w(t) + w(\delta)) + \ldots \,,$$

concluding that

$$w(t+\delta) - w(t) | G_t \;=\; w(\delta) \;.$$

This means that $w(t)$ is a continuous (by Loewner's theorem) stochastic process with independent stationary increments. Thus for *a random curve satisfying (B')* the driving force $w(t)$ has to be a Brownian motion with a certain speed $\kappa \in [0, \infty)$ and drift $\alpha \in \mathbb{R}$:

$$w(t) = \sqrt{\kappa} B_t + \alpha t \;.$$

Applying (A) with anti-conformal reflection $\phi(u+iv) = -u+iv$ or with stretching $\phi(z) = 2z$ shows that α vanishes. So one logically arrives at the definition of SLE and the following

Schramm's principle. *A random curve satisfies (A) and (B) if and only if it is given by* SLE(κ) *for some* $\kappa \in [0, \infty)$.

The discussion above is essentially contained in Oded Schramm's paper [32] for the radial version, when slit is growing towards a point inside and the Loewner differential equation takes a slightly different form. To make this principle a rigorous statement, one has to require the curve to be almost surely an allowed slit.

4. SLE as a scaling limit

4.1. Strategy. In order to use the above principle one still has to show the *existence* and *conformal invariance* of the scaling limit, and then calculate some observable to pin down the value of κ. For percolation one can employ its locality or Cardy's formula for crossing probabilities to show that $\kappa = 6$. Based on this observation Oded Schramm concluded in [32] that *if* percolation interface has a conformally invariant scaling limit, it must be SLE(6).

But it is probably difficult to show that some interface has a conformally invariant scaling limit without actually identifying the latter.

To identify a random curve in principle one needs "infinitely many observables," e.g. knowing for any finite number of points the probability of passing above them. This seems to be a difficult task, which is doable for percolation since the locality allows us to create many observables from just one (crossing probability), see [38].

Fortunately it turns out that even in the general case if an observable has a limit satisfying analogues of (A) and (B), one can deduce convergence to SLE(κ) (with κ determined by the values of the observable).

This was demonstrated by Greg Lawler, Oded Schramm and Wendelin Werner in [23] in establishing the convergence of two related models: of loop erased random walk to SLE(2) and of uniform spanning tree to SLE(8).

They described the discrete curve by a Loewner Evolution with unknown random driving force. Stopping the evolution at times t and s and comparing the values of the observable, one deduces (approximate) formulae for the conditional expectation and variance of the increments of the driving force. Skorokhod embedding theorem is then used to show that driving force converges to the Brownian motion. Finally one has to prove a (stronger) convergence of the measures on curves. The trick is that knowing just one observable (but *for all domains*) after conditioning translates to a continuum of information about the driving force.

We describe a different approach with the same general idea, which is perhaps more transparent, separating "exact calculations" from "a priori estimates". The idea is to first get a priori estimates, which imply that collection of laws is precompact in a suitable space of allowed slits. Then to establish the convergence it is enough to show that limit of any converging subsequence coincides with SLE. To do that we describe the subsequential limit by Loewner Evolution (with unknown random driving force $w(t)$) and extract from the observable enough information to evaluate expectation and quadratic variation of increments of $w(t)$. Lévy's characterization implies that $w(t)$ is the Brownian motion with a particular speed κ and so our curves converge to SLE (κ).

As an example we discuss below an alternative proof of convergence to SLE (6) in the case of percolation, which uses crossing probability as an observable. For a domain Ω with boundary points a, b superimpose triangular lattice with mesh ϵ and Dobrushin boundary conditions. We obtain an interface γ_ϵ (between open vertices on one side and closed on another) running from a to b, see Fig. 1, i.e. a measure μ_ϵ on random curves running from a to b.

4.2. Compactness. First we note that the collection $\{\mu_\epsilon\}$ is precompact (in weak-$*$ topology) in the space of continuous curves that are Loewner allowed slits.

The necessary framework for precompactness in the space of continuous curves was suggested by Michael Aizenman and Almut Burchard [2]. It turns out that appropriate bounds for probability of an annulus being traversed k times imply tightness: a curve has a Hölder parameterization with stochastically bounded norm. Hence $\{\mu_\epsilon\}$ is precompact by Prokhorov's theorem: a (uniformly controlled) part of μ_ϵ is supported on a compact set (of curves with norm bounded by M), and so such parts are weakly precompact by Banach-Alaoglu theorem, whereas the mass of the remainder tends uniformly to zero as $M \to \infty$.

The curves on the lattice are simple, so they cannot have transversal self-intersections even after passing to the limit. So to check that for any weak limit of μ_ϵ's almost every curve is an allowed slit, one has to check that as we grow it the tip is always visible and moves continuously when viewed from infinity. Essentially, one has to rule out two scenarios: that the curve passes for a while inside already visited set, and that the curve closes a loop, and then travels inside before exiting. Both are reduced to probabilities of annuli traversing.

In the case of percolation one uses the Russo-Seymour-Welsh theory [31, 36] together with Michael Aizenman's observation [1] (that in the limit interface can visit no point thrice – "no 6 arms") to obtain the required estimates.

Since the collection of interface laws $\{\mu_\epsilon\}$ is precompact (in weak-$*$ topology) in the space of continuous curves that are Loewner allowed slits, to show that as mesh goes to zero the interface law converge to the law of SLE (6), it is sufficient to show that the limit of any converging subsequence is in fact SLE (6).

Take some subsequence converging to a random curve in the domain Ω from a to b. We map conformally to a half-plane $\mathbb{C}_+$, obtaining a curve γ from 0 to ∞ with law μ. We must show that μ is given by SLE (6).

By a priori estimates γ is almost surely an allowed slit. So we can describe γ by a Loewner Evolution with a (random) driving force $w(t)$. It remains to show that $w(t) = \sqrt{6}B_t$. Note that at this point we only know that $w(t)$ is an almost surely continuous random function – we do not even have a Markov property.

4.3. Martingale observable. Given a topological rectangle (a simply connected domain Ω with boundary points a, b, c, d) one can superimpose a lattice with mesh ϵ onto Ω and study the probability $\Pi_\epsilon\left(\Omega,[a,b],[c,d]\right)$ that there is an open cluster joining the arc $[a,b]$ to the arc $[c,d]$ on the boundary of Ω. It is conjectured that there is a limit $\Pi := \lim_{\epsilon\to 0}\Pi_\epsilon$, which is conformally invariant (depends only on the conformal modulus of the configuration Ω, a, b, c, d), and satisfies Cardy's formula (predicted by John Cardy in [9] and proved in [37]) in half-plane:

$$\Pi\left(\mathbb{C}_+,[1-u,1],[\infty,0]\right) \;=\; \frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)}\,u^{1/3}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};u\right) \;=:\; F(u)\;. \quad (5)$$

Above ${}_2F_1$ is the hypergeometric function, so one can alternatively write

$$F(u) \;=\; \int_0^u \left(v(1-v)\right)^{-2/3}dv \bigg/ \int_0^1 \left(v(1-v)\right)^{-2/3}dv\;.$$

Particular nature of the function is not important, we rather use the fact that there is an explicit formula for half-plane with four marked boundary points and hence by conformal invariance for an arbitrary topological rectangle. The value $\kappa = 6$ will arise later from some expression involving derivatives of F.

Assume that for some percolation model we are able to prove the above conjecture (for critical site percolation on the triangular lattice it was proved in [37, 38]).

Add two points on the boundary, making Ω a topological rectangle $axby$ and consider the crossing probability $\Pi_\epsilon\left(\Omega,[a,x],[b,y]\right)$ (from the arc ax to the arc by on a lattice with mesh ϵ).

Parameterize the interface γ_ϵ in some way by time, and draw the part $\gamma_\epsilon[0,t]$. Note that it has open vertices on one side (arc $\gamma_\epsilon(t)a$) and closed on another (arc $a\gamma_\epsilon(t)$). Then any open crossing from the arc by to the arc ax inside Ω is either disjoint from $\gamma_\epsilon[0,t]$, or hits its "open" arc $\gamma_\epsilon(t)a$. In either case it produces an open crossing from the arc by to the arc $\gamma_\epsilon(t)x$ inside $\Omega\setminus\gamma_\epsilon[0,t]$, and converse also holds. Therefore one sees that for every realization of $\gamma_\epsilon[0,t]$ the crossing probability conditioned on $\gamma_\epsilon[0,t]$ coincides with crossing probability in the slit domain $\Omega\setminus\gamma_\epsilon[0,t]$:

$$\Pi_\epsilon\left(\Omega,[a,x],[b,y]|\gamma_\epsilon[0,t]\right) = \Pi_\epsilon\left(\Omega\setminus\gamma_\epsilon[0,t],[\gamma_\epsilon(t),x],[b,y]\right)\;, \quad (6)$$

– an analogue of the Markov property (B). Alternatively this follows from the fact that Π can be understood in terms of the interface as the probability that it touches the arc xb before the arc by. For example, on Figure 1 there is no horizontal grey (open) crossing (there is a vertical white crossing instead), and interface traced from the left upper corner a touches the lower side xb before the right side by.

Stopping the curve at times $t < s$ and using (6) we can write by the total probability theorem for every realization of $\gamma_\epsilon[0,t]$

$$\begin{aligned}&\Pi_\epsilon\left(\Omega\setminus\gamma_\epsilon[0,t],[\gamma_\epsilon(t),x],[b,y]\right)\\ &\quad= \mathbb{E}_{\gamma_\epsilon[t,s]}\left(\Pi_\epsilon\left(\Omega\setminus\gamma_\epsilon[0,s],[\gamma_\epsilon(s),x],[b,y]\right)|\gamma[0,t]\right)\;. \end{aligned} \quad (7)$$

The same a priori estimates as in the previous subsection show that the identity (7) also holds for the (subsequential) scaling limit μ (strictly speaking there is an error term in case the interface touches the arcs $[ax]$ or $[ya]$ before time s, but it decays very fast as we move x and y away from a). We know that the scaling limit $\Pi := \lim_{\epsilon\to 0}\Pi_\epsilon$ of the crossing probabilities exists and is conformally invariant, so we can rewrite (7) for the curve γ with Loewner parameterization as

$$\begin{aligned}&\Pi\left(\mathbb{C}_+ \setminus \gamma[0,t], [\gamma(t), x], [\infty, y]\right)\\ &\quad = \mathbb{E}_{\gamma[t,s]}\left(\Pi\left(\mathbb{C}_+ \setminus \gamma[0,s], [\gamma(s), x], [\infty, y]\right) | \gamma[0,t]\right)\ ,\end{aligned} \tag{8}$$

for almost every realization of $\gamma[0,t]$. Moreover we can plug in exact values of the crossing probabilities, given by the Cardy's formula. Recall that the domain $\mathbb{C}_+ \setminus \gamma[0,t]$ is mapped to half-plane by the map $g_t(z)$ with $\gamma(t) \mapsto w(t)$. Then the map $z \mapsto \frac{g_t(z)-g_t(y)}{g_t(x)-g_t(y)}$ also maps it to half-plane with $\gamma(t) \mapsto \frac{w(t)-g_t(y)}{g_t(x)-g_t(y)}$, $y \mapsto 0$, $x \mapsto 1$. Using conformal invariance and applying Cardy's formula we write

$$\begin{aligned}\Pi\left(\mathbb{C}_+ \setminus \gamma[0,t], [\gamma(t), x], [\infty, y]\right) &= \Pi\left(\mathbb{C}_+, \left[-\frac{g_t(y)-w(t)}{g_t(x)-g_t(y)}, 1\right], [\infty, 0]\right)\\ &= F\left(\frac{g_t(x)-w(t)}{g_t(x)-g_t(y)}\right)\ ,\end{aligned} \tag{9}$$

for Cardy's hypergeometric function F.

4.4. Conformally invariant martingale.

Plugging (9) into both sides of (8) we arrive at

$$F\left(\frac{g_t(x)-w(t)}{g_t(x)-g_t(y)}\right) = \mathbb{E}_{\gamma[t,s]}\left(F\left(\frac{g_s(x)-w(s)}{g_s(x)-g_s(y)}\right) | \gamma[0,t]\right)\ . \tag{10}$$

Remark 7. Denote by $x_t := g_t(x) - w(t)$ and $y_t := g_t(y) - w(t)$ trajectories of x and y under the random Loewner flow. Then (10) essentially means that $F\left(\frac{x_t}{x_t-y_t}\right)$ is a martingale.

Since we want to extract the information about $w(t)$, we fix the ratio $x/(x-y) := 1/3$ (anything not equal to $1/2$ would do) and let x tend to infinity: $y := -2x$, $x \to +\infty$. Using the normalization $g_t(z) = z + 2t/z + \mathcal{O}(1/z^2)$ at infinity, writing Taylor expansion for F, and plugging in values of derivatives of F at $1/3$, we obtain the following expansion for the right-hand side of (10):

$$\begin{aligned}
\ldots &= F\left(\frac{x-w(t)+2t/x+\mathcal{O}(1/x^2)}{(x+2t/x+\mathcal{O}(1/x^2))-(-2x+2t/(-2x)+\mathcal{O}(1/x^2))}\right)\\
&= F\left(\frac{1}{3}-\frac{w(t)}{3}\frac{1}{x}+\frac{t}{3}\frac{1}{x^2}+\mathcal{O}\left(\frac{1}{x^3}\right)\right)\\
&= F\left(\frac{1}{3}\right)-\frac{w(t)}{3}F'\left(\frac{1}{3}\right)\frac{1}{x}+\left(\frac{t}{3}F'\left(\frac{1}{3}\right)+\frac{w(t)^2}{3^2\cdot 2}F''\left(\frac{1}{3}\right)\right)\frac{1}{x^2}+\mathcal{O}\left(\frac{1}{x^3}\right)\\
&= F\left(\frac{1}{3}\right)-\frac{1}{x}\frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)}\frac{3^{1/3}}{2^{2/3}}\mathbb{E}\,w(t)\\
&\quad -\frac{1}{x^2}\frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)}\frac{1}{3^{2/3}2^{5/3}}\mathbb{E}\left(w(t)^2-6t\right)+\mathcal{O}\left(\frac{1}{x^3}\right)\\
&=: A-\frac{1}{x}B\,\mathbb{E}\,w(t)-\frac{1}{x^2}C\,\mathbb{E}\left(w(t)^2-6t\right)+\mathcal{O}\left(\frac{1}{x^3}\right)\ ,
\end{aligned}$$

where we plugged in values of the derivative for hypergeometric function. Using similar reasoning for the right-hand side of (10) we arrive at the following identity:

$$\begin{aligned}
&A-\frac{1}{x}B\,\mathbb{E}\,w(t)-\frac{1}{x^2}C\,\mathbb{E}\left(w(t)^2-6t\right)+\mathcal{O}\left(\frac{1}{x^3}\right)\\
&\quad = A-\frac{1}{x}B\,\mathbb{E}_{\gamma[t,s]}\left(w(s)|\gamma[0,t]\right)-\frac{1}{x^2}C\,\mathbb{E}_{\gamma[t,s]}\left(w(s)^2-6s|\gamma[0,t]\right)+\mathcal{O}\left(\frac{1}{x^3}\right).
\end{aligned}$$

Equating coefficients in the series above, we conclude that

$$\mathbb{E}_{w[t,s]}\left(w(s)|w[0,t]\right)=0\ ,\quad \mathbb{E}_{w[t,s]}\left(w(s)^2-6s|w[0,t]\right)=w(t)^2-6t\ . \tag{11}$$

Thus $w(t)$ is a continuous (by Loewner's theorem) process such that both

$$w(t)\quad\text{and}\quad w(t)^2-6t$$

are martingales so by Lévy's characterization of the Brownian motion $w(t)=\sqrt{6}B_t$, and therefore SLE (6) is the scaling limit of the critical percolation interface.

The argument will work wherever Cardy's formula and a priori estimates are available, particularly for triangular lattice. More generally, any conformally invariant martingale will do, with value of κ arising from its Taylor expansion.

Remark 8. The scheme can also be reversed to do calculations for SLE's, if an observable is a martingale (e.g. crossing probability). Indeed, writing the same formulae with $x/(x-y)=a$ we conclude that the coefficient by $1/x^2$, namely

$$C=\frac{2a(1-2a)}{1-a}tF'(a)+\frac{a}{2}\mathbb{E}\left(w(t)^2\right)F''(a)\ ,$$

vanishes. Since for $w(t) = \sqrt{6}B(t)$ one has $\mathbb{E}\left(w(t)^2\right) = 6t$, we arrive at the differential equation

$$\frac{2(1-2a)}{3(1-a)}F'(a) + F''(a) = 0.$$

With the given boundary data it has a unique solution, which is Cardy's hypergeometric function.

5. Ising model and beyond

The martingale method as described above shows that to construct a conformally invariant scaling limit for some model we need a priori estimates and a non-trivial martingale observable with a conformally invariant scaling limit.

5.1. A priori estimates. A priori estimates are necessary to show that collection of interface laws is precompact in weak-$*$ topology (on the space of measures on continuous curves which are allowed slits).

If we follow the same route as for percolation (via the work [2] of Michael Aizenman and Almut Burchard), we only need to evaluate probabilities of traversals of an annulus in terms of its modulus. For percolation such estimates are (almost) readily available from the Russo-Seymour-Welsh theory. For uniform spanning tree and loop erased random walk one can derive the estimates using random walk connection and the known estimates for the latter (a "branch" of a uniform spanning tree is a loop erased random walk), see [3, 32].

For the Ising model the required estimates do not seem to be readily available, but a vast arsenal of methods is at hand. Essentially all we need can be reduced by monotonicity arguments to spin correlation estimates of Bruria Kaufman, Lars Onsager and Chen Ning Yang [14, 47].

For general random cluster or loop models such exact results are not available, but we actually need much weaker statements, and many of the techniques used by us for the Ising model (like FKG inequalities) are well-known in the general case.

So this part does not seem to be the main obstacle to construction of scaling limits, though it might require very hard work. Moreover, following the proposed approach we actually get that interfaces have a Hölder parameterization with uniformly stochastically bounded norm. Thus rather weak kinds of convergence of interfaces would lead to convergence in uniform norm (or rather weak-$*$ convergence of measures on curves with uniform norm).

It also appears that the same a priori estimates can be employed to show observable convergence in the cases concerned, and hopefully they will be sufficient for other models. So a more pressing question is how to construct a martingale observable.

5.2. Conformally covariant martingales. Suppose that for every simply connected domain Ω with a boundary point a we have defined a random curve

γ starting from a. Mark several points $b, c, \dots$ in Ω or on the boundary. Remark 7 suggests the following definition:

Definition 9. We say that a function (or rather a *differential*) $F(\Omega, a, b, c, \dots)$ is a *conformal (covariant) martingale* for a random curve γ if

$$\begin{aligned} F \text{ is conformally covariant}: \quad & F\,(\Omega, a, b, c, \dots) = \\ = & F\,(\phi(\Omega), \phi(a), \phi(b), \phi(c), \dots) \cdot \phi'(b)^\alpha \bar{\phi}'(b)^\beta \phi'(c)^\gamma \bar{\phi}'(c)^\delta \dots, \end{aligned} \tag{12}$$

and

$$F(\Omega \setminus \gamma[0,t], \gamma(t), b, c, \dots) \quad \text{is a martingale} \tag{13}$$

with respect to the random curve γ drawn from a (with Loewner parameterization).

Introducing covariance at $b, c, \dots$ we do not ask for covariance at a, since it always can be rewritten as covariance at other points. And applying factor at a would be troublesome: once we started drawing a curve the domain becomes non-smooth in its neighborhood, creating problems with the definition.

If the exponents $\alpha, \beta, \dots$ vanish, we obtain an invariant quantity. While the crossing probability for the percolation was invariant, many quantities of interest in physics are covariant differentials, e.g. cluster density at c would scale as a lattice step to some power (depending on the model), so we would arrive at a factor

$$|\phi'(c)|^\delta = \phi'(c)^{\delta/2} \bar{\phi}'(c)^{\delta/2} .$$

There are other possible generalizations, e.g. one can add the Schwarzian derivative of ϕ to (12). The two properties in Definition 9 are analogues of (A) and (B), and similarly combined they show that for the curve γ mapped to half-plane from any domain Ω so that $a \mapsto 0$, $b \mapsto \infty$, $c \mapsto x$ (note that the image curve in $\mathbb{C}_+$ might depend on Ω – we only know the conformal invariance of an observable, not of the curve itself) we have an analogue of (B'), which was already mentioned in Remark 7 for percolation. Namely

$$F\,(\mathbb{C}_+, 0, \infty, g_t(x), \dots) \cdot g_t'(x)^\gamma \bar{g}_t'(x)^\delta \dots,$$

is a martingale with respect to the random Loewner evolution (covariance factor at $b = \infty$ is absent, since $g_t'(\infty) = 1$).

The equation (10) can be written for this F, and if we can evaluate F exactly, the same machinery as one used by Greg Lawler, Oded Schramm and Wendelin Werner in [23] or as the one discussed above for percolation proves that our random curve is SLE. So one arrives at a following generalization of Oded Schramm's principle:

Martingale principle. *If a random curve γ admits a (non-trivial) conformal martingale F, then γ is given by* SLE *with κ (and drift depending on modulus of the configuration) derived from F.*

Remark 10. In chordal situation we consider curves growing from a towards another boundary point b in a simply connected domain. But the same conclusion would hold on general domains or Riemann surfaces with boundary once we find a covariant martingale (for appropriate generalizations of Loewner Evolutions see e.g. the book [4]). The only difference is that driving force of the corresponding Loewner Evolution will be a Brownian motion with drift depending on conformal modulus of the configuration $\Omega, a, b, c, \dots$, leading to SLE generalizations. Starting from lattice models with various boundary conditions and conditioned on various events, one can see which drifts will be of interest for SLE generalizations.

5.3. Discrete analyticity.

Passing to the lattice model, we want to find a discrete object, which in the limit becomes a conformally covariant martingale.

Martingale property is actually more accessible in the discrete setting. For example, functions which are defined as observables (like probability of the interface going through a vertex, edge density for the model, etc.) have the martingale property built in, and so only conformal covariance must be established.

Alternatively, one can work with a discrete function $F(\Omega, a, b, c)$ (a priori not related to lattice models) which has a conformally covariant scaling limit by construction. Then we need to connect it to a particular lattice model, establishing a martingale property (13). In the discrete case it is sufficient to check the latter for a curve advanced by one step. Assume that once we have drawn the part γ' of the interface from the point a to point a', it turns left with probability $p = p(\Omega, \gamma', a', b)$ creating a curve $\gamma_l = \gamma \cup \{a_l\}$ or right with probability $(1-p)$ creating a curve $\gamma_r = \gamma \cup \{a_r\}$. Then it is enough to check the identity

$$F(\Omega \setminus \gamma', a', b, c) = pF(\Omega \setminus \gamma_l, a_l, b, c) + (1-p)F(\Omega \setminus \gamma_r, a_r, b, c) \ , \qquad p = p(\Omega, \gamma', a', b) \ . \tag{14}$$

Actually our proof for the Ising model can be rewritten that way, with F defined as a solution of an appropriate discretization of the Riemann Boundary Value Problem (17) – the observable nature of F never comes up.

Moreover, starting with F one can define a random curve by choosing "turning probabilities" p so that identity (14) is satisfied, obtaining a model with conformally invariant scaling limit by "reverse engineering." For example, starting with a harmonic function of c with boundary values 1 on the arc ba and 0 on the arc ab, one obtains a unique discrete random curve, which has it as a martingale. Note that such a function is a particular case $\alpha = 1$ of the martingale (15) below, corresponding to $\kappa = 4$ (or rather its integral). In [35] Oded Schramm and Scott Sheffield introduced this curve with a nicer "Harmonic Explorer" definition, and utilizing the mentioned observable showed that it indeed converges to SLE (4). It seems that in this way one can use the solutions to the problem (17) to construct models converging to arbitrary SLE's, however it is not clear though whether they would similarly have "nicer" definitions.

Anyway, for either approach to work we need a *discrete conformal covariant* with a scaling limit. We have tried discretizations of many conformally invariant objects (extremal length, capacity, solutions to variational problems, ...) and the most promising in this context seem to be discrete harmonic or analytic functions in additional variable(s) (in $c, \dots$). Firstly, all other invariants can be rewritten in this way. Secondly, discretization of harmonic and analytic functions is a nice and very well studied (especially in the case of harmonic ones) object. Thirdly, one can obtain very non-trivial invariants by just checking local conditions: harmonicity or analyticity inside plus some boundary conditions (Dirichlet, Neumann, Riemann-Hilbert, etc.). The most natural candidate would be a harmonic function solving some Dirichlet problem.

Note that such an observable is known for the Brownian motion. A classical theorem [13] of Shizuo Kakutani states that in a domain Ω exit probabilities for Brownian motion started at z are harmonic functions in z with easily determinable boundary values. Though Kakutani works directly with Brownian motion, one can do the same for the random walk (which is actually much easier, since discrete Laplacian of the exit probability is trivially zero), and then passing to a limit deduce statements about Brownian motion, including its conformal invariance.

5.4. Classification of conformal martingales. Before we start working in the discrete setting, we might want to investigate which functions are conformal martingales for SLE curves, and so can arise as scaling limits of martingale observables for lattice models.

As discussed in Remark 8, one can write partial differential equations for SLE conformal martingales. For small number of points those equations can be solved, and in such a way one computes dimensions, scaling exponents and other quantities of interest. For any particular value of κ we can see which martingales have the simplest form and so are probably easier to work with. Also if they have a geometric SLE interpretation (like probability of SLE curve going to one side of a point, etc.) we can study similar quantities for the lattice model.

It turns out that only for $\kappa = 4$ one obtains a nice harmonic martingale with Dirichlet boundary conditions. In that case the probability of SLE (4) passing to one side of a point z is harmonic in z and has boundary conditions 0 and 1, see [33]. Oded Schramm and Scott Sheffield [35] constructed a model which has this property on discrete level built in. Unfortunately the property was not yet observed in any of the classical models conjecturally converging to SLE (4), though results of Kenyon [16] show it holds for double-domino curves in Temperley domains (i.e. a domain with the boundary satisfying a certain local condition).

In the case of mixed Dirichlet-Neumann conditions, it becomes possible to work with some other values of κ, including uniform spanning tree $\kappa = 8$, which is exploited in [23]. There are also covariant candidates for a few other values of κ (notably 8/3 which corresponds to self-avoiding random walk), but they were not yet observed in lattice models.

Thus to study general models, one is forced to utilize more general boundary value problems with a Riemann(-Hilbert) Boundary Value Problem being the natural candidate. Besides harmonic function it involves its harmonic conjugate, and so is better formulated in terms of analytic functions. Moreover, discrete analyticity involves a first order Cauchy-Riemann operator, rather than a second order

Laplacian, and so it should be easier to deal with than harmonicity.

As discussed above, we can classify all analytic martingales. For chordal SLE and F with three points a, b, z as parameters we discover two particularly nice families. The following Proposition will be discussed in [40, 42] and our subsequent work:

Proposition 11. *Let Ω be a simply connected domain with boundary points a, b. Let $\Phi(z) = \Phi(\Omega, a, b, z)$ be a mapping of Ω to a horizontal strip $\mathbb{R} \times [0, 1]$, such that a and b are mapped to $\mp\infty$. Then*

$$F(\Omega, a, b, z) = \Phi'(z)^{\alpha} \quad \text{with} \quad \alpha = \frac{8}{\kappa} - 1\ , \tag{15}$$

is a martingale for SLE(κ). *Let $\Psi(z) = \Psi(\Omega, a, b, z)$ be a mapping of Ω to a half-plane $\mathbb{C}_+$, such that a and b are mapped to ∞ and 0 correspondingly. Then*

$$G(\Omega, a, b, z) = \Psi'(z)^{\alpha} \Psi'(b)^{-\alpha} \quad \text{with} \quad \alpha = \frac{3}{\kappa} - \frac{1}{2}\ , \tag{16}$$

is a martingale for SLE(κ).

These martingales make most sense for $\kappa \in [4, 8]$ and $\kappa \in [8/3, 8]$ correspondingly, and are related to observables of interest in Conformal Field Theory (which was part of our motivation to introduce them). Note that both functions are covariant with power α (which is the spin in physics terminology), and solve the Riemann boundary value problems

$$\operatorname{Im}\left(F(z)\tau(z)^{\alpha}\right) = 0, \quad z \in \partial\Omega, \tag{17}$$

where $\tau(z)$ is the tangent vector to $\partial\Omega$ at z.

The problem is to observe these functions in the discrete setting, and some intuition can be obtained from their geometric meaning for SLE's. For example, F is roughly speaking (one has to consider an intermediate scale to make sense of it) an expectation of SLE curve passing through z taken with some complex weight depending on the winding.

5.5. Height models and Coulomb gas.

The above-mentioned expectation actually makes more sense (and is immediately well-defined) in the discrete setting and one arrives at the same object with the same complex weight via several different approaches.

One way is to consider the Coulomb gas arguments (cf. [29] by Bernard Nienhuis) for the loop representation. In the random cluster case at criticality, the weight of a loop is $\sqrt{q}$ – recall (2). We randomly and independently orient the loops, and introduce the height function h which whenever a loop is crossed changes by ± 1 (depending on loop direction – think of a topographic map). One could weight oriented loops by $\sqrt{q}/2$, obtaining essentially the same model. However it makes sense to consider a complex weight instead. When q is in the $[0, 4]$ range, there is a complex unit number $\mu = \exp(k \cdot 2\pi i)$ such that

$$k = \frac{1}{2\pi} \arccos\left(\sqrt{q}/2\right) \quad \text{or} \quad \mu + \bar{\mu} = \sqrt{q}. \tag{18}$$

We (independently and randomly) orient all loops, prescribing weight μ per counterclockwise and $\bar{\mu}$ per clockwise loop.

Forgetting orientation of loops reconstructs the original model. Unfortunately the new partition function is complex and no longer leads to a probability measure (moreover, its variation blows up as the lattice step goes to zero), but it can be defined locally, making it much more accessible.

Indeed, going around a cycle, and turning by Δ_z at vertex z, the total sum of turns $\sum_{z\in\text{cycle}}\Delta_z$ is $\pm 2\pi$ depending on whether the cycle is counter or clockwise. So the weight per cycle can be written as $\prod_{z\in\text{cycle}}\exp(ik\cdot\Delta_z)$ and the total weight of the configuration is $\prod_{z\in\Omega}\exp(ik\cdot\Delta_z)$, which can be computed locally (without reference to the global order of cycles). The same weight can also be written in terms of the gradient of height function.

The interface is always oriented from a to b, so that the height function is always equal to 0 on the arc ab and to 1 on the arc ba. From physics arguments the interface curve (being “attached” to the boundary on both sides) should be weighted differently from loops, namely by $\exp(i(2k-1/2)\cdot\Delta_z)$ per turn. When interface runs between two boundary points (being oriented from a to b), these factors do not matter, since total turn from a to b is independent of the configuration.

However, if we choose a point z on an interface and reverse the orientation of one of its halves (so that it is oriented from a to z and from z to b), the interface inputs a non-constant complex factor. This orientation reversal has a nice meaning: after it the height function acquires a $+2$ monodromy at z: when we go around z we cross two curves (halves of the interface) incoming into it.

All the loops (when we forget their orientation) still contribute the same $\sqrt{q}$ per loop, and the complex weight can be expressed in terms of the interface winding (total turn expressed in radians) from b to z, denoted by $w(\gamma, b\to z)$. So one logically arrives at the partition function Z for our model with $+2$ monodromy at z:

$$F(\Omega,a,b,z) := Z_{+2 \text{ monodromy at } z} = \mathbb{E}\chi_{z\in\gamma}\exp\left(i(4k-1)w(\gamma,b\to z)\right). \qquad (19)$$

This function is clearly a martingale, and there are strong indications (both from mathematics and physics points of view) that it is discrete analytic.

This follows from the fact that the interface can arrive at a boundary point z from b with a unique winding equal to the winding of the boundary from b to z, so we can express it in terms of the tangent vector $\tau(z)$. Writing this down, we discover that the function F solves a discrete version of the Riemann Boundary Value Problem (17) with $\alpha = 1-4k$.

Remark 12. The continuum problem was solved by the function (15), so if we establish discrete analyticity it only remains to show that a solution to a discrete Riemann Boundary Value Problem converges to its continuum counterpart. Moreover, combining identity $\alpha = 1-4k$ with (15) and (18) we obtain the relation between κ and q stated in Conjecture 4.

This convergence problem seems to be difficult (and open in the general case). The way we solve it in the Ising case is sketched below.

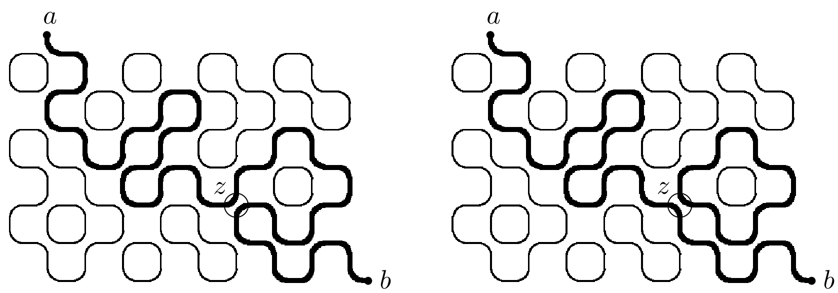

Figure 8. Rearrangement at a point z: we only change connections inside a small circle marking z. Either interface does not pass through z in both configurations, or it passes in a way similar to the pictured above. On the left the interface (in bold) passes through z twice, on the right (after the rearrangement) it passes once, but a new loop through z appears (also in bold). The loops not passing through z remain the same, so the weights of configurations differ by a factor of $\sqrt{q}$ because of the additional loop on the right. To get some linear relation on values of F, it is enough to check that any pair of such configurations makes equal contributions to two sides of the relation.

There are other indications that this function is nice to work with. Indeed, the easiest form of discrete analyticity involves local partial difference relations, and to prove those we should count configurations included into our expectation. To obtain relations, we need some bijections in the configuration space, and the easiest ones are given by local rearrangements (we worked with global rearrangements for percolation [37], but such work must be more difficult for non-local models).

The easiest rearrangement involves redirecting curves passing through z, see Figure 8, and we have a good control over relative weights of configurations whenever they are defined through windings. Counting how much a pair of configurations contributes to values of F at neighbors of z, we get some relations. Moreover, a careful analysis shows that the maximal number of relations is attained with the complex weight (19).

5.6. Ising model.

We finish with a sketch of our proof for the random cluster representation of the Ising model (i.e. $q = 2$) on the square lattice $\epsilon\mathbb{Z}^2$ at the critical temperature. As before consider loop representation in a simply connected domain Ω with two boundary points a and b and Dobrushin boundary conditions.

Consider function $F = F_\epsilon(\Omega, a, b, z)$ given by (19) which is the expectation that interface from a to b passes through a vertex z taken with appropriate unit complex weight. Note that for Ising $q = 2$, so $k = 1/8$ and the weight is Fermionic (which of course was expected): a passage in the same direction but with a 2π twist has a relative weight -1, whereas a passage in the opposite direction with a counterclockwise π twist has a relative weight $-i$.

As discussed F automatically has the martingale property when we draw γ starting from a, so only conformal invariance in the limit has to be checked.

Color lattice vertices in chessboard fashion, and to each edge e prescribe orientation such that it points from a black vertex to a white one, turning it into a vector, or equivalently a complex number e. Denote by $\ell(e)$ the line passing through the origin and $\sqrt{\bar{e}}$ – the square root of the complex conjugate to e (the choice of the square root is not important). Careful analysis of the rearrangement on Figure 8 shows that F satisfies the following relation: for every edge $e \in \Omega$ orthogonal projections of the values of F at its endpoints on the line $\ell(e)$ coincide. We denote this common projection by $F(e)$ as it would also be given by the same formula (19) with z taken on the edge e (to be exact one has to divide by $2\cos(\pi/8)$ to arrive at the same normalization).

It turns out to be a form of *discrete analyticity*, and implies (but does not follow from) the common definition. The latter asks for the discrete version of the Cauchy-Riemann equations $\partial_{i\alpha} F = i\partial_\alpha F$ to be satisfied. Namely for every lattice square the values of F at four corners (denoted u, v, w, z in the counter-clockwise direction) should obey

$$F(z) - F(v) = i\,(F(w) - F(u)) \ .$$

Remark 13. In the complex plane *holomorphic* (i.e. having a complex derivative) and *analytic* (i.e. admitting a power series expansion) functions are the same, so the terms are often interchanged. Though the term *discrete analytic* is in wide use, in discrete setting there are no power expansions, so it would be more appropriate to speak of *discrete holomorphic* (or *discrete regular*) functions.

As discussed above, F solves a discrete version of the Riemann Boundary Value Problem (17) with $\alpha = 1 - 4k = 1/2$, which was solved in the continuum case by $\sqrt{\Phi'}$. It remains to show that as the lattice step goes to zero, properly normalized F converges to the latter.

A logical thing to do is to integrate F^2 to retrieve Φ. Unfortunately, the square of a discrete analytic function is no longer discrete analytic and so cannot be integrated. However it turns out that there is a unique function $H = \mathrm{Im} \int F^2 dz$, which is defined on the dual lattice by

$$H(b) - H(w) \ = \ |F(e)|^2 \ , \tag{20}$$

where edge e separates the centers of two adjacent squares, black b and white w.

After writing (20), one checks that

1. H is well defined and unique up to an additive constant,
2. H restricted to white (black) squares is super (sub) harmonic,
3. $H = 1$ on (counterclockwise) boundary arc ba and $H = 0$ on (counterclockwise) boundary arc ab,
4. The (local) difference between H restricted to white and to black squares tends uniformly to zero.

The properties 1, 2 are consequences of discrete analyticity: 1 a rather direct one, while 2 follows from the identity

$$\Delta H(u) \;=\; \pm\left|F(x)-F(y)\right|^2 \;,$$

where u is a center of white (black) square with two opposite corner vertices x and y (particular choice is unimportant). Definition of F implies the property 3. The property 4 easily follows from a priori estimates (namely Kaufman-Onsager-Yang results [14, 47]). In principle it should also directly follow from the discrete analyticity of F and the property 3.

We immediately infer that H converges to $\mathrm{Im}\Phi$, and after differentiating and taking a square root we obtain the following

Proposition 14. *Suppose that the lattice mesh ϵ_j goes to zero and a lattice domain Ω_j with boundary points a_j, b_j converges (in a weak sense, e.g. in Carathéodory metric) to a domain Ω with boundary points a, b as $j\to\infty$. Then away from the boundary there is a uniform convergence:*

$$\frac{1}{\sqrt{\epsilon_j}}\,F(\Omega_j,a_j,b_j,z) \rightrightarrows \sqrt{\Phi'(\Omega,a,b,z)}$$

Since by Proposition 11 the function on the right is a martingale for SLE (16/3), convergence of the interface to the Schramm-Löwner Evolution with $\kappa = 16/3$ follows.

6. Conclusion

At the moment the approach discussed above works only for a (finite) number of models. Another notable case when it works is the usual spin representation of the Ising model at critical temperature on the square lattice, pictured on Figure 2, where considering a similar observable (partition function with $+1$ monodromy, cf. (19)) leads to the martingale (16) and to Schramm-Loewner Evolution with $\kappa = 3$. Interestingly, exactly the same definition of discrete analyticity arises.

Analogously, examination of partition function with $+1$ monodromy at z for hexagonal loop models (for all values of n at criticality) suggests its convergence to conformal martingale (16). These considerations lead to a new explanation of the Nienhuis' Conjecture 2 for the critical value of x. In this case we firmly believe that our method works all the way for $n = 1$ constructing conformally invariant scaling limits for the $O(1)$ model, but convergence estimates still have to be verified.

Two parallel methods, with observables related to (15) and (16), seem specially adapted to the square lattice and the hexagonal lattice correspondingly. However, the main arguments work for a large family of four- and trivalent graphs correspondingly. So we advance towards establishing the universality conjectures.

Though only for a few models the conformal invariance was proved, the only essential missing step for the remaining ones is discrete analyticity, and it can be attacked in a great number of ways.

So from our point of view, the perspectives for establishing conformal invariance of classical 2D lattice models are quite encouraging. Moreover, we can start discussing reasons for universality, and try to construct the full loop ensemble starting from the discrete picture. The approach discussed above is rigorous, but what makes it (and the whole SLE subject) even more interesting is that while borrowing some intuition from physics, it gives a new way to approach these phenomena.

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Existence of a discrete analytic function in the Ising spin model which has potential to imply convergence of interfaces to SLE (3) was first noticed by Rick Kenyon and the author based on the dimer techniques applied to the Fisher lattice. However at the moment the Riemann Boundary Value Problem seemed beyond reach. John Cardy independently observed that (the classical version) of discrete analyticity holds for the function (19) restricted to edges.

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